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ON THE HOMOGENEOUS COUNTABLE BOOLEAN CONTACT ALGEBRA

Abstract. In a recent paper, we have shown that the class of Boolean contact algebras (BCAs) has the hereditary property, the joint embedding property and the amalgamation property. By Fraïssé's theorem, this shows that there is a unique countable homogeneous BCA. This paper investigates this algebra and the relation algebra generated by its contact relation. We first show that the algebra can be partitioned into four sets $\{0\}$, $\{1\}$, K, and L, which are the only orbits of the group of base automorphisms of the algebra, and then show that the contact relation algebra of this algebra is finite, which is the first non-trivial extensional BCA we know which has this property.

Keywords: pointless geometry, Boolean contact algebra, homogeneous structure, contact relation algebra, region connection calculus

1. Introduction

The study of connection structures started from Whitehead's 'extensional connectedness' relation [30], which itself was based on earlier work of de Laguna [4] and Nicod [25]. Since then, various axiomatizations of

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Whitehead's connectedness relation have been proposed (see e.g. Grzegorczyk [13], Clarke [3], Gerla [12], and Randell *et al.* [26]) in the context of 'pointless geometry'. Here, as in Tarski's work [28] on the geometry of solids, the basic entities are some form of spatial regions, and the usual points of geometry are obtained by second order constructions.

The present paper focuses on Boolean contact algebras proposed in Düntsch and Winter [9] as well as Dimov and Vakarelov [5], which are Boolean algebras with a binary predicate of "being in contact". A contact relation on a Boolean algebra B is, loosely speaking, a symmetric and reflexive relation $\mathcal{C}$ on the nonzero elements of B with additional compatibility properties. A BCA is called *connected* if each element $c \in B \setminus \{0, 1\}$ is in contact with its complement, and is called *extensional* if the mereological "part of" relation P — which is the Boolean order in a BCA — can be defined by $\mathcal{C}$. Models of the Region Connection Calculus (RCC) (called RCC algebras or Boolean connection algebras [27]) are special BCAs which are connected and extensional.

Using P and $\mathcal{C}$, many other relations (including the well-known RCC-8 relations) can be defined in a BCA B. We call the binary relation algebra (BRA) generated on B by $\mathcal{C}$ and P the *contact relation algebra (CRA) of* B. It is then natural to ask "How many relations are there in the CRA of B?" If B is the atomless countable Boolean algebra, and $\mathcal{C}$ is the overlap relation, it is easy to see that the CRA of B contains only seven atoms. The situation is quite complicated if the BCA is connected, especially when it is an RCC algebra. Düntsch et al. [7] show that the CRA of any RCC algebra contains at least 25 relations, which are jointly exhaustive and pairwise disjoint (JEPD), and Mormann [24] then shows that some of these relations can be further split. Li *et al.* [21] show that the CRA of the standard RCC model defined on the n-dimensional Euclidean space contains infinitely many relations for any $n \geq 1$. However, it remains open if there is an RCC algebra with a finite CRA.

In this paper we will shed some light on the above question by considering the countable homogeneous BCA. Let $\mathbf{K}$ and $\mathbf{K}_c$ be, respectively, the class of finite BCAs and the class of finite connected BCAs. In a recent work, we have shown that $\mathbf{K}$ has the amalgamation property, but $\mathbf{K}_c$ does not [6]. By Fraïssé's theorem, this implies that there is a unique countable homogeneous BCA, but no countable homogeneous connected BCA, where a BCA B is homogeneous if every isomorphism between finite substructures of B can be extended to an automorphism of B.

In this paper we investigate this homogeneous BCA and the relation algebra generated by its contact relation.

The remainder of this paper is structured as follows. We introduce basic definitions and notation that will be used in this paper in Section 2, and recall Boolean contact algebras in Section 3. Then in Section 4 we analyze the homogeneous countable BCA, and, in particular, we show that the algebra can be partitioned into four sets $\{0\}$, $\{1\}$, K, and L, which are the only orbits of the group of base automorphisms of the algebra. We investigate the contact relation algebra $\mathfrak{A}$ of the homogeneous BCA in Section 5, where we prove that the Galois closure of $\mathfrak{A}$ has 112 atoms and show that these atoms split the RCC-25 relations introduced in [7]. An outlook to further work concludes the paper.

2. Definitions and notation

For a set U we denote its group of permutations by $\mathrm{Sym}(U)$. If $R \subseteq U \times U'$ is a binary relation, we let $\mathrm{ran}(R) := \{y \in U' : xRy \text{ for some } x \in U\}$ be the *range of* R, and $\mathrm{dom}(R) := \{x \in U : xRy \text{ for some } y \in U'\}$ be the *domain of* R.

Suppose that $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ are partially ordered sets. A pair $\langle f, g \rangle$ of mappings $f\colon P \to Q$, $g\colon Q \to P$ is called a *Galois connection between P and Q* [2] if for all $x, x' \in P, y, y' \in Q$,

1. $x \leq_P x'$ implies $f(x') \leq_Q f(x)$.
2. $y \leq_Q y'$ implies $g(y') \leq_P g(y)$.
3. $x \leq_P g(f(x))$, $y \leq_Q f(g(y))$.

The operations on a Boolean algebra B are usually denoted by $+$ (join), $\cdot$ (meet) and $-$ (complement). Its constants are 0 (minimum) and 1 (maximum). B^+ is the set of all nonzero elements of B, and $\mathrm{Ult}(B)$ its set of ultrafilters.

Suppose that $\mathcal{L}$ is a countable first order language. If $\mathfrak{A}, \mathfrak{B}$ are $\mathcal{L}$-structures and $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, we write $\mathfrak{A} \leq \mathfrak{B}$. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures and $f\colon \mathfrak{A} \to \mathfrak{B}$ is a homomorphism. If $\varphi(x_0, \ldots, x_n)$ is an $\mathcal{L}$-formula with free variables among $\{x_0, \ldots, x_n\}$, then f *preserves* φ, if $\mathfrak{A} \models \varphi(a_0, \ldots, a_n)$ implies that $\mathfrak{B} \models \varphi(f(a_0), \ldots, f(a_n))$ for all $a_0, \ldots, a_n \in A$. We will only need the following instance of this concept:

LEMMA 2.1 ([14, Theorem 2.4.3]). *If f is an automorphism of the $\mathcal{L}$-structure $\mathfrak{A}$, then f preserves all $\mathcal{L}$-formulas.*

An $\mathcal{L}$-structure $\mathfrak{A}$ is called *homogeneous*, if any isomorphism between two finitely generated substructures of $\mathfrak{A}$ can be extended over all of $\mathfrak{A}$. The countable atomless Boolean algebra is atomless, and so are the rational numbers with the natural ordering. For a survey on homogeneous structure we invite the reader to consult [22]. We say that a class $\mathbf{K}$ of $\mathcal{L}$-structures has the

1. *Hereditary property* (HP) if $\mathbf{K}$ is closed under substructures.
2. *Joint embedding property* (JEP) if for any $A, B \in \mathbf{K}$, there is some $C \in \mathbf{K}$ such that A and B are embeddable into C.
3. *Amalgamation property* (AP) if for all $A, B, C \in \mathbf{K}$ such that C is (isomorphic to) a common substructure of A and B, say, with embeddings $h_B \colon C \hookrightarrow B$ and $h_A \colon C \hookrightarrow A$, there are some $D \in \mathbf{K}$ and embeddings $e_A \colon A \hookrightarrow D$ and $e_B : B \hookrightarrow D$ such that $e_A; h_A = e_B; h_B$. D is called an *amalgamated product of B and C over A*.

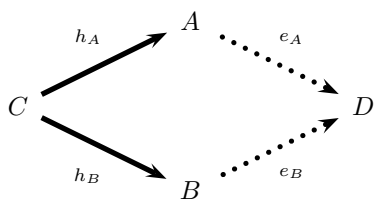


Figure 1. Amalgamation property

The amalgamation property is a prominent tool in algebraic logic and model theory. While in the former it is an algebraic counterpart of interpolation properties of a logic [23], it is used in model theory to construct countable structures with very strong properties by amalgamating (isomorphs of) its finite substructures. More formally, if $\mathfrak{D}$ is an $\mathcal{L}$-structure, we define the *age* of $\mathcal{D}$, written as $Age(\mathcal{D})$ as the class of all finitely generated $\mathcal{L}$-structures which can be embedded into $\mathcal{D}$. The following results, due to Fraïssé, describe the situation:

THEOREM 2.2 ([14, Theorem 7.1.2.]). *If $\mathbf{K}$ is a nonempty countable set of finitely generated $\mathcal{L}$-structures which has HP, JEP, and AP, then the isomorphism class of $\mathbf{K}$ is the age of some countable homogeneous $\mathcal{L}$-structure $\mathfrak{D}$ which is unique up to isomorphism.*

Such $\mathfrak{D}$ is often called the *Fraïssé limit of* $\mathbf{K}$.

Theorem 2.3 ([14, Theorem 7.1.7.]). *If $\mathfrak{D}$ is a countable homogenous $\mathcal{L}$-structure, then Age($\mathfrak{D}$) has HP, JEP, and AP.*

A theory of $\mathcal{L}$ is called *κ-categorical*, if all its models of cardinality κ are isomorphic. By Morley's celebrated result a theory which is κ-categorical in some uncountable cardinal κ is categorical in all uncountable cardinals. A single $\mathcal{L}$-structure $\mathfrak{D}$ is called κ-categorical if its first order theory is κ-categorical.

A class $\mathbf{K}$ of $\mathcal{L}$-structures is called *uniformly locally finite* if there is a function $f\colon \omega \to \omega$ such that every n-generated structure in $\mathbf{K}$ has cardinality at most $f(n)$. The class of finite Boolean algebras is uniformly locally finite, witnessed by the function $n \mapsto 2^{2^n}$.

A connection between Fraïssé limits and ω-categorical structures is given by the following:

Theorem 2.4 ([14, Theorem 7.4.1.]). *Let $\mathfrak{D}$ be the Fraïssé limit of a countable uniformly locally finite class $\mathbf{K}$ of finite $\mathcal{L}$-structures which has HP, JEP and AP. Then, $\mathfrak{D}$ is ω-categorical.*

2.1. Binary relations and their algebras

For any nonempty set U, the *full algebra of binary relations on U* is the Boolean powerset algebra of $U \times U$ with additional distinguished operations of *composition* ;[1] and *relational converse* $^\smile$. These operations are called the *RA operations*. Distinguished constants are the empty relation $\varnothing$, the universal relation V, and the identity relation $1'$. With some abuse of notation we will identify the full algebra of binary relations on U with its universe $\mathrm{Rel}(U)$.

An *algebra of binary relations* (BRA) is a subalgebra of $\mathrm{Rel}(U)$ for some nonempty U. We will denote BRAs by $\mathfrak{A}$, $\mathfrak{B}$, ... with respective universes A, B, ...

If $\mathcal{R} \subseteq \mathrm{Rel}(U)$ we denote the set of all elements of the smallest BRA over U containing all relations in $\mathcal{R}$ by $\langle \mathcal{R} \rangle$. We say that $S \in \mathrm{Rel}(U)$ is *RA-definable from* $\mathcal{R}$ if it is an element of $\langle \mathcal{R} \rangle$. The following result is decisive for relation algebras and relates RA-definability to a fragment of first order logic:

[1] We follow the work of Jónsson and Tarski [16] in writing ; instead of the often used $\circ$. The symbol ; arises by adding a "," below the absolute multiplication symbol "$\cdot$". Analogously, Jónsson and Tarski denote relative addition by $\dagger$.

THEOREM 2.5 ([29]). *If $\mathcal{R} \subseteq Rel(U)$, then $\langle \mathcal{R} \rangle$ is the set of all binary relations on U which are definable in the (language of the) relational structure $\langle U, \mathcal{R} \rangle$ by first order formulas using at most three (possibly reused) variables, two of which are free.*

If $R \in \mathrm{Rel}(U)$, $x, y \in U$, and $f \in \mathrm{Sym}(U)$, we let $R^f := \{\langle f(x), f(y)\rangle : xRy\}$. If $R = R^f$, we say that R *is invariant under* f. If $\mathfrak{A}$ is a BRA on U, the set $\{f \in \mathrm{Sym}(U) : R^f = R \text{ for all } R \in A\}$ is called the *group of base automorphisms of* A, denoted by A^ρ.

If $\mathfrak{A}$ is generated by, say, $\langle R_0, \ldots, R_n\rangle$, then the base automorphisms of $\mathfrak{A}$ are exactly the automorphisms of the first order structure $\langle A, R_0, \ldots, R_n\rangle$, since the RA operations are first order definable. The reason for using the term *base automorphism* is to distinguish these from the automorphism of the relation algebra.

If H is a group of permutations of U, and $x, y \in U$, we let

$$H_{x,y} := \{\langle f(x), f(y)\rangle : f \in H\}. \tag{2.1}$$

The sets $H_{x,y}$ are the orbits of the action of H on $U \times U$. Let H^σ be the BRA on U generated by the relations $\{H_{x,y} : x, y \in U\}$. It is well known that H^σ is a complete and atomic Boolean algebra with the atoms $H_{x,y}$ [15]. The pair $\langle \rho, \sigma\rangle$

$$\begin{aligned} {}^\rho &: \text{Subalgebras of } \mathrm{Rel}(U) \to \text{Subgroups of } \mathrm{Sym}(U), \\ {}^\sigma &: \text{Subgroups of } \mathrm{Sym}(U) \to \text{Subalgebras of } \mathrm{Rel}(U) \end{aligned}$$

is a Galois connection, and $\mathfrak{A} \leq \mathrm{Rel}(U)$ is called *Galois closed* if $\mathfrak{A} = \mathfrak{A}^{\rho\sigma}$ [15]. As far as definability and $\mathfrak{A}^\rho$ is concerned, we mention

LEMMA 2.6 ([1]). *Let $\mathfrak{A}$ be a finite subalgebra of* $\mathrm{Rel}(U)$*, and $M \subseteq U$ be not empty. Then, M is (first order) definable in $\langle U, A\rangle$ if and only if M is a union of orbits of $\mathfrak{A}^\rho$.*

LEMMA 2.7. *Suppose that $\mathfrak{A}$ is a finite subalgebra of* $\mathrm{Rel}(U)$*. Then, each first order definable binary relation over the elements of A is a union of atoms of its Galois closure $\mathfrak{A}^{\rho\sigma}$.*

PROOF. The atoms of the Galois closure $\mathfrak{A}^{\rho\sigma}$ are the orbits of the action of $\mathfrak{A}^\rho$ on $U \times U$. Suppose that R is the truth set of a first order formula $\varphi(x, y)$, i.e. $R = \{\langle a, b\rangle : \mathfrak{A} \models \varphi(a, b)\}$. Let $\langle a, b\rangle \in R$. If f is a base automorphism of $\mathfrak{A}$, then $\mathfrak{A} \models \varphi(f(a), f(b))$, i.e. $\langle f(a), f(b))\rangle \in R$. Thus, R completely contains the orbit of $\mathfrak{A}^\rho$ generated by $\langle a, b\rangle$. □

If $R, S \in \mathrm{Rel}(U)$, then the *right residual of R with respect to S*, denoted by $R \diagdown_{\mathrm{rr}} S$, is the largest binary relation T on U with $R \,;\, T \subseteq S$. It can be shown that $R \diagdown_{\mathrm{rr}} S$ is relationally definable by $R \diagdown_{\mathrm{rr}} S = -(R^{\smile} \,;\, -S)$; here $-T$ is the complement of T in $U \times U$. Furthermore,

$$x(R \diagdown_{\mathrm{rr}} S)y \iff R^{\smile}(x) \subseteq S^{\smile}(y), \tag{2.2}$$

R is called a *functional element* if $R^{\smile} \,;\, R \subseteq 1'$. The following will be helpful in the sequel:

Theorem 2.8 ([17]). *Suppose that $\mathfrak{A}$ is an atomic BRA, and P is an atom of $\mathfrak{A}$. Then we have:*

1. *$P^{\smile}$ is an atom of $\mathfrak{A}$.*
2. *If Q is a functional element of $\mathfrak{A}$ and $\mathrm{ran}(P) \cap \mathrm{dom}(Q) \neq \varnothing$, then $P \,;\, Q$ is an atom of $\mathfrak{A}$.*

3. Boolean contact algebras

A *Boolean contact algebra* $\mathfrak{B}$ (BCA) is a Boolean algebra $\langle B, +, \cdot, -, 0, 1\rangle$ — usually denoted by just B — together with a binary relation $\mathcal{C}$ on B so that the following hold for all $x, y, z \in B$:

C_0. $0(-\,\mathcal{C})x$
C_1. $x \neq 0$ implies $x\ \mathcal{C}\ x$ (domain reflexivity)
C_2. $x\ \mathcal{C}\ y$ implies $y\ \mathcal{C}\ x$ (symmetry)
C_3. $x\ \mathcal{C}\ y$ and $y \leq z$ implies $x\ \mathcal{C}\ z$. (monotonicity)
C_4. $x\ \mathcal{C}\ (y + z)$ implies ($x\ \mathcal{C}\ y$ or $x\ \mathcal{C}\ z$) (distributivity)

If $a\ \mathcal{C}\ b$, we say that *a is in contact to b* or *a is connected to b*. Additional properties appearing in the literature include

C_5. $\mathcal{C}(x) = \mathcal{C}(y)$ implies $x = y$ (extensionality)
C_6. If $(\forall z)(x\ \mathcal{C}\ z$ or $y\ \mathcal{C}\ -z)$ then $x\ \mathcal{C}\ y$ (interpolation)
C_7. $x \neq 0$ and $x \neq 1$ implies $x\ \mathcal{C}\ -x$. (connection)

The smallest contact relation is given by $x\ \mathcal{C}_{\min}\ y \iff x \cdot y \neq 0$. Usually, $\mathcal{C}_{\min}$ is called the *overlap relation*, denoted by $\mathcal{O}$; clearly, $\mathcal{O}$ satisfies C_5. The largest contact relation is $\mathcal{C}_{\max} = B^+ \times B^+$; it satisfies C_7. Given a BCA $\langle B, \mathcal{C}\rangle$ and $R \in \mathrm{Rel}(B)$ we say that R *is RA definable* if R is in the BRA generated by $\mathcal{C}$.

(3.1)	$P := \leq \cap\ (B \setminus \{0\} \times B \setminus \{0\})$	part of
(3.2)	$PP := P \cap -1'$	proper part of
(3.3)	$O := P^{\smile}\ ;\ P$	overlap
(3.4)	$T := P\ ;\ P^{\smile}$	underlap
(3.5)	$PO := O \cap -(P \cup P^{\smile})$	partial overlap
(3.6)	$EC := \mathcal{C} \cap -O$	external contact
(3.7)	$TPP := PP \cap (EC\ ;\ EC)$	tangential proper part
(3.8)	$NTPP := PP \cap -TPP$	non-tangential proper part
(3.9)	$DC := -\mathcal{C}$	disconnected
(3.10)	$DD := -(O \cup T \cup 1')$	complementation
(3.11)	$ECD := \mathcal{C} \cap DD$	externally connected and $x + y = 1$
(3.12)	$ECN := EC \cap -ECD$	$xECy$ and $x + y \neq 1$.
(3.13)	$HL := ECN \cap (ECN \diagdown_{\mathrm{rr}} O)$	hole relation

Table 1. Relations defined from $\mathcal{C}$ and $\leq$

Let $\overline{B} := B \setminus \{0, 1\}$. It was shown in [8] that $\{\langle 0, 1\rangle\}$ and $\{\langle 1, 0\rangle\}$ are RA definable in any BCA, and thus, so is $\overline{B} \times \overline{B}$. Table 1 shows some relations RA definable from $\mathcal{C}$ and $\leq$.

We say that x *has a hole* if $x \in \mathrm{ran}(HL)$; otherwise, we call x *solid* [24].

If B is a finite Boolean algebra, and R is a reflexive and symmetric relation on $At(B)$, then R determines a contact relation $\mathcal{C}$ on B by $a\ \mathcal{C}\ b$ if and only if there are atoms u, v such that $u \leq a,\ v \leq b$ and uRv, and each contact relation on B arises in this way [10, 11].

EXAMPLE 1 ([20]). Let B_0 be the Boolean algebra generated by its set of atoms $At(B_0) = \{a, b, c, d, e\}$. Suppose that R is the relation on $At(B_0)$

$$\{\langle a, b\rangle, \langle b, e\rangle, \langle c, e\rangle, \langle d, e\rangle\}.$$

Let $x\ \mathcal{C}_0\ y$ if and only if there are atoms s, t such that $s \leq x,\ t \leq y$, and $\langle s, t\rangle \in R \cup R^{\smile} \cup 1'$. Then, $\mathfrak{B}_0 := \langle B_0, \mathcal{C}_0\rangle$ is a BCA which satisfies C_7, none of the relations in Table 1 is empty, and $aHLb$, i.e. a is a hole of b. Furthermore, each BCA which satisfies C_5 and C_7 contains a substructure isomorphic to $\mathfrak{B}_0$. A sketch of $\mathfrak{B}_0$ is shown in Figure 2,

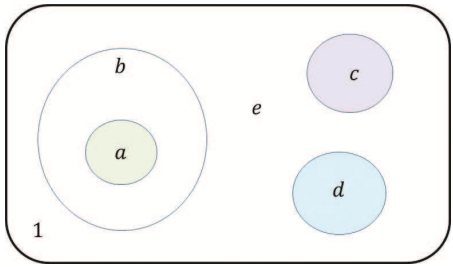


Figure 2. The example from Li and Ying [20]

where a, b, c, d, e are nonempty regular closed sets in the real plane $\mathbb{R}^2$, which is also the largest element 1 of B_0. □

The following results will be useful in the sequel:

Lemma 3.1 ([8]). *Suppose that $\langle B, \mathcal{C}\rangle$ is a BCA. The following are equivalent:*

1. *$\mathcal{C}$ satisfies C_5.*
2. *For all $x \in B, x \neq 1$, there is some $y \in B, y \neq 0$, such that $x(-\mathcal{C})y$.*
3. *The Boolean order $\leq$ is the residual $\mathcal{C} \setminus_{\mathrm{rr}} \mathcal{C}$.*

Lemma 3.2 ([18]). *Suppose that $\langle B, \mathcal{C}\rangle$ is a BCA, and set $A := \{x \in B : x(-\mathcal{C}) - x\}$. Then, A is a Boolean subalgebra of B, and the restriction of $\mathcal{C}$ to A is the overlap relation.*

The following facts are well known for BCAs satisfying C_5 and C_7; here, we generalize them to arbitrary BCAs:

Lemma 3.3. *Let $\langle B, \mathcal{C}\rangle$ be a BCA.*

1. *If $a \lesssim b$ and $b(-\mathcal{C}) - b$, then $aNTPPb$. Furthermore, b is solid.*
2. *If $b\ \mathcal{C} - b$, then $aECNb \iff aTPP - b$.*
3. *If $a \lesssim b$, then $aNTPPb \iff aDC - b$.*
4. *$aHLb \iff aECNb$ and $aNTPP(a + b)$.*

Proof. 1. Assume that $aTPPb$; then, there exists some $s \in B$ such that $aECsECb$. Now, $sECb$ implies sCb and $s \cdot b = 0$, i.e. $s \leq -b$. By C_3 we have $-b\ \mathcal{C}\ b$, contradicting the hypothesis. If $sHLb$ for some s, then, in particular, $sECNb$ which contradicts $-b(-\mathcal{C})b$.

2. "⇒": Since $aECNb$ and $bEC - b$ by the hypothesis, we obtain $a \lesssim -b$ and $aECbEC - b$, which implies $aTPP - b$.

"⇐": Suppose that $aTPP-b$. Then, $a \neq -b, a \cdot b = 0$, and $aECsEC-b$ for some $s \in B$. Now, $s \leq b$ and $aECs$ together with C_3 imply $a \mathcal{C} b$. Since $a \leq\!\!\!\!\neq -b$, we have $a+b \neq 1$ and $a \cdot b = 0$.

3. "⇒": Let $aNTPPb$. If $a = 0$, then $aDC-b$ by C_0. Suppose that $a \neq 0$. If $bDC-b$, then $a \leq b$ and C_3 imply $aDC-b$. If $b \mathcal{C} -b$, assume that $a \mathcal{C} -b$. Then, $a \neq 0$ and $a \leq b$ imply $aEC-b$. Thus, $aEC-bECb$, and it follows that $aTPPb$, contradicting our assumption.

"⇐": Let $a \leq b$ and $aTPPb$. Then, there is some $s \in B$ such that $aECsECb$. It follows from $aECs$ that aCs, and $sECb$ implies $s \leq -b$. Thus, C_3 implies that $a \mathcal{C} -b$.

4. By definition of the residual, $xHLy \iff xECNy$ and $EC(x) \subseteq O(y)$.

"⇒": Let $aHLb$, and assume that $aTPP(a+b)$; then, there is some $s \in B$ such that $aECsEC(a+b)$. It follows from $sEC(a+b)$ that $s \cdot b = 0$, and, together with $aECs$ we obtain $s \in EC(a)$ and $s \notin O(b)$, contradicting that $aHLb$.

"⇐": Suppose that $aECNb$, $aNTPP(a+b)$, and assume that $aECs$ and $s \cdot b = 0$ for some $s \in B$. Then, $sEC(a+b)$ by $aECs$ and C_3, and thus, $aECsEC(a+b)$. It follows that $aTPP(a+b)$, contradicting the hypothesis. Hence, $EC(a) \subseteq O(b)$, and therefore, $aHLb$. □

4. The countable homogeneous BCA

In [6] we have shown

THEOREM 4.1. *The class* $\mathbf{K}^0$ *of Boolean contact algebras has HP, JEP, and AP.*

PROOF. We just indicate the construction for AP, details can be found in [6]. Suppose that $\mathfrak{A} = \langle A, \mathcal{C}_A \rangle$ and $\mathfrak{B} = \langle B, \mathcal{C}_B \rangle$ are two BCAs and $\mathfrak{C} = \langle C, \mathcal{C}_C \rangle$ is isomorphic to a common substructure of $\mathfrak{A}$ and $\mathfrak{B}$. Let D be the Boolean amalgamated free product of A and B over C (see [19]). Suppose w.l.o.g. that $A, B \leq D$; then, $A \cap B = C$. Define a binary relation R on $\mathrm{Ult}(D)$ as follows:

$$\begin{aligned} \langle H, H' \rangle \in R \iff & (H \cap A) \times (H' \cap A) \subseteq \mathcal{C}_A \text{ and} \\ & (H \cap B) \times (H' \cap B) \subseteq \mathcal{C}_B. \end{aligned}$$

For $x, y \in D$ let $x \; \mathcal{C}_D \; y$ if and only if there are $H, H' \in \mathrm{Ult}(D)$ such that $x \in H$, $y \in H'$, and $\langle H, H' \rangle \in R$. Then, $\mathcal{C}_D$ is a contact relation on D which extends both $\mathcal{C}_A$ and $\mathcal{C}_B$. □

Since the axioms of BCAs are universal, $\mathbf{K}^0$ has the HP, and since the two element BCA is embeddable into each BCA, the class $\mathbf{K}^0$ also has the JEP.

Consider the class $\mathbf{K}$ of all finite BCAs. Since each n-generated BCA has cardinality at most 2^{2^n}, $\mathbf{K}$ is uniformly locally finite. Furthermore, there are infinitely many finite isomorphism types of BCAs. Therefore, by Fraïssé's Theorem 2.2 and Theorem 2.4 we obtain

Theorem 4.2. *There is a countable homogeneous and ω-categorical BCA $\mathfrak{B} = \langle B, \mathcal{C} \rangle$ such that each at most countable BCA is isomorphic to a substructure of $\mathfrak{B}$.*

In the sequel we let $\mathfrak{B}$ be this algebra, and H its group of automorphisms. Furthermore, set

$$V := B \times B, \quad K := \{x \in B : x \; \mathcal{C} \; -x\},$$
$$L := \{x : x \neq 0, 1 \text{ and } x(-\mathcal{C})-x\}.$$

Below we suppose that the finite BCAs we consider are substructures of $\mathfrak{B}$. We start by exhibiting the orbits of H:

Lemma 4.3. *The orbits of H are $\{0\}, \{1\}, K$, L, and both K and L are infinite.*

Proof. Since 0 and 1 are constants, $\{0\}$ and $\{1\}$ are orbits of H; furthermore, $\{\{0\}, \{1\}, K, L\}$ is a partition of B. Let $x, y \in K$, $\mathfrak{B}_0$ be the substructure of $\mathfrak{B}$ with universe $\{0, 1, x, -x\}$ and $\mathfrak{B}_1$ be the substructure of $\mathfrak{B}$ with universe $\{0, 1, y, -y\}$. Since $x \; \mathcal{C} \; -x$ and $y \; \mathcal{C} \; -y$, the mapping induced by $x \mapsto y$ is an isomorphism between $\mathfrak{B}_0$ and $\mathfrak{B}_1$. Since $\mathfrak{B}$ is homogeneous, there is some $f \in H$ such that $f(x) = y$. A similar argument can be used when $x, y \in L$, and, clearly, no automorphism can map an element of K to an element of L.

L is infinite since $\mathbf{K}$ contains each finite Boolean algebra with the overlap relation $\mathcal{C}_{\min}$ in which no element is connected to its complement. K is infinite, since $\mathbf{K}$ contains each finite Boolean algebra with contact relation $\mathcal{C}_{\max}$ in which each element $x \notin \{0, 1\}$ is in contact with its complement. □

Observe that $B_L := L \cup \{0, 1\}$ is the universe of a Boolean subalgebra of B by Lemma 3.2, and that the restriction of $\mathcal{C}$ to B_L is the overlap relation. Denote this substructure of $\mathfrak{B}$ by $\mathfrak{B}_L$. K is, of course, not a Boolean algebra. Instead, we have the following result:

THEOREM 4.4. *K generates B as a Boolean algebra.*

PROOF. Let $\mathfrak{B}_0$ be the Boolean algebra with atoms a, b, c and let $\mathcal{C}_0$ be induced by $a\ \mathcal{C}_0\ b$. Then, $a + c, b + c \in K$, and $(a + c) \cdot (b + c) = c \in L$. If $\varphi(x)$ says that $x(-\ \mathcal{C}) - x$ and there are y, z such that both y and z are in contact with their respective complements and $y \cdot z = x$, then $\mathfrak{B}_0 \models \varphi(c)$, and thus, $\mathfrak{B} \models \varphi(u)$ for each $u \in L$. Hence, each element of L is the product of two elements of K. □

Recall that a subset M of a Boolean algebra is called *dense*, if for every $a \in B, a \neq 0$, there is some $m \in M, m \neq 0$, such that $m \leq a$.

LEMMA 4.5. *Both K and L are dense in $\mathfrak{B}$.*

PROOF. If $a \in K$, consider the following formula $\varphi(x)$:

$$(\exists y)[\neg(y = 0) \wedge (y < x) \wedge \neg(y\ \mathcal{C} - y)].$$

Clearly, $\varphi(x)$ says that there is a nonzero element of L strictly below x which is not in contact to its complement. Let $\mathfrak{B}_0$ be a subalgebra of $\mathfrak{B}$ with atoms $\{r, s, t, u\}$, and $\mathcal{C}_0$ be the contact relation on B_0 defined by

$$x\ \mathcal{C}_0\ y \iff x = y \text{ or } (r \leq x \text{ and } s \leq y) \text{ or } (s \leq x \text{ and } r \leq y).$$

Then, $(r + t)\ \mathcal{C}_0\ -(r + t)$, $0 \lneq t \lneq r + t$, and $t(-\ \mathcal{C}_0) - t$. Thus, $\mathfrak{B} \models \varphi(r + t)$ and since K is an orbit of G, we also have $\mathfrak{B} \models \varphi(a)$. For $a \in L$ use the formula $\psi(x)$

$$(\exists y)[\neg(y = 0) \wedge (y < x) \wedge (y\ \mathcal{C} - y)].$$

Then, $\mathfrak{B} \models \psi(r + s)$, and thus, again by homogeneity, $\mathfrak{B} \models \psi(a)$. □

Next, we will show that $\mathcal{C}$ satisfies C_5. This will follow at once from the following observation:

LEMMA 4.6. *If $x \in K$, there is some $y \in K$ such that $x(-\ \mathcal{C})y$.*

Proof. Let $\mathfrak{B}_0$ be the BCA of Example 1 considered w.l.o.g. as a substructure of $\mathfrak{B}$. Here, c is in contact to $-c$ and b is connected to its complement, but not connected to c. Therefore, if $\varphi(x)$ is the formula

$$x\ \mathcal{C}\ -x \wedge (\exists y)[y \neq 0 \wedge y \neq 1 \wedge y\ \mathcal{C}\ -y \wedge \neg(x\ \mathcal{C}\ y)],$$

then $\mathfrak{B}_0 \models \varphi(b)$, and therefore, $\mathfrak{B} \models \varphi(b)$ since $\varphi(x)$ is existential. By homogeneity and the fact that K is an orbit of G, each $u \in K$ satisfies $\varphi(u)$. □

Theorem 4.7. *$\mathcal{C}$ satisfies C_5.*

Proof. We will use Lemma 3.1(2). If $a \in L$, then a is not in contact with its complement, and if $a \in K$ there is some b with $a(-\mathcal{C})b$ by the preceding lemma. □

Observe that $\mathcal{C}$ does not satisfy C_7, since $L \neq \varnothing$. Furthermore, the Boolean order is RA-definable by Lemma 3.1.

Recall that an element of B is called *solid*, if it is not in the range of the relation HL defined in (3.13).

Theorem 4.8. *Each element of L is solid and each element of K has a hole.*

Proof. If a is not solid, then it must be connected to its complement; hence, each element of L is solid. Suppose that $\varphi(x)$ says that x has a hole, i.e. $x \in \mathrm{ran}(HL)$; then, $\varphi(x)$ can be chosen to be existential. Looking again at the BCA $\mathfrak{B}_0$ of Example 1, we see that $\mathfrak{B}_0 \models \varphi(b)$, and therefore, $\mathfrak{B} \models \varphi(b)$. Homogeneity implies that $\mathfrak{B} \models \varphi(u)$ for all $u \in K$. □

Call an element $b \in B$ *disconnected* if there are $a_0, a_1 \in B$ such that $a_0 + a_1 = b$ and $a_0 DC a_1$.

Theorem 4.9. *Each element of B is disconnected.*

Proof. Clearly, 0 is disconnected, and, since $L \neq \varnothing$, so is 1. Consider $\varphi(x)$: $(\exists y, z)[y + z = x \wedge yDCz]$. If $\mathfrak{B}_0$ is the BCA with three atoms a, b, c and the overlap relation, then $\mathfrak{B}_0 \models \varphi(a+b)$, and thus $\mathfrak{B}_0 \models \varphi(u)$ for all elements of L. If $\mathcal{C}_0$ is the contact relation on B_0 induced by $a\ \mathcal{C}_0\ b$, then $\mathfrak{B}_0 \models \varphi(b+c)$, and $b + c \in K$. Thus, each element of K is also disconnected. □

5. The relation algebra generated by $\mathcal{C}$

Let $\mathfrak{A} = \langle A, \cap, \cup, -, \varnothing, B \times B, \ ; \ , \breve{} , 1', \langle \mathcal{C} \rangle \rangle$ be the BRA generated on B by $\mathcal{C}$; in other words, the Boolean operations on $\mathfrak{A}$ are the set operations, its smallest element is $\varnothing$, and its largest element is $B \times B$. We first show that the group of base automorphisms of $\mathfrak{A}$ is just the automorphism group H of $\mathfrak{B}$:

THEOREM 5.1. $\mathfrak{A}^\rho = H$.

PROOF. Let $f \in \mathfrak{A}^\rho$. Then, f preserves $\mathcal{C}$, and we need to show that f is a Boolean homomorphism. Since $\mathcal{C}$ satisfies C_5 by Theorem 4.7, the Boolean order $\leq$ is RA definable from $\mathcal{C}$ by Lemma 3.1. Hence, f is an order isomorphism, and therefore it preserves all Boolean operations. Conversely, each automorphism f of $\mathfrak{B}$ is in particular a homomorphism with respect to $\mathcal{C}$. This means that P and $\mathcal{C}$, hence all relations in $\mathfrak{A}$, are invariant under f. Therefore, $f \in \mathfrak{A}^\rho$. □

THEOREM 5.2. $\mathfrak{A}$ *is finite.*

PROOF. Since $\mathfrak{B}$ is ω-categorical, its group H of automorphisms has only finitely many orbits in its action on each B^n by the Engeler, Ryll-Nardzewski, Svenonius Theorem (see e.g. [14, Theorem 7.3.1]). In particular, the action of H on $B \times B$ has only finitely many orbits, and thus, H^σ is finite. The claim now follows from Theorem 5.1 and the fact that $\mathfrak{A} \leq \mathfrak{A}^{\rho\sigma} = H^\sigma$. □

In view of the fact that each countable BCA can be embedded into $\mathfrak{B}$, and there are such algebras in which the relation algebra generated by the contact relation is infinite [21], the fact that $\mathfrak{A}$ is finite is somewhat surprising. Such a situation, however, is not uncommon: For example, the relation algebra generated by the natural order $\leq$ on $\mathbb{Q}$ is finite, while the restriction of $\leq$ to $\mathbb{Z}$ generates an infinite relation algebra.

The aim of this section is to give a closer examination of the atoms of $\mathfrak{A}^{\rho\sigma}$. By $\mathfrak{A}^\rho = H$ (Theorem 5.1) we know each atom of $\mathfrak{A}^{\rho\sigma}$ has the form $H_{x,y}$ with $x, y \in B$. Take $a \in K$ and $b \in L$. Then from the description of the orbits of H in Lemma 4.3 we conclude that

$$H_{0,0} = \{\langle 0,0\rangle\},\ H_{1,1} = \{\langle 1,1\rangle\},\ H_{0,1} = \{\langle 0,1\rangle\},\ H_{1,0} = \{\langle 1,0\rangle\} \tag{5.1}$$

$$H_{0,a} = \{0\}\times K,\ H_{a,0} = K\times\{0\},\ H_{0,b} = \{0\}\times L,\ H_{b,0} = L\times\{0\} \tag{5.2}$$

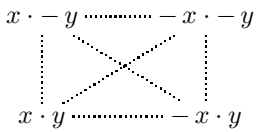


Figure 3. Atomic configurations

$$(5.3)\quad H_{1,a} = \{1\}\times K,\ H_{a,1} = K\times\{1\},\ H_{1,b} = \{1\}\times L,\ H_{b,1} = L\times\{1\}$$

are all atoms of $\mathfrak{A}^{\rho\sigma}$ and are contained in $\mathfrak{A}$. All that is left is to exhibit the atoms of $\mathfrak{A}^{\rho\sigma}$ below $\overline{B} \times \overline{B}$. To this end, we introduce for each atomic relation $H_{x,y}$ a unique graph representation, and show that two atomic relations are the same if and only if they have the same graph representations.

5.1. Atomic configurations

Given $x, y \in \overline{B}$, let $B_{x,y}$ be the (Boolean) subalgebra of B generated by $\{x, y\}$, and $\mathfrak{B}_{x,y}$ the substructure of $\mathfrak{B}$ induced by $\mathcal{C}$. Note that each element in $N(x, y) := \{x \cdot y, -x \cdot y, x \cdot -y, -x \cdot -y\}$ is either 0 or an atom of $\mathfrak{B}_{x,y}$ and $\mathfrak{B}_{x,y}$ has no other atoms. Furthermore, since a contact relation on a finite Boolean algebra is determined by its action on the atoms, $\mathfrak{B}_{x,y}$ can be described by a graph on the nodes $N(x, y)$, where two points are connected by an edge if and only if they are (externally) connected in $\mathcal{C}$, see Figure 3. Write $G(x, y)$ for the graph as in Figure 3, and $E(x, y)$ for the set of edges of $G(x, y)$. We note that $E(x, y)$ contains at most six elements. In what follows, we call $G(x, y)$ the *atomic configuration* of $H_{x,y}$. We will show that each atom $H_{x,y}$ of $\mathfrak{A}^{\rho\sigma}$ is uniquely determined by its atomic configuration.

THEOREM 5.3. *Suppose a, b, a', b' are elements in B. Then $H_{a,b} = H_{a',b'}$ if and only if there is a graph isomorphism $f\colon G(a, b) \to G(a', b')$ such that $a^t \cdot b^t = 0$ if and only if $a'^t \cdot b'^t = 0$ and $f(a^t \cdot b^t) = a'^t \cdot b'^t$ for $a^t \in \{a, -a\}$ and $b^t \in \{b, -b\}$.*

PROOF. If $H_{a,b} = H_{a',b'}$, then there is some $f \in \mathfrak{A}^\rho = H$ such that $f(a) = a'$ and $f(b) = b'$. Since f is an automorphism of $\mathfrak{B}$ it follows that $f\colon \mathfrak{B}_{a,b} \to \mathfrak{B}_{a',b'}$ is an isomorphism. It is clear that f, when restricted to $N(a, b)$, is a graph isomorphism that satisfies the condition in the theorem.

On the other hand, suppose there exists a graph isomorphism $f\colon G_{a,b} \to G_{a',b'}$ such that $a^t \cdot b^t = 0$ if and only if $a'^t \cdot b'^t = 0$ and $f(a^t \cdot b^t) = a'^t \cdot b'^t$ for $a^t \in \{a, -a\}$ and $b^t \in \{b, -b\}$. It is clear that f can be uniquely extended to an isomorphism between $\mathfrak{B}_{a,b}$ and $\mathfrak{B}_{a',b'}$. Also write f for this isomorphism between $\mathfrak{B}_{a,b}$ and $B_{a',b'}$. Then f satisfies the condition $f(a) = a'$ and $f(b) = b'$. By the homogeneity of $\mathfrak{B}$, we know f can be further extended to an automorphism f of $\mathfrak{B}$. Thus, $\langle a', b'\rangle \in H_{a,b}$ and therefore, $H_{a,b} = H_{a',b'}$. □

By the above theorem, each atom is uniquely determined by its atomic configuration. For any $x = y \in \overline{B}$, exactly two nodes are 0, i.e. $x \cdot y^*$ and $x^* \cdot y$. There are only two atomic configurations in this case, both with nodes $\{x, x^*\}$. If x is connected to $-x$, then both x and y are elements in K; otherwise, both are elements in L. Therefore, the two corresponding atomic relations in $\mathfrak{A}^{\rho\sigma}$ split the identity relation $1'$ (when restricted to $\overline{B}$) into $(K \times K) \cap 1'$ and $(L \times L) \cap 1'$. Similarly, for any $x = -y \in \overline{B}$, there are only two atomic configurations in this case, both with nodes $\{x, y\}$, and the two corresponding atomic relations in $\mathfrak{A}^{\rho\sigma}$ split the relation DD into $ECD = (K \times K) \cap DD$ and $(L \times L) \cap DD$. In fact, these four atoms can be directly computed as follows:

$$\begin{aligned} H_{a,a} &= (K \times K) \cap 1', \quad H_{b,b} = (L \times L) \cap 1', \\ H_{a,-a} &= (K \times K) \cap DD, \quad H_{b,-b} = (L \times L) \cap DD, \end{aligned} \tag{5.4}$$

where $a \in K$ and $b \in L$. It is clear that these relations are also in $\mathfrak{A}$.

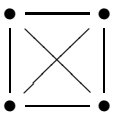
There are $2^{\binom{3}{2}} \times 4 = 32$ atomic configurations with exactly three nodes, and $2^{\binom{4}{2}} = 64$ atomic configurations with four nodes. Putting all of this together results in the following:

THEOREM 5.4. *$\mathfrak{A}^{\rho\sigma}$ has 112 atoms, where exactly 100 are below $\overline{B} \times \overline{B}$.*

Before giving a closer examination of these atoms below $\overline{B} \times \overline{B}$, we first summarize some useful properties of atomic configurations.

NOTATION. When representing an atomic configuration G, we often omit the names of the four nodes, and address the four nodes as, respectively, the bottom left node, the bottom right node, the top left node, the top right node, and the six edges as, respectively, the left edge, the right edge, the top edge, the bottom edge, the major diagonal, and the minor

diagonal of G. For example, in case all the six edges are in the graph, i.e. every two elements are connected, we have the following representation:



Let $a, b \in \overline{B}$. Recall that $H_{a,b}$ is the atomic relation in $\mathfrak{A}^{\rho\sigma}$ that contains (a, b). Then

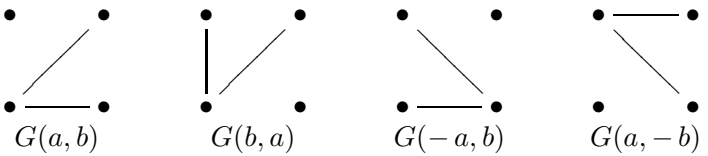
$$H_{b,a} = H_{a,b}{}^{\smile} \tag{5.5}$$

$$H_{-\,a,b} = DD; H_{a,b} \tag{5.6}$$

$$H_{a,-\,b} = H_{a,b}; DD \tag{5.7}$$

As far as configurations are concerned, $G(b, a)$ can be obtained by mirroring $G(a, b)$ along the major diagonal; $G(-a, b)$ can be obtained by mirroring $G(a, b)$ along the left edge; and $G(a, -b)$ can be obtained by mirroring $G(a, b)$ along the bottom edge (see (5.8) for illustrations). Four other atomic relations $H_{-\,a,-\,b}$, $H_{b,-\,a}$, $H_{-\,b,a}$, and $H_{-\,b,-\,a}$ can be obtained from $H_{a,b}$ similarly. Therefore, given $G(a, b)$, we can define seven other atomic relations using composition (with DD) and conversion.

(5.8)



Proposition 5.5. *Given $a, b \in \overline{B}$, if $H_{a,b}$ is a relation in $\mathfrak{A}$, then so are $H_{-\,a,b}$, $H_{a,-\,b}$, $H_{-\,a,-\,b}$, $H_{b,a}$, $H_{b,-\,a}$, $H_{-\,b,a}$, and $H_{-\,b,-\,a}$.*

Recall $K = \{a \in \overline{B} : a\ \mathcal{C}\ -a\}$ and $L = \{a \in \overline{B} : a(-\mathcal{C}) - a\}$. Given an atomic relation $H_{a,b}$ in $\mathfrak{A}^{\rho\sigma}$, we can determine if a (b, resp.) is in K by examining the atomic configuration $G(a, b)$.

Proposition 5.6. *Let $a, b \in \overline{B}$. Suppose $G(a, b)$ is the atomic configuration of the atom $H_{a,b}$. Then*

(1) *$a \in K$ if the bottom edge or the top edge is in $E(a, b)$;*
(2) *$b \in K$ if the left edge or the right edge is in $E(a, b)$;*
(3) *$a, b \in K$ if the major diagonal or the minor diagonal is in $E(a, b)$.*

As a corollary, we know

COROLLARY 5.7. *Let $a, b \in \overline{B}$. Suppose $G(a, b)$ is the atomic configuration of the atom $H_{a,b}$. Then*

(1) *$a \in L, b \in K$ if and only if $E(a, b)$ is nonempty and contains at most the left and the right edges.*
(2) *$a \in K, b \in L$ if and only if $E(a, b)$ is nonempty and contains at most the bottom and the top edges.*
(3) *$a, b \in L$ if and only if $E(a, b)$ contains no edge.*

Let

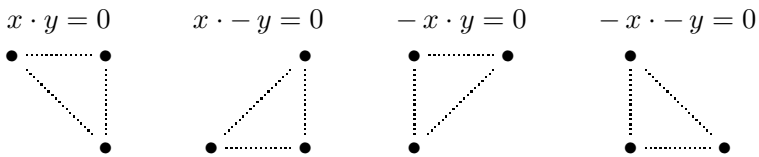
$$R = ECN \,;\, TPP, \tag{5.9}$$

$$S = (ECN \,;\, TPP) \,;\, ECD. \tag{5.10}$$

It is easy to prove that $S = TPP; TPP^{\smile}$ and $S^{\smile} = TPP^{\smile}; TPP$. Düntsch et al. [7] defined a set of 25 necessary relations in every RCC algebra (see Table 2). In what follows, we call these the RCC-25 relations. In the following two subsections, we examine the atomic configurations with three and, respectively, four nodes. For each atomic configuration $G(a, b)$, we will decide the RCC-25 relation that related a to b.

5.2. Atomic configurations with three nodes

If an atomic configuration has three nodes, then either $x \cdot y = 0$, or $x \cdot -y = 0$, or $-x \cdot y = 0$, or $-x \cdot -y = 0$. That is, each atomic configuration has one of the following forms



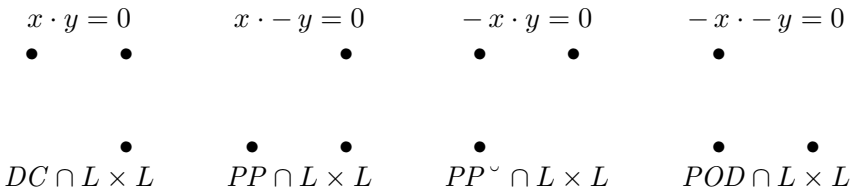
Since there are eight different configurations in each case, we have in total $4 \times 8 = 32$ atomic configurations that have three nodes. We now examine these configurations in detail.

Recall that $L \times L$, $L \times K$, $K \times L$, and $K \times K$ are all relations in $\mathfrak{A}$, hence relations in $\mathfrak{A}^{\rho\sigma}$.

$$
\begin{aligned}
1' & \\
NTPP & \\
NTPP^{\smile} & \\
DD & \\
DC & \\
ECNA &= ECN \cap S \\
ECNB &= ECN \cap -S \\
TPPA &= TPP \cap R \\
TPPB &= TPP \cap -R \\
TPPA^{\smile} &= TPP^{\smile} \cap R^{\smile} \\
TPPB^{\smile} &= TPP^{\smile} \cap -R^{\smile} \\
PODYA &= POD \cap -(ECN; NTPP) \cap (TPP^{\smile}; TPP) \\
PODYB &= POD \cap -(ECN; NTPP) \cap -(TPP^{\smile}; TPP) \\
PODZ &= POD \cap (ECN; NTPP) \\
PONXA1 &= PON \cap R \cap R^{\smile} \cap S \cap S^{\smile} \\
PONXA2 &= PON \cap R \cap R^{\smile} \cap S \cap -S^{\smile} \\
PONXB1 &= PON \cap R \cap R^{\smile} \cap -S \cap S^{\smile} \\
PONXB2 &= PON \cap R \cap R^{\smile} \cap -S \cap -S^{\smile} \\
PONYA1 &= PON \cap -R \cap R^{\smile} \cap S \cap S^{\smile} \\
PONXA2 &= PON \cap -R \cap R^{\smile} \cap S \cap -S^{\smile} \\
PONYA1^{\smile} &= PON \cap R \cap -R^{\smile} \cap S \cap S^{\smile} \\
PONYA2^{\smile} &= PON \cap R \cap -R^{\smile} \cap S \cap -S^{\smile} \\
PONYB &= PON \cap -R \cap R^{\smile} \cap -S \\
PONYB^{\smile} &= PON \cap R \cap -R^{\smile} \cap -S \\
PONZ &= PON \cap -R \cap -R^{\smile}
\end{aligned}
$$

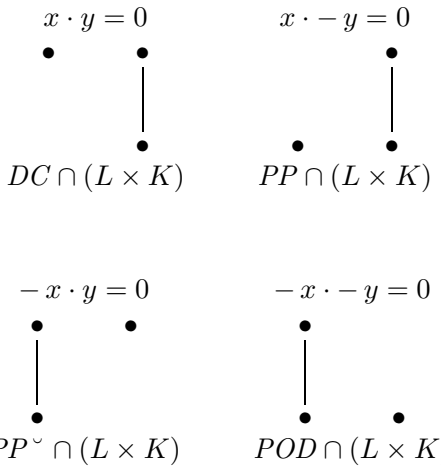
Table 2. RCC-25 relations

We first consider those atoms of $\mathfrak{A}^{\rho\sigma}$ which are contained in $L \times L$. Let $H_{a,b}$ be an atomic relation of $\mathfrak{A}^{\rho\sigma}$ that is contained in $L \times L$. If $G_{a,b}$ contains only three nodes, then because $a, b \in L$, we have by Corollary 5.7 that $E(a, b)$ contains no edge. Therefore, there are exactly four atomic configurations in this case:



It is clear that these four atomic relations are the intersections of $L \times L$ with, respectively, DC, PP, $PP^{\smile}$, and POD, hence are all relations in $\mathfrak{A}$.

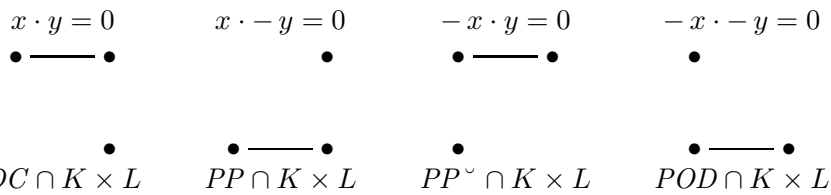
Second, we consider those atoms which are contained in $L \times K$. Let $H_{a,b}$ be an atomic relation of $\mathfrak{A}^{\rho\sigma}$ that is contained in $L \times K$. If $G_{a,b}$ contains only three nodes, then because $a \in L$ and $b \in K$, we have by Corollary 5.7 that $E(a, b)$ is nonempty and contains at most the left edge or the right edge. Therefore, there are exactly four atomic configurations in this case:



These four atomic relations are the intersections of $L \times K$ with, respectively, DC, PP, $PP^{\smile}$, and POD, hence are all relations in $\mathfrak{A}$.

Third, we consider those atoms which are contained in $K \times L$. Let $H_{a,b}$ be an atomic relation of $\mathfrak{A}^{\rho\sigma}$ that is contained in $K \times L$. If $G_{a,b}$ contains only three nodes, then because $a \in K$, $b \in L$, we have by Corollary 5.7 that $E(a, b)$ is nonempty and contains at most the left edge

or the right edge. Therefore, there are exactly four atomic configurations in this case:

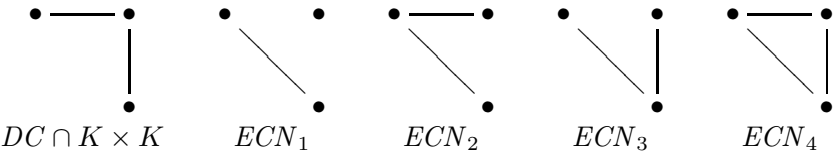


These four atomic relations are the intersections of $K \times L$ with, respectively, DC, PP, $PP^{\smile}$, and POD, hence are all relations in $\mathfrak{A}$.

Last, we consider those atoms which are contained in $K \times K$. Let $H_{a,b}$ be an atomic relation of $\mathfrak{A}^{\rho\sigma}$ that is contained in $K \times K$. If $G_{a,b}$ contains only three nodes, then because $a, b \in K$, we have by Corollary 5.7 that $E(a, b)$ contains the major diagonal, or the minor diagonal, or contains the non-diagonal edges. Therefore, there are exactly twenty atomic configurations in this case.

Case 1. Suppose $x \cdot y = 0$. Then $xDCy$ or $xECNy$.

(5.11)



Note that if the minor diagonal is not in $E(x, y)$, then $xDCy$. The other four atomic configurations correspond to ECN relations. Recall that $xDC - x \cdot -y$ if and only if $xNTPP(x + y)$. Consider the second graph, in which we have $xNTPP(x + y)$, and $yNTPP(x + y)$, hence $xHLy$ and $yHLx$. On the other hand, if $xHLy$ and $yHLx$, then we have $xDC - y \cdot -y$ and $yDC - x \cdot -y$. That is, $G(x, y)$ is the same as the second graph. Therefore, we know

$$ECN_1 = ECN \cap HL \cap HL^{\smile}. \tag{5.12}$$

Similarly, we can show that

$$ECN_2 = ECN \cap -HL \cap HL^{\smile} \tag{5.13}$$

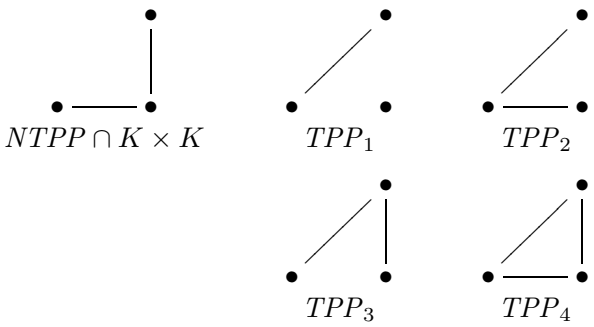
$$ECN_3 = ECN \cap HL \cap -HL^{\smile} \tag{5.14}$$

$$ECN_4 = ECN \cap -HL \cap -HL^{\smile}. \tag{5.15}$$

Therefore all relations in (5.11) are in $\mathfrak{A}$. Note if $(x, y) \in ECN_4$, then we have $x\,TPP(x + y)$ and $y\,TPP(x + y)$. Therefore $(x, y) \in ECN \cap (TPP; TPP^{\smile}) := ECNA$. On the other hand, if $(x, y) \in ECNA$, then we can also obtain that $x\,TPP(x+y)$ and $y\,TPP(x+y)$, which implies that $G(x, y)$ is the same as the atomic configuration of ECN_4. So we know $ECN_4 = ECNA$, and $ECNB = ECN_1 \cup ECN_2 \cup ECN_3$.

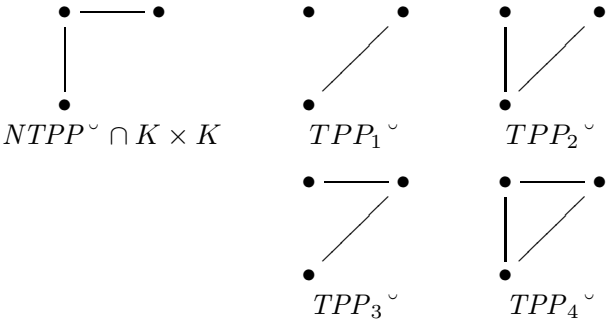
Case 2. Suppose $x \cdot - y = 0$. Then $xPPy$. There are also five atomic configurations, which can be obtained by mirroring the configurations in (5.11) along the bottom edge. It is easy to see that the first configuration defines the relation $NTPP \cap K \times K$. Furthermore, we can prove that $TPP_i = ECN_i; ECD$ for $i = 1 \sim 4$. Therefore, these relations are all in $\mathfrak{A}$.

(5.16)



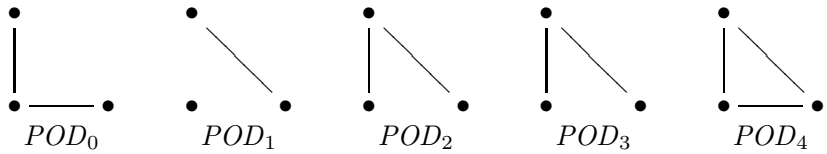
Similar to ECN relations, we can show $TPPA = TPP_4$, and $TPPB = TPP_1 \cup TPP_2 \cup TPP_3$.

Case 3. Suppose $-x \cdot y = 0$. Then $xPP^{\smile}y$. There are also five atomic configurations, which are converses of the five configurations in (5.16). The atomic relations of $\mathfrak{A}^{\rho\sigma}$ represented by these graphs are therefore also in $\mathfrak{A}$.



In this case, we have $TPPA^{\smile} = TPP_4{}^{\smile}$, and $TPPB^{\smile} = TPP_1{}^{\smile} \cup TPP_2{}^{\smile} \cup TPP_3{}^{\smile}$.

Case 4. Suppose $-x \cdot -y = 0$. Then $x + y = 1$ and $POD(x, y)$. There are also five atomic configurations. It is straightforward to show that $POD_0 = POD \cap DD$; $NTPP$, and $POD_i = ECD$; TPP_i for $i = 1 \sim 4$. Therefore, these relations are all in $\mathfrak{A}$.



In this case, we have $PODZ = POD_0$, $PODYA = POD_4$, and $PODYB = POD_1 \cup POD_2 \cup POD_3$.

In summary, all the 32 atomic configurations with three nodes represent atomic relations in $\mathfrak{A}$.

5.3. Atomic configurations with four nodes

Let $a, b \in \overline{B}$. It is clear that $G(a, b)$ has four nodes if and only if $(a, b) \in PON$. In this case, $G(a, b)$ contains four nodes, and has at most six edges. Therefore, there are $2^6 = 64$ atomic configurations. If $a, b \in L$, we know, by Corollary 5.7 again, that $E(a, b)$ is empty, and the corresponding atom in $\mathfrak{A}^{\rho\sigma}$ is the restriction of PON to $L \times L$. If $a \in L$, $b \in K$, then $E(a, b)$ is nonempty and contains at most the left edge and the right edge. There are three atomic configurations that satisfy this condition. The case when $a \in K$ and $b \in L$ is the converse of the second case. Therefore, there remains $64 - 1 - 3 - 3 = 57$ atoms of $\mathfrak{A}^{\rho\sigma}$ that are contained in $K \times K$.

We summarize our results in the following theorem.

THEOREM 5.8. *In $\mathfrak{A}^{\rho\sigma}$, there are 64 PON atomic relations, including 19 PONXA1 relations, 3 PONXA2 relations, 3 PONXB1 relations, 9 PONXB2 relations, 3 PONYA1 (PONYA1$^\smile$) relations, 1 PONYA2 (PONYA2$^\smile$) relation, 3 PONYB (PONYB$^\smile$) relations, and 18 PONZ relations.*

We next discuss when a PON atomic relation is in an RCC-25 relation, e.g. $PONXA1$. To this end, we introduce the following relations in $\mathfrak{A}$:

$$LT := PON \cap ECN \diagdown_{\mathrm{rr}} O \tag{5.17}$$

$$LB := PON \cap (ECN \diagdown_{\mathrm{rr}} O)\ ;\ DD \tag{5.18}$$

$$RT := PON \cap DD\,;(ECN \setminus_{\mathrm{rr}} O) \tag{5.19}$$

$$RB := PON \cap DD\,;(ECN \setminus_{\mathrm{rr}} O)\,;DD \tag{5.20}$$

$$BR := PON \cap (ECN \setminus_{\mathrm{rr}} O)^{\smile} \tag{5.21}$$

$$BL := PON \cap DD\,;(ECN \setminus_{\mathrm{rr}} O)^{\smile} \tag{5.22}$$

$$TR := PON \cap (ECN \setminus_{\mathrm{rr}} O)^{\smile}\,;DD \tag{5.23}$$

$$TL := PON \cap DD\,;(ECN \setminus_{\mathrm{rr}} O)^{\smile}\,;DD \tag{5.24}$$

We have the following characterizations of the above relations.

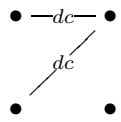
PROPOSITION 5.9. *Let x, y be two elements in $\overline{B}$ with $xPONy$. Then*

$$\begin{aligned}
xLTy &\text{ iff } xDC - x \cdot - y.\\
xLBy &\text{ iff } xDC - x \cdot y.\\
xRTy &\text{ iff } -xDCx \cdot - y.\\
xRBy &\text{ iff } -xDCx \cdot y.\\
xBRy &\text{ iff } yDC - x \cdot - y.\\
xBLy &\text{ iff } yDCx \cdot - y.\\
xTRy &\text{ iff } -yDC - x \cdot y.\\
xTLy &\text{ iff } -yDCx \cdot y.
\end{aligned}$$

PROOF. We take the case of LT relation as an example. The other cases are similar. Because $xPONy$, we have $0 \neq -x \cdot -y \neq -x$. Suppose $xLTy$. Then by the definition of the LT relation we know $xECNz$ only if yOz for any $z \in \overline{B}$. Let $z_0 = -x \cdot -y$. Then by $y - Oz_0$ we know $x - ECNz_0$. Because $x \cdot z_0 = 0$ and $z_0 \neq -x$, this implies that $xDCz_0$. On the other hand, suppose $xDC - x \cdot -y$. We show $xLTy$. For any z with $xECNz$, we need only show $z \cdot y > 0$. Suppose this is not true. Then $z \leq -y$. Note that $z \leq -x$ for $xECNz$. We have $z \leq -x \cdot -y$. So $xDCz$ for $xDC - x \cdot -y$. This is a contradiction. Therefore we have $x \cdot y > 0$, i.e. xOy. □

These relations have very nice graphical representations. Take LT as an example. Suppose $G(x, y)$ is the graph that represents $H_{x,y}$. Then $(x, y) \in LT$ if and only if $G(x, y)$ contains neither the top edge nor the major diagonal. In other words, if $(x, y) \notin LT$ if and only if $G(x, y)$ contains either the top edge or the major diagonal.

In what follows, we represent the relation LT as a graph



where the nodes are explained in the same way as an atomic configuration, but an edge indicates that the two nodes are disconnected. Because LT is a relation in $\mathfrak{A}$, each atom $H_{x,y}$ of $\mathfrak{A}^{\rho\sigma}$ is either contained in LT or disjoint from LT. The graphic representation of LT as given above provides an easy way for characterizing the atoms $H_{x,y}$ contained in LT. In fact, we have the following proposition.

Proposition 5.10. *Let $x, y \in \overline{B}$. Then $H_{x,y}$ is contained in LT if and only if $G_{x,y}$, the atomic configuration of $H_{x,y}$, contains neither the top edge nor the major diagonal.*

In this way, we regard the graph  as a set of atomic configurations, and say an atomic configuration $G_{x,y}$ is an instance of  , written $G_{x,y} \in$, if $H_{x,y} \subseteq LT$. Note that $H_{x,y} \cap LT = \varnothing$ if $G_{x,y}$ contains either the top edge or the major diagonal. Similar explanations and illustrations apply to the other seven relations.

Using these relations, we are able to give a graphical characterization for the relation $PON \cap \alpha$ for each $\alpha \in \{-R, -S, -R^{\smile}, -S^{\smile}\}$.

Proposition 5.11. *For R, S as defined in (5.9-5.10), we have*

$$\begin{aligned} PON \cap -R &= LB \cup TR, \\ PON \cap -R^{\smile} &= BL \cup RT, \\ PON \cap -S &= LT \cup BR, \\ PON \cap -S^{\smile} &= TL \cup RB. \end{aligned}$$

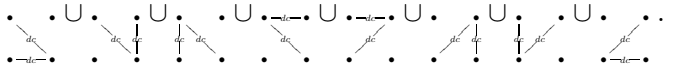
In other words, for $x, y \in \overline{B}$ and $xPONy$, we have

$$\begin{aligned} (x,y) \in -R &\text{ iff } xDC - x \cdot y \text{ or } -yDC - x \cdot y \\ (x,y) \in -R^{\smile} &\text{ iff } -xDCx \cdot -y \text{ or } yDCx \cdot -y \\ (x,y) \in -S &\text{ iff } xDC - x \cdot -y \text{ or } yDC - x \cdot -y \\ (x,y) \in -S^{\smile} &\text{ iff } -xDCx \cdot y \text{ or } -yDCx \cdot y. \end{aligned}$$

We now examine for each RCC-25 relation α the atomic relations $H_{x,y}$ of $\mathfrak{A}^{\rho\sigma}$ belonging to α.

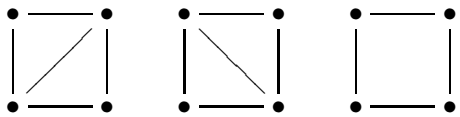
5.3.1. *PONXA1* graphs

Suppose $(x, y) \in PON$. By $PONXA1 = PON \cap R \cap R^{\smile} \cap S \cap S^{\smile}$ and Proposition 5.11, we know $(x, y) \in PONXA1$ if and only if (x, y) is not an instance of any of LB, TR, BL, RT, LT, BR, TL, RB, i.e. $G(x, y)$ satisfies the following equation

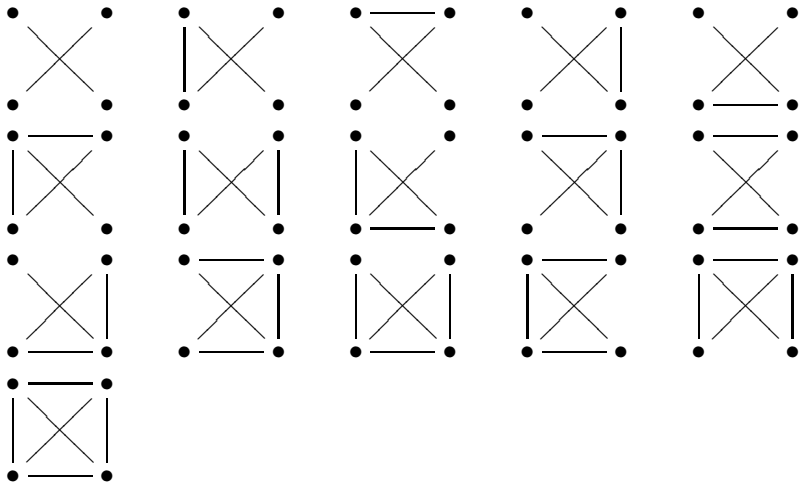
(5.25) $G(x, y) \notin$ 

If $G(x, y)$ does not contain the major diagonal, then by (5.25) $G(x, y)$ contains each of the four non-diagonal edges; similarly, if $G(x, y)$ does not contain the minor diagonal, then by (5.25) $G(x, y)$ contains each of the four non-diagonal edges. On the other hand, if $G(x, y)$ contains the major and the minor diagonals, then $G(x, y)$ satisfies the above condition and hence is a *PONXA1* graph. Therefore, *PONXA1* has the following 19 graphs:

(*PONXA1* graphs that contain at most one diagonal)

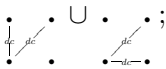


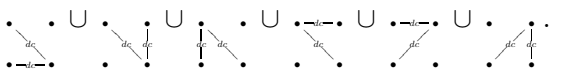
(*PONXA1* graphs that contain the diagonals)



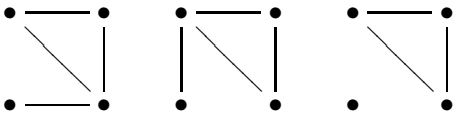
5.3.2. *PONXA2* graphs

Suppose $(x, y) \in PON$. By $PONXA2 = PON \cap R \cap R^{\smile} \cap S \cap -S^{\smile}$ and Proposition 5.11, we know (x, y) is an instance of $PONXA2$ if and only if $G(x, y)$ satisfies the following two equations

(5.26) $G(x, y) \in$  ;

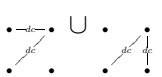
(5.27) $G(x, y) \notin$  .

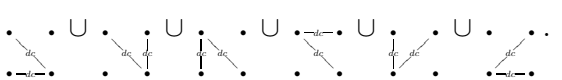
By (5.26), $G(x, y)$ does not contain the main diagonal. Hence, by (5.27), it contains the top and the right edges. If $G(x, y)$ also contains the minor diagonal, then all constraints are satisfied; if not, then by (5.27), $G(x, y)$ also contains the bottom and the left edges, which violates (5.26). Therefore, PONXB2 has the following three graphs



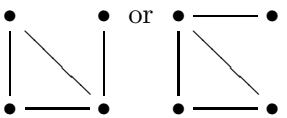
5.3.3. *PONXB1* graphs

Suppose $(x, y.) \in PON$. By $PONXB1 = PON \cap R \cap R^{\smile} \cap -S \cap S^{\smile}$ and Proposition 5.11, we know (x, y) is an instance of $PONXB1$ if and only if $G(x, y)$

(5.28) $G(x, y) \in$ 

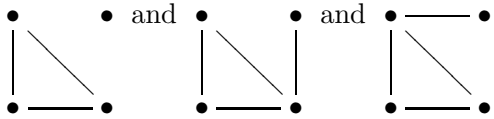
(5.29) $G(x, y) \notin$  .

(5.28) implies that $G(x, y)$ is a subgraph of



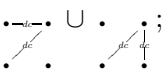
Because the major diagonal is not in these graphs, we have by (5.29) that the left and the bottom edges are in such a graph. Note if $G(x, y)$ does not contain the minor diagonal, then it should contain all non-diagonal edges, which is contradicts (5.28). Hence $G(x, y)$ contains the minor diagonal. Therefore, by (5.29) again, we know $PONXB1$ has three

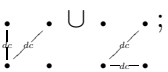
graphs, i.e.

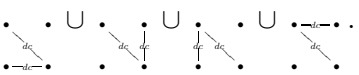


5.3.4. *PONXB2* graphs

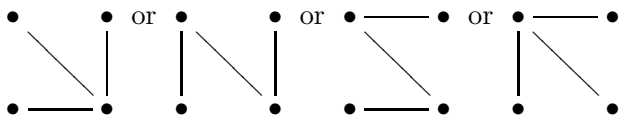
Suppose $(x, y) \in PON$. By $PONXB2 = PON \cap R \cap R^{\smile} \cap -S \cap -S^{\smile}$ and Proposition 5.11, we know (x, y) is an instance of *PONXB1* if and only if

(5.30) $G(x, y) \in$ 

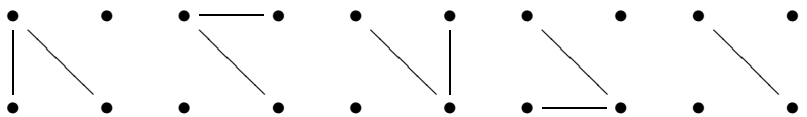
(5.31) $G(x, y) \in$ 

(5.32) $G(x, y) \notin$ 

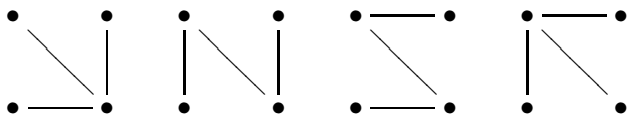
The first two conditions imply that each PONXB2 graph is a subgraph of



If the minor diagonal is not an edge in $G(x, y)$, then by (5.32) we know $G(x, y)$ contains the four non-diagonal edges, which is however impossible. So we know each PONXB2 graph $G(x, y)$ contains the minor diagonal, and is a subgraph of one of the above four graphs. Therefore, *PONXB2* has the following nine graphs:
(*PONXB2* graphs that contain one or two edges)



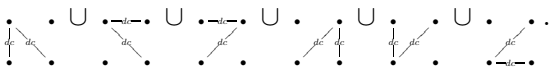
(PONXB2 graphs that contain three edges)



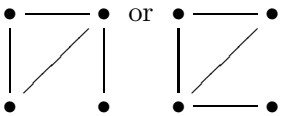
5.3.5. *PONYA1* graphs

Suppose $(x, y) \in PON$. By $PONYA1 = PON \cap -R \cap R^{\smile} \cap S \cap S^{\smile}$ and Proposition 5.11, we know (x, y) is an instance of *PONYA1* if and only if $G(x, y)$ satisfies the following conditions

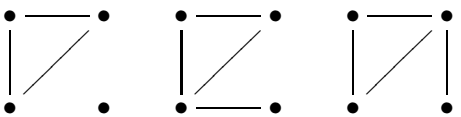
(5.33) $G(x, y) \in$ 

(5.34) $G(x, y) \notin$ 

By (5.33) we know $G(x, y)$ is a subgraph of

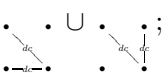


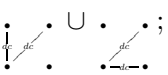
Because $G(x, y)$ does not contain the minor diagonal, by (5.34), it contains the left and the top edges. It is also clear that such a graph should contain the major diagonal: otherwise it should contain all the other non-diagonal edges hence not a subgraph of the above two graphs. Therefore, there are only three *PONYA1* graphs, i.e.

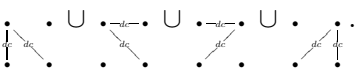


5.3.6. *PONYA2* graphs

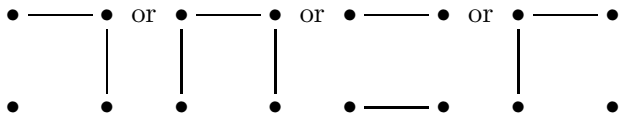
Suppose $(x, y) \in PON$. By $PONYA2 = PON \cap -R \cap R^{\smile} \cap S \cap -S^{\smile}$ and Proposition 5.11, we know (x, y) is an instance of *PONYA1* if and only if $G(x, y)$ satisfies the following conditions

(5.35) $G(x, y) \in$ 

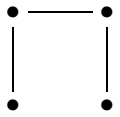
(5.36) $G(x, y) \in$ 

(5.37) $G(x, y) \notin$ 

(5.35) and (5.36) imply that $G(x, y)$ is a subgraph of

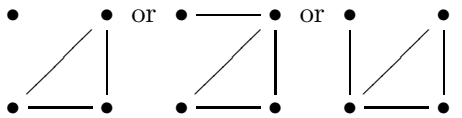


It is now easy to see that only the second graph satisfies (5.37). Moreover, no subgraph of the second graph satisfies (5.37). Therefore, *PONYA2* has only one graph, i.e.



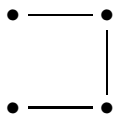
5.3.7. *PONYA1*$^\smile$ graphs

Suppose $(x, y) \in PONYA1^\smile$. By the symmetry between $G(x, y)$ and $G(y, x)$, each atomic configuration of $PONYA1^\smile$ is obtained by mirroring an atomic configuration of *PONYA1* along the major diagonal. Therefore, $PONYA1^\smile$ has only three atomic configurations



5.3.8. *PONYA2*$^\smile$ graphs

Suppose $(x, y) \in PONYA2^\smile$. By the symmetry between $G(x, y)$ and $G(y, x)$, each atomic configuration of $PONYA2^\smile$ is obtained by mirroring an atomic configuration of *PONYA2* along the major diagonal. Therefore, $PONYA2^\smile$ has only one atomic configuration, i.e.



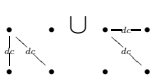
5.3.9. *PONYB* graphs

Suppose $(x, y) \in PON$. By $PONYB = PON \cap -R \cap R^\smile \cap -S$ and Proposition 5.11, we know (x, y) is an instance of *PONYB* if and only if

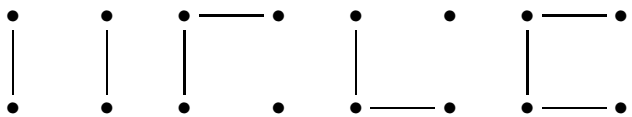
$G(x, y)$ satisfies the following conditions

(5.38) $G(x, y) \in$ 

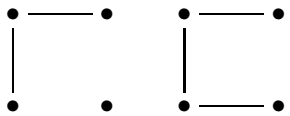
(5.39) $G(x, y) \in$ 

(5.40) $G(x, y) \notin$ 

By (5.38) and (5.39), we know $G(x, y)$ is a subgraph of either of the following four graphs

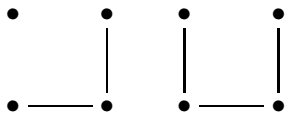


Because (5.40), we know $G(x, y)$ contains the minor diagonal or contains both the left and the top edges. Therefore, *PONYB* contains only two graphs



5.3.10. *PONYB*$^\smile$ graphs

Suppose $(x, y) \in PONYB^\smile$. By the symmetry between $G(x, y)$ and $G(y, x)$, each atomic configuration of $PONYB^\smile$ is obtained by mirroring an atomic configuration of *PONYB* along the major diagonal. Therefore, $PONYB^\smile$ has only two atomic configurations

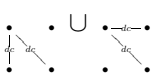


5.3.11. *PONZ* graphs

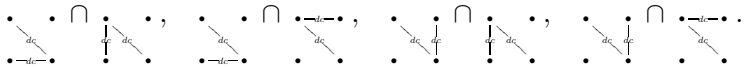
Lastly, we consider *PONZ* graphs. Suppose $(x, y) \in PON$. By $PONZ = PON \cap -R \cap -R^\smile$ and Proposition 5.11, we know (x, y) is an instance

of *PONZ* if and only if $G(x, y)$ satisfies the following conditions

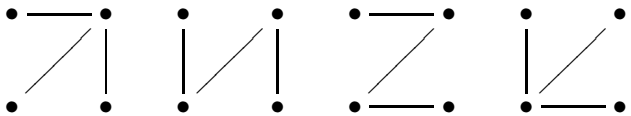
(5.41) $G(x, y) \in$ 

(5.42) $G(x, y) \in$ 

By (5.41) and (5.42), we know $G(x, y)$ is contained in either of:

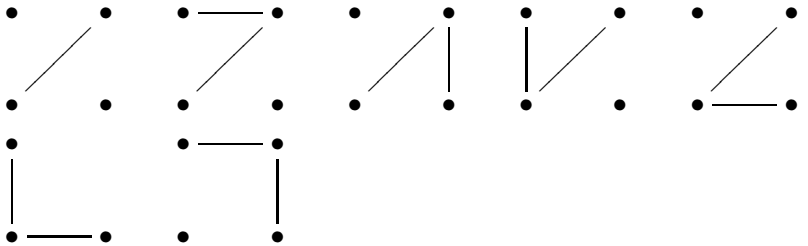


This means that $G(x, y)$ is a subgraph of the following graphs:
(Maximal *PONZ* graphs)

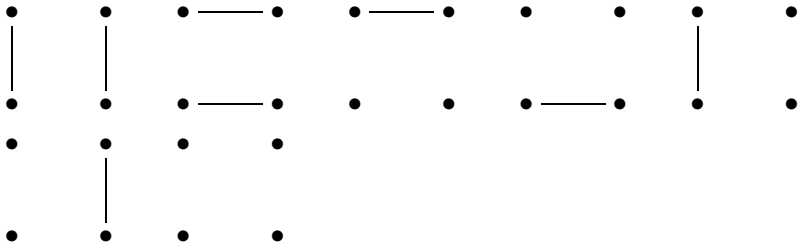


Except these, $G(x, y)$ could be the following graphs:

(Connected *PONZ* graphs with one or two edges)



(Disconnected *PONZ* graphs)



In summary, there are all together 18 *PONZ* relations, 11 contained in $K \times K$, and seven are disconnected.

Theorem 5.12. *In $\mathfrak{A}^{\rho\sigma}$, there are 64 PON atomic relations, including 19 PONXA1 relations, 3 PONXA2 relations, 3 PONXB1 relations, 9 PONXB2 relations, 3 PONYA1 (PONYA1$^\smile$) relations, 1 PONYA2 (PONYA2$^\smile$) relation, 3 PONYB (PONYB$^\smile$) relations, and 18 PONZ relations.*

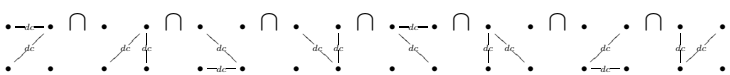
5.4. When is an atom of $\mathfrak{A}^{\rho\sigma}$ in $\mathfrak{A}$

From §§5.1 and 5.2 we know that each atomic configuration with three or less nodes represents an atomic relation in $\mathfrak{A}$ (see (5.1–5.3), (5.4), and the equations in §5.2). In this subsection we consider the *PON* graphs. Note that *PONYA2* has only one atomic configuration. This means that *PONYA2* itself is an atom of $\mathfrak{A}^{\rho\sigma}$, hence an atom of $\mathfrak{A}$. How about the other graphs? Are they also atoms of $\mathfrak{A}$?

Given a *PON* graph $G(x,y)$, to show $H_{x,y}$ (the atomic $\mathfrak{A}^{\rho\sigma}$ relation represented by $G(x,y)$) is a relation in $\mathfrak{A}$, our strategy is to show that it can be generated by the eight relations LT, BR, LB, TR, RT, BL, RB, and TL in $\mathfrak{A}$ defined in (5.17–5.24). Recall that each of the eight relations has a graphical representation, which can be regarded as a set of atomic configurations. For example, LT is the set of atomic configurations which do not contain the major diagonal and the top edge. In what follows, for an atomic configuration G, we write $\downarrow G$ for the set of subgraphs of G. Note that each subgraph of G is also an atomic configuration. We say an atomic configuration G is *representable* (w.r.t. the relations in (5.17–5.24)) if $\downarrow G$ is the intersection of a subset of the eight relations in (5.17–5.24)); we say G is *strongly representable* if G and all its proper subgraphs are all representable. We have the following proposition.

Proposition 5.13. *Suppose $(x,y) \in PON$. Then $H_{x,y}$ is a relation in $\mathfrak{A}$ if $G(x,y)$ is strongly representable.*

Proof. We prove this by using induction on the number of edges contained in $G(x,y)$. Suppose $G(x,y)$ is strongly representable. If $G(x,y)$ contains no edge, i.e. $E(x,y) = \varnothing$, then $G(x,y)$ is the unique graph contained in

(5.43) 

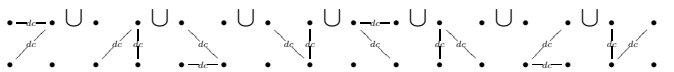
In general, suppose $G(x,y)$ has $k > 0$ edges and each proper subgraph of $G(x,y)$ represents a relation in $\mathfrak{A}$. Note that the union of these rela-

tions, written α_1, is also a relation in $\mathfrak{A}$. Moreover, because $G(x, y)$ is representable, the union of all the atoms in $\mathfrak{A}^{\rho\sigma}$ that are represented by $G(x, y)$ or one of its proper subgraphs, written α, is also a relation in $\mathfrak{A}$. Since $H_{x,y}$ is exactly the difference of α and α_1, we know $H_{x,y}$ is also a relation in $\mathfrak{A}$. It is clear that it is also atomic in $\mathfrak{A}$. □

So, when is an atomic configuration strongly representable? We have the following result:

PROPOSITION 5.14. *Suppose $(x, y) \in PON$. Then $G(x, y)$ is strongly representable if and only if it is not a PONXA1 graph, i.e. if it contains the two diagonals or contains at least the four non-diagonal edges.*

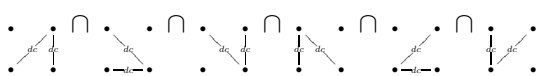
PROOF. Suppose $G(x, y)$ is a *PONXA1* graph. Then it is clear that $G(x, y)$ is not in any of the eight relations defined in (5.17–5.24), i.e.

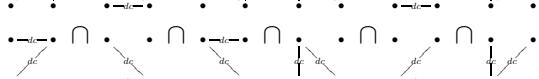
(5.44) $G(x, y) \notin$ 

This shows that $G(x, y)$ is not representable.[2] On the other hand, we show each non-*PONXA1* graph is strongly representable. Note that if $G(x, y)$ is not a *PONXA1* graph, then none of its proper subgraphs is a *PONXA1* graph. Therefore, to show all non-*PONXA1* graphs are strongly representable, we need only show that they are representable.

We first consider graphs with fewer edges. In the proof of Proposition 5.13, we have seen that the empty graph G_0 is the intersection of all the eight relations in (5.17–5.24). There are six graphs which have only one edge. We write these graphs as G_l, G_t, G_b, G_r, G_{mj}, and G_{mn}, where for example G_l is the graph which contains the left edge, and G_{mn} is the graph which contains the minor diagonal. The interpretations of the other graphs are clear. Then we have

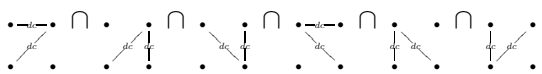
$\downarrow G_l$ = 

$\downarrow G_t$ = 

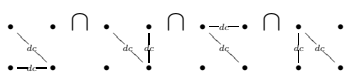
$\downarrow G_r$ = 

[2] Actually, each atomic relation of $\mathfrak{A}^{\rho\sigma}$ contained in *PONXA1* is not in the Boolean algebra generated by the eight relations in (5.17–5.24).

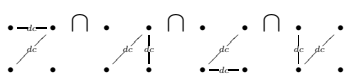
$\downarrow G_b$ =



$\downarrow G_{mj}$ =



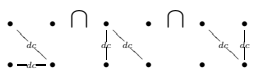
$\downarrow G_{mn}$ =



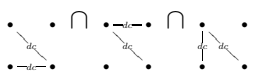
That is, each graph with one edge is representable.

Suppose $G(x, y)$ is an atomic configuration that has two edges. Note that $G(x, y)$ is a *PONXA1* graph only if it contains the two diagonals. Write $G_\times$ for this graph, and write, for instance, G_{mjt} for the graph that contains the major diagonal and the top edge. Then we have

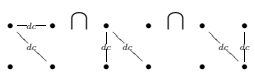
$\downarrow G_{mjt}$ =



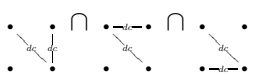
$\downarrow G_{mjr}$ =



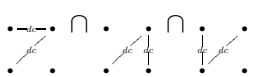
$\downarrow G_{mjb}$ =



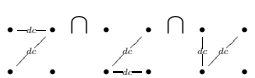
$\downarrow G_{mjl}$ =



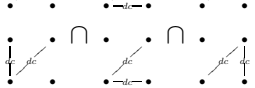
$\downarrow G_{mnb}$ =



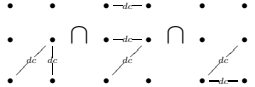
$\downarrow G_{mnr}$ =



$\downarrow G_{mnt}$ =

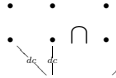


$\downarrow G_{mnl}$ =

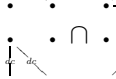


$\downarrow G_{bl}$ =

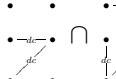
$\downarrow G_{tl}$ =



$\downarrow G_{tr}$ =



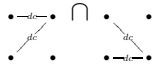
$\downarrow G_{br}$ =



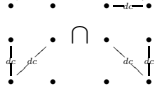
Rels.	Conv.	atoms in $K \times K$	atoms in $K \times L$	atoms in $L \times K$	atoms in $L \times L$
$1'$	$1'$	1	0	0	1
DC	*DC*	1	1	1	1
DD	*DD*	1	0	0	1
NTPP	$NTPP^\smile$	1	1	1	1
ECN	*ECN*	4	0	0	0
TPP	$TPP^\smile$	4	0	0	0
POD	*POD*	5	1	1	1
PONXA1	*PONXA1*	19	0	0	0
PONXA2	*PONXA2*	3	0	0	0
PONXB1	*PONXB1*	3	0	0	0
PONXB2	*PONXB2*	9	0	0	0
PONYA1	$PONYA1^\smile$	3	0	0	0
PONYA2	$PONYA2^\smile$	1	0	0	0
PONYB	$PONYB^\smile$	2	0	0	0
PONZ	*PONZ*	11	3	3	1
		79	7	7	7

Table 3. Atomic relations in $\mathfrak{A}^{\rho\sigma}$ that are contained in $\overline{B} \times \overline{B}$

$$\downarrow G_{lr} \quad = \quad$$



$$\downarrow G_{bt} \quad = \quad$$



Therefore, all non-*PONXA1* graphs with two edges are representable. Note that a non-*PONXA1* graph contains at most four edges. The cases when $G(x, y)$ has three or four edges are similar. We omit the details here. □

5.5. Summary

Above, we have shown that $\mathfrak{A}^{\rho\sigma}$ has 100 atoms below $\overline{B} \times \overline{B}$, where seven are below $L \times L$, seven are below $L \times K$, seven below $K \times L$, and 79 are below $K \times K$ (see Table 3 for details.) Except the 19 atoms contained in *PONXA1*, the other atoms are all known to be atoms of $\mathfrak{A}$. It is still open if the atoms of *PONXA1* are also atoms of $\mathfrak{A}$.

6. Conclusion and outlook

In this paper, we have investigated the countable homogeneous BCA $\mathfrak{B}$ and the contact relation algebra $\mathfrak{A}$ of $\mathfrak{B}$. In particular, we have shown that B can be partitioned into four sets $\{0\}$, $\{1\}$, K, and L, which are the only orbits of the group of automorphisms of $\mathfrak{B}$. Moreover, we have shown that both K and L are dense in $\mathfrak{B}$, $L \cup \{0,1\}$ is a subalgebra of $\mathfrak{B}$, and K generates $\mathfrak{B}$. As for the contact relation algebra $\mathfrak{A}$ of $\mathfrak{B}$, we have shown that $\mathfrak{A}^{\rho\sigma}$, the Galois closure of $\mathfrak{A}$, has 112 atoms. This seems to be the first known infinite BCA with non-minimal $\mathcal{C}$ which has a finite contact relation algebra. We have also introduced a graphic representation for each atom of $\mathfrak{A}^{\rho\sigma}$ that is below $\overline{B} \times \overline{B}$. Based upon this graphical representation, we have shown how these atoms in $\mathfrak{A}^{\rho\sigma}$ refine the 25 relations defined in Düntsch *et al.* [7].

There are several problems we leave open: Which, if not all, *PONXA1* graphs represent relations in $\mathfrak{A}$? Are the 79 relations below $K \times K$ realizable in the real plane (or in any extensional and connected BCA, i.e. RCC algebra)? Is there an extensional and connected BCA with more than two elements and a finite CRA?

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