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STRUCTURAL FEATURES IN ERNST SCHRÖDER'S WORK. Part II

Abstract. In this paper (the second of two parts) we propose a structural interpretation of Schröder's work, pointing out his insistence on the priority of a whole in comparison with its parts. The examples are taken from the diverse areas in which Schröder was active, with a particular interest in his project of an *absolute algebra*.

Keywords: structure, universal algebra, relation.

Résumé

This paper is the second half of a two-part text on Schröder's *philosophy of mathematics*. In the previous part [Bon11a], we showed the importance of particularization and generalization in the everyday work of scientists. On one side, a scientist contextualizes his concepts in a *determinate* situation, in order that abstract objects can acquire their *semantic meaning*. This particularization make so the scientific objects context-dependent, in the sense that scientific objects have a meaning only in a determinate state of affairs.

One would be tempted to say that scientists need *models* for their theories. This is correct if the expression *model* is not meant with its usual technical meaning. On the contrary, the situation giving meaning to scientific concepts is often a vague totality: *vague* not because it is not clear, but because its boundary are fuzzy. Notwithstanding, this fuzziness doesn't prevent a theory from being well articulated in itself. As a matter of fact, a concept receives its meaning only in a web of relations with other concepts belonging to the same context. It is this

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bundle of relations in which a concept is inserted and which it satisfies which gives a real meaning to the concept.

On the other side, the scientist tries to generalize his concepts. It was Banach who generalized many particular spaces with the concept of *Banach functional space*. Many mathematicians at the beginning of 20th century worked with functional spaces. For example, von Neumann who elaborated the Quantum Mechanics formalism relying on Hilbert Spaces with operators. The problem was that most mathematicians didn't grasp that *their* functional spaces were only particular examples of a more general space. Banach's insight was to see instead of many *unrelated* spaces, an unique space admitting different exemplifications.

Then mathematicians walk along two paths: one from the general to the particular (contextualization) and one from the particular to the general (formalization). Schröder clearly exemplified these two ways. In particular, the main result of Schröder's work was to build up a formal theory whose elements have no meaning at all. They are simple signs on paper. In this situation, how can an element of a formal theory have a sense? If the semantic context is put aside, the relations between the elements persist. Are relations to give a formal meaning to formal elements, because a formal element *is* what it *is* only through standing in certain determinate relations? In other words, is its position in the relational web of the theory that determines its meaning?

It makes sense to speak of a *structural* philosophy with Schröder, because formal elements acquire their meaning according to the *structure*[1] in which they are included. The formal whole which constitutes the relational net of a formal theory is not an *indistinct* totality, in which there are no differences. Elements are distinguishable one from other in a precise (albeit not *quantitative*) way, because or they fulfil the same relations (being then the *same* object) or there is a relation which an element satisfies and another one doesn't.

Furthermore, in the preceding part of this paper we noted that Schröder saw no essential difference between the various fields of mathematics and between mathematics and logic. Schröder didn't ever consider logic as an *independent* discipline, but as a field tied in some way (unspecified) to mathematics. For this reason Schröder tackled problems

[1] For the detailed meaning of *structure* I refer the reader to the fourth footnote in the previous part. In this paper, I use the term throughout in a non technical way. A structure is simply an *ordered* whole.

in different mathematical and logical areas in the same way. For example a concept which Schröder treated uniformly was that of a *solution problem* which will be the subject of the next subsection.[2]

3.6. Solution Problem

Let us start from Schröder's first logical booklet *Der Operationskreis in dem Logikkalkuls* [Sch66a] where he states explicitly:

> The sentence 20 [i.e. the theorem by which we can solve the equations of the calculus] is the *principal* theorem [Haupttheorem], in which the whole logical calculus culminates [gipfelt]. [Sch66a, p. 22]

As said above, Schröder needs a solution problem because it is the unique method which makes deduction possible in this context.[3] Once more, Kolmogorov rightly writes:

> Mathematical logic developed in the 19th century primarily in the form of logical algebra. The analogy leading to the creation of logical algebra lay in the fact that each solution of a problem, *via* the setting up and solution of an equation, is in essence the derivation of consequences from the statement of the problem. [KY01, p. 33]

In other words, any premise has an equational form. We collect our premises in a unique equation, then we solve it. The solutions of this equation are the consequences of the premises. Note also the particular form of the solution. In [Sch66a] this solution is expressed as a linear combination of two opposite classes. In fact, theorem 20 relies on theorem 14 stating that:

> Any class B[4] can be expressed in a linear and homogeneous form by **any** other [class] A:
>
> $$B = (X \cap A) \cup (Y \cap -A),$$

[2] Indeed, in our opinion Schröder was *not* a logician, despite his investigations in logic. The focus of his work was the project of building up a formal theory modelled on familiar algebra. A standard approach to algebra is present in almost every paper of his, thus distinguishing his *pure algebraic* way of thought from a *logic-algebraic* one.

[3] Obviously, given the algebraic structure of the calculus.

[4] Schröder means with B not a class, but a *symbol of class*: *Object of the logical operations are letters* [Buchstaben], *which* (. . .) *indicate* symbols of classes [Klassensymbole] [Sch66a, pp. 1–2]. Nevertheless, in the text at issue Schröder is not consistent with this choice, sometimes using the expression *class* and sometimes the expression *symbol of class*.

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> where X, Y are not completely determined[5] *symbols of classes*, which can equal $\emptyset$ or V.[6] [Sch66a, p. 14]

Schröder is here suggesting a sort of *Orthocomplementation* Theorem which has a well known analogue in modern linear algebra:

> Given any vector $p \in R_m$[7] and any subspace $S \in R_m$, p has a unique representation of the form $p = p_s + p_\perp$, where $p_s \in S$ and $p_\perp$ is orthogonal to S in the sense that $(p_\perp, x) = 0$ for all $x \in S$.[8] [Shi96, p. 222]

Of course I am conscious of the difference between the calculus in [Sch66a] and modern vectorial algebra. I am only presenting a metaphor to help us to decipher Schröder's hidden thought. What Schröder had in mind was the idea of expressing an object as a sum of two opposite other things. The comparison will aid the reader in grasping the way by which Schröder formulated such a sum.

3.6.1. Poetry and Rhetoric

Schröder is for diverse reasons a sort of *black box*. We have no biography of him other than [Lür02] which paraphrases [Eck01]. His contemporaries ignored him. For Peano Schröder was only a source of bibliographic references (see [Pea94, p. 3]).

Schröder himself left no exhaustive profile on his life and on his work. Furthermore, he never married, had no brothers and no *true* friend.

In this situation a scholar must make use of comparisons, metaphors, allusions, and other rhetorical tools in order to grasp his thought. The fact that these devices have no *scientific* validity is not an appropriate objection.[9] The black box that is *Ernst Schröder* must be opened in or-

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[5] I.e. arbitrary.

[6] The emboldening is mine. For the explanation of this emboldening see the Appendix.

[7] Obviously the subscript m indicates the number of dimensions of the space R.

[8] $(p_\perp, x)$ represents the cos of the angle θ between $p_\perp$ and x. In such case, being $\cos\theta = 0$, θ must be equal to π.

[9] As said in the previous section, the qualitative features of the context don't hinder the work of a mathematician. On the contrary they provide an adequate justification of the ways by which they formulate their results. Furthermore, many of our modern requirements of rigour are absent in the mathematical literature of past centuries. For example, Klaus Jänich states in 2004 that, (...) [a] *function is not really completely defined without* [the specification of] *its domain* [Jän05, p. 7]. In fact, while we may not express the range of a mapping, we cannot not indicate its domain.

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der to cast light on its contents. If we don't own a key to open it, we must try other tools. For example, a good way in might be to smash it open.

It is no matter if we choose the wrong way to break into the box. As in some methods of approximation of a fixed point, it can happen that we start with a wrong input. Nevertheless, we eventually reach the searched-for approximation. It is only the length of the computation that is different. If the starting input is a good one, the length of the calculus is shorter than with a bad input. The result is identical.

3.7. A fixed idea

The idea of expressing an *object* as a linear combination of two opposite objects was a sort of *fixed idea*. In fact, in the third volume of the *Lectures*, the solution problem is tackled in same way. The famous equation which Peirce didn't understand expressed just the fact that a relation (unknown) can be formulated as an opportune linear combination:[10]

Nevertheless, Schröder in [Sch70a] takes it for granted that he is speaking of functions with complex arguments without further explanation. Mathematical papers before the 20th century lack the *theoretical* precision that we ask of contemporary scientists. It is a positivistic idea that mathematics must be disjoint from metaphysics. Cantor saw in his concept of *infinity* a real counterpart of the infinitude of God. Plato and the Pythagorean school associated mathematics and music without fear. Man made mathematics without our modern sense of precision and often with much more poetry than today. Perhaps the necessity to explain their own work in a standard manner and in the most precise way are obstacles to creative insights, because they put an heavy charge on the mathematical minds. This is not a complaint against rigour. It is necessary. But it must not hinder creativity and quality. I recall Dedekind's most celebrated words: (...) *numbers are free creations of the human mind* (...) [Ded96b, p. 791]. In the same vein Dedekind observes: (...) *the greatest and most fruitful advances in mathematics and other sciences have invariably been made by the* ***creation*** *and introduction of new concepts* (...) [Ded96b, p. 792, the emboldening is mine]. Furthermore, *it appears to me all the more beautiful that, without any notion of measurable quantities and simply by a finite system of simple steps of thought, man can advance to the* ***creation*** *of the pure continuous number-domain; and only by this means is it in my opinion possible for him to render the notion of continuous space clear and definite* [Ded96b, pp. 793–794, the emboldening is mine]. Obviously, this doesn't undermine the value of rigour. In fact, Dedekind states: (...) *in* [the] (...) *possibility of reducing such truths to others more simple, no matter how long and apparently artificial the series of inferences, I recognize a convincing proof that their possession* (...) *is never given by inner intuition but is always gained only by a* (...) *repetition of the individual inferences* [Ded96b, p. 791]. Quality and quantity are not enemies but intervene at different moments in the construction of a scientific theory.

[10] [Sch66b, p. 166]. Where $\Lambda^2 = -V^2$.

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$$\{f(S)=\emptyset\} \to (\{f(R)=\emptyset\}$$
$$\Leftrightarrow \exists T\{R = S \cap \overbrace{(V^2 \circ f(R) \circ V^2)}^{\alpha} \vee T \cap \overbrace{(\Lambda^2 \bullet -f(R) \bullet \Lambda^2)}^{-\alpha}\})$$

3.7.1. Variations

In this context, I go back to our first discourse on the δ-function and the fixed points theorems. There we noted that it is the context to make determinate the meaning of a mathematical concept. In fact, a mathematical concept has a meaning only in a web of relations. We deduced this state of affairs from different statements of the same concept. We could rewrite Schröder's general solution this way:

$$\{R(\vec{a}) = \Lambda^2\} \to$$
$$(\{R(\vec{x}) = \Lambda^2\} \Leftrightarrow \exists\vec{b}[\vec{x} = \lceil\vec{a}\cap\{V^2\circ R(\vec{b})\circ V^2\}\rceil \vee \lceil\vec{b}\cap\{\Lambda^2\bullet -R(\vec{b})\bullet\Lambda^2\}\rceil])$$

While in the previous formulation our aim was to stress the orthogonal linearization of the general solution and not to focus on the precise nature of the relations, in this formulation we stress the *vectorial* character of the relations, generalizing the solution to n-ary relations.[11] Other formulations are possible such as:

$$\forall x_1, x_2 \neg(x_1 R x_2) \to \forall x_1, x_2(\neg(x_1 R x_2) \Leftrightarrow$$
$$\exists c, d[x_1 R x_2 = <a, b> \cap \{x_1, x_2 | \exists x_3, x_4(x_1 V^2 x_3 \wedge x_3 R x_4 \wedge x_4 V^2 x_2)\} \vee$$
$$<c, d> \cap \{x_1, x_2 | \forall x_3, x_4(x_1 \Lambda^2 x_3 \vee \neg(x_3 R x_4) \vee x_4 \Lambda^2 x_2)\}])$$

This time we have used a set-theoretic language which allows us to enter into the structure of the relations and of the operations involved.

3.8. Individuum

Let us go back to the idea of expressing a class as an orthogonal linear combination. Schröder in a opening lecture in 1890 (13 years after [Sch66a]), introduces the concept of *individual* as an object which cannot stay in two disjoint classes (see [Sch90b, p. 21]):

$$(I \neq \emptyset) \to \forall X\{(I \subseteq X) \vee (I \subseteq -X)\}$$

[11] A relation being viewed as a generic vector, the sum and time of relations corresponding to the vectorial-sum and -time, respectively. The composition of relations is a composition of vectors.

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That is, if I is not empty, then I can be formulated as a linear composition with any class X whatever.[12] Using Schröder's words:

> (...) a point [i.e. the individuum] cannot be split up; it cannot be projected [hineinragen] in two separate (disjoint) sets [Gebiete] in the same time (...)[13] [Sch66c, p. 320]

That is, the essence of an individuum lays in its *indivisibility*. Furthermore, an individuum is a *limit* concept:[14]

> One may [express the definition of *point*] in words: a set I is a *point* if and only if, **without vanishing or being an empty set**, it never overlaps with a set and its complement in the same time.[15] [Sch66c, p. 221]

In this definition is essential the clause *in the same time* [zugleich]. Nothing prevents a set from being included in the set S at time t_i and in $-S$ at time t_j, for $i \neq j$, without being a point. Formally, this definition would require a quantification over a set of instants.

Obviously, we can argue that Schröder in this context is following Peirce's definition of individual as an entity which cannot satisfy two opposite properties. I have stressed this in another work [Bon01, p. 89, p. 107 and ff.]. Now I note simply the *omnipresence* of the idea of expressing an object as a linear combination of two opposite objects, an unique formal concept which Schröder interpreted in many ways in mathematics and logic.

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[12] Note that I is a *set*!

[13] R. Dipert translates the second part of this quotation in this way: (...) *no individual straddles two mutually exclusive domains* [Dip90, p. 156]. I think that my translation is more faithful than Dipert's. The word *Gebiet* has for Schröder many meanings according to context. A literal translation which doesn't take into account the real intention of the author will be misleading. For Schröder, generally, *Gebiet* is equivalent to *class*, *set*, *manifold* or *collection*. In [Sch70a] it is synonymous of *neighbourhood*. Above we translated *Gebiet* with *domain* or *area*.

[14] I refer the reader to the earlier discussion of Leśniewski and Lebesgue. Note how close is Schröder's definition of individual to Lebesgue's set of measure zero in the following quotation.

[15] The emboldening is mine. These two quotations are from the second volume of the *Vorlesungen* where Schröder devotes an entire lecture (the 22th) to the concept of *individual*. I remember that [Sch66c] provides an interpretation in terms of propositional logic for the calculus exposed in [Sch66d].

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3.8.1. Individuum and relations

This definition of *individual* is not the unique definition of *individual* which we encounter in Schröder's *Opera*. In fact, in a lecture held at the first international mathematical congress in Zurich, he proposed this version (see [Sch98b, p. 155]):[16]

$$\begin{cases} (\text{num}.A = I) & = (A \text{ is an element, an individual, a constant}) = \\ & (\mathsf{Di} \circ A \circ V^2 = -A) = \\ & (\mathsf{Di} \circ A = -A) = (A \nsubseteq \mathsf{Di} \circ A) = (A^{-1} \circ (\mathsf{Id} \bullet -A)). \end{cases}$$

I have quoted the entire excerpt because once again we are facing the n-th array of equivalent formulations. This approach is mereological inasmuch Schröder takes for granted a totality (the relation A) and then collapses it. This is what Schröder calls a *Point-relation* [Einaugerelativ][17], i.e. a relation whose matrix has one only entry:

> After all, the equation (...):
>
> $$\mathsf{Id} \bullet -R \bullet \mathsf{Id} = R \circ V^2 \cup V^2 \circ R$$
>
> is to be considered the (...) concisest definition of *individuum in the second universe of thought* $[V^2]$. We want build up the complete theory of individual with extreme scientific rigour relying on this fundamental definition. [Sch66b, p. 432]

Note the absence of any individual variable and of any quantifier. The individuum is obtained manipulating three relative, R, Id and V^2. We can *mime* this definition with modern topological tools:

> Be l a simply closed curve, which is contained in a figure M. A curve l can *contract*, if one can shrink it (in M) to a point. [BE86, p. 91]

An individual in the relational setting is a like a curve contracting to a point. This point is the individuum and the curve is a relation R. We can erase from the matrix of R step by step any entry, until R includes only one entry. Obviously, we can consider this entry as an ordered pair. In the usual notation of matrix calculus, it is denoted by: a_{ij}. Nevertheless for Schröder we must regard it as a unique object.[18]

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[16] Remember that a relation R is a set *iff* $R \circ V^2 = R$.

[17] [Sch66b, p. 424 and ff.]. The German *Ein-Auge-Relativ* means *relation with only one point* [Auge].

[18] Take care to observe that we are in V^2 where we have neither individual constants nor individual variables. This explains the absence of first-order quantifiers.

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3.9. Two further examples

Going back to our main concern to exemplify Schröder's various recastings of the solution problem, I propose two other quotations: one from an algebraic context and one from a functional one. In the *Lehrbuch* of 1873 we read:

> The task now consists in *solving* any equation, i.e. in finding these determined [bestimmten] numbers,[19] which inserted as values of x [i.e. the unknown] satisfy the equation [in issue]–[obviously] in the case that such values exist [i.e. in the case the equation is solvable].
>
> [Sch73, p. 115].

The task [Aufgabe] is the same also in *Ueber iterirte Functionen*, culminating in Schröder's celebrated equation:[20]

> Be $f(z) = z$ a mapping, whose n-th iteration [Wiederholung] $f^n(z) = z^n$ is known, and $g(u) = u$ another map, of whose we search the n-th iteration $g^n(u) = u^n$ $(\dots)$; the problem is solved if we are able to find a function h such that, if
>
> $$h(z) = u,$$
>
> i.e. $h^{-1}(u) = z$, then
>
> $$h(z) = z.$$
>
> Generally it will be also true that:
>
> $$h(z^n) = u^n,$$
>
> and one can always formulate [bilden] the map:
>
> $$g^n(u) = h(f^n(h^{-1}(u)));$$

[19] With the expression *bestimmten Zahlen* Schröder refers implicitly to the distinction between *determined* numbers and *undetermined* numbers. The first are numbers, the second are *variables*. Schröder considered, as did his contemporaries, a variable as a undetermined number; i.e. a non-specified number. I refer the reader to [Fre08a] where Frege is explaining the essence of the concept of *function* starting from the fact that usually by *function* a undetermined number was understood: *Usually one means with the word* function *an expression in which a number by the letter* x *is indicated only in an undetermined way* $(\dots)$ [Fre08a, p. 5]. Furthermore: *There are not variable numbers and this is confirmed by the fact that we don't possess any proper noun for variable numbers.* $(\dots)$ *Do we not indicate with* x, y, z *variable numbers? One makes use often of this way of speaking* [Redeweise]; *but these letters are not proper nouns of variable numbers, in the same way* 2 *and* 3 *are proper nouns of constant numbers* $(\dots)$ [Fre08b, p. 63]. On the contrary for Schröder there was no *essential* difference between variable and constant numbers.

[20] See below Figure 1.

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$$
\begin{aligned}
& z, u \in \mathbb{C}, \\
& \qquad Dom(f), Dom(g), Dom(h) \subseteq \mathbb{C} \\
& f(z) = z \text{ and } f^n(z) = z^n && (1) \\
& g(u) = u \text{ and } g^n(u) = u^n? && (2) \\
& \qquad \text{What is the value of } u^n? \\
& u = h(z) \text{ and } z = h^{-1}(u) && (3) \\
& u^n = h(z^n) \qquad (3) && (4) \\
& u^n = h(f^n(z)) \qquad (1), (4) && (5) \\
& u^n = h(f^n(h^{-1}(u))) \qquad (3), (5) && (6) \\
& g^n(u) = h(f^n(h^{-1}(u))) \qquad (2), (6) \qquad \square && (7)
\end{aligned}
$$

$$
\begin{aligned}
& g^n = h(f^n(h^{-1})) = h^{-1} \circ f^n \circ h \qquad (7) && (8) \\
& g = h(f(h^{-1})) = h^{-1} \circ f \circ h \qquad (1), (2), (3), (8) && (9) \\
& h \circ g = \lceil f \circ h = h \circ g \rceil \qquad (9) \qquad \blacksquare && (10)
\end{aligned}
$$

Figure 1. The formula enclosed by the delimeters $\lceil, \rceil$ is the *Schröder Equation*. The domains of f, g, h are included in $\mathbb{C}$ and not, for example, in $\mathbb{C} \otimes \mathbb{C}$ because f, g, h have only one argument.

or using a conciser writing:

$$g^n = h^{-1} \circ f^n \circ h.$$ [21]

In other words, the solution problem expressed in this quotation is solved if we find an appropriate value for the *unknown* function h. In fact, by the process explained above, we are able to express the solution of $g^n(u) = u^n$ (where u^n is the unknown) as a function of another function h. For the details of this procedure which ends with the formulation of *Schröder Equation* I refer to Figure 1.

In both [Sch73] and in [Sch70a] the solution problem is tackled in same way: in both cases it is to find a number which is solution of the equation in question. The unique difference is that in the second problem, Schröder shows that this solution can be expressed as a function

[21] [Sch70a, pp. 300–301]. Although this paper is *also* about fixed points, its main concern is the study of functional iteration. It is curious that Simeon Reich (in a personal conversation) admitted not knowing [Sch70b]. It is *this* text that is devoted mainly to fixed point results.

of a particular map. We might say that Schröder approached the solution problem always from an algebraic and formal point of view.

3.9.1. Möbius transformations

Incidentally, I observe that Schröder introduced the functions f, g, h as recursive ones. In particular they satisfy the *substitution schema* for recursion: $f(z) = z$ and $\forall n, f^n(z) = f(f^{n-1}(z))$ (see [Sch70a, p. 296], [Vel06, p. 280] and [MB93, pp. 632–633]). This is highly interesting, because elsewhere Schröder proved that the concept of *chain* translated in his algebra of relatives coincides with that of *reflexive-transitive closure of a relation* iterating binary relations (see [Bon07, p. 40 and ff.] and [Sch66b, p. 361]). The (*infinite*) iteration is fundamental also for [Sch70b].

Another thing deserving to be mentioned is Schröder's own solution to his equation:

$$h(u) = \frac{au + b}{cu + d}, \qquad h^{-1}(u) = \frac{-du + b}{cu - a}.$$ [22]

In this passage Schröder is using as the value for h a *bilinear transformation*. As is well known, bilinear transformations are *conformal* mappings[23] and Schröder stated his solution problem in a chapter just devoted to this type of functions:

> If the function h fulfils the condition (13) [i.e. Schröder Equation], then the plane of the points $u^1, \ldots, u^n$ represents a conformal map of the plane $z^1, \ldots, z^n$. [Sch70a, p. 301]

In other words, the Möbius transformation h maps a complex plane into another one, in a conformal way. Once again, Schröder is not investigating a complex plane in itself, but a *relationship* (via a conformal mapping) between a plane $\mathbb{C}_i$ and another plane $\mathbb{C}_j$. The focus is on a particular type of transformations (Möbius' one) which puts in relations as a bridge the original plane with that transformed. In other words Schröder is studying the distortion allowed by a transformation. He was

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[22] [Sch70a, p. 307]. See [BN10, p. 177] and [Bea05, p. 254, Chapter 13]. I refer the German reader to [Wey08, p. 29, p. 35 and ff.]. I didn't find in Schröder's paper the *essential* requirement that $ad - bc \neq 0$. *This condition ensures that h is neither identically constant nor meaningless* [BN10, p. 177].

[23] Weyl called them *winkeltreue* [preserving the angles] [Wey08, p. 26].

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interested neither in the original plane nor in the distorted plane, but in the *way* to transform the first in the second.

Perhaps, this explains why Schröder didn't cite Abel in [Sch70a]. The matter sounds strange, because there is a strict relation between the Schröder Equation and the Abel Equation as Adamard stressed (see [Had44, p. 67]):

> The question is connected with Abel's functional equation
>
> $$\phi(f(x)) = 1 + \phi(x) \quad (\ldots). \tag{1}$$
>
> Instead of (1), one can introduce Schröder's equation
>
> $$\psi(f(x)) = k\psi(x)$$
>
> (k a constant) in which ψ is connected with the unknown ϕ of (1) by $\psi = k^{\phi}$, and with the help of which the solution would be expressed by
>
> $$\psi(f^n(x)) = k^n\psi(x).$$[24]

Notwithstanding this, Schröder didn't refer to Abel in any way.[25] The context in which Schröder works is very different from Abel's one. For Schröder his equation is functional to solve a solution problem. This is the main goal of [Sch70a]. In pursuing this aim Schröder introduced a class of transformations which highlighted his love for relations and symmetries.

According to our interpretation Schröder investigated functional and complex objects from an *algebraic* and *formal* point of view. However,

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[24] Yet in 1884 Koenigs wrote: *In his investigations, mr. Schröder met with a functional equation, from which one may deduce Abel's one making the logarithm of the two members. To solve the Abel Equation or Schröder's one is then the same problem* [Koe84, p. 4]. The relation between the Schröder Equation and Abel's Equation was also stressed by Pincherle: *If one substitutes* $\varphi(x)$ *with* $\log \varphi(x)$*, one obtains from the equation* (84) [i.e. Abel's Equation] *the following one:* $\varphi(\alpha(x)) = c\varphi(x)$ *or* $S_\alpha \varphi = c\varphi$*, which is called* [with the name of] E. Schröder; *the determination and the domain of validity of the solutions of* (84) *let derived from these of* (86) [i.e. Schröder's Equation] [Pin16, pp. 791–792]. Recently Reich et al. generalized the Schröder Equation obtaining the *Abel-Schröder Equation* underpinning the tight link between the two equations: *Let* Δ *be the open unit disc of the complex plane* $\mathbb{C}$*. The equation* $f \circ \varphi = \psi \circ f$*, where* φ *and* ψ*, which belong to* $Hol(\Delta, \mathbb{C})$*, are given, is called the* Abel-Schröder equation. *In the particular case where* $\varphi \in Hol(\Delta)$ [i.e. φ is an holomorphism on Δ with values in $\mathbb{C}$] *fixes* 0*, that is,* $\varphi(0) = 0$*, and* $\psi = \psi'(0) = \lambda$, (1.8) [i.e. the Abel-Schröder Equation] *becomes Schröder's equation* $f \circ \varphi = \lambda f, \quad \varphi(0) = 0$ [RSK03, p. 69].

[25] At any rate, the process to transform the Schröder Equation into the Abel Equation was known to Schröder, who describes it further in his [Sch70a, p. 302].

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the idea of rephrasing a solution as a function of objects is not typical of his work. Above in the subsection 3.1 (first part) we saw Schröder formulating the solution of an equation as a function of another unknown this way: $\sigma(x) = x$. The solution σ of the equation is itself a function. In the trivial case when the solution is a constant, we have a 0-ary mapping coinciding with its fixed point.

3.9.2. Towards Skolem

The idea that a solution is itself a function is belonging not only to algebraic or functional contexts, but also to logical ones. For example, in [Sch66a] Schröder states:

> (...) *we are now able to compute* not only a single symbol of class, but *any logical function* $f(A, B, C, \ldots)$ *of a demanding set* $\{A, B, C, \ldots\}$ *of such symbols of classes*, **expressing it by an arbitrary set of other symbols of class.**[26] [Sch66a, p. 24]

That is, we can find the value of a unknown, expressing it as function of other unknowns. In this situation, we express the map $f(x, y, z, \ldots)$ with a *functional* $\omega(u) = f(x, y, z \ldots)$, searching for the possible values of u. We are no longer facing a manifold of unknowns $x, y, z, \ldots$, but only a single variable u. With this procedure we have reduced the number of variables.

In the eleventh lecture of [Sch66b], 18 years after [Sch66a], Schröder, generalizing the distributivity, introduced a procedure enabling the rewriting of a variable as a function of the other variables. I will not explain here how Schröder accomplished this task, because it is too involved and not our main topic.

4. Elegance

We have shown how a structural approach arises from the context and from the formalism. In the first case, the context provided a structured situation in which and only in which a theorem or a scientific sentence has a meaning. The object in question has a meaning only in relation with another object belonging to the context. Only then are we authorized to speak of a *structural* meaning. I think that the example of chess can be of use in understanding the power of the situation in assigning a meaning to particular elements.

[26] The emboldening is mine.

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In the second case, we do not have a *semantic* context. We have a purely *formal* context, represented by the lattice of relations building up a theory in place of the previous semantic context. While in the first case the context is semantic, in the second one is syntactical. Nevertheless, in both cases it is a set of relations that structure the meaning of our concepts. Schröder adopted a formal context, leaving undetermined the possible semantic situations which could exemplify it.

This interest in the relations between the elements of a totality has also an aesthetic flavour. How often we encounter symmetries or ordered structures in art! A movement of a sonata is rich in the relationships that hold between the various musical elements. For example, the key of the *ripresa* in a sonata-tempo is in an appropriate relation for the key of the exposition. In *serial* music the relations are fundamental: we have a series which admits a retrograde, a mirror-counterpart, and many-transposed version of it. The work results from the interrelationships of all these series. Even a single series can be in itself structured, being divisible in many groups which are in a particular relation to each other.

A structural approach can cast light on many musical or artistic works as it is easily seen in the modern literature on them. We can also represent these relations graphically, as when a soprano voice is raising and a bass voice is descending. The world is split up: we can choose the rising path or the descending path. In Schröder this splitting up is indicated by a vertical (and an horizontal) stroke. We can enter into the original world or into its dual.

In the dual world there are not more or fewer things than in our ordinary world. It is a perfect copy of our world because any object of our world is mapped bivocally in the dual world. Obviously this copy is a little distorted; the relations between the creatures of this world are not the same those holding between the creatures of the ordinary one. It is possible that some object occupies in the dual world an opposite position with respect to its *Urbild* [original image].

Schröder is then asking: *what type of relation obtains among these possible worlds?* This is a question which Schröder had already posed in his youth as the following quotation witnesses:

> At this point it is permissible to generalize the two following sentences, which until now were valid only when p [or α] was divisible by q [or β] in $\mathbb{N}$:[27]

[27] I.e. if $\frac{p}{q} \in \mathbb{N}$. This quotation is from [Sch62, p. 60].

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If one stretches a[ny] regular side p [belonging to a p-gon circumscribing a circle C] until any side [of the p-gon] intersects its following q-th side, one obtains q regular [sides of length] $\frac{p}{q}$, which circumscribe the same circle.	If one connects any angle α [belonging to an α-gon inscribed in a circle C] with the β-th angle following it, one obtains β [α?] regular angles of measure $\frac{\alpha}{\beta}$, which are all inscribed in the same circle.

4.1. Explanation

Although it is not our intention to explain the proposed examples in order to privilege their appearance, the quotation above deserves a word of comment. We will illustrate the statement at left of the vertical stroke with an *hexagon*[28] which circumscribes a circle C. We stretch a side p_1 to infinity. We then choose a value for q. Let $q = 2$. Then we stretch in the same way the $p_{1+q} = p_3$ side. They intersect at a point p_1'. Now it is the turn of p_2. Both p_2 and $p_{2+q} = p_4$ are stretched until they intersect at a point p_2'. Continuing the same process, we ultimately have a starlike 12-gon with tops $p_1', p_2', p_3', p_4', p_5', p_6'$. This 12-gon circumscribes the same circle which the previous hexagon circumscribes. Furthermore, we have exactly $q = 2$ sides of length $\frac{p}{q} = \frac{6}{2} = 3$-gons, that is 2 triangles.

Let us pass to the right of the vertical stroke. Now the hexagon is inscribed in the circle C. We indicate the angles of the 6-gon with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. Let $\beta = 2$ as above. We connect α_1 with $\alpha_{1+\beta} = \alpha_3$, α_2 with $\alpha_{2+\beta} = \alpha_4$, etc. We obtain again a starlike 12-gon, inscribed in C and with tops in $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. There are $\alpha = 6$ angles of measure $\frac{\alpha}{\beta} = \frac{6}{2} = 3$.[29] In [Sch62] Schröder has the two drawings of Figure 2 to illustrate both statements with a *pentagon*.

4.2. Geometrical symmetries

This example shows the first use of the vertical stroke by Schröder to formulate two correlated possibilities. We can work inside the circle, or we can work outside the circle. The circle is the same and is the *trait d'union* of these two possibilities. Note that if we apply the right statement after the left statement has been applied to a p-gon circumscribing C we obtain again a p-gon, although not circumscribing C.

[28] I.e. a p-gon, for $p = 6$. Its six sides are $p_1, p_2, p_3, p_4, p_5, p_6$.

[29] Schröder states that there are β angle of measure $\frac{\alpha}{\beta}$. In the example in issue we would have 2 angles of length 3. This is impossible, because the measures of the 6 angles at the tops are identical. For this reason I inserted an α with a question mark into the quotation.

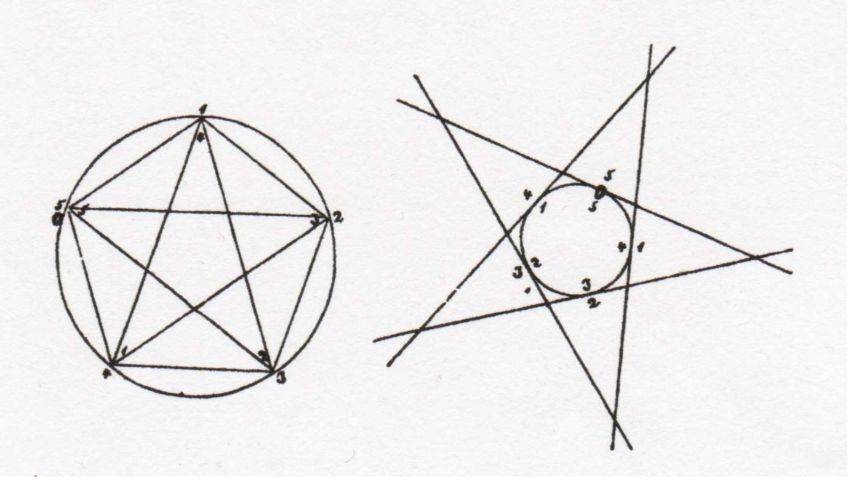

Figure 2. [Sch62, p. 61]. The *left* drawing refers to the *right* statement and the *right* drawing to the *left* statement. $p\ [\alpha] = 5$ and $q\ [\beta] = 2$.

These two statements are only two facets of a broader geometrical situation. Schröder used the same letters p, q on either side to stress their relation. We have changed p with α and q with β on the right in order to facilitate understanding. We might say that we have another type of context: a *graphical* context. In this quotation it is the graphical context that gives meaning to the statements. The layout highlights the correlations between the two possibilities. It suffices to *see* page 60 of [Sch62] to understand that there *must* be a relation between the sentences.

4.2.1. A mental experiment

Let us propose a little *experiment* in order to prove the power of the graphical context in shaping the meaning of a concept (see Figure 3). We re-write the two sentences above in runic characters, maintaining the vertical stroke. Obviously, now the text makes no sense at all. We retain the letters p, q on the right side of this graphical arrangement.

No one can deny the *hypnotic* character of the vertical stroke. It seems to put in order confused material, suggesting a division. Because the symbols p, q, C occur on both sides, one thinks that there must be a relation between the two sides. The right side cannot be a translation of the left one, because on both sides the same letters seem to occur. In ei-

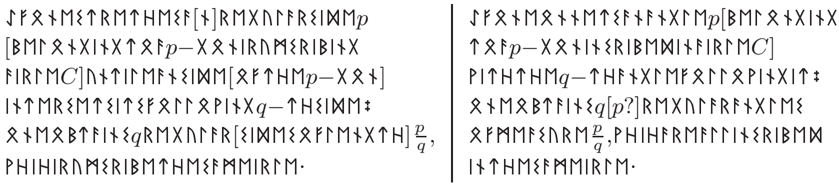

Figure 3.

ther column the first letter is ᛇ and the final letter is ᛗ. This suggests another organization. What about $q-$ᛏᚺ? We don't know its meaning but we are able to understand that it is not a typo, as it appears more times in the text. This makes us think that it constitutes a *pattern* of signs.

I think that this example represents in the clearest way Schröder's formal thinking, where there are only signs without significance. The meaning is deduced by taking into consideration the relations in which an element of the structure stands. In the example above, where we observe a structure of signs without trying to understand them, a sign ᚨ receives its peculiar [eigen] meaning through satisfying a particular relation with other signs of the language. The graphical context is the *visible* counterpart of the formal context.

4.3. More on *Über die formalen Elemente*

Let us examine a quotation from [Sch74b]:

> If one connects [verknüpft] progressively three numbers a, b, c with two of the three fundamental operations [i.e. time and division], *eighteen elementary expressions* arise which can be ordered in three groups of six elements, according to our principles of permutations [Vertauschungprincipien] (...) in such a way that in [any group] the letters can be permuted again. [Sch74b, pp. 12–13]

Schröder expressed graphically these three groups of six operations by drawing three pictures (see Figure 4). These drawings are not mere aids to facilitate the understanding. They express the *relations* obtaining among the elements of any group. Any operation has not a meaning in itself, but only in relation to the other operations belonging to the same group. In this case, the relations are permutations. In this way, the drawings of the groups show that a group is a structured totality.

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As a matter of fact, the attention of the reader is first attracted by the boundary of the hexagons, then by the inscribed rectangles,[30] by the inscribed triangles[31] and finally by the arrows. The reader's mind is drawn principally by the general shape of the single figures, which means that the totality is graphically more important than its parts. Of course, an interested and competent reader will bypass the attractiveness of the figure to focus on the meanings of the relations involved. The fact that this requires an effort on the part of the reader means that the graphical context is more natural and fundamental than the elements to which it gives meaning.

4.3.1. Understanding Figure 4

I refer the reader to Figure 1 in the first part above in which Schröder states in a more usual way the possible permutations of an operation. We shall try to clarify the meaning of the first hexagon of operations: $a(bc) = (cb)a$. Because '=' means a double implication, we can assert equivalently $[a(bc) \to (cb)a] \wedge [(cb)a \to a(bc)]$. When the left implication coincides with the right implication (i.e. one uses the same permutation to pass from left to right and vice versa) Schröder draws a *dotted* line and a letter c indicating the particular permutation used. The *full* line *inside* the hexagon indicates that the permutation by which we pass from left to right in an equation is *different* from that enabling us to pass from right to left. Schröder writes over these lines the name of the used permutation with a letter c to which is added an arrow. This arrow indicates the direction of the permutation. For example we might need a permutation c_i to pass from left to right and a permutation c_j to pass from right to left. Finally the full lines of the perimeter indicate that we can move from any angle of the perimeter to another one both clockwise and counter-clockwise.

4.4. Schröder Equation

The previous examples were from a geometrical and an algebraic paper, respectively, to show that the graphic context is determinant not only in a visually-based discipline (as geometry is), but also in an abstract setting (as Schröder's formal algebra was). Now we shall consider an

[30] In the case of the the first hexagon: $\langle a(bc).\frac{a}{(\frac{b}{c})}.(cb)a.\frac{(\frac{c}{b})}{a}\rangle$.

[31] In the first hexagon: $(\frac{c}{b}).a(\widehat{bc).a :}\, (b:c)$ and $(c:b)\, \widehat{: a.\frac{a}{(\frac{b}{c})}}.(cb)a$.

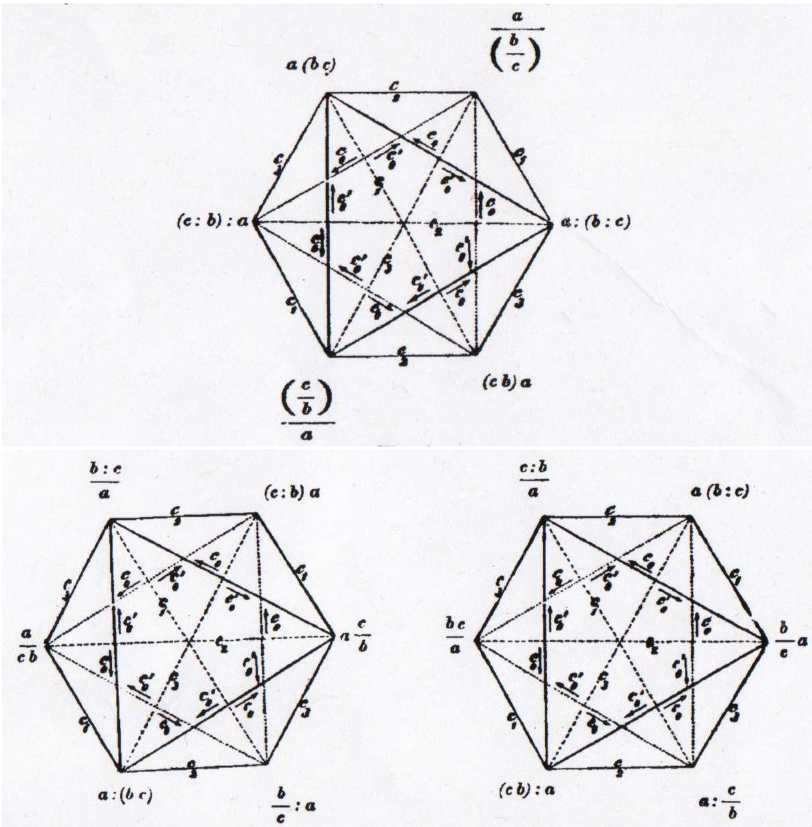

Figure 4. At the angles of these polygons we find the *elementary operations* linked by segments. These segments, indicated by the letter c represent the *principles of permutation* [algorithms] which transform an elementary operation into another elementary operation. The direction of c is indicated with a little arrow. I have cleaned up these images but the letters c are still not clear. It is almost impossible to decipher what type of sub- or super-script the various c have.

example from a paper on functional analysis [Sch70a]. In this case we have no vertical stroke, but an arrangement of the formulas forming the vertices of an hypothetical and invisible square (see [Sch70a, p. 301]):[32]

$$\begin{cases} g = h^{-1} \circ f \circ h, & g \circ h = h \circ f, \\ g \circ h^{-1} = h^{-1} \circ f, & h \circ g \circ h^{-1} = f, \end{cases}$$

[32] Remember that f, g and h are complex functions. See above Section 3.9.

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As one can easily see, there is no need at all to introduce four formulas. A single formula of this *square* suffices to deduce the other three. Nevertheless, remembering also Schröder's combinatorial interests, it is clear that Schröder would attract the reader *not* only to one result, but to the relations between these results. The graphical arrangement is the totality of which the single formulas are parts. Once more the graphical context helps to illustrate a structured totality.

I claim that such a structure imposes itself on the reader with its psychological and hypnotic features, as my *runic* experiment showed. It is not essential that a reader be versed in mathematics in order to grasp that these signs are not written at random. Everybody understands that we are in presence of a pattern. The formulas in issue are not to be read one by one, from left to right or from above to below, but as an unique great composite sign.

Of course for one engaged in this field, one formula can be more useful than other one. In my first book on Schröder I proved a theorem by Peirce using only de Morgan's K Theorem (see [Bon07, pp. 49–50]). To be honest, I actually exploited a formula equivalent to it. The following is de Morgan's celebrated theorem [dM66, p. 224]:

K THEOREM. $R \circ S \subseteq T \iff R^{-1} \circ -T \subseteq -S \iff -T \circ S^{-1} \subseteq -R.$

Peirce affirmed that any reflexive and transitive relation R is equivalent to $R \bullet (-R)^{-1}$ (see [Pei33, p. 65]). For this, the K Theorem in its original formulation was not particularly *inspiring*. I tackled the problem relying on *one of the sixty* sentences by Schröder equivalent to the K Theorem (see [Sch66b, p. 243]):

$$R \circ S \subseteq T \Leftrightarrow R \subseteq T \bullet (-S)^{-1} .$$

Although previously we stated the necessity of reading Schröder's blocks of formulas as a single sign, this doesn't mean that the differences in the formulation are not relevant.

> (...) so one [can] see, that we have at our disposal for any such subsumption [i.e. the K Theorem] immediately 12+24+24 = 60 expressions and one recognizes how terrific[33] multiform [ungeheuer vielgestaltig] (highly multiform) our discipline [i.e. the algebra of relatives] is. Is this

[33] Schröder translated *ungeheuer* with *highly*. I preferred *terrific*, because *ungeheuer* has also a psychological flavour, being synonymous with *schrecklich* [Dud07, p. 1762] (terrible, dreadful [Pon08, p. 753 of the German section]). Schröder is here indicating that the algebra of relatives is so highly multiform as to strike one with fear.

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> an advantage? In any case it is a state of affairs with which we must be satisfied and familiarize ourself! [Sch66b, p. 244]

The totality of which we continue to speak is not an indistinct totality, a *night in which all cows are black*,[34] but a totality structured in itself. I don't know personally what is there, beyond the boundary of the context (semantic, formal or graphical). The boundary is shifty... For this reason I restrict myself to the study of the context. Although the structure has a *gestalt* character, it is a composed totality. It is a sort of *organism* like the human body which is *one* and *many* in the same time. Obviously, the unity of the body is more important than its parts, because any part finds its raison d'être only in the context of the body.

The body is a lattice of relations, in which a component has meaning only in relation to other components. A *liver*, say, detached from the body to which it belonged is only a piece of flesh. It is no more a liver. A liver in itself has no meaning. It acquires its meaning being in a determinate position of the body and fulfilling determinate relations, which characterize a liver as a liver.

We can remove a liver from a body and insert in its place a stone. Unfortunately the stone will not stand in the relations which the liver did. A stone cannot become a liver. Note that what is essential is the bundle of the relations the organ stands in (or not), not the matter or nature of the organ. If there could exist a thing satisfying the same relations which the liver does and no other relation which the liver doesn't satisfy, then we must consider it as a liver.

In this sense, a structural approach focused on the whole not only does *not* put aside the relevance of its parts, but justifies the being of the parts with their peculiar differences. The parts are different inasmuch they are in a particular place in the relational lattice building up the structure.

4.5. Die Umformungsregeln für algebraische Ausdrücke

In 1871 Schröder wrote a little paper entitled *The Transformations Rules for Algebraic Equations* [Sch71] where he proposed the three arrays from Fig. 5 (see [Sch71], pp. 410, 413 and 414, respectively).

---

[34] *To oppose such unique knowledge, that everything is identical in the Absolute, to the distinguished and fulfilled knowledge or to that searching and demanding ful-*

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$$
\text{I.}\begin{cases}
a-b=(a+n)-(b+n),\\
\frac{a}{b}=\frac{a\cdot n}{b\cdot n},\\
\sqrt[b]{a}=\sqrt[b\cdot n]{a^n},\\
\log_b=\log_{b^n}(a^n).
\end{cases}
$$

$$
\text{II.}\begin{cases}
a-b & =(a+n)-(b+n)=(a-n)-(b-n)=(n-b)-(n-a)\\
& =(a-n)+(n-b),\\
a+b & =(a+n)+(b-n)=(a-n)+(b-n)=(a+n)-(n-b)\\
& =(b+n)-(n-a),\\
\frac{a}{b} & =\frac{an}{bn}=\frac{a}{n}:\frac{b}{n}=\frac{n}{b}:\frac{n}{a}=\frac{a}{n}\cdot\frac{n}{b}=\log_{n^b}(n^a)=\log_{\sqrt[a]{n}}\sqrt[b]{n},\\
a\cdot b & =(an)\cdot\frac{b}{n}=\frac{a}{n}\cdot(bn)=(an):\frac{n}{b}=(bn):\frac{n}{a}=\\
& =\log_{\sqrt[b]{n}}(n^a)=\log_{\sqrt[a]{n}}(n^b),\\
\sqrt[b]{a} & =\sqrt[bn]{a^n}=\sqrt[\frac{b}{n}]{\sqrt[n]{a}}=(\sqrt[n]{a})^{\frac{n}{b}}=\sqrt[\log_a n]{\sqrt[b]{n}}=(\sqrt[b]{n})^{\log_n a},\\
\log_b a & =\log_{b^n}(a^n)=\log_{\sqrt[n]{b}}\sqrt[n]{a}=\log_b n\cdot\log_n a=\frac{\log_n a}{\log_n b}=\frac{\log_b n}{\log_a n},\\
a^b & =(a^n)^{\frac{b}{n}}=(\sqrt[n]{a})^{bn}=\sqrt[\frac{n}{b}]{a^n}=(n^b)^{\log_n a}=\sqrt[\log_a n]{n^b}.
\end{cases}
$$

$$
\text{III.}\begin{cases}
(n-a)-(b+n)=(n-b)-(a+n)=-(a+b),\\
\frac{n}{a}:bn=\frac{n}{b}:an=\log_{n^a}\sqrt[b]{n}=\log_{n^b}\sqrt[a]{n}=\frac{1}{a\cdot b}.
\end{cases}
$$

Figure 5. The three arrays

It is curious that in [Sch73], notwithstanding its larger dimensions, there is no such extended grouping. This love for large arrays is by no means justified by the matter in hand. Once the meaning of the seven fundamental operations has been explained, it is pointless writing down the various forms they can assume. We must not forget that Schröder approached mathematics from a combinatorial and formal point of view. This example shows just this.

Incidentally, the first four rows of the second group are present also in the logical setting of [Sch66a, p. 34] to prove once more the priority of the structure with its possible interpretations. However we noted above that Schröder approached logic in a combinatorial vein, showing that an algebraic passage from [Sch73] could be translated into a logical passage

filment, — or to pass its Absolute *as the night, where, how man usually says, all the cows are black, is the naivety of an empty knowledge* [Heg07, pp. 65–66].

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from [Sch90a] (see Subsection 3.1 (the first part)). Schröder re-elaborates and enlarges these groups in [Sch74a, pp. 21–23][35] rotating two pages for lack of horizontal space. One could ask: *why? He could break the lines*. Yes. He could. But this shows how much interest Schröder laid in the graphical layout.

4.6. Schröder on the K Theorem

At this point I cannot *not* mention the 60 equivalent formulations to the K Theorem whose importance Peirce didn't understand. Using Maddux's words:

> It is the system sketched in this paper [i.e. [Pei86]] which Schröder develops into 649 pages, perhaps thereby incurring Peirce's assessment that Schröder *brought out its glaring defect of involving hundreds of merely formal theorems without any signifiance.*[36] [Mad91, p. 425]

I agree with Peirce on the fact that many theorems by Schröder are purely *formal*. Nevertheless, this doesn't imply that they are meaningless. As a matter of fact, a formal sentence has at least a formal context which assigns to it a (formal) meaning. By not listing any formal modifications of the K theorem, its contextual meaning isn't fixed. The theorem acquires its meaning only in relation to other theorems. Schröder was not much interested in a result alone, detached from the context, but in the relational net in which a theorem is embedded.

Although there are many similarities between Peirce and Schröder, they had a different approach to the *same* calculus. I have italicized 'same' because Peirce's and Schröder's calculi were only apparently similar. The philosophy in the background and the context were dissimilar. Peirce had no particular interest in combinatorics, in a formal theory, or in the layout, as opposed to Schröder. Schröder's and Peirce's algebra of relatives were however different *things* with the same *dress*.

[35] Please note that the text in question is [Sch74a] and *not* [Sch73] which the *Outlines of Arithmetic and Algebra* abridge for the students in 1874. I stress this because in the literature [Sch74a] and [Sch73] are often confused. They are not the same book. Unfortunately in many catalogues they are given the same name.

[36] See also [Mad01, p. 7]. In [Pei86] Peirce doesn't mention the name of Schröder. Further proof of the low view that Peirce had of Schröder.

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Now let us introduce the K Theorem as Schröder did (see [Sch66b, p. 242–244]):[37]

> The *first Inversion Problem* requires the solution of any of the four [following] inclusions in the unknown X:
>
> $$1. \begin{cases} X \circ S \subseteq R & R \subseteq X \bullet S \\ S \circ X \subseteq R & R \subseteq S \bullet X. \end{cases}$$
>
> The solution is found (...) by a group of sentences, which I call *the first Inversion Theorems* and which are easy to remember in view of their symmetry and of the *cyclic permutation of the three letters* occuring in them. These theorems determine [statuiren] the equivalence of the following subsumptions equating each other:
>
> $$2. \begin{cases} (S \circ T \subseteq -R^{-1}) \leftrightarrow (T \circ R \subseteq -S^{-1}) \leftrightarrow (R \circ S \subseteq -T^{-1}) \leftrightarrow \\ \quad \leftrightarrow (T \subseteq -S^{-1} \bullet -R^{-1}) \leftrightarrow (S \subseteq -R^{-1} \bullet -T^{-1}) \leftrightarrow (R \subseteq -T^{-1} \bullet -S^{-1}) \leftrightarrow \\ \quad \leftrightarrow (T^{-1} \circ S^{-1} \subseteq -R) \leftrightarrow (R^{-1} \circ T \subseteq -S) \leftrightarrow (S^{-1} \circ R^{-1} \subseteq -T) \leftrightarrow \\ \quad \leftrightarrow (T^{-1} \subseteq -R \bullet -S) \leftrightarrow (S^{-1} \subseteq -T \bullet -R) \leftrightarrow (R^{-1} \subseteq -S \bullet -T), \\ \\ (-R^{-1} \subseteq S \bullet T) \leftrightarrow (-S^{-1} \subseteq T \bullet R) \leftrightarrow (-T^{-1} \subseteq R \bullet S) \leftrightarrow \\ \quad \leftrightarrow (-S^{-1} \circ -R^{-1} \subseteq T) \leftrightarrow (-R^{-1} \circ -T^{-1} \subseteq S) \leftrightarrow (-T^{-1} \circ -S^{-1} \subseteq R) \leftrightarrow \\ \quad \leftrightarrow (-R \subseteq T^{-1} \bullet S^{-1}) \leftrightarrow (-S \subseteq R^{-1} \bullet T^{-1}) \leftrightarrow (-T \subseteq S^{-1} \bullet R^{-1}) \leftrightarrow \\ \quad \leftrightarrow (-R \circ -S \subseteq T^{-1}) \leftrightarrow (-T \circ -R \subseteq S^{-1}) \leftrightarrow (-S \circ -T \subseteq R^{-1}). \end{cases}$$
>
> (...) For application purposes, by the way, the less symmetric expression of our [previous] theorems may often be more manageable:[38]
>
> $$3. \begin{cases} (R \circ S \subseteq T) \leftrightarrow (-T^{-1} \circ R \subseteq -S^{-1}) \leftrightarrow (S \circ -T^{-1} \subseteq -R^{-1}) \leftrightarrow \\ \quad \leftrightarrow (R \subseteq T \bullet -S^{-1}) \leftrightarrow (S \subseteq -R^{-1} \bullet T) \leftrightarrow (-T^{-1} \subseteq -S^{-1} \bullet -R^{-1}) \leftrightarrow \\ \quad \leftrightarrow (S^{-1} \circ R^{-1} \subseteq T^{-1}) \leftrightarrow (R^{-1} \circ -T \subseteq -S) \leftrightarrow (-T \circ S^{-1} \subseteq -R) \leftrightarrow \\ \quad \leftrightarrow (R^{-1} \subseteq -S \bullet -T) \leftrightarrow (S^{-1} \subseteq T^{-1} \bullet -R) \leftrightarrow (-T \subseteq -R \bullet -S), \\ \\ (R \subseteq S \bullet T) \leftrightarrow (-S^{-1} \subseteq T \bullet -R^{-1}) \leftrightarrow (-T^{-1} \subseteq -R^{-1} \bullet S) \leftrightarrow \\ \quad \leftrightarrow (-S^{-1} \circ R \subseteq T) \leftrightarrow (R \circ -T^{-1} \subseteq S) \leftrightarrow (-T^{-1} \circ -S^{-1} \subseteq -R^{-1}) \leftrightarrow \\ \quad \leftrightarrow (R^{-1} \subseteq T^{-1} \bullet S^{-1}) \leftrightarrow (-S \subseteq -R \bullet T^{-1}) \leftrightarrow (-T \subseteq S^{-1} \bullet -R) \leftrightarrow \\ \quad \leftrightarrow (R^{-1} \circ -S \subseteq T^{-1}) \leftrightarrow (-T \circ R^{-1} \subseteq S^{-1}) \leftrightarrow (-S \circ -T \subseteq -R). \end{cases}$$
>
> Note also that any of the *twelve* inclusions equivalent each other in 2 or 3 can be reformulated in order to have V^2 as subject [i.e. antecedent] or Λ^2

[37] It is curious that Schröder does not cite de Morgan formulating these equivalences.

[38] I have coloured in red the K Theorem and in blue the formula which I used to prove Peirce's theorem.

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as predicate [i.e. consequent]. Furthermore according to Grassmann's Theorems[39] they may be converted into an equation in two ways — for example,

$$(R \circ S \subseteq T) \leftrightarrow (V^2 \subseteq (-R \bullet -S) \cup T) \leftrightarrow ((R \circ S) \cap -T \subseteq \Lambda^2) \leftrightarrow$$
$$\leftrightarrow ((R \circ S) \cap T = R \circ S) \leftrightarrow ((R \circ S) \cup T = T)$$

— so one [can] see, that we have at our disposal for any such subsumption immediately $12 + 24 + 24 = 60$ expressions (...).[40]

First of all a little explanation. Schröder lists the inversion theorems in two arrays, numbered 2 and 3. Each array contains two sub-groups of twelve formulas. I have separated these groups by inserting additional spaces. This way, we have four groups of formulas, two in 2 and two in 3. Each formula is only equivalent to any other formula belonging to the *same* group. For example, $(S \circ T \subseteq -R^{-1})$ (first formula of the first group in 2) is not equivalent to, say, $(-S \subseteq R^{-1} \bullet T^{-1})$ (second formula in the third row of the second group). In fact, Schröder distinguishes the first from the second group of any array by putting a comma at the end of the first group.

Maddux rightly orders the inversion theorems in four groups observing that:

> Any two formulæ in the same group are equivalent.
>
> [Mad91, p. 435], [Mad01, p. 12]

Schröder's sentence *these theorems determine the equivalence of the following subsumptions equating each other* is a little ambiguous if we do not pay attention to the last phrase *equating each other*. Only the theorems equating each other are equivalent.

Another little observation must be made about the number of these theorems. Any sentence admits five formulations:

1. the original one,
2. the sentence with antecedent V^2,
3. the sentence with subsequent Λ^2,
4. the sentence obtained by the first Grassmann Theorem,
5. the sentence obtained by the second Grassmann Theorem.

[39] I.e. $A \subseteq B \leftrightarrow A \cap B = A$ and $A \subseteq B \leftrightarrow A \cup B = B$ [Sch66a, p. 25]. In the next formula I have coloured in red the sentences corresponding to the *Grassmann's Theorems* in the algebra of relatives.

[40] I have introduced additional brackets in the last formula to show the order of execution of the operations.

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Then we have, for any group, 12 theorems plus 12 theorems with antecedent V^2, 12 theorems with subsequent Λ^2 (24), 12 theorems reformulated by the first Grassmann Theorem, and 12 theorems reformulated by the second Grassmann Theorem (24). In total we have $12+24+24 = 60$ theorems for any group, as Schröder says. If we operate in the same way with any groups, we obtain $60 + 60 + 60 + 60 = 240$ inversion theorems of which the K Theorem is only one instance. In any case Schröder is able to generalize the K Theorem from only 3 sentences to 60 ones. So there are 240 inversion theorems can be used to solve the first array.

4.6.1. A compendium

This long example from [Sch66b] demonstrates the main features of Schröder's work: the solution problem which is the *source* and the *goal* of the 240 theorems; the combinatorial vein exemplified in the search for any possible (in some cases equivalent) recasting of any formula; the structural approach to the solution problem. Any inversion theorem has meaning only in relation to the others and to the problem. The totality has the priority over the particular. This does not mean that a single formula is not relevant. The whole is not undifferentiated.

In this (Schröder's) night, *not* all the cows are black. In fact six are red and one blue (pun intended). In other words, any formula has particular features which can be turn out fundamental according to the context or the problem in question. As I said before I have used only one of these 240 formulas. The form is important for suggesting or making obvious something hidden in a different formulation.

Bob Coecke uses category theory to envisage a new and more perspicuous symbolic language for quantum mechanics. He affirms that the discovery of *teleportation* was made so late because quantum scientists used the complicated von Neumann formalism:

> Why did it take us 50 years since the birth of the quantum mechanical formalism to discover that unknown quantum states cannot be cloned? Yet, the proof of the *no-cloning theorem* is easy (...). Similarly, why did it take us 60 years to discover the conceptually intriguing and easily derivable physical phenomenon of *quantum teleportation*? We claim that the quantum mechanical formalism doesn't support our intuition, nor does it elucidate the key concepts that govern the behaviour of the entities that are subject to the laws of quantum physics (...). Using

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> a technical term from computer science, the quantum mechanical formalism is *low level*.[41] [Coe09, p. 1, abstract]

Moreover (see [Coe05, p. 1]):

> Why did discovering quantum teleportation take 60 year[s]? We claim that this is due to a *bad quantum formalism* (bad $\neq$ wrong) and this badness is in particular due to the fact that the formalism is *too low level* cf.
>
> $$\frac{\texttt{"GOOD QM"}}{\texttt{von Neumann QM}} \simeq \frac{\texttt{HIGH LEVEL language}}{\texttt{low-level language}}$$

I think that Coecke exaggerates but it is not this our actual concern. Coecke is saying that *form matters*. He entitled one of his papers *Kindergarten Quantum Mechanics* to show that category theory makes quantum mechanics so intuitive that a child could understand it at first sight. Until now Coecke has not yet proved that his theory is equivalent to the usual von Neumann C^*-algebra. If proven we would have two equivalent formulations of the same matter, albeit otherwise different in so many ways!

Schröder himself attached always a great importance to the symbolic language in which to express a theory and engaged in a struggle with Dedekind, Frege and Peano. Unfortunately, pursuing this topic would take us too far afield. We want only make clear the importance of the form. In this sense the 240 theorems above could be all equivalent but would express the same thing in a different way.

4.6.2. Another facet of the previous example

The long excerpt from [Sch66b] shows again Schröder's interest in analysing in how many ways three elements can be connected. We have mentioned [Sch73] and [Sch90a].[42] These inversion theorems state all the possible ways to combine three relation R, S, T. I think that it was this feature of the K Theorem that inspired Schröder's investigations. de Morgan combined three relations in only three ways. Was it not possible to find other combinations?

This sentence of Schröder is interesting from this point of view:

---

[41] By the way Coecke is a computer scientist and this explains his approach to quantum mechanics.

[42] See above subsection 3.1 (first part).

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> During my occupation with the writing of an elementary Handbook[43] of interest to me was the question: in how many ways actually can a sum of n elements (or also a product of n factors) be written?
>
> [Sch70c, p. 361]

In one way or another a combinatorial approach emerges from most of Schröder's papers.

4.7. Well ordering

During his last years Schröder focused on set-theory and on the principle of well-ordering just before Zermelo introduced his *Axiom of Choice.*[44] To this topic Schröder devoted his last paper [Sch01]. The matter clearly interested him deeply because he spoke about it at the congress of philosophy in Paris in 1900 (see [Lov01, pp. 176–177]).[45]

This article was written in French and contains the nth array of formulas by Schröder (see [Sch01, p. 238]):[46]

$$R = R^{-1}, (R \circ R) = R, (\mathsf{Id} \cap S) = (\mathsf{Id} \cap R),$$
$$(-R \circ R) \subseteq (S^{-1} \cap -R), \quad (R \circ -R) \subseteq (S \cap -R).$$

$$\left\{ \begin{array}{llll} T \subseteq \mathsf{Di}, & R \cap T^{-1} = \Lambda^2, & T \circ S \subseteq T, & (R \cup T) \circ (R \cup T) \subseteq (R \cup T), \\ T^{-1} \subseteq \mathsf{Di}, & R^{-1} \circ T \subseteq T, & R \circ T \subseteq T, & \qquad T \circ T \subseteq T. \end{array} \right.$$

$$R = S \cap S^{-1} \cap -T \cap -T^{-1}, \quad T \cup T^{-1} = S \cap S^{-1} \cap -T, \quad T \cap T^{-1} = \Lambda^2.$$

I have reduced the height of the font for the sake of space. In this I have strictly followed Schröder who asked to his publisher to reduce the character of the text when a very large list was to be typeset. I regret having split up the first formula of this group but our modern notation is more cumbersome than Schröder's one.

As in the previous examples not all the formulas are *strictly* necessary.

---

[43] Schröder must be referring to [Sch73] or [Sch74a].

[44] Zermelo wrote for the first time on well-ordering in a letter to Hilbert in 1904 [Zer67b], just two years after Schröder's death. See also [Zer67a]. I refer the reader to [Zer04] and [Zer08], respectively (note: the texts are in German).

[45] This is one of the neglected (or unknown) papers in the literature on Schröder.

[46] Incidentally, T is the relation named *gradation* by Schröder.

5. Jakob Lüroth

This love for nice groupings also conquered Jakob Lüroth during his stay with Schröder. What is following is an excerpt of an early paper by Lüroth written in *Carlsruhe* [sic!] where Schröder taught at the time. Note the *schröderean* vertical stroke (see [Lür96, p. 151]):[47]

Now we define:

An imaginary element ABA_1B_1 *goes in a real element* s*, when any element* $AA_1BB_1\ldots$ *of the fundamental structure goes in* s.	*An imaginary element* ABA_1B_1 *lies in a real element* u*, when any element* $AA_1BB_1\ldots$ *of the fundamental structure lies in* u.

This definition applied to special cases yields:

An imaginary point lies on a real line, when the involution expressing the point lies on the line.	An imaginary plane goes into a real line, when the involution expressing the plane has any line as axis.

In this case, the vertical stroke doesn't separate two propositions that are the duals of each other, but two possibilities. At any rate the stroke is a way to distinguish and unify at the same time. It tells us that there are two alternatives which are in a mutual relationship. So the stroke embeds the multiplicity in the unity, articulating it.

What is is interesting is that yet in 1847 von Staudt used vertical strokes throughout his *Geometrie der Lage* [Geometry of Position] and

[47] It is not my intention to explain this excerpt; I refer the reader to the appropriate literature. About the concept of *imaginary point* Lüroth wrote: *Analytic geometry indicating an imaginary point, for example, with a group of three complex numbers (i.e. actually six real points), one can always choose a structure which allows a definition* [Bestimmung] *with six real numbers to express a point* [Lür96, p. 146]. For von Staudt's original formulation see [vS56a, p. 76]: *If one joins with an involutory uniform* [einförmig] *structure,* $AA_1.BB_1\ldots$ *(which has not any order element) a determinate* [structure] ABA_1 *included in it, one obtains an imaginary element* ABA_1B_1 [of] *first specie, namely an imaginary point, an imaginary line* [of] *first specie or an imaginary plane, depending on if the uniform structure is a set of points* [Punktgebilde], *a bundle of rays* [i.e. the set of all rays which have a midpoint in common and lay on the same plane. See [vS47, p. 9]] *or a bundle of planes* [i.e. the set of all planes which intersect at the same line. See [vS47, p. 9]]. For the definition of *fundamental structure* I refer again to von Staudt: *The straight line, the bundle of rays* (...) *and the bundles of planes are to call uniform fundamental structures or fundamental structures of first specie* [vS47, p. 10].

throughout his successive works as [vS56a] and [vS56b]. Schröder used it for the first time as student in Heidelberg in 1862. Did he know von Staudt at the time? It is impossible to be sure, given the meagre information about Schröder. However, this next example is from [vS47, pp. 31–32]:

If A, B, C, D are four points, and the lines $\overline{AB}, \overline{CD}$ intersect, then the four points lie on a plane whatever; therefore also the lines $\overline{AC}, \overline{BD}$, and the lines $\overline{AD}, \overline{BC}$ intersect.	If A, B, C, D are four planes, and the lines $\overline{AB}$, $\overline{CD}$ intersect, then the four planes go through the same point; therefore also the lines $\overline{AC}$, $\overline{BD}$, and $\overline{AD}, \overline{BC}$ intersect.

In any case Schröder extends the meaning of the vertical stroke. For von Staudt and Lüroth it was only an handy tool to put in evidence symmetries. For Schröder it is something more: it graphically structures the context. It is the visual representation [Vorstellung][48] of his formalism. A stroke for Schröder means the arising of a relational world. It is matter not only of symmetries, but of all relations whatsoever. I don't exclude the possibility that Schröder took up this stroke from the geometrical world, but Schröder transforms in the phenomenological world a mere way of writing into the *Offenbarung* [revelation] of a formal structure.

5.1. Lüroth on Schröder

It is interesting how in the same paper by Löroth, the author, referring to [Sch73], writes:

> (...) now the premises from which to deduce the definition of addition and subtraction are proved, from which all the remaining properties of these operations [**NB**] are purely formal consequences, which one can

[48] The German verb *vor-stellen* from which the noun *Vorstellung* derives is synonymous with *für-stellen* [not more used in contemporary German] [Dud07, p. 1872], that is *to stand for*. The graphical context *stands for*, is an image, of Schröder's formal theory. It is the *phenomenical* appearance, the *Erscheinung* of formalism. If we consult the dictionary by the brothers Grimm [GGss, online research with the query `fürstellen`], we find as explanation of *für-stellen*: *vor Augen stellen* [to put in front of eyes], *præsentare* [from Latin; to show], *ostendere* [from Latin; to make visible — for example *aciem ostenderem* (to show the army ready to fight) Liv.], *im Bilde gegenwärtig sein machen* [to make present with an image], *vor dem Geist erscheinen machen* [to make appear in front of the mind], etc. All these meanings have in common an act which make visible or present in the phenomenon a thing.

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> derive, **without paying attention to the special nature of the connection** [Verknüpfung].[49] [Lür96, p. 191]

Here Lüroth is underpinning Schröder's formalism of *Ueber die formalen Elemente*, inasmuch in the deduction he exploits a pure formal calculus without entering in the particular nature of the relations between the involved propositions. What does *ohne auf die specielle Natur der Verknüpfung einzugehen* mean? That we must abstract from the particular character of a relation. For example, in the case of the relation of inclusion, we neglect its set-theoretical facets and we take it into account only as a pre-order relation. In this sense the relation $\subseteq$ (considered **only** as a pre-order relation) is formally equivalent to $\leq$ or to $\rightarrow$. Schröder spoke not by chance of a general theory of connections.[50]

Referring to the French congress of philosophy which Schröder attended and to his speech on the well-ordering, Macfarlane said:

> The algebra of relations serves as a connecting link between the symbolic logic and the various branches of mathematics such as the calculus of operations and the geometrical calculus. [Lov01, p. 177]

For this reason at a certain point of his life Schröder started to investigate a possible theory of relations which culminated in the third volume of the *Lectures on the Algebra of Logic*. This amounts to saying that all Schröder's work had a formal theory of relations as a goal. Not as a logical or mathematical investigation into the concept of *relation*, but as combinatorial and formal investigations which could constitute the basis for a formal theory, a skeleton to be interpreted in various way according to the topic in question. The project of a possible universal algebra must be regarded as a point of departure to understand Schröder's efforts in any field of research as Macfarlane notes above.

I don't want deny Schröder's valuable results in the algebra of logic, but want rather to stress the fact that for Schröder they were only a possible semantic context for his formal theory of connections. At times, Schroder put aside his mathematical interests to focus on the area of logic. Nevertheless for him logic was only a method subordinate to the main scope of a formal theory. In fact he approached logic in his usual

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[49] The emboldening is mine.

[50] Eine allgemeine *Theorie der Verknüpfung* [Eck01, p. 2], i.e. what Schröder called *absolute Algebra*.

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combinatorial way, as he didn't perceive the difference between algebra and logic. They were only two different facets of the same object.

It is then not surprising that Schröder, once he had arrived at formal concepts of *function* and *set*, put aside logic to focus on the new set theory. Let us look again at Lüroth:

> I began then to consider the applications of this calculus [i.e. the calculus of relatives] to the mapping of sets, as it was introduced by Schröder in the third volume of his exact logic, and pick out the main theorems. [Lür04, p. 73]

It is curious that in this article Lüroth doesn't consider graphically the symmetries between diverse formulas. Although the argument in the same as with the third volume of the *Lectures*, this article has its own style. For example, the layout in general is more modern and less *cumbersome*. Furthermore Lüroth didn't make any use of the symbolism of his friend.

> The *negation* of R, which Schröder indicated with $-R$, is denoted [by us] with $\mathcal{N}R$, to facilitate [the understanding of] the sentences. [Lür04, p. 74].

For the converse of a relation, instead of writing R^{-1},[51] Lüroth wrote $\mathcal{C}R$ (see [Lür04, p. 80]). Nevertheless, if not faithful to Schröder' symbolism, Lüroth was faithful to Schröder's main goal of investigating set-theory as a possible interpretation of a formal theory built up with the aid of the theory of relations. In only 13 pages Lüroth defines the concept of *relation*, the main operations with relations and introduces the theory of quantification. At this points he has the tools to face the most important application of the theory of relatives: the *function theory* of sets.

> Of the calculus of relatives only an application will be made on the *Functions Theory* of sets. [Lür04, p. 90]

The same Schröder at the beginning of [Sch66b] stressed the importance of Dedekind for the calculus of relations, referring to the theory of chains and implicitly to injective maps. As said above, once he had solved the solution problem in the relation calculus Schröder passed on to investigate the concepts of *individuum* (a particular type of set), of *function* (injective and bijective) and of *set*. His successive papers focused on a

[51] Schröder wrote $\breve{a}$.

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set-theoretical definition of the natural numbers $0, 1, 2, 3$[52] on the concept of *finite set*[53] and on well-ordering (see [Sch01]). Schröder was so struck by Cantor's set-theory that he devoted his last years to it. In fact today we associate Schröder mainly with his proof (albeit invalid) of the Cantor-Bernstein-Schröder Theorem: an important set-theoretic result (see [Sch98c, p. 309, formula 4]).

Lüroth follows Schröder in devoting the principal part of his paper to a set-theoretic application of the calculus of relatives. This does not mean that Schröder tried to found set-theory on his calculus of relations. Set-theory is only a possible semantics for a formal calculus. Lüroth is clear on this point: set-theory is *only* an application of the calculus of relations.

6. Finale

6.1. Semantic Context

In section 2 (first part) we showed the importance of a semantic context in assigning references to mathematical concepts. The idea was that a concept receives its precise *determination* only in a *precise* context. A mathematical object such as a mollifer[54] can acquire or lose features depending on the place in which it is inserted. We can see an object from right or from left, from behind or from before; every time we perceive the *same* object but from a different perspective. It is neither feasible or nor very useful from a technical point of view to regard a thing from every possible perspective at the same time. In art this is possible and it is interesting, but in a scientific discipline it would be puzzling and confusing.

6.1.1. James Joyce

Let us consider as an example the following excerpt from Joyce's last masterpiece *Finnegans Wake*, where the author adopts a "cubist" style of writing, showing an object from many points of view simultaneously:

[52] [Sch98a]. The definition of *individuum* is used in this book to define the sets in question. See [Sch98a, p. 367].

[53] [Sch98c]. This paper and the preceding one were read by Schröder at the Leolpoldinish-Carolinish Academy in 1896, exactly **one** year after the publication of [Sch66b].

[54] Let us recall that *mollifers* are certain kind of functions.

> The fall (bababadalgharaghtakamminnarronnkonnbronntonner-
> ronntuonnthunntrovarrhounawnskawntoohoohoordenenthur-
> nuk!) of a once wallstrait oldparr is related early in bed and later on life down through all christian minstrelsy.[55] [Joy92, p. 3]

According to [McH91, p. 3] in the long bracketed word, the following expressions are condensed and distorted: *Babel*, *karak* [thunder; Hindustani], *kaminari* [thunder; Japanese], *brontaô* [I thunder; Greek], *tonnerre* [thunder; French], *tuono* [thunder; Italian], *ṫrovão* [thunder, Portuguese], *tun* [thunder; Old Romanian], *åska* [thunder; Swedish], *tordenen* [the thunder; Danish], *tórnach* [thunder; Irish]. Then, *Wall Street*, *Old Parr* [English centenarian accused of incontinence], *parr* [young salmon], *père* [father; French], *retailed*, *Christy's Minstrels*, and *ministry*.

Using Melchiori's words:

> (...) in the fiction no character (...) [is] only *an unique* character, (...) [encompassing] in itself a crowd of characters far each other in time and space. [Joy82, p. xxxi]

Melchiori is stating that in *Finnegans*[56] *Wake* there is an overlap between different contexts. While this is highly interesting from an artistic point of view, it would be problematic in a scientific setting. In science we must specify every time the context in question according to our purposes. For example, if we are considering the differential calculus, it would be misleading to introduce the mollifier as a δ-function *as Dirac did*, also if possible. At most we can add a footnote about the original context in which the mollifier was introduced. We have stated that an object or a theorem has meaning only in a context, but this context must be determinate and unique.

6.2. Formal Context

We have demonstrated the idea that an object is context-dependent with the concept of *mollifier* and of *fixed point theorem*. Actually there is not a single fixed point theorem. We can formulate a fixed point theorem in diverse ways depending on our project. Schröder thought just this. He tried to generalize scientific objects in order to stress their formal features. This paved the way for investigations into a possible semantics for

[55] I have followed Joyce's original hyphenation.

[56] Please note that it is Finnegans Wake without an apostrophe not Finnegan's Wake.

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them. At the same time this showed the dependence of the meaning of an object on a semantic situation. We have also on one side a purely *formal* concept and on the other side its possible *semantic* interpretations. We have called a possible semantic interpretation a *semantic context*.

In the effort to generalize a scientific theory Schröder envisaged a formal theory, whose elements have no meaning at all.[57] Their meaning arises from the overlapping of all relations which they satisfy inside the theory. It is the totality of these connections that give meaning to the elements, because an element has meaning only inasmuch it occupies a precise place in the relational web of the formal theory. This net is the *formal context* of a formal element which we dealt with in Section 3 (first part).

So, a scientific object admits of two possible meanings, one determined from the semantic context and one determined from the formal one. In any case, it is the totality of the semantic or formal relations that give references to the concepts. The relation itself is a whole which founds the objects fulfilling it. The context is a whole in which everything is entangled. For this reason the process of abstraction of an object belonging to this whole can end only with a thing by fuzzy boundaries. It is not possible to cut out sharply an object from the whole to which it belongs, individual objects of a theory being only *limit* entities.

6.3. Mereology

This way of thought locates Schröder inside a typical German philosophy focused on the centrality of the whole in comparison with its parts. The parts have their *raison d'être* only inside this whole. Then, the fundamental question is: *taking for granted that are the relations to give sense to the elements of a theory, what is meant by* relation*?* Note that in both cases, i.e. in the semantic and in the formal context, the relations are crucial. Schröder tried to answer to this question by developing a theory of relations.

In this way, Schröder's calculus of relations is not a generic possible field of research, but it is the pivot of all his activity. Understanding the concept of *relation* is necessary in order to develop a pure formal theory of connections which admits more possible interpretations. Despite its modest proportions, [Sch74b] shows this most clearly. A scholar

[57] The elements of a formal theory do not have a *semantic* meaning, but a *formal* one.

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must surpass the differences between mathematics and logic and between the various branches of mathematics stressing their common features to build up a theory of which logic and mathematics are only interpretations. For example, the formal operation of $\sqcup$ could be interpreted as the logical connective $\vee$, as the sum of complex numbers, as the geometrical addition of points, as von Staudt's addition of dices, etc.

6.3.1. Foundationalism?

I therefore disagree with the claim that Schroder founded mathematics on a more fundamental theory (calculus of relatives), mathematics being only a possible semantics. There is no relation of foundation or dependence of mathematics on the calculus of relatives. The calculus of relatives is a syntactical theory of which mathematics is a semantics. There is no dependence between syntax and semantics, because they exist in different *worlds*. Who could assert that a theory founds the models which satisfy them?

At any rate, Schröder was ambiguous about the rôle of logic (of relatives); logic seen as a realization of a formal theory of connections is a *syntactical* object, of which mathematics is a *sort* of model. Nevertheless, the same logic, from another point of view, is an *interpretation* of a formal algebra. In this *last* sense, logic and mathematics are *not* dependent one from other, because they are on the same level, being side by side as two possible semantics of the same formal theory. This ambiguity is present in [Sch66b] where at the beginning logic is investigated as an interpretation (i.e. logic is used to sound the meaning of *relation*) to become after the ninth lecture only a *Zeichensprache*, a syntactical theory in which some set-theoretical concepts are recast.

Obviously, I don't deny that in [Sch66b] Schröder stated explicitly:

> The *ultimate* goal of the work [i.e. the work on Dedekind's theory of chains] is: to achieve a rigorous [streng] logical *definition* of the *relational* concept *number of-*, from which we will derive all sentences relating to this concept in a pure deductive way.[58]
>
> [Sch66b, pp. 349–350]

First of all, does it suffice to develop a theory of finite (arithmetic) numbers in order to deduce from it the entire mathematics?

[58] Schröder adds that *the concept of number applies only to* finite *sets* [Mengen] (...) [Sch66b, p. 350]. The expression *in a pure deductive way* means that the process of deduction is formal, not taking in account the meanings of the letters involved.

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> (...) it appears as something self-evident and not new that every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers — a declaration I have heard repeatedly from the lips of Dirichlet.[59]
>
> [Ded96b, p. 792] and [Ded32b, p. 338]

These quotations *seem* to indicate a sort of foundationalism in Schröder. I believe that they are misleading. Schröder refers *only* to the work on chain theory and *not* to the more general work on relations. The investigations into chain theory are only *one* of the topics treated in [Sch66b]. What is not stressed in this famous excerpt by Schröder is the *formal* character of the deduction on numbers. Schröder generalized the concept of *finite number* to achieve a formal definition of it.[60] Schröder's work on numbers is syntactical, with ordinary number theory being only an interpretation of it.

Recall that Schröder used a language of signs [Zeichensprache] and not a language of formulas. Schröder in this passage is asking: *can number theory be a semantics for the calculus of relations?* A good question. If we are able to show that number theory has as a syntactical counterpart the calculus of relations, the problem is solved.

6.4. Graphic context

If the section 3 was devoted to the formal context in Schröder, in sections 4 and 5 we reflected on the importance of the layout in Schröder's papers and books. As a matter of fact the layout is the door by which the formalism enters into the phenomenon. The drawings, the choice of signs and their arrangement make clear the formal relations obtaining between the formal concepts represented by signs. For this reason we borrowed from Schelling the word *Offenbarung*,[61] because Schröder's lay-

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[59] Dedekind refers in this preface also to [Dir68, paragraphs 159, 160 and 163].

[60] We can paraphrase Schröder's words saying that while in [Sch74b] the German mathematician developed a formal algebra, in the third volume of the *Lectures on the Algebra of Logic* [Sch66b] he tried to build up a *formal arithmetic* of which the usual arithmetic is only an interpretation, as usual algebra is only an interpretation of the absolute algebra of [Sch74b].

[61] *Offenbaren* derives from the mittelhochdeutsch *offenbæren* which means *what until now was hidden, to show, to unveil* [Dud07, p. 1228]. If we consult [GGss, query of research `offenbaren`] we find: *to uncover, to make visible.* With the same query we find for *Offenbarung*: *the expression of a inner state. Expression* [Äußerung, in modern German writing] in its turn means *significatio*, i.e. *the act to make known*

$$
\begin{tikzcd}
gcs & & \\
\mathcal{GC} \arrow[u, Rrightarrow] \arrow[r] \arrow[d] & \mathcal{FC} \arrow[r, Rrightarrow] \arrow[dl, Rightarrow] & fcs \\
\mathcal{SC} \arrow[d, Rrightarrow] & & \\
scs & &
\end{tikzcd}
$$

Figure 6. $\alpha \to \beta$ means that α *reveals* β; $\alpha \Rightarrow \beta$ means that β is an *interpretation* for α; $\alpha \Rrightarrow \beta$ means that α *founds* β. Capital letters indicate *wholes*, small letters indicate *elements*. $\mathcal{GC}$ stands for *graphical context*, gcs for *elements belonging to* $\mathcal{GC}$; $\mathcal{FC}$ stands for *formal context*, fcs for *elements belonging to* $\mathcal{FC}$; finally, $\mathcal{SC}$ stands for *semantic context*, and scs for *elements belonging to* $\mathcal{SC}$.

out *reveals* the structure of his formalism, making it *visible* or an *object of experience*. I have illustrated the links between the various contexts, *semantic*, *formal*, and *graphic* in Figure 6.

6.5. Stretta

To sum up: Schröder had a *structural* outlook. In particular, he saw no crucial difference between the various fields of mathematics and logic. They are only possible semantics for a unique formal structure, i.e. the theory of relations or connections. According to this approach, the concepts are fluid, because they can be carried over from one branch of mathematics to another,[62] or from mathematics (in particular, algebra) to logic. We have illustrated this way of working by Schröder by focusing on the solution problem, on his combinatorial investigations, on his building up an absolute algebra and on his search for graphical elegance.

All this demonstrates the importance Schröder ascribed to a structured totality and to a mere syntactical theory, susceptible of many interpretations. Although Schröder contributed to the rise of logic as an

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something [GGss, query of research `äuszerung`]. In any case, an Offenbarung [revelation] makes visible or clear what is concealed. Obviously Schelling used this word with its religious signifiance.

[62] I refer, for example, to the concept of *iteration* which was recast by Schröder in an algebraic [Sch73], in a functional [Sch70a] and in a logical setting [Sch66b].

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independent discipline,[63] he saw no difference between mathematics and logic. As a matter of fact Schröder was not able to differentiate logic from mathematics and his search for a more general and abstract theory shows just this.

Appendix

In paragraph 3.6 quoting from Schröder's *Operationskreis* we highlighted the expression [jede], because it *makes* a theorem wrong which is otherwise correct:

> Any class B can be expressed in a linear and homogeneous form by **any** other [class] A:
> $$B = (X \cap A) \cup (Y \cap -A),$$
> where X, Y are not completely determined *symbols of classes*, which can equal $\emptyset$ or V. [Sch66a, p. 14]

It is not true that *any class B can be expressed as a linear and homogeneous composition with* ***any other*** *class A*. What it is true is that *for any class B* ***at least*** *a class A exists, such that B can expressed as a linear and homogeneous composition with A*. Walter Carnielli, in a private conversation, formulated a counter-example to Schröder's original formulation. Just take $A \subset B$ and X or $Y = \emptyset$. We set $Y = \emptyset$; theorem 14 becomes:

$$B = (X \cap A) \cup (\emptyset \cap -A) = (X \cap A) \cup \emptyset = (X \cap A)$$

But it is impossible that $B = X \cap A$ under our assumptions. In fact, for hypothesis we have that $A \subset B$. Clearly, $(X \cap A) \subseteq A$. $[(X \cap A) \subseteq A] \wedge (A \subset B)$ imply that $(X \cap A) \subset B$ for transitivity. $(X \cap A) \subset B$ contradicts $(X \cap A) = B$ because the inclusion in B is *strict* $[\subset]$. Then, from our assumptions we derive a contradiction. I believe that Schröder was led astray, lacking of a theory of quantification. As a matter of fact, Schröder wrote:

$$\forall B \forall A \exists X, Y (B = (X \cap A) \cup (Y \cap -A)),$$

instead of:

$$\forall B \exists A, X, Y (B = (X \cap A) \cup (Y \cap -A)).$$

[63] Walter Carnielli in a private conversation.

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We must not forget that Schröder introduced the theory of quantification (borrowed from Peirce) 10 years later. At any rate, the main argument doesn't change. It is a fact that Schröder had in mind the possibility of expressing one object as a function of two other objects opposite each other. The comparison with modern linear algebra is so not odd. It is a metaphor to decipher Schröder's thought. Let me stress again that we are not in possession of any detailed biography or study on the German mathematician. He had no friends,[64] and no other scholar was really interested in him. Neither Peirce who underestimated Schröder, nor Peano who took into account Schröder's books only for their references, nor the historian Moritz Cantor. Schröder passed almost unnoticed in his time. The use of the *metaphor* is thus indispensable as a tool for understanding Schröder's philosophy.

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[64] I don't believe that the friendship between Schröder and Jakob Lüroth was a great one, despite what Lüroth wrote in his obituary: *with him has passed away a good and friendly man, a competent mathematician, and for me a dear and loyal friend (with whom for forty years I was united by a close relationship)* [Lür02, p. 249]. I think that these words are a bit exaggerated, as Schröder didn't give him the task of putting in order his *Nachlass* after his death. He chose Eugen Müller.

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