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INTUITIONISTIC OVERLAP STRUCTURES

Abstract. We study some connections between two kinds of *overlap* relations: that of point-free geometries in the sense of Grzegorczyk, Whitehead and Clarke, and that recently introduced by Sambin within his constructive approach to topology. The main thesis of this paper is that the overlap relation in the latter sense is a necessary tool for a constructive and intuitionistic development of point-free geometry.

Keywords: Overlap algebras, connection structures, mereological fields, constructive reasoning.

Introduction

The relation of "overlap" between regions is a basic notion in mereology, in mereotopology and in several other point-free descriptions of geometry and topology. A similar idea, together with the related notion of "positivity", has recently gained importance in certain intuitionistic approaches to topology and mathematics in general, such as Sambin's [11, 12]. This paper aims to be a first step towards an exchange between these two traditions.

In Section 1, we start by presenting the so-called *overlap algebras* introduced by Sambin [12]. By means of some examples, we show how they can be employed to obtain intuitionistic versions of classical theorems about complete Boolean algebras. In particular, we address the problem of an intuitionistically sound definition of "atom" for a frame and we discuss a couple of possible solutions.

Then, in Section 2, we use overlap algebras to construct intuitionistic versions of various kinds of *connection structures* [1, 2]. In particular, we

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propose a notion of *intuitionistic mereological field* which is based on the notion of an overlap algebra rather than on that of a complete Boolean algebra. Finally we prove intuitionistic versions of some classical results on the conncetion structures associated to the regular open sets of a topology.

All proofs and constructions in this work are carried on by means of intuitionistically sound arguments. Therefore, though being valid also under a classical reading, our results remain true also in more general frameworks such as topos-valid mathematics.

1. Overlap algebras

What are the algebraic properties that characterize the collection $\mathrm{Pow}(S)$ of all subsets of a given set S? Although natural the question is, the answer is by no means trivial. First of all, it depends on the language one choose. In the language of lattices, for instance, one can characterize powersets as atomic, complete Boolean algebra. On the other hand, the answer heavily relies on the foundational assumptions one is inclined to make and, specifically, on the kind of *logic* one uses at the metalanguage. It is a fact of life that $\mathrm{Pow}(S)$ is no longer a Boolean algebra as long as you look at it from an intuitionistic point of view. Are we able to select *all* those properties of powersets that hold intuitionistically?

Before proposing an answer to this question (following Sambin [12]), we would try to motivate our interest in intuitionistic logic. There is certainly someone who feels intuitionistic logic to be more natural than classical one. There can even exist someone believing that only intuitionistic logic is "true". Surely there are many who are simply curious about the possibility of adopting a different logic. However, besides these kinds of subjective motivations, there is also a more mathematical reason for developing Mathematics on intuitionistic basis. In that way, in fact, all definitions make sense and all results are valid not only in the usual intended set-theoretic interpretation, but also within the interal language of any *topos* [10].

For every set S, $\mathrm{Pow}(S)$ is a complete Heyting algebra (with respect to the obvious operations) and this remains true also intuitionistically. Is this all what we can say intuitionistically about powersets? Certainly not; at least for two reasons. First, one has to find a suitable notion of "atom" in such a way that every powerset wll turn out to be atomic.

Second, one needs a way for expressing when an element is "inhabited", a statement which is intuitionistically stronger than merely asserting non-emptyness. As we are now going to see, these two problems are deeply linked.

If $(X, \leq, 0)$ is an arbitrary poset with a bottom element, then one wants atoms to be minimal among the elements that are different from 0. This same idea can be formalized in several ways which are equivalent classically, but not intuitionistically. For instance, any one of the following conditions is a candidate for defining when $a \in X$ is an atom.

$$a \neq 0 \quad \& \quad \neg(\exists x \in X)(x \neq 0 \;\&\; x < a) \tag{1}$$
$$a \neq 0 \quad \& \quad (\forall x \in X)(x < a \Rightarrow x = 0) \tag{2}$$
$$a \neq 0 \quad \& \quad (\forall x \in X)(x \leq a \Rightarrow x = 0 \vee x = a) \tag{3}$$
$$a \neq 0 \quad \& \quad (\forall x \in X)(x \neq 0 \;\&\; x \leq a \Rightarrow x = a) \tag{4}$$

Which of these is the more convenient from an intuitionistic point of view? None! They all look too restrictive because either singletons in $\mathrm{Pow}(S)$ cannot be proven to satisfy them, as it happens in the cases of (3) and (4), or it is impossible to prove that every subset satisfying them is a singleton. As an example, let us analyze the case of (4). Let $\{a\}$ be a singleton and assume that $X \subseteq S$ is such that $\emptyset \neq X \subseteq \{a\}$. Why cannot we conclude that $X = \{a\}$? Surely we could if we knew X to be *inhabited*. In fact, if that were the case, that is, if there existed an element $b \in X$, then $b = a$ because $X \subseteq \{a\}$ and hence $a \in X$, that is, $\{a\} \subseteq X$ as wished. Note also that X being inhabited is a necessary condition for X to coincide with $\{a\}$. So the question is: are the hypotheses sufficient for finding an element $b \in X$? From an intuitionistic point of view, the answer is: "No"! We only know $X \neq \emptyset$, that is, $\neg\forall b \neg(b \in X)$. This is intuitionistically weaker than $\exists b(b \in X)$.

In view of this discussion, it should be clear that the problem of finding a good definition of atom comes together with the need for expressing inhabitedness. So a possible solution is to add a new primitive predicate to the language of lattices, namely an *overlap* relation.

Definition 1.1 (Sambin [12]). Let $\mathcal{P}$ be a complete lattice. An *overlap* relation $\between$ on $\mathcal{P}$ is a binary relation on $\mathcal{P}$ such that

O1 $x \between y \iff y \between x$

O2 $(x \wedge z) \between y \iff x \between (z \wedge y)$

O3 $x \between (\bigvee_{i \in I} y_i) \iff (\exists i \in I)(x \between y_i)$

O4 $x \leq y \iff \forall z\big((x \between z) \Rightarrow (y \between z)\big)$
for all $x, y, z \in \mathcal{P}$ and every set-indexed family $\{y_i \mid i \in I\} \subseteq \mathcal{P}$.
We call *overlap algebra* (or simply *o-algebra*) a complete latttice with overlap.

Among the consequences of the axioms O1-O4 (see [4,5]), we want to mention only one, perhaps surprising: *every o-algebra is a frame*, that is, binary meets distribute over arbitrary joins. In fact, for any $z \in \mathcal{P}$, one has: $(x \wedge \bigvee_{i\in I} y_i) \between z$ iff $\big(\bigvee_{i\in I} y_i\big) \between (x \wedge z)$ iff $(\exists i \in I)\big(y_i \between (x \wedge z)\big)$ iff $(\exists i \in I)\big((x \wedge y_i) \between z\big)$ iff $\big(\bigvee_{i\in I}(x \wedge y_i)\big) \between z$. This shows that $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$ by O4.

With classical logic, $x \between y$ becomes equivalent to $x \wedge y \neq 0$ and o-algebras turn out to be just complete Boolean algebras [4, 5]. This fact suggests a general idea: in order to prove an intuitionistic version of a classical result about complete Boolean algebras, replace them by o-algebras.

We say that an element x in an o-algebra $\mathcal{P}$ is *positive* (or *inhabited*), and we write $\mathsf{Pos}(x)$, if $x \between x$ holds. O-algebras can be presented also in terms of this positivity predicate Pos. In fact, it is easy to check that an o-algebra is just a frame equipped with a unary predicate Pos such that:
(5)
$$\mathsf{Pos}(\bigvee_i y_i) \Leftrightarrow \exists i \mathsf{Pos}(y_i) \quad \text{and} \quad \forall z\big(\mathsf{Pos}(x \wedge z) \Rightarrow \mathsf{Pos}(y \wedge z)\big) \Rightarrow x \leq y\ .$$

In this case the overlap relation $x \between y$ is defined as $\mathsf{Pos}(x \wedge y)$.

The positivity predicate Pos is precisely what is needed in order to obtain an intuitionistically sound definition of atom. What we are going to give (following [12]) is just a "positive" rendering of (4).

DEFINITION 1.2. Let $\mathcal{P}$ be an o-algebra.[1] We say that $a \in \mathcal{P}$ is an *atom* if

$$\mathsf{Pos}(a)\ \ \&\ \ \forall x\big(\mathsf{Pos}(x)\ \&\ x \leq a \Longrightarrow x = a\big)\ . \tag{6}$$

An o-algebra $\mathcal{P}$ is *atomic* if every $x \in \mathcal{P}$ is a join of atoms, that is, $x = \bigvee\{a \in \mathcal{P} \mid a \leq x\ \&\ a \text{ is an atom}\}$.

[1] This same definition makes sense also for more general structures. In [4] it was used for formal topologies equipped with a positivity predicate, that is, a predicate Pos satisfying a weakened form of (5). From the point of view of Locale Theory [8], these structures coincides with the so-called *open* (or *overt*) locales.

In the following paragraph we will show that this definition of atom provides a solution to the problem of characterizing powersets in an intuitionistically sound way. We end this section by an elegant characterization of the notion of atom (see [4] for a detailed proof).

Proposition 1.3. *Let $\mathcal{P}$ be an o-algebra. Then $a \in \mathcal{P}$ is an atom if and only if*

$$a \leq x \iff a \between x \qquad \text{for all } x \in \mathcal{P} . \tag{7}$$

1.1. Atomic o-algebras

Till now, we have not explained yet why the notion of o-algebra (and the corresponding notion of atom) solves our initial problem of characterizing powersets. First of all, one should check that $\mathrm{Pow}(S)$ is an o-algebra for every set S. The overlap relation $X \between Y$ between subsets $X, Y \subseteq S$ is defined by the formula: $(\exists a \in S)(a \in X \cap Y)$.[2] Therefore the corresponding positivity predicate $\mathsf{Pos}(X)$ is given by $(\exists a \in S)(a \in X)$ and hence it says precisely that X is inhabited.

It is easy to check that $\mathrm{Pow}(S)$ is an example of an atomic o-algebra. Its atoms are precisely the singletons. Moreover, every atomic o-algebra is isomorphic to a powerset [12]. More explicitly, if $\mathcal{P}$ is atomic and S is the set of all its atoms, then the map $x \mapsto \{a \in S \mid a \leq x\}$ is an order isomorphism between $\mathcal{P}$ and $\mathrm{Pow}(S)$ that "respects overlap", that is, $x \between y$ holds iff the corresponding subsets of atoms have an element in common.

Summing up, from an intuitionistic point of view, powersets are precisely the atomic o-algebras. If one reads this result with a classical eye, then it is clear that atomic o-algebras must coincide with atomic, complete Boolean algebras. Examples of non-atomic, even atom-less, o-algebras are given in [3, 5].

1.1.1. Other approaches to atomicity

There exist at least two other ways for characterizing powersets intuitionistically. One is developed in [9] where powersets, seen as discrete locales, are characterized in terms of maps between locales. A predicative account of such a method is developed in [4]. Another approach is as follows.

[2] Sambin uses the symbol $X \between Y$ for the overlap relation between subsets.

Given a poset $(\mathcal{P}, \leq)$ and an element $p \in \mathcal{P}$, let us write $\downarrow p$ for the sub-poset of $\mathcal{P}$ whose carrier is the set $\{x \in \mathcal{P} \mid x \leq p\}$. As a poset, $\downarrow p$ always has a top element, namely p. Moreover, it inherits much of the structure of $\mathcal{P}$. For instance, if $\mathcal{P}$ has arbitrary joins, then also $\downarrow p$ has arbitrary joins (and they are computed in $\mathcal{P}$).

If $(\mathcal{P}, \leq, 0)$ is a poset with bottom element, then $a \in \mathcal{P}$ is a minimal non-zero element if and only if $\downarrow a$ is order-isomorphic to $\{0, 1\}$. The same idea can be used also intuitionistically as long as $\{0, 1\}$ is replaced in a suitable way. What is the characteristic property of the poset $\{0, 1\}$? One possible answer is: $\{0, 1\}$ is the *initial* object in the category of frames. Another one is: $\{0, 1\}$ is (isomorphic to) the powerset of a singleton set. From an intuitionistic point of view, the initial frame can be presented as $\mathrm{Pow}(\{*\})$ even though this frame is far away from containing just two elements! Thus we are led to the following alternative definition of atom.

DEFINITION 1.4. Let $\mathcal{P}$ be a poset with bottom element. An element $a \in \mathcal{P}$ is an atom if $\downarrow a$ is isomorphic to the initial frame.

This definition seems to work well also intuitionistically. In fact, it is clear that every singleton in $\mathrm{Pow}(S)$ is an atom and so $\mathrm{Pow}(S)$ is an atomic frame. Moreover we can show (intuitionistically) that also the converse holds. First, we need a couple of lemmas.

LEMMA 1.5. *Let $\mathcal{P}$ be a poset with zero, $a \in \mathcal{P}$ be an atom and $\{x_i \mid i \in I\} \subseteq \mathcal{P}$. If $\bigvee_{i \in I} x_i$ exists and $a = \bigvee_{i \in I} x_i$, then $a = x_i$ for some $i \in I$.*

PROOF. If $a = \bigvee_{i \in I} x_i$, then each x_i belongs to $\downarrow a$. Let f be the frame isomorphism between $\downarrow a$ and $\mathrm{Pow}(\{*\})$ which exists by hypothesis. By applying f to both sides of the equation $a = \bigvee_{i \in I} x_i$, we obtain $f(a) = \bigcup_{i \in I} f(x_i)$. Since a is the top element of $\downarrow a$, it must be $f(a) = \{*\}$. So $* \in \bigcup_{i \in I} f(x_i)$ and hence $* \in f(x_i)$ for some $i \in I$. For this same index i, the subset $f(x_i)$ must coincides with the whole $\{*\}$ and so $a = f^{-1}(\{*\}) = f^{-1}(f(x_i)) = x_i$, as wished. q.e.d.

LEMMA 1.6. *Let $\mathcal{P}$ be a frame and let $a, b \in \mathcal{P}$ be two atoms. Under these assumptions, if $a \leq b$, then $a = b$.*

PROOF. Let f_a and f_b be the two isomorphisms from $\downarrow a$ and, respectively, $\downarrow b$ to the initial frame $\mathrm{Pow}(\{*\})$. Moreover, let $g : (\downarrow b) \to (\downarrow a)$ be the map defined by $x \mapsto a \wedge x$. So $g(a) = g(b)$. Since $\mathcal{P}$ is a frame, this map turns out to be a frame homomorphism. So the composition

$g \circ {f_b}^{-1}$ is a frame morphism from the initial frame to $\downarrow a$. By the universal property of the initial object in a category, it must be $g \circ {f_b}^{-1} = {f_a}^{-1}$, that is, $g = {f_a}^{-1} \circ f_b$. In particular, g is injective and so $a = b$ follows from $g(a) = g(b)$. q.e.d.

PROPOSITION 1.7. *Let $\mathcal{P}$ be an atomic frame and let S be the set of atoms of $\mathcal{P}$. Then $\mathcal{P}$ is isomorphic to* $\mathrm{Pow}(S)$.

PROOF. Let $f \colon \mathcal{P} \to \mathrm{Pow}(S)$ be the map defined by $f(x) = \{a \in S \mid a \leq x\}$. If $x, y \in \mathcal{P}$ and $x \leq y$, then clearly $f(x) \subseteq f(y)$ and so f is order-preserving. We claim that the order-preserving map $\{a_i \mid i \in I\} \mapsto \bigvee_{i \in I} a_i$ from $\mathrm{Pow}(S)$ to $\mathcal{P}$ is the inverse of f. Clearly, $x = \bigvee f(x)$ because $\mathcal{P}$ is atomic. On the other hand, the inclusion $\{a_i \mid i \in I\} \subseteq f(\bigvee_{i \in I} a_i)$ is clear because $a_i \leq \bigvee_{i \in I} a_i$ for all $i \in I$. As for the other inclusion, let $a \in f(\bigvee_{i \in I} a_i)$, that is, $a \leq \bigvee_{i \in I} x_i$ and a is an atom of $\mathcal{P}$. Since $\mathcal{P}$ is a frame, we can write $a = a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I}(a \wedge a_i)$. By the first lemma above, we obtain $a = a \wedge a_i$ for some $i \in I$; so $a \leq a_i$ and hence $a = a_i$ by the second lemma. q.e.d.

2. O-algebras and connection structures

Besides being the key feature of o-algebras, the idea of an "overlap relation" is also typical of all approaches to point-free geometry. So the question arises as to whether this is a mere coincidence of terms or something more. We are going to show that deep links exist between o-algebras and the so-called "connection structures" (for which we refer mainly to [1, 2, 6]).

2.1. Intuitionistic mereological fields

Several kinds of connection structures exist. According to [1], which is based on the works of Whitehead and Clarke, a connection structure is given by an inhabited set R together with a reflexive and symmetric binary relation C on it such that:

- for every $x, y \in R$, if $\forall z(xCz \Leftrightarrow yCz)$, then $x = y$;
- every inhabited subset $X \subseteq R$ has a *fusion*,

where $f(X) \in R$ is a fusion of X if $\forall z\big(f(X)Cz \Leftrightarrow (\exists x \in X)xCz\big)$.

The first condition ensures that the pre-order defined by

$$x \leq y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \forall z(xCz \Rightarrow yCz) \tag{8}$$

is in fact a partial order. The second condition says that $f(X)$ is the join of the subset X with respect to this partial order. In fact, for every y, one has: $f(X) \le y$ iff $\forall z(f(X)Cz \Rightarrow yCz)$ iff $\forall z((\exists x \in X)xCz \Rightarrow yCz)$ iff $\forall z(\forall x \in X)(xCz \Rightarrow yCz)$ iff $(\forall x \in X)\forall z(xCz \Rightarrow yCz)$ iff $(\forall x \in X)(x \le y)$. Thus a connection structure is equivalent to a triple $(R, \le, C)$ where:

- $(R, \le)$ is a poset such that every inhabited subset of R has a join,
- C is a reflexive and symmetric binary relation on R,
- $x \le y \iff (\forall z \in R)(xCz \Rightarrow yCz)$ for all $x, y \in R$ and
- $(\bigvee X)Cy \iff (\exists x \in X)(xCy)$ for every inhabited $X \subseteq R$.

The relation C is called the *connection relation*. In every connection structure, an overlap relation O can be defined by putting:

$$xOy \overset{\text{def}}{\iff} (\exists z \in R)(z \le x \;\&\; z \le y)\ . \tag{9}$$

Clearly, xOy always implies xCy.

A classical example of a connection structure is given by a *mereological field* with $x \not\le -y$ as the connection relation. Recall that a mereological field is the algebraic structure obtained by deleting the bottom element from a complete Boolean algebra. As it is, this definition looks suspicious from an intuitionistic point of view. Is just discarding 0 what we really want to do? Or would we rather like to select “inhabited” elements? Not surprisingly, I suggest to call an *intuitionistic mereological field* the structure obtained by selecting all positive elements of an o-algebra. For $\mathcal{P}$ an o-algebra, we thus put:

$$\mathcal{P}^+ \quad \overset{\text{def}}{=} \quad \{x \in \mathcal{P} \mid \mathsf{Pos}(x)\} \quad = \quad \{x \in \mathcal{P} \mid x \between x\}\ . \tag{10}$$

It is clear that $\mathcal{P}^+$ is a connection structure with C given by the restriction of $\between$ to $\mathcal{P}^+$. Moreover, the connection relation C and the overlap relation O coincide in $\mathcal{P}^+$, as we know show. Let $x, y \in \mathcal{P}^+$ such that xCy, that is, $x \between y$; we claim that xOy. To this aim, it is sufficient to check that $x \wedge y \in \mathcal{P}^+$. To see this, we first rewrite the hypothesis $x \between y$ as $(x \wedge x) \between (y \wedge y)$. Then, by the axioms O1 and O2 in Definition 1.1, we get $(x \wedge y) \between (x \wedge y)$, that is, $\mathsf{Pos}(x \wedge y)$.

2.2. Grzegorczyk’s connection structures

A different approach to pointfree geometry was proposed by Grzegorczyk in [7], where a notion of “being separated” is assumed as primitive in addition to the structure of a mereological field. Following [2], we prefer to assume the complementary notion instead, namely that of "being connected".

DEFINITION 2.1. An *intuitionistic* (*Grzegorczyk's*) *connection structure* is given by an intuitionistic mereological field R together with a reflexive and symmetric relation C on R such that $x \leq y \Longrightarrow (\forall z \in R)(xCz \Rightarrow yCz)$ for all $x, y \in R$.

For every o-algebra $\mathcal{P}$, the structure $(\mathcal{P}^+, \between)$ is clearly a connection structure in which C and O coincides (they both coincides with $\between$).

We are now going to enrich the structure of an o-algebra by means of "operators" in order to obtain intuitionistic models of connection structures in which the overlap relation does not boil down to the connection relation. The first idea that comes to mind is to add to an o-algerba $\mathcal{P}$ a *closure operator*, that is a function $\overline{(\ )} : \mathcal{P} \to \mathcal{P}$ satisfying the following conditions

$$x \leq \overline{x}\ , \quad \overline{\overline{x}} = \overline{x} \quad \text{and} \quad x \leq y \Rightarrow \overline{x} \leq \overline{y} \tag{11}$$

for all $x, y \in \mathcal{P}$. A connection structure is then obtained by considering on $\mathcal{P}^+$ the connection relation $\overline{x} \between \overline{y}$. In this kind of models, the relation O still coincides with $\between$, while C is strictly weaker than them, in general.

An important notion, which is usually employed in the definition of points and which becomes interesting only when C and O do not coincide, is that of *non-tangential inclusion*. Recall that the non-tangential inclusion relation $\ll$ is defined in the following way:

$$x \ll y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \forall z(xCz \Rightarrow yOz)\ . \tag{12}$$

It is easy to check that $x \ll y$ implies $\overline{x} \leq y$ in those connection structures that are obtained from o-algebras with closure operators as above. In fact if $\overline{x} \between z$, then also $\overline{x} \between \overline{z}$, that is xCz, and hence yOz, that is $y \between z$. So $\overline{x} \leq y$ by O4 of Definition 1.1. Are we able to construct models in which also the converse holds? The answer is known to be affirmative for models constructed by using regular open sets [7]. Here we are going to follow the same idea but, instead of considering topological spaces, we will employ a more general and abstract notion, that of *o-topology* [5].

DEFINITION 2.2. An *o-topology* is an overlap algebra $\mathcal{P}$ together with a function $(\ )^\circ : \mathcal{P} \to \mathcal{P}$ (the *interior operator*) satisfying

$$x^\circ \leq x\ , \quad x^{\circ\circ} = x^\circ\ , \quad x \leq y \Rightarrow x^\circ \leq y^\circ \quad \text{and} \quad (x \wedge y)^\circ = x^\circ \wedge y^\circ \tag{13}$$

for all $x, y \in \mathcal{P}$.

An o-topology is an intuitionistic version of what is usually called an *interior algebra*. Similarly to what is usually done for interior algebras, every o-topology can be equipped with a closure operator $\overline{(\ )}$ too. Contrary to the classical case, however, $\overline{y}$ is not defined as $-(-y)^\circ$ but in the following (classically equivalent) way instead:

$$\overline{y} \stackrel{\text{def}}{=} \bigvee\{x \mid \forall z((x \between z^\circ) \Rightarrow (y \between z^\circ))\} . \tag{14}$$

It follows by O3 that $\overline{y} \between z^\circ \Rightarrow y \between z^\circ$ for every y, z. This is precisely the *compatibility* condition of [12]. Given this, it is easy to prove that $x \leq \overline{y}$ if and only if $x \between z^\circ \Rightarrow y \between z^\circ$ for all $z \in \mathcal{P}$. Therefore $\overline{(\ )}$ is a closure operator, that is, the three conditions displayed in (11) are satisfied.

It is easy to check that the mapping $x \mapsto \overline{x}^\circ$ is idempotent, besides being monotone. Its fixed points are the *regular* open elements of $\mathcal{P}$. We write $\mathcal{P}_{reg}$ for the class $\{x \in P \mid x = \overline{x}^\circ\}$ and $\mathcal{P}^+_{reg}$ for the positive elements of $\mathcal{P}_{reg}$.

PROPOSITION 2.3. *Let $\mathcal{P}$ be an o-topology and let $\mathcal{P}^+_{reg}$ be the class of all positive, regular elements of $\mathcal{P}$. If xCy is defined as $\overline{x} \between \overline{y}$, then $(\mathcal{P}^+_{reg}, C)$ is a connection structure (in the sense of Definition 2.1) in which $x \ll y \Leftrightarrow \overline{x} \leq y$.*

PROOF. We refer to [5] for the proof that $\mathcal{P}_{reg}$ is an o-algebra. We only recall that $\leq$, $\wedge$ and $\between$ between regular elements are those inherited from $\mathcal{P}$; on the contrary, joins in $\mathcal{P}_{reg}$ are given by $\overline{(\ )}^\circ$-closure of those in $\mathcal{P}$. As a consequence, $\mathcal{P}^+_{reg}$ is an intuitionistic mereological field in the sense of the previous section. Since the axioms required on C are easy to check, it only remains to prove that $\overline{x} \leq y$ implies $x \ll y$. So let xCz, that is, $\overline{x} \between \overline{z}$. Therefore $y \between \overline{z}$ because $\overline{x} \leq y$ and hence $y \between z$ because y is open (recall the compatibility condition above), that is, yOz. q.e.d.

Note that the equivalence $x \leq y \Leftrightarrow \forall z(xCz \Rightarrow yCz)$ does not generally hold in such a kind of models. Can we find connection structures in which this holds but at the same time O and C do not coincide (in general)? As suggested in [1], a solution is to consider regular spaces. As above we move to the more general framework of regular o-topologies [5].

DEFINITION 2.4. An o-topology is *regular* if for all x, y:

$$x \between y^\circ \quad \Longrightarrow \quad \exists z(x \between z^\circ \ \& \ \overline{z^\circ} \leq y^\circ) . \tag{15}$$

This definition is justified by the following argument. Thanks to the axiom O3 in Definition 1.1, an o-topology is regular if and only if $y^\circ = \bigvee\{z^\circ \mid \overline{z^\circ} \leq y^\circ\}$ for every y. This expresses in an algebraic way the following well-known characterization of a regular space: *a topological space X is regular if and only if each open set $Y \subseteq X$ is a union of open sets whose closure is contained in Y.*

PROPOSITION 2.5. *Let $\mathcal{P}$ be an o-topology and let $(\mathcal{P}^+_{reg}, C)$ be the associated connection structure on its positive, regular elements. If $\mathcal{P}$ is regular, then $x \leq y \Leftrightarrow \forall z(xCz \Rightarrow yCz)$ holds in $(\mathcal{P}^+_{reg}, C)$.*

PROOF. For fixed but arbitrary $x, y \in \mathcal{P}^+_{reg}$, let us assume that xCz implies yCz for all $z \in \mathcal{P}^+_{reg}$. We fist claim that $x \leq \overline{y}$. Thanks to the definition of $\overline{(\ )}$, to prove the claim it is sufficient to show that $(\forall p \in \mathcal{P})(x \between p^\circ \Rightarrow y \between p^\circ)$. So let $x \between p^\circ$. By regularity, there exists z such that $x \between z^\circ$ and $\overline{z^\circ} \leq p^\circ$. Put $t = \overline{z^\circ}^\circ$ so that t is regular. Note that $z^\circ \leq t$. Moreover, t is positive; in fact, $x \wedge z^\circ$ is positive because $x \between z^\circ$ and so $\mathsf{Pos}(t)$ holds because $x \wedge z^\circ \leq z^\circ \leq t$. So t belongs to $\mathcal{P}^+_{reg}$. From $x \between z^\circ$ it also follows that $x \between t$ and hence, a fortiori, xCt. By assumption, we then get yCt, that is, $\overline{y} \between \overline{t}$. Since the map $\overline{(\ )}^\circ$ is idempotent, the last condition becomes $\overline{y} \between \overline{z^\circ}$. From the hypothesis $\overline{z^\circ} \leq p^\circ$, it now follows that $\overline{y} \between p^\circ$. Hence we can conclude $y \between p^\circ$ (by the compatibility condition), which proves the claim. To complete the proof we must check that $x \leq y$ follows from $x \leq \overline{y}$ together with the assumptions that both x and y are regular. This is easy because $x \leq \overline{y}$ yields $\overline{x} \leq \overline{y}$ by the properties of a closure operator and so $x = \overline{x}^\circ \leq \overline{y}^\circ = y$. q.e.d.

Summing up, for the connection structure associated to the positive, regular elements of a regular o-topology we know that:

- the overlap relation xOy is given by $x \between y$ in the underlying o-algebra,
- the connection relation xCy is $\overline{x} \between \overline{y}$,
- the inclusion relation $x \leq y$ is equivalent to $\forall z(xCz \Rightarrow yCz)$ and
- the non-tangential inclusion $x \ll y$ coincides with $\overline{x} \leq y$.

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