Logic and Logical Philosophy
Volume 22 (2013), 139–143
DOI: 10.12775/LLP.2013.008

Cristina Coppola
Giangiacomo Gerla

SPECIAL ISSUE ON POINT-FREE GEOMETRY AND TOPOLOGY
An Introduction

Abstract. In the first section we briefly describe the methodological assumptions of point-free geometry and topology. We also outline the history of geometrical theories based on the notion of a *region*. The second section is devoted to a concise presentation of the content of the LLP special issue on point-free theories of space.

Keywords: mereology, Boolean algebras, point-free geometry, point-free topology, intuitionism, category theory, multi-valued logic, decidability.

1. Mereology and point-free geometry

In a point-free foundation of geometry and topology the notion of *solid body* or *region* along with some kind of relation or relations (such as *inclusion* and *contact*) are assumed as primitive. More precisely, in spite of the fact that sometimes the word “solid body” is used, the notion of *region* means a locality, not a physical object. The crucial issue is to define immaterial entities as points, lines and surfaces from the “concrete” entities that regions are. This is usually done by suitable “abstraction processes”.

It seems that the very first step in this direction was Lobachevsky’s famous book *New principles of geometry with complete theory of parallels* [9]. Indeed, this book, usually mentioned only as a mile-stone in the development of non-Euclidean geometry, is interesting also for the fact

Special Issue on Mereology and Point-free Geometry. Edited by C. Coppola and G. Gerla

ISSN: 1425-3305

that solid bodies, rather than points, are assumed as primitives. Surfaces are defined in terms of sections of solid bodies, curves arise as sections of surfaces and so forth. This is in accordance with Lobachevsky's idea according to which geometry is the study of forms occurring in the physical universe and not only an abstract theory, which is mathematical in nature. Unfortunately, this interesting aspect of Lobachevsky's work was completely ignored by subsequent researchers. One of the reasons could be that his definitions of surface, line and point are somewhat obscure. Moreover, it turns out that points, lines and surfaces are the true primitives in the book, since the main definitions and the axioms refer directly to these entities, while it is evident that in a genuine point-free approach to geometry the axioms should refer directly to properties of regions.

Afterwards, a more explicit and interesting analysis appeared in two books by Whitehead ([11, 12]). These works are focused on the notions of *event* (or *region*) and *part of*. As observed for example in Casati and Varzi ([1]), the approach proposed in these books is a basis for a *mereology* rather that for a geometry since the inclusion relation is set-theoretical and not geometrical in nature. Moreover, the choice of the *part of* relation generates several technical difficulties. For example, it is a very hard challenge to give a suitable definition of a point (see for example [6]). So, it is not surprising that a couple of years after publication of these two books, Whitehead in *Process and Reality* ([13]) proposed a different idea in which the topological notion of *contact* is assumed as a primitive one. This idea was suggested by de Laguna ([4]). Furthermore, the notion of *oval* is proposed to define geometrical notions such as the one of *straight segment*.

Whitehead's books contain deep philosophical analysis but they are not mathematical in nature. The construction of mathematical theories was carried out by other mathematicians and mathematically oriented philosophers. For example, Clarke ([2, 3]) proposed a precise axiom system for Whitehead's contact relation (see also Grzegorczyk's paper, [7]) and Tarski [10] developed a theory based on three primitive notions – alongside the notion of a *region* and mereological relation of *inclusion* he assumes the metrical notion of a *sphere*. Survey papers describing various aspects of and historical remarks on region-based theory of space are [5] and [8].

2. This issue

In the previous section we described the origin of point-free geometry in a very compact way. As a matter of fact a comprehensive historical investigation of the subject is still an open issue. An important step in this direction is the opening paper of Bélanger and Marquis in which the authors look at how the idea of point-free topology evolved by referring to the definitions of topological space given by Menger and Nöbeling.

The papers by Maietti and Sambin and by Ciraulo consider problems in point-free geometry from an intuitionistic perspective. The first gives solid arguments demonstrating that the point-free approach to formal topology, introduced by Martin-Löf and Sambin in the 1980s, is compulsory and not just an option. This gives a new, strong and unexpected support to point-free geometry.

In the second, by Ciraulo, structures which are not based on a Boolean algebra (in accordance with classical logic) but on a frame (in accordance with intuitionistic logic) are considered. The importance of giving an adequate definition of a point (namely of an *atom*) in these structures is evident. Ciraulo considers and examines several possible definitions.

The paper by Duntsch and Li is an important application of model theoretical tools to the Boolean contact algebras. The existence of a (unique) countable homogeneous Boolean contact algebra is ascertained and this basic algebraic structure is examined by focusing attention on the algebra of the binary relations generated by the contact relation.

Borgo's paper, which opens the second part of the issue, refers to Tarski's approach from [10] through a consideration of adopting primitive notions other than that of a *sphere*. Hyper-cubes and regular simplexes are proposed. It is shown that taking the notion of *sphere* as a primitive is not the best choice from a technical point of view.

The possibility of introducing a dynamic version of relational mereotopology is considered in Nenchev's paper where a mereotopological relation R can appear in two versions. The first "stable version", formalizes the situation in which a pair of changing objects is "always" in the relation R, while the second "unstable version" is one in which a pair of changing objects is "sometimes" in the relation R.

The paper by Bucalo and Rosolini concerns a categorical approach to point-free topology. In spite of the fact that this theory is far from the tradition of point-free geometries, it represents a very complete and tested example of a point-free approach to mathematics.

The final paper by Hsing-chien Tsai focuses on the very important logical problem of decidability. The finite inseparability and therefore undecidability of a large class of mereotopological theories is proved.

To conclude, at beginning of the development of point-free geometries the contributions to the discipline were infrequent. In recent times the literature on this subject have increased in quantity and in quality. Currently point-free geometry is a well-founded topic in both logic and mathematics and in our opinion the works we hereby present testify to this well.

References

[1] Casati R., and A. Varzi, "Spatial entities", in *Spatial and Temporal Reasoning*, Oliviero Stock (ed.), Dordrecht, Kluwer, 1997, pp. 73–96 .

[2] Clarke, B., "A calculus of individuals based on connection", *Notre Dame Journal of Formal Logic*, 22 (1981), 3: 204–218.

[3] Clarke, B., "Individuals and points", *Notre Dame Journal of Formal Logic* 26 (1985), 1: 61–75.

[4] De Laguna, T., "Point, line and surface as sets of solids", *The Journal of Philosophy*, 19 (1922): 449–461.

[5] Gerla, G., "Pointless geometries", in *Handbook of Incidence Geometry*, F. Buekenhout (ed.), Elsevier Science, 1994, pp. 1015–1031.

[6] Gerla, G., and B. Paolillo, "Whitehead's point-free geometry and diametric posets", *Logic and Logical Philosophy*, 19 (2010): 289–308.

[7] Grzegorczyk, A., "Axiomatization of geometry without points", *Synthese*, 12 (1960): 228–235.

[8] Gruszczyński, R., and A. Pietruszczak, "Full development of Tarski's geometry of solids", *The Bulletin of Symbolic Logic*, 14 (2008), 4: 481–540.

[9] Lobachevskii, N. I., "New principles of geometry with complete theory of parallels" (in Russian), *Polnoe Sobranie Socinenij*, 2 (1835).

[10] Tarski, A., "Foundations of the geometry of solids", in *Logic, Semantics, Metamathematics*, J. Corcoran (ed.), Oxford University Press, Oxford, 1956, pp. 24–30. Translation of: "Les fondaments de la géométrie des corps", in *Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego*, 3 (1929), pp. 29–33.

[11] Whitehead, A. N., *An Inquiry Concerning the Principles of Natural Knowledge*, Univ. Press. Cambrige, 1919.

[12] Whitehead, A. N., *The Concept of Nature*, Univ. Press. Cambrige, 1920.
[13] Whitehead, A. N., *Process and Reality*, Macmillan, 1929, N. Y.

Cristina Coppola and Giangiacomo Gerla
Dipartimento di Matematica
Università degli Studi di Salerno
via Ponte don Melillo (SA) Italy
{ccoppola,ggerla}@unisa.it