Logic and **L**ogical **P**hilosophy
Volume 22 (2013), 21–43
DOI: 10.12775/LLP.2013.002

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THE ABUNDANCE OF THE FUTURE
A Paraconsistent Approach to Future Contingents*

Abstract. Supervaluationism holds that the future is *undetermined*, and as a consequence of this, statements about the future may be *neither true nor false*. In the present paper, we explore the novel and quite different view that the future is *abundant*: statements about the future do not lack truth-value, but may instead be *glutty*, that is *both true and false*. We will show that (1) the logic resulting from this "abundance of the future" is a *non-adjunctive paraconsistent* formalism based on *subvaluations*, which has the virtue that all classical laws are valid in it, while no formula like $\phi \wedge \neg\phi$ is satisfiable (though both ϕ and $\neg\phi$ may be true in a model); (2) The peculiar behaviour of abundant logical consequence has an illuminating analogy in *probability logic*; (3) abundance preserves some important features of classical logic (not preserved in supervaluationism) when it comes to express those important *retrogradations of truth* which are presupposed by the argument *de praesenti ad praeteritum*.

Keywords: Future contingents, supervaluationism, gluts, subvaluations, retrogradation of truth.

* The authors wish to thank George Masterton for many detailed comments and for insightful remarks on probabilities, as well as Heinrich Wansing and Greg Restall who read and commented a previous version of this work. We also take the opportunity to thank Mauro Mariani who introduced both of us to temporal logic and its philosophical applications. This work was carried out while Roberto Ciuni was a Humboldt Postdoctoral Fellow with the project 'A Tempo-Modal Logic for Responsibility Attribution in Many-Step Actions' (2011–2013) and while he was a collaborator of the project 'Logiche tempo-modali per gli agenti deontici' (University of Padova, Departments of Philosophy and Mathematics, 2011–2013, Progetto di

Received April 18, 2012; *Revised* October 8, 2012
 ISSN: 1425-3305

1. Introduction

The future — so it is often said — is *undetermined*. If nothing today makes it inevitable that there will be a sea battle, or inevitable that there will never be any such battle, then the course of the events is open to both options.

It is often claimed, as an immediate consequence of this, that sentences like "There will be a sea battle" are *neither true nor false*: they are *unsettled*. Obviously, not all the sentences about the future encounter this fate: "At least once in the future there is a sea battle or there is no such battle" is true at any moment m, since no possible developments of the events includes a later moment m' where "Either there is a sea battle or there is no sea battle" (or any other tautology) fails to hold. Also, "Either there will be a sea battle or there will be no such battle" (in symbols, $\mathrm{F}\,\phi \vee \neg\,\mathrm{F}\,\phi$) is true at any moment m, since for every possible development, either there is a later moment where ϕ holds, or there is no such moment. This view — nowadays known as *supervaluationism* — has always had a good number of proponents and presupposes a conception of truth that — for our purposes — can be stated as follows: "Truth and falsehood imply *settledness*; without settledness, there is lack of truth-value".[1]

Here, we want to investigate the "mirror" of supervaluationism: the view that the future is — as we shall say — *abundant*: future truths and falsehoods are not absent, rather they abound. Not only "There will be a sea battle" is *true* today, but also "There will be no sea battle" is, since the world is open to develop either way. We shall call this

ricerca di Ateneo Bando 2010 — CPDA109451). During this period Carlo Proietti was a Postdoctoral Fellow at Lund University with the project 'Ceteris paribus modal logic for Epistemology and Philosophy of Science' (financed by the Swedish Research Council, 2009–2011) and subsequently within the 'Lund University Knowledge and Information Quality Research Group' (2012).

[1] The technique of *supervaluations*, which equates *truth* to *settledness*, has been first introduced by Bas van Fraassen in [van Fraassen, 1966]. It has been applied to a wide range of philosophical and logical problems, such as vagueness and more specific questions about it (see for instance [Fine, 1975], [Kamp, 1981], [Thomason, 1972]), presuppositions ([van Fraassen, 1969]), lack of denotation ([van Fraassen, 1966], [Skyrms, 1968]), quantum physics ([Bigaj, 2001], [Lambert, 1969]). The foremost application of supervaluations to future contingents is due to Richmond Thomason, in [Thomason, 1970]. Here, we shall abuse terminology a bit and refer to the supervaluationist semantics for temporal logic as "supervaluationism" or "supervaluationist semantics" simply.

approach the *abundance of the future*. To the best of our knowledge, this view has never been deeply investigated, and this is probably due to its contradictory flavor. Indeed, this position is naturally embodied by a *paraconsistent* semantics (Section 4). Can this possibly make sense or it is just a "weird" conceptual alternative among all the possible temporal logics? The question becomes more pressing since this alternative seems to have been left unexplored during more than two thousand years, whereas supervaluationism seemingly counts Aristotle (*On Interpretation* chap. 9) as its most eminent forerunner. Possible motivations for asymmetry could depend either on the *prima facie* unnaturalness of paraconsistency — the best case — or on the relative intrinsic lack of "utility" of this logical approach — in the worst case. In order to defeat criticisms of the second kind one should give a possible application, or at least a natural interpretation of this logic.

Abundance has at least some intuitive grounding in our linguistic use: most of the times, when we say "tomorrow it is going to rain" we do not claim that this is certain, only that there are reasons to assert it, but since there may also be reasons for asserting its negation, both seem to be tenable. In general, a truth in the abundant sense is to be read as something that one is *entitled to claim* or to defend. Otherwise, a proposition is true if the event corresponding to it has at least *some degree of probability*, i.e. if we can rationally *bet* on it. Notwithstanding the fact that the semantics we present here is purely *qualitative*, a probabilistic interpretation of it will shed light on its paraconsistent features, especially on some peculiar properties of *abundant logical consequence* such as the failure of multiple-premises classical consequence. The most significant result in this respect is condensed by Theorem 4 (Section 6).

The goal of this paper is, therefore, twofold. First, it aims at giving a clear formal presentation of a paraconsistent abundant temporal logic, which is in turn based on the technique of *subvaluations*. Second, it aims at presenting the reasons that could make this logic an option for future contingents. The work is structured as follows: Section 2 introduces the classical problem of *future contingents*, or contingent statements *de futuro* and presents the standard Ockhamist semantics based on branching-time structures. Section 3 gives an overview of supervaluationist semantics. Section 4 introduces the abundance of the future and its semantics, and shows how it applies to contingent statements *de futuro* in a branching-time setting. In that section, we also show that classical laws are preserved by our approach. Section 5 illustrates that, appearances notwithstanding, abundance turns out to be on a par with supervaluationism w.r.t. preservation of classical logic. In particular, we shall show that the two approaches have similar troubles when it comes to preservation of *classical rules of inference*. Section 6 explores the interesting links between abundance and the logic of probability. In Section 7, we show that abundance is somehow "closer" to the classical inferential apparatus than supervaluationism in case we supply our logical language with devices for expressing *retrogradation of truths*. Section 8 resumes the content of the paper and discusses some of its possible openings.

2. Future Contingents and Ockhamist Semantics

Many formal approaches have been devised in order to deal with the logic of *future contingents*, each one encoding a specific philosophical view.[2] The basis of most of them, including ours, is A. Prior's *Ockhamist semantics*,[3] which gives formal expression to the idea that sentences about *future contingents* are either true or false, without however *ipso facto* being *necessarily true* or *necessarily false*. This is accomplished by relativizing truth not only to a moment, but also to histories, i.e. specific courses of events. The consequence of this take is that "There will be a sea battle tomorrow" is true *now* w.r.t. a given history, and yet it may well be false *now* w.r.t. another one. Ockhamist semantics is motivated by the purpose of keeping bivalence while at the same time resisting the so-called *Master Argument* by Diodorus Cronus, which — briefly put — was supposed to prove that if propositions concerning the future are *either true or false*, then every future event will happen *by necessity* (or is already *settled*).[4]

[2] The debate on which view on *future contingents* proves more suitable from the conceptual point of view traces at least back to Aristotle, whose stance on the topic is to be found in the very famous chapter IX of *On Interpretation* ([Aristotle, 1941]).

[3] Ockhamist semantics was introduced in [Prior, 1967] (for a detailed analysis of it see also [Thomason, 1984] and [Hasle and Øhrstrøm, 1995], p. 14 and 212–215). However, it must be noticed that there are reasons to doubt that the semantical device put forth under this name can actually be an adequate formal rendering of Ockham's views on future contingents (on this point see [Øhrstrøm, 2009]).

[4] An immediate consequence of such an argument is that there would be no place for *contingency* in the future, nor for *free choice* or *decision* about it. This would in turn open the way to a full-blooded *fatalism*, with consequent philosophical indesiderata which are easy to imagine.

Let us now introduce the semantics proper. The Ockhamist language $\mathcal{L}_{\mathrm{Ock}}$ consists in a set of atomic formulas p, q, ... (intuitively representing immediate present tense sentences), the boolean operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, the temporal operators F ("it will be the case that") and P ("it has been the case that") and an additional operator $\Box$, whose intuitive reading is "it is necessary that"[5] or "it is determined that" or again "it is settled that"; ϕ, ψ, ... stands for arbitrary formulas. Future contingent statements like "there will be a sea battle" can thus be expressed by formulas like $\mathrm{F}\,p$. The structures employed by Ockhamist formalisms are *branching-time* structures $\mathcal{T}$ of the form $\langle T, < \rangle$, where T is a nonempty set of moments[6] and $<$ is a strict ordering relation (i.e. irreflexive, transitive and asymmetrical), where the $<$-predecessors of any point m from T are totally ordered by $<$. A *history* h is a *maximal* *chain* in $\mathcal{T}$ for the relation $<$. The set of histories in $\mathcal{T}$ will be called $\mathrm{H}(\mathcal{T})$. Given a moment m, H_m will designate the set of all histories containing it, and h_m will be a shortening for "h such that contains m". A structure of this kind is represented in Figure 1, where right-branching represents the different possible future courses of time (or development

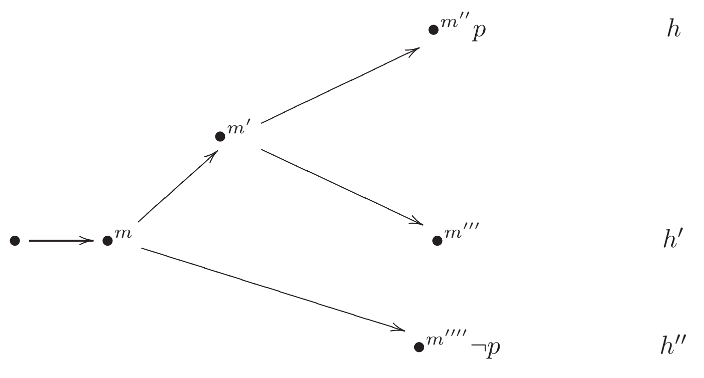

Figure 1. A model for branching time

[5] This notion of necessity is not a *logical* or *absolute* one, but temporal. Medieval philosophers named it necessity *per accidens*. It is that kind of necessity that can be attributed to past events as already *settled*.

[6] The cardinality of T does not matter here. However, T is usually assumed to be *continuous* (with $<$ being a *dense* ordering), since this seems the most adequate representation of time.

of the events). We then define Ockhamist models $\mathcal{M} = \langle T, <, V \rangle$, where V is an evaluation function assigning to every propositional variable a subset of $T \times \mathrm{H}(\mathcal{T})$. As clear from below, the evaluation of propositions is *classical* (and hence *bivalent*), but relative to a moment $m \in T$ and a history $h \in \mathrm{H}(\mathcal{T})$:

$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} p \Leftrightarrow \langle m, h \rangle \in V(p)$
$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \neg\phi \Leftrightarrow \mathcal{M}, \langle m, h \rangle \not\models_{\mathrm{Ock}} \phi$
$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \phi \wedge \psi \Leftrightarrow \mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \phi$ and $\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \psi$
$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \phi \vee \psi \Leftrightarrow \mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \phi$ or $\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \psi$
$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} P\phi \Leftrightarrow \exists m' < m$ such that $\mathcal{M}, \langle m', h \rangle \models_{\mathrm{Ock}} \phi$
$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} F\phi \Leftrightarrow \exists m' > m$ such that $\mathcal{M}, \langle m', h \rangle \models_{\mathrm{Ock}} \phi$

In addition, the following condition must be imposed:

IPP $\langle m, h \rangle \in V(p) \Rightarrow \forall h'(h' \in H_m \Rightarrow \langle m, h' \rangle \in V(p))$.

which states the *inevitability of the past and the present* (whence the label). Using a standard notation, $\mathcal{M}, m \models_{\mathrm{Ock}} \phi$ will stand for truth of a formula ϕ in an Ockhamist model at a moment m for every history in H_m. *Global truth* of a formula in an Ockhamist model will be $\mathcal{M} \models_{\mathrm{Ock}} \phi$ and truth in every Ockhamist model, i.e. Ockhamist validity, will be $\models_{\mathrm{Ock}} \phi$.

In order to express *settledness*, or *necessity per accidens*, Ockhamist semantics employs the operator $\Box$, which is defined by the following clause:

$$\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \Box\phi \Leftrightarrow \forall h'(h' \in H_m \Rightarrow \mathcal{M}, \langle m, h' \rangle \models_{\mathrm{Ock}} \phi)$$

By **IPP** and the truth-clause of $\Box$ it follows that whenever a sea battle *is taking place* (*took place*), it is then *settled* that it is taking place (took place).[7]

The Ockhamist truth-relation and the notion of settledness are key to understanding how Ockhamist semantics retains bivalence and yet resists the inference from *truth* to *settledness* imposed by the Master Argument: on the one side, $\langle m, h \rangle \in V(\phi) \Leftrightarrow \langle m, h \rangle \notin V(\neg\phi)$, while on the other side, checking the truth of $\mathrm{F}\,p$ and $\Box\,\mathrm{F}\,p$ at $\langle m, h \rangle$ in the model in Figure 1 is enough to see that the step from the truth of $\mathrm{F}\,\phi$ to the truth of $\Box\,\mathrm{F}\,\phi$ is blocked.

[7] This is expressed by $\models_{\mathrm{Ock}} p \to \Box p$ and $\models_{\mathrm{Ock}} \mathrm{P}\,p \to \Box\,\mathrm{P}\,p$, where p is clearly not to be intended as freely substitutable.

3. Supervaluationism

Though successful in resisting the Master Argument, Ockhamist semantics seems to encounter a serious conceptual problem: in a nutshell, while it is clear which moment of evaluation we are referring to (usually the "moment of assertion" itself, when we make a given utterance), it cannot be clear at all which *history* of evaluation we would refer to.[8] Thus, a semantics which primarily takes statements *de futuro* to be evaluated only w.r.t. moments seems to be conceptually more tenable.[9]

To this effect, supervaluationism assumes that when we state "There will be a sea battle", we are not stating that the sea battle will happen in a given history h: we are just saying that it will happen, full stop. In order to formally render this idea, supervaluationism intervenes on the semantical notion of truth. Truth of ϕ at m — this is the general claim — involves *per se* a universal quantification over possible histories and the same holds for falsehood. In order to distinguish the supervaluationist notion of truth and falsehood from the Ockhamist ones expressed by $\models_{\text{Ock}}$ and $\not\models_{\text{Ock}}$, we will follow the standard use and call them "supertruth" and "superfalsehood".

Given an Ockhamist model $\mathcal{M} := \langle T, <, V \rangle$ we can define a corresponding supervaluationist model $\mathcal{M}_{\text{Sup}} := \langle T, <, V^+, V^- \rangle$, where T and $<$ are as in $\mathcal{M}$ and V^+ (supertruth) and V^- (superfalsehood) are supervaluationist evaluation functions from propositional variables to *sets of moments* which are defined as follows:

$$m \in V^+(\phi) \Leftrightarrow \forall h(h \in H_m \Rightarrow \mathcal{M}, \langle m, h \rangle \models_{\text{Ock}} \phi)$$
$$m \in V^-(\phi) \Leftrightarrow \forall h(h \in H_m \Rightarrow \mathcal{M}, \langle m, h \rangle \not\models_{\text{Ock}} \phi)$$

Truth at a moment m in a supervaluationist model $\mathcal{M}_{\text{Sup}}$ can be also denoted in the usual way by $\mathcal{M}_{\text{Sup}}, m \models_{\text{Sup}}$ letting the following relations

[8] This is due to the fact that, while the moment m of evaluation is in principle distinguishable from the earlier and later ones, the many possible histories of evaluation passing through m are in principle indistinguishable at m, since they verify exactly the same statements on past and present and differ only in their evaluations of statements about the future. Thus, believing that we can distinguish them proves incompatible with any full-blooded form of indeterminism.

[9] Many philosophers seem oriented toward this conclusion. In [Prior, 1967], Prior himself points out that truth relative to a history is only "*prima facie* truth". Also [MacFarlane, 2003] claims that the notion of *truth-at-m* is indeed *primary*, from the perspective of natural language understanding, w.r.t. truth-at-$\langle m, h \rangle$.

to hold:

$\mathcal{M}_{\text{Sup}}, m \models_{\text{Sup}} \phi$ iff $m \in V^{+}(\phi)$
$\mathcal{M}_{\text{Sup}}, m \models_{\text{Sup}} \neg\phi$ iff $m \in V^{-}(\phi)$

Truth of a formula in a supervaluationist model ($\mathcal{M}_{\text{Sup}} \models \phi$), truth in every model (supervaluationist validity, $\models_{\text{Sup}} \phi$) and satisfiability are defined as usual. Here, the truth of ϕ is intended as truth in all possible histories, i.e. supervaluationism equates the truth of ϕ with that of $\Box\phi$ in Ockhamist semantics.[10] It is easy to see that **IPP** extends to supervaluationist semantics.[11]

4. The abundance of the future

Contrary to the Ockhamist semantics, supervaluationism assumes a *history-independent* notion of truth, i.e. a notion of truth which is relative to a moment m but to no particular history. Our "abundant" approach agrees with supervaluationism on this point, and also presents a notion of truth which is *indeterminist*, insofar as it is essentially rooted on a branching-time semantics. The core difference is that abundance takes satisfaction in one single history to suffice for truth. From the formal point of view, the abundance of the future is defined by the technique of *subvaluations*, which are in turn the dual of supervaluations.[12] The main effect of this choice is that *truth-value gluts* are the key feature of abundance.[13]

[10] The technique of supervaluationism consists indeed in defining a notion of truth (namely, supertruth) which is equivalent to "holding w.r.t. to *all* the relevant parameters of a given kind". When applied to branching-time, histories are such parameters.

[11] As a consequence, when a formula contains no occurrence of F, truth relative to a history h (i.e. Ockhamist truth) coincides with truth relative to all histories h' (i.e. supertruth). As a consequence, the discrepancy between not being supertrue and being superfalse arises only when future tense sentences are considered. The fact that **IPP** holds also in the supervaluationist semantics follows from the fact that $m \models_{\text{Sup}} \phi$ can be defined as $\forall h_m(\langle m, h\rangle \models_{\text{Ock}} \phi)$.

[12] For some detailed studies on the duality between supervaluations and subvaluations see [Varzi, 1995], [Varzi, 1999] [Varzi, 2003].

[13] The view we are devising here is unprecedented, at least to our knowledge: while supervaluations have been frequently applied to the problem of future contingents, the applications of subvaluations have not included such a problem, while focusing on other philosophical issues such as vagueness (for such an application, see [Hyde, 1997]).

Subvaluationist semantics are also based on Ockhamist ones: given an Ockhamist model $\mathcal{M} := \langle T, <, V \rangle$, a corresponding abundant model has the form $\mathcal{M}_{\mathrm{Ab}} := \langle T, <, V^+_{\mathrm{Ab}}, V^-_{\mathrm{Ab}} \rangle$, where T and $<$ are as in $\mathcal{M}$ and V^+_{Ab} and V^-_{Ab} are functions from propositional variables to *sets of moments* such that:

$$m \in V^+_{\mathrm{Ab}}(\phi) \Leftrightarrow \exists h(h \in H_m \text{ and } \mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \phi),$$
$$m \in V^-_{\mathrm{Ab}}(\phi) \Leftrightarrow \exists h(h \in H_m \text{ and } \mathcal{M}, \langle m, h \rangle \not\models_{\mathrm{Ock}} \phi).$$

In order to distinguish the abundant notion of truth from the Ockhamist and supervaluationist one, we shall also speak of "subtruth" for V^+_{Ab} and "subfalsehood" for V^-_{Ab}. From the above, it is clear that $m \in V^+_{\mathrm{Ab}}(\neg\phi) \Leftrightarrow m \in V^-_{\mathrm{Ab}}(\phi)$. Abundance then guarantees the minimal assumption that *subfalsehood* of a formula ϕ equates with *subtruth* of its negation $\neg\phi$. This is also a feature of supervaluationism: given the definitions of V^+ and V^- in Section 2.1, it is easy to check that $m \in V^+(\neg\phi) \Leftrightarrow m \in V^-(\phi)$ (*superfalsehood* of a formula ϕ equates with *supertruth* of its negation $\neg\phi$).

Furthermore, $m \notin V^+_{\mathrm{Ab}}(\phi) \Rightarrow m \in V^-_{\mathrm{Ab}}(\phi)$ also holds. Thus, the subvaluationist apparatus of abundance is *bivalent*: if a formula fails to be subtrue, then it is subfalse.[14] This is not a feature of supervaluationism: once again by the definitions in Section 2.1, it is clear that $m \notin V^+(\phi)$ and $m \notin V^-(\phi)$ may be the case for some sentence ϕ: there are at least a sentence ϕ and an Ockhamist model such that $\langle m, h \rangle \models_{\mathrm{Ock}} \phi$ and $\langle m, h' \rangle \models_{\mathrm{Ock}} \neg\phi$ (with $h \neq h'$), to the effect that neither ϕ and $\neg\phi$ are supertrue.

The semantics of abundance is *not consistent*, though. It is easy to check that $m \in V^+_{\mathrm{Ab}}(\phi) \not\Rightarrow m \notin V^+_{\mathrm{Ab}}(\neg\phi)$: the Ockhamist model above—where $\langle m, h \rangle \models_{\mathrm{Ock}} \phi$ and $\langle m, h' \rangle \models_{\mathrm{Ock}} \neg\phi$ hold (with $h \neq h'$)—is enough to show that both ϕ and $\neg\phi$ may be subtrue.[15] This is indeed the source of paraconsistency in abundance (as in any subvaluationist approach).

Truth in a model and at a moment is denoted by $\mathcal{M}_{\mathrm{Ab}}, m \models_{\mathrm{Ab}} \phi$. Truth of a formula in an abundant model ($\mathcal{M}_{\mathrm{Ab}} \models \phi$), and truth in

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[14] Easy to check, if a formula fails to be false, then it is true: $m \notin V^-_{\mathrm{Ab}}(\phi) \Rightarrow m \in V^+_{\mathrm{Ab}}(\phi)$.

[15] However, notice that the converse holds: if $\neg\phi$ holds relative to no history, ϕ holds for at least some histories (more precisely, for all histories). Thus, $m \notin V^+_{\mathrm{Ab}}(\neg\phi) \Rightarrow m \in V^+_{\mathrm{Ab}}(\phi)$. It is also easy to check that that $m \notin V^+_{\mathrm{Ab}}(\phi) \Rightarrow m \in V^+_{\mathrm{Ab}}(\neg\phi)$.

every abundant model (abundant validity, $\models_{\mathrm{Ab}} \phi$) are defined by the usual universal quantifications. Satisfiability is defined as usual. As for supervaluationism we can write:

$\mathcal{M}_{\mathrm{Ab}}, m \models_{\mathrm{Ab}} \phi$ iff $m \in V^{+}_{\mathrm{Ab}}(\phi)$
$\mathcal{M}_{\mathrm{Ab}}, m \models_{\mathrm{Ab}} \neg\phi$ iff $m \in V^{-}_{\mathrm{Ab}}(\phi)$

It is then clear that abundance equates the truth of ϕ with the truth of $\neg\Box\neg\phi$ (briefly, $\Diamond\phi$) in Ockhamist semantics, and that $m \models_{\mathrm{Ab}} \phi$ is nothing but $\exists h_m(\langle m, h\rangle \models_{\mathrm{Ock}} \phi)$. As it was already clear from our last remark, the definition of $\models_{\mathrm{Ab}}$ results in the presence of *truth-value gluts* (the model in Figure 1 indeed represents a situation where $\mathcal{M}, m \models_{\mathrm{Ab}} \mathrm{F}\, p$ and $\mathcal{M}, m \models_{\mathrm{Ab}} \neg\, \mathrm{F}\, p$). The possibility of *gluts* is, however, confined to statements about the future. Indeed, it is easy to see that **IPP** transmits from Ockhamist semantics to abundance, to the effect that $m \in V^{+}_{\mathrm{Ab}}(p) \Leftrightarrow m \notin V^{+}_{\mathrm{Ab}}(\neg p)$ and $m \in V^{+}_{\mathrm{Ab}}(\mathrm{P}\, p) \Leftrightarrow m \notin V^{+}_{\mathrm{Ab}}(\neg\, \mathrm{P}\, p)$, and similarly for any formula not containing any occurrence of F. This matches what happens in supervaluationism, where the notions of supertruth and Ockhamist truth coincide in case of sentences which contains no F. Thus, when it comes to sentences that contain no future operator, **IPP** restores *consistency* for abundance and *bivalence* for supervaluationism. This is a desideratum, since we may reasonably want to limit departure from consistency to problematic cases — namely, future contingents.

5. Abundance and Logical Consequence

[Hyde, 1997] has shown that — contrary to received opinion — supervaluationist and subvaluationist semantics are on a par w.r.t. "preservation" of classical propositional logic. In other words, none of them preserves all classical inferences, but their deviations are "specular".[16] After introducing the notion of "logical consequence", which we hold to be *preservation of truth* relative to the notion of truth of our logic, we will show that (i) *if single premises and multiple conclusions* are at stake, abundant

[16] While subvaluations fail to preserve inferences with multiple premises, supervaluation cannot preserve multiple conclusion classical inferences. The reason why conformity with classical logic matters so much is that supervaluationism and subvaluationism have been devised to avoid some indesiderata in specific philosophical fields (vagueness, future contingents, and so on) *without* at the same time imposing a dramatic departure from the rules and the laws of classical reasoning.

consequence and Ockhamist consequence coincide (Theorem 1), and (ii) the coincidence does not extend to the case of *multiple premises and single conclusions* (Theorem 2).

The notion of *logical consequence* relative to our logic *Ab* is defined as follows:

(LC) A set Δ of formulas is a logical consequence of a set Γ ($\Gamma \models_{\mathrm{Ab}} \Delta$) *iff* whenever *all* formulas in Γ are *subtrue* then at least *some* formula in Δ is *subtrue*.

LC encodes the idea that logical consequence is essentially *preservation of subtruth*, that is preservation of truth in the sense of "truth" endorsed by abundance.[17] *Logical validity* is to be understood in the usual way, i.e. ϕ is *logically valid* iff $\emptyset \models_{\mathrm{Ab}} \phi$ (or simply, $\models_{\mathrm{Ab}} \phi$).

According to this definition, it is easy to see that abundance preserves classical validities, in the sense that $\models_{\mathrm{Ab}} \phi$ whenever ϕ has the form of a tautology of classical propositional logic. A more general result can be proved by following the same line of [Hyde, 1997], namely:

Theorem 1. *For any formulas $\phi, \psi_1, \ldots, \psi_k$ of the purely temporal language (i.e. without additional alethic operators $\Box$, $\Diamond$) the following equivalence holds:*

$$\phi \models_{\mathrm{Ock}} \psi_1, \ldots, \psi_k \quad \textit{iff} \quad \phi \models_{\mathrm{Ab}} \psi_1, \ldots, \psi_k.$$

Proof. ($\Rightarrow$) Suppose that (i) $\phi \models_{\mathrm{Ock}} \psi_1, \ldots, \psi_k$ and (ii) $m \models_{\mathrm{Ab}} \phi$. By (ii), there is a h such that $\langle m, h\rangle \models_{\mathrm{Ock}} \phi$; then it follows from (i) that $\langle m, h\rangle \models_{\mathrm{Ock}} \psi_i$ for some $1 \leqslant i \leqslant k$ and thus $m \models_{\mathrm{Ab}} \psi_i$.

($\Leftarrow$) Suppose that $\phi \not\models_{\mathrm{Ock}} \psi_1, \ldots, \psi_k$. Then there is a model $\mathcal{M}$ and a point m, h such that $\mathcal{M}, \langle m, h\rangle \models_{\mathrm{Ock}} \phi$ and $\mathcal{M}, \langle m, h\rangle \not\models_{\mathrm{Ock}} \psi_i$ for all $1 \leqslant i \leqslant k$. Consider now a model $\mathcal{M}'$ obtained from $\mathcal{M}$ by eliminating all histories except h. It is easy to prove (by induction on the complexity) that for every *purely temporal* formula ψ:

$$\mathcal{M}, \langle m, h\rangle \models_{\mathrm{Ock}} \psi \quad \text{iff} \quad \mathcal{M}', \langle m, h\rangle \models_{\mathrm{Ock}} \psi.$$

It then follows that $\mathcal{M}', m \models_{\mathrm{Ab}} \phi$ and, since h is the only history in H_m, $\mathcal{M}', m \not\models_{\mathrm{Ab}} \psi_i$ and the result is proved. $\dashv$

[17] This is just one among many possible definitions of *consequence*. The reason why we are adopting it is that it gives us the most natural expression of *preservation* relative to subtruth. A detailed study of the many possible notions of logical consequence for supervaluationism and subvaluationism can be found in [Varzi, 2007].

We should notice that the left-to right implication ($\Rightarrow$) is independent from the language we adopt, therefore it also holds for languages with $\Box$ and $\Diamond$. Thus, every classically valid single premise inference is valid in abundant semantics. By contrast, the opposite direction ($\Leftarrow$) depends on the fact that the model restriction preserves modal satisfaction. This may not be true for more expressive languages and we should leave this as a conjecture.

This result raises a more general question: what happens in the case of multiple premises consequences? Failure of *modus ponens* in many paraconsistent logics — e.g. discussive logics — is a sufficient reason to suspect that the same can hold in our logic. The suspicion is indeed correct, since

$$\Gamma, \phi, \phi \to \psi \not\models_{\mathrm{Ab}} \psi$$

fails: this can be tested in the model in Figure 1.[18] However, an immediate consequence of Theorem 1 is that *single premise Modus Ponens* holds in abundance, i.e. $\phi \wedge (\phi \to \psi) \models_{\mathrm{Ab}} \psi$.

Once again, we can prove a general result, to the effect that classically valid schemata for single conclusions from many premises are not sound for the abundant approach. In order to prove it, we first define the concept of *mutual independence* and *essentiality*. Given a set $\Gamma = \{\psi_1, \dots, \psi_n\}$ of formulas, we shall say that these formulas are *mutually independent* if there is a model $\mathcal{M}$ and a point m such that for each ψ_i there is a $h_i \in H_m$ such that $\langle m, h_i \rangle \models_{\mathrm{Ock}} \psi_i$ and, for any $j \neq i$, $\langle m, h_i \rangle \not\models_{\mathrm{Ock}} \psi_j$. Given a consequence $\Gamma \models_{\mathrm{Ock}} \phi$ we say that $\psi_i \in \Gamma$ is *essential* for the derivation of ϕ if and only if $\Gamma \setminus \{\psi_i\} \not\models_{\mathrm{Ock}} \phi$. We can then state the following.

Theorem 2. *Given a consequence $\Gamma \models_{\mathrm{Ock}} \phi$, if $\Gamma = \{\psi_1, \dots, \psi_n\}$ and $\psi_1, \dots, \psi_n$ are all* mutually independent *and* essential *for the derivation of ϕ, then $\Gamma \not\models_{\mathrm{Ab}} \phi$.*

Proof. By mutual independence we have a model $\mathcal{M}$ and a point m such that $\mathcal{M}, m \models_{\mathrm{Ab}} \psi_i$ for all $\psi_i \in \Gamma$. Since the premises are essential, their mutual independence at m implies that $\mathcal{M}, \langle m, h \rangle \not\models_{\mathrm{Ock}} \phi$ for all $h \in H_m$. It then follows that $\mathcal{M}, m \not\models_{\mathrm{Ab}} \phi$. $\dashv$

[18] If we replace there ϕ with $\mathrm{F}\,p$ and ψ with $\mathrm{F}\,q$, we will have $m \models_{\mathrm{Ab}} \mathrm{F}\,p$ and $m \models_{\mathrm{Ab}} \mathrm{F}\,p \to \mathrm{F}\,q$ (because the antecedent is false somewhere) but $m \not\models_{\mathrm{Ab}} \mathrm{F}\,q$.

From Theorem 2, it follows as an immediate corollary that the *rule of adjunctivity* fails:

$$\Gamma, \phi, \psi \not\models_{\mathrm{Ab}} \phi \wedge \psi$$

Abundance thus belongs to the family on *non-adjunctive paraconsistent* formalisms.

6. Abundance and the Logic of Probability

Theorem 2 shows a noteworthy divergence from classical reasoning. However, abundance is far from being an isolated phenomenon in this respect. Indeed, when it comes to reasoning with *uncertain premises*, the logic of probability (see [Adams and Levine, 1975] and [Adams, 1999]) displays the same departure from classical reasoning. A typical example is presented by the lottery paradox, which shows the failure of *adjunctivity* in probabilistic reasoning. Consider a lottery with 1000 tickets and the propositions ϕ_i = "ticket i will not win" which we can assume to have a probability of 0.999 and, therefore, an uncertainty of 0.001. Clearly $\phi_1, \ldots, \phi_{1000} \models \phi_1 \wedge \cdots \wedge \phi_{1000}$ is a classically valid inference, where the consequent is to be read as "no ticket will win", but when the premises are not certain, as it is the case here, the conclusion can have probability 0 as it does.

The affinity between abundance and the logic of probability goes beyond a mere conformity in the failure of classical inferences, and seems to point at a common source of the failure: as we prove in Theorem 4, validity of an abundant consequence is relevantly connected with the relation between the *degree of uncertainty* of its premises and that of its conclusions — a relation which is assessed w.r.t. every probability function that does not attach probability 0 to any of the relevant histories. Before proving the result, we introduce some preliminary facts.

Given any probability distribution P on the different *state descriptions* of a propositional language, for a classically valid inference there is a connexion between the number of premises and the *uncertainty* $U(x) = 1 - P(x)$ of the conclusion, whose *upper bound* is given by the following expression

$$U(\phi) \leqslant U(\phi_1) + \cdots + U(\phi_n)$$

where ϕ is the conclusion of a valid inference and $\phi_1 \ldots \phi_n$ are its premises (see [Adams, 1999] chap. 2).

A similar result can be proved for Ockhamist semantics when we define a probability distribution on the set of *possible future histories of any moment* m. Indeed, consider a point m in an Ockhamist model $\mathcal{M}$ and the set H_m. We can assign a probability distribution P_m to the event space H_m such that $0 \leqslant P_m(h) \leqslant 1$ for every $h \in H_m$ and $\Sigma_{h \in H_m} P_m(h) = 1$ and define

$$P_m(\phi) := P_m(\{h \in H_m \mid \langle m, h\rangle \models_{\mathrm{Ock}} \phi\})$$

It is easy to check that this definition satisfies the Kolmogorov axioms, i.e. for any ϕ and ψ:

K1 $0 \leqslant P_m(\phi) \leqslant 1$
K2 If $\models_{\mathrm{Ock}} \phi$ then $P_m(\phi) = 1$
K3 If $\phi \models_{\mathrm{Ock}} \psi$ then $P_m(\phi) \leqslant P_m(\psi)$
K4 If ϕ and ψ are *Ock*-inconsistent then $P_m(\phi \vee \psi) = P_m(\phi) + P_m(\psi)$

By **K1**–**K4** we can easily extend to Ockhamist semantics the *Uncertainty Sum Theorem* holding for Propositional Logics (see [Adams, 1999] chap. 2, Theorem 11).

Theorem 3 (Uncertainty Sum Theorem). *If* $\Gamma \models_{\mathrm{Ock}} \phi$ *then for any model* $\mathcal{M}$, *any* m *and any* P_m *defined on it*

$$U_m(\phi) \leqslant \textstyle\sum_{\psi \in \Gamma} U_m(\psi)$$

Proof. Suppose that $\psi_1, \dots, \psi_n \models_{\mathrm{Ock}} \phi$. Then, by K3, $P_m(\psi_1 \wedge \cdots \wedge \psi_n) \leqslant P_m(\phi)$ and therefore $U_m(\phi) \leqslant U_m(\psi_1 \wedge \cdots \wedge \psi_n)$. The stated result follows by proving, by induction on n, that $U_m(\psi_1 \wedge \cdots \wedge \psi_n) \leqslant U_m(\psi_1) + \cdots + U_m(\psi_n)$ (see [Adams, 1999] p. 32). $\dashv$

More important for our case, is an analogous result which characterizes abundant consequence relative to the class Π of all the probability functions P_m (defined on some m of some $\mathcal{M}$) such that $P_m(h) > 0$ for any $h \in H_m$.

Theorem 4. *Given a set* $\Gamma = \{\psi_1, \dots, \psi_n\}$ *and a formula* ϕ, *the following are equivalent*:

(a) $\Gamma \models_{\mathrm{Ab}} \phi$
(b) *Given any* $\mathcal{M}$ *and any* m, *for all* $P_m \in \Pi$, *if* $U_m(\psi_i) < 1$ *for every* $1 \leqslant i \leqslant n$ *then* $U_m(\phi) < 1$

Proof. ($\Rightarrow$) Suppose that (b) is not the case. Then there is a $P_m \in \Pi$ such that (i) $U_m(\psi_1), \dots, U_m(\psi_n) < 1$ and (ii) $U_m(\phi) = 1$. By (i) it follows immediately that $\mathcal{M}, m \models_{\mathrm{Ab}} \Gamma$. By (ii) we have that $P_m(\phi) = 0$

and since every h has positive probability this means that ϕ does not hold in any $h \in H_m$ and thus $\mathcal{M}, m \not\models_{\mathrm{Ab}} \phi$.

($\Leftarrow$) If (a) does not hold we can find a $\mathcal{M}$ and a m in it such that for all $\psi_i \in \Gamma$ there is a $h_i \in H_m$ such that $\mathcal{M}, \langle m, h_i \rangle \models_{\mathrm{Ock}} \psi_i$ but for no h $\mathcal{M}, \langle m, h \rangle \models_{\mathrm{Ock}} \phi$. Three cases are possible: (i) H_m is *finite*; (ii) H_m is *countable*; (iii) H_m is *more than countable*. Cases (i) and (ii) are straightforward. In case (i), if k is the cardinality of H_m then we can easily define a suitable *uniform* distribution P_m by setting $P_m(h) = \frac{1}{k}$ for all $h \in H_m$. In case (ii) we cannot have a uniform P_m, but it is always possible to find a *non uniform* one. Case (iii) is more complex because no probability distribution over *more than denumerable* sample spaces can assign a probability higher than 0 to *all* the objects in the sample space. Nevertheless, since in our language satisfaction of a formula only depends the history of evaluation, we can easily *prune* our model $\mathcal{M}$ into an adequate one. Let us define $\mathcal{M}' = (\mathcal{T}', V')$ as a restriction of $\mathcal{M} = (\mathcal{T}, V)$ such that $m \in T'$, $H_m = \{h_1, \ldots h_n\}$ and V' be the restriction of V to $\mathcal{T}'$. Now, for every formula in our language $\mathcal{M}, \langle m, h_i \rangle \models_{\mathrm{Ock}} \phi$ if and only if $\mathcal{M}', \langle m, h_i \rangle \models_{\mathrm{Ock}} \phi$; so our premises are still satisfied at m but not the conclusion. Moreover H_m is now finite and we can refer back to case (i) to define a uniform P_m on it. $\dashv$

From this result we can infer that classical reasoning fails in abundance and the logic of probability for similar reasons. Indeed, abundant consequence holds when the positive probability of the premises transfers to the conclusion, but it may fail, as in the lottery paradox, when this is not the case. Also, in analogy with the logic of probability, where an increase of the number of premises is positively connected to the uncertainty of the conclusion, abundant inference is *non-monotonic*.

7. Retrogradation of truth and Classical Rules of Inference

We have seen that *supervaluationism* and *abundance* preserve different features of classical logic. But is there any reason for possibly favouring *abundance* over *supervaluationism* in temporal reasoning? A possible reason is connected to the so-called argument for *retrogradation of truth*, also called *de praesenti ad praeteritum*, and to the fact that, in order to express the intuitions underlying the argument, supervaluationism has to be augmented with devices that force the abandonment of some important (classical) rules of inference, while abundance may be endowed with devices that enable them to be kept unaltered. *De praesenti ad praeteritum* is an argument "from the present to the past" which consists in stating *retrospectively* "the statement $\mathrm{F}\,\phi$ *was* true". A clear summary of it is given by J. MacFarlane:

> But now what about someone who is assessing Jake's utterance [there will be a sea-battle tomorrow] from some point in the future? Sally is hanging onto the mast, deafened by the roar of the cannon. She turns to Jake and says 'your assertion yesterday turned out to be true'. Sally's reasoning seems to be unimpeachable:
> Jake asserted yesterday that there would be a sea-battle today.
> There is a sea-battle today.
> So Jake's assertion was true.
> When we take this retrospective view, we are driven to assign a determinate truth-value to Jake's utterance. [MacFarlane, 2003], p. 322

Referring again to our model in Figure 1, we can see that when a sea battle really takes place, as happens at point m'', supervaluations allow the retrogradation of truth and this is reflected by the fact that $m'' \models_{\mathrm{Sup}} \mathrm{P}\,\mathrm{F}\,p$. Such a retrogradation has a role in important activities such as gambling, expecting, forecasting and the like,[19] and a suitable logic of temporal statements should give a viable rendering of it.[20]

The supervaluationist apparatus is designed to let gappy sentences turn out to have been true. In a nutshell, when evaluating $\mathrm{F}\,p$ at m we should take into account all the histories in H_m, while when evaluating $\mathrm{P}\,\mathrm{F}\,p$ at $m' > m$, we should restrict our consideration to all the histories in $H_{m'}$ throughout all the evaluation process. Thus, we can have $\mathrm{F}\,p$ gappy at m and $\mathrm{P}\,\mathrm{F}\,p$ true at m', since some of the histories in $H_m \setminus H_{m'}$ include no moment where p is supertrue, while all the histories that are both in H_m and $H_{m'}$ are such that they include a moment where p is

[19] When we win after gambling, we are given the reward exactly because the content of our gambling — say "Ribot will win the next race" — turned out to be true. After a scientific experiment is accomplished, we can state whether our forecast on the experiment proved correct, by saying that it was true (or false). After our expectations are fulfilled, we can state that their content — say "My partner will be on time" — turned out to be true.

[20] A study that explicitly insists on the importance of retrogration of truth in temporal logic is [Thomason, 1970], pp. 267-268.

supertrue (namely m').[21] However, retrogradations of truth determine a problem for the supervaluationist semantics, since, at any two moments $m < m'$, the three following hypotheses are incompatible:

(i) $m \not\models_{\mathrm{Sup}} \mathrm{F}\, p$
(ii) $m' \models_{\mathrm{Sup}} \mathrm{P}\,\mathrm{F}\, p \Leftrightarrow m \models_{\mathrm{Sup}} \mathrm{F}\, p$
(iii) $m' \models_{\mathrm{Sup}} \mathrm{P}\,\mathrm{F}\, p$

Supervaluationism is characterized precisely by the rejection of (ii), a usual assumption about *temporal dependence*: the fact that something turned out to be true *retrospectively* does not imply it was true before. But then the question arises how we can possibly express, from the point of view of m', the fact that "there will be a sea battle" was in fact indeterminate before (for instance, at m). The only way to do it seems to be the introduction of the Ockhamist operator $\Box$ in the language of supervaluationism. Indeed, contrary to (iii), we can easily notice that in our model $m' \not\models_{\mathrm{Sup}} \mathrm{P}\,\Box\,\mathrm{F}\, p$: it is not true at m' that, at m, all the histories in H_m are barred with ϕ. With $\Box$, then, the supervaluationist has the chance of distinguishing *within her language* the situation expressed by "Your statement *turned out* to be true" from "Your statement *was settled* true". However, when expanding the language this way, many classical inferential rules cease to be valid in supervaluationism:[22]

THEOREM 5. *The following inference rules are not sound.*

$$\Gamma, \phi \models_{\mathrm{Sup}} \psi \Rightarrow \Gamma \models_{\mathrm{Sup}} \phi \to \psi \qquad \textit{(Conditional Proof)}$$

$$\Gamma, \phi \models_{\mathrm{Sup}} \psi \Rightarrow \Gamma, \neg\psi \models_{\mathrm{Sup}} \neg\phi \qquad \textit{(Modus Tollens)}$$

$$\Gamma, \phi \models_{\mathrm{Sup}} \psi \text{ and } \Gamma, \zeta \models_{\mathrm{Sup}} \psi \Rightarrow \Gamma, \phi \vee \zeta \models_{\mathrm{Sup}} \psi \qquad \textit{(Argument by cases)}$$

$$\Gamma, \phi \models_{\mathrm{Sup}} \psi \wedge \neg\psi \Rightarrow \Gamma \models_{\mathrm{Sup}} \neg\phi \qquad \textit{(Reductio ad Absurdum).}$$

PROOF: see Appendix. ⊣

[21] Notice that this is no *ad hoc* move by the supervaluationist. On the contrary, it is deeply rooted in the conception underlying the formal rendering of supervaluationism. Indeed, for the supervaluationist a sentence is true if it holds relative to every history passing through the moment of evaluation. Now take $m' \models_{\mathrm{Sup}} \mathrm{P}\,\mathrm{F}\, p$. It equates with $\forall h_{m'} \exists m < m'$ such that $\langle m, h\rangle \models \mathrm{F}\, p$. Obviously, all histories in $H_{m'}$ satisfy this condition in our model, since they all include m', where p holds.

[22] The following result and proof is readapted from [Varzi, 2007].

Thus, by adding $\Box$ supervaluationism rids its troubles with the retrogradation of truth, but loses some important rules of inference of classical logic.

The same *does not happen* when *abundance* faces the challenge of retrogradation. Here, the problem with expressivity is dual to the one we have noticed for supervaluationism: while we needed there to secure the "passage" from a gappy formula to a supertrue one, here we need to secure the "passage" from a glutty formula to a non-glutty one. Indeed, when a sea battle happens at m', we are allowed to say that "it was not true, before, that a sea battle would not take place", although at m "A sea battle will take place" is glutty. In other words, $\mathrm{P}\,\mathrm{F}\,p$ is true, yet not subtrue, at m', though it is subtrue at m.[23] Notice that abundance can secure the falsehood of $\mathrm{P}\neg\mathrm{F}\,p$ at m' out of the glutty $\mathrm{F}\,p$ at m in pretty much the same way as supervaluationism: the histories in H_m are considered when evaluating $\mathrm{F}\,\phi$ at m, while our consideration is restricted to all the histories in $H_{m'}$ when evaluating $\mathrm{P}\,\mathrm{F}\,\phi$ at $m' > m$.[24] Once this is done, we still remain with the question: "How can we express, from the point of view of m', the fact that 'there will be a sea battle' was *glutty* before (for instance, at m)?". The only way seems to be by using the operator $\Diamond$, which corresponds to our notion of truth and to $\neg\Box\neg$ in Ockhamist semantics. With its help we can rephrase the past truth of "there won't be a sea battle", with $\mathrm{P}\Diamond\neg\mathrm{F}\,p$, since $m' \models \mathrm{P}\Diamond\neg\mathrm{F}\,p$. Thus, modalities seem to be needed also for the abundance of the future, at least for expressivity reasons.

Coming to inferential rules, the expansion of our language by $\Diamond$ leads to less *indesiderata* w.r.t. the previous list of inference rules than the expansion of the supervaluationist language above: though *Modus Tollens* is clearly lost,[25] *Conditional Proof*, *Argument by Cases* and *Reductio ad Absurdum* are preserved:

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[23] This holds, obviously, if p holds at m' *and* h is such that $\neg p$ never holds. These conditions set up a very idealised model, but are important to understand the dynamics of our example, where the truth of $\mathrm{P}\,\mathrm{F}\,p$ at $\langle m', h\rangle$ is due to the subtruth of $\mathrm{F}\,p$ at $\langle m, h\rangle$, and $\mathrm{P}\,\mathrm{F}\neg p$ is simply false at $\langle m', h\rangle$. Notice that, by **IPP**, if a non-tensed statement holds at m, then it holds at every history in H_m, and thus it is true, yet not glutty — since its negation will hold in no history in H_m.

[24] Exactly as with supervaluationism, this is something that depends on the very conception of truth endorsed by abundance: $\mathrm{P}\,\mathrm{F}\,p$ behaves like Ockhamist $\Diamond\,\mathrm{P}\,\mathrm{F}\,p$, not like $\Diamond\,\mathrm{P}\Diamond\,\mathrm{F}\,p$, and thus it may turn out that a sentence *was* simply true — *retrospectively* — although *at the moment of evaluation* it was *glutty*.

[25] It is indeed easy to see that $\Diamond\,\mathrm{F}\,p \models_{\mathrm{Ab}} \mathrm{F}\,p$, but clearly $\neg\,\mathrm{F}\,p \not\models_{\mathrm{Ab}} \neg\Diamond\,\mathrm{F}\,p$.

Theorem 6. *The following inference rules are valid:*

$\Gamma, \phi \models_{\mathrm{Ab}} \psi \Rightarrow \Gamma \models_{\mathrm{Ab}} \phi \to \psi$

$\Gamma, \phi \models_{\mathrm{Ab}} \psi$ and $\Gamma, \zeta \models_{\mathrm{Ab}} \psi \Rightarrow \Gamma, \phi \vee \zeta \models_{\mathrm{Ab}} \psi$

$\Gamma, \phi \models_{\mathrm{Ab}} \psi \wedge \neg\psi \Rightarrow \Gamma \models_{\mathrm{Ab}} \neg\phi$.

Proof: see Appendix ⊣

8. Discussion

In this paper, we have proposed a non-adjunctive paraconsistent formalism for future contingents, interpreted on branching-time structures. The semantics is paraconsistent since it allows a formula $\mathrm{F}\,\phi$ and its negation $\neg\,\mathrm{F}\,\phi$ to be both true, i.e. we have *gluts* of truth-values. Formally, gluts obtain because a sentence about the future is true if and only if *there is a history* relative to which the statement holds. The proposed semantics is dual w.r.t. the supervaluationist one, which is one of the most well known and philosophically sustained tools for interpreting contingent statements *de futuro*. Preservation of classical features is a *desideratum* for both abundance and supervaluationism, and the latter is usually claimed to be closer to classical logic than any paraconsistent semantics. Here, we have shown that — on the contrary — they are much on a par in preserving and loosing important inferential rules of classical logic. We have also shown that differences arise when *retrogradations of truth* are taken into account. The success of abundance in modeling retrogradations of truth without worsening its conformity with classical logic, though maybe not decisive, makes this view a noteworthy alternative to the supervaluationist approach to temporal logic and future contingents.

We hinted at the fact that truth in the abundant sense is something that one is *entitled to claim* or to defend, due to the fact that the proposition she is stating has *some degree of probability*, or due to the fact that we have probabilistic reasons to *bet* on it, in a context where neither full uncertainty, nor full certainty, are at stake. From a probabilistic perspective, the logical machinery we introduced gains some interest. Indeed, the results in section 6 essentially explain abundant logical consequence (and its non-standard features) as something that one is entitled to claim, or to bet on, whenever she is entitled to claim Γ. Under this reading, *sound modes of inference* of this logic are rules for preserving this *transmission of plausibility*.

Some final words are worth spending on abundance and *norms of assertion*: those conditions at which we are entitled to assert a statement. Some believe that the norm of assertion is truth, others believe the norm is knowledge or strong evidence.[26] In either case, supervaluationism seems to fit: if $\mathrm{F}\,\phi$ is *supertrue* (or is *evident* since it holds in all the histories I may consider) this may constitute an *uncontroversial* ground for the assertion of a statement about the future, for it rules out $\neg\,\mathrm{F}\,p$. What about abundance? Here *subtruth* is the notion at stake, which equates with possibility. Thus, saying that the norm of assertion is subtruth means that we are entitled to assert $\mathrm{F}\,\phi$ if the current state of the world does not rule out $\mathrm{F}\,\phi$. If we go for an epistemic norm, this means that we are entitled to assert $\mathrm{F}\,p$ whenever we cannot exclude it with certainty. This can sound strange: all in all, assertion seems to require something more than mere possibility. But if we look at real-life *dialogues* this would be a too strict *desideratum*; after all, the essence of a dialogue is that both ϕ and $\neg\phi$ may be, in most cases, asserted as opposing theses, each endorsed by a different agent. Abundance builds on the very same idea: the feasibility of both $\mathrm{F}\,\phi$ and $\neg\,\mathrm{F}\,\phi$. As with discussive logics, if we adopt the idea that lies behind abundance, *feasibility* becomes a sufficient condition for *assertibility*. In these cases neither supertruth nor strong evidence will be the right notion, because they are exactly what is supposed to lack when opposing theses are both assertible.

Extending the links with probabilistic reasoning and assertion is not our present purpose and could be the topic for an independent paper. It is enough for this work to indicate a natural interpretation for an "immediate" conceptual alternative that has been left unexplored thus far.

Appendix

Proof of Theorem 5. *Conditional Proof*: let ϕ be $\mathrm{F}\,p$ and ψ be $\Box\,\mathrm{F}\,p$. $\Gamma, \mathrm{F}\,p \models_{\mathrm{Sup}} \Box\,\mathrm{F}\,p$ holds. Indeed, for any arbitrary model $\mathcal{M}_{\mathrm{Sup}}$ and mo-

[26] If one follows the first option, she concludes that we are entitled to assert ϕ only if ϕ is true, while if one follows the second option, she concludes that we are entitled to assert ϕ only if we *know that* ϕ, or — alternatively — only if we have some strong *evidence* in favour of ϕ.

ment m, $\mathrm{F}\,p$ and all formulas in Γ being supertrue equates to $\forall h_m(\langle m,h\rangle \models \Gamma$ and $\exists m' > m$ such that $\langle m',h\rangle \models p)$. This implies that $\forall h_m(\exists m' > m$ such that$\langle m',h\rangle \models p)$. By the truth-clauses of $\Box$, this equates with $\langle m,h\rangle \models \Box\,\mathrm{F}\,p$.

However, $\Gamma \models_{\mathrm{Sup}} \mathrm{F}\,p \to \Box\,\mathrm{F}\,p$ does not hold. To see this, consider that, for any arbitrary $\mathcal{M}_{\mathrm{Sup}}$ and moment m, $m \models_{\mathrm{Sup}} \mathrm{F}\,p \to \Box\,\mathrm{F}\,p$ equates to $\forall h_m((\exists m' > m$ such that $\langle m',h\rangle \models p) \Rightarrow \forall h'_m(\exists m'' > m$ such that $\langle m'',h\rangle \models p))$. However, there is at least one model $\mathcal{M}'$, where $\exists h_m((\exists m' > m$ such that $\langle m',h\rangle \models p$ and $\exists h'_m(\exists m'' > m$ such that $\langle m'',h\rangle \models \neg p))$. This proofs that $\Gamma \models_{\mathrm{Sup}} \mathrm{F}\,p \to \Box\,\mathrm{F}\,p$ is not a valid inferential pattern,[27] and together with the first part of the proof, this proves that Conditional Proof is not valid in a supervaluationist language endowed with $\Box$.

Modus Tollens: let once again ϕ be $\mathrm{F}\,p$ and ψ be $\Box\,\mathrm{F}\,p$. $\Gamma, \mathrm{F}\,p \models_{\mathrm{Sup}} \Box\,\mathrm{F}\,p$ holds. However, $\Gamma, \neg\Box\,\mathrm{F}\,p \models_{\mathrm{Sup}} \neg\,\mathrm{F}\,p$ does not: there is a model $\mathcal{M}'$ where $\langle m,h\rangle \models \mathrm{F}\,p$, but $\langle m,h'\rangle \not\models \mathrm{F}\,p$. By this and the truth-clause of $\Box$ and those of F, $\Box\,\mathrm{F}\,p$ is superfalse (since its superfalsehood equates with Ockhamist falsehood), but $\mathrm{F}\,\phi$ is not, since is made true at least in $\langle m,h\rangle$.

Argument by Cases: let ϕ be $\mathrm{F}\,p$, ζ be $\neg\,\mathrm{F}\,p$, and ψ be $\Box\,\mathrm{F}\,p \vee \Box\neg\,\mathrm{F}\,p$. $\Gamma, \mathrm{F}\,p \models_{\mathrm{Sup}} \Box\,\mathrm{F}\,p \vee \Box\neg\,\mathrm{F}\,p$ and $\Gamma, \neg\,\mathrm{F}\,p \models_{\mathrm{Sup}} \Box\,\mathrm{F}\,p \vee \Box\neg\,\mathrm{F}\,p$ clearly hold (by the semantics of $\vee$, $\neg$ and $\Gamma, \mathrm{F}\,p \models_{\mathrm{Sup}} \Box\,\mathrm{F}\,p$). However, it is easy to see that there is a model $\mathcal{M}$ where the supertruth of $\mathrm{F}\,p \vee \neg\,\mathrm{F}\,p$ at an arbitrary m is compatible with both $m \notin V^+(\mathrm{F}\,p)$ and $m \notin V^-(\mathrm{F}\,p)$. Then, in $\mathcal{M}'$ we have $\Gamma, \mathrm{F}\,p \vee \neg\,\mathrm{F}\,p$ supertrue at m, but $\Box\,\mathrm{F}\,p \vee \Box\neg\,\mathrm{F}\,p$ is not supertrue, for an arbitrary Γ.

Reductio ad Absurdum: let ϕ be $\mathrm{F}\,p \wedge \neg\Box\,\mathrm{F}\,p$ and ψ be $\Box\,\mathrm{F}\,p$. $\Gamma, \mathrm{F}\,p \wedge \neg\Box\,\mathrm{F}\,p \models_{\mathrm{Sup}} \Box\,\mathrm{F}\,p \wedge \neg\Box\,\mathrm{F}\,p$ holds. However, there are models where, given an arbitrary m, $m \not\models_{\mathrm{Sup}} \neg(\mathrm{F}\,p \wedge \neg\Box\,\mathrm{F}\,p)$ for any arbitrary Γ (for instance, take the model $\mathcal{M}'$ in the previous sub-proof). Thus, Reductio fails. $\dashv$

[27] Another way to prove it is by the relations between supervaluationist validity and Ockhamist validity: $\Box(\mathrm{F}\,p \to \Box\,\mathrm{F}\,p)$ is not valid in Ockhamist semantics, and hence $\mathrm{F}\,p \to \Box\,\mathrm{F}\,p$ cannot be valid in supervaluationist semantics. Since validity is logical consequence from the empty set of premises, we have falsified $\Gamma \models_{\mathrm{Sup}} \mathrm{F}\,p \to \Box\,\mathrm{F}\,p$ with $\Gamma = \{\emptyset\}$, and hence the validity of the rule of inference. For the failure of Conditional Proof, see also [Thomason, 1970], p. 275–276.

Proof of Theorem 6. *Conditional Proof*: Suppose $\Gamma, \phi \models_{\mathrm{Ab}} \psi$. We have then two possible cases: (1) for some $\langle m, h\rangle$ we get $\langle m, h\rangle \not\models \phi$. But then, $\langle m, h\rangle \models \phi \to \psi$ and, by the definition of $\models_{\mathrm{Ab}}$, $m \models_{\mathrm{Ab}} \phi \to \psi$. This proves the result for the first case. (2) for all $\langle m, h\rangle$ we get $\langle m, h\rangle \models \phi$. In this case, we obviously have $m \models_{\mathrm{Ab}} \phi$ and hence, by hypothesis, $m \models_{\mathrm{Ab}} \psi$. The latter implies $m \models_{\mathrm{Ab}} \phi \to \psi$: if the consequent is true at m in at least on history, the whole conditional will be true at n in the same history. This proves the result for the second case.

Argument by Cases: Suppose that $\Gamma, \phi \models_{\mathrm{Ab}} \psi$ and $\Gamma, \zeta \models_{\mathrm{Ab}} \psi$ and that $m \models_{\mathrm{Ab}} \phi \vee \zeta$ for an arbitrary moment m which satisfies Γ. By the definition of $\models_{\mathrm{Ab}}$, it follows that we have $\langle m, h\rangle \models \phi \vee \zeta$ for some $\langle m, h\rangle$. We then have two possible cases: (1) $\langle m, h\rangle \models \phi$. By definition, this implies $m \models_{\mathrm{Ab}} \phi$, and hence, by hypothesis, $m \models_{\mathrm{Ab}} \psi$. (2) $\langle m, h\rangle \models \zeta$. By the same reasoning, we have $m \models_{\mathrm{Ab}} \psi$. Both cases prove the result.

Reductio ad Absurdum: Suppose $\Gamma, \phi \models_{\mathrm{Ab}} \psi \wedge \neg\psi$. Now, any moment m which satisfies Γ, is such that it satisfies also $\neg\phi$ ($\Gamma \models_{\mathrm{Ab}} \neg\phi$). If it was not so, by the hypothesis we would have $m \models_{\mathrm{Ab}} \psi \wedge \neg\psi$, but as we have shown, $\psi \wedge \neg\psi$ cannot be satisfied at any moment. $\dashv$

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