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MODELS OF POSSIBILISM AND TRIVIALISM

‘I can’t believe THAT!’ said Alice.
‘Can’t you?’ the Queen said in a pitying tone.
‘Try again: draw a long breath, and shut your eyes.’
Alice laughed. ‘There’s not use trying,’
she said: ‘one CAN’T believe impossible things.’
‘I daresay you haven’t had much practice,’ said the Queen.

Abstract. In this paper I probe the idea that neither possibilism nor trivialism could be ruled out on a purely logical basis. I use the apparatus of relational structures used in the semantics for modal logics to engineer some models of possibilism and trivialism and I discuss a philosophical stance about logic, truth values and the meaning of connectives underlying such analysis.

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1. Introduction

I will call *possibilism* the thesis according to which anything is possible and *trivialism* the idea that every proposition is true or more generally, that every proposition (or what one takes as truth bearer) has a designated value.[1] The thesis I want to try to make plausible here is that

[1] I introduce this more general version of trivialism because I am not going to leave out the possibility that there were more than two truth values and other values besides truth the preservation of which makes an inference valid (or whose posses-

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possibilism and trivialism, or at least certain versions of them, cannot be dismissed on purely logical grounds. Note that I am not saying that trivialism is true as is done in [7], but I am making the weaker claim that nothing in our logical notions rules out that it could be so.

This idea is not entirely new. On one hand, some authors have pointed out that the same strategy followed by dialetheists to make room for true contradictions in logic can be mimicked by trivialists, and trivialism cannot be rejected on a purely logical basis if neither dialetheism is (cf. [3], [19]). On the other hand, João Marcos [8] has showed how certain "degenerate" logics arise from a very abstract notion of consequence. Between logics where nothing follows from nothing and logics where everything follows from everything, there are the well-known cases of consequence where only something follows from something (and perhaps from nothing). Nonetheless, the philosophical foundations of such a general view of logical consequence might not be neat enough. Something similar happens with the case of the internal logic of degenerate toposes. Logical notions, from truth values to zero-, first- and higher-order connectives can be defined in toposes. However, in degenerate toposes (categories in which for every mathematical purpose one can say that there is only one object) every proposition is (the same morphism as) *true*. This is not due to a change in the definition of logical notions, but is a special case of them (cf. [4]). To my knowledge, there has been no philosophical reflection on this. In contrast, the most sustained defense of trivialism to date ([7]) lacks a formal apparatus which enables the discussion of some subtle points. I hope this paper contributes to clarification both in the philosophical and the formal sides.

My defense of the claim that neither possibilism nor trivialism can be ruled out by pure logic rests on a very important assumption, namely semantic minimalism. According to it, not every element related to a logical notion contributes to its meaning. For example, when it is said that the meaning of a connective is given by its truth conditions, most of the times what the authors are taking as truth conditions is a mixture of several elements. Think of any common truth table. In it you will find:

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sion under any valuation makes a formula a logical truth). I will manage a similar generalization for other thesis like consistentism (the idea that there are no designated contradictions), dialetheism (the thesis that some but not all contradictions are designated), etc.

- Truth conditions properly speaking, a rule that says which values is going to take a formula, but also:
  - a minimum number of truth values, usually two (true and false),
  - a maximum number of truth values (again usually two).
- A very determinate notion of validity, which presupposes
  - a very determinate way of separating truth values.

In the case of a relational semantics, truth conditions are mixed with

- The structure of frames.

Roughly, a *maximalist* thinks that every element related to a logical notion, as those listed above for the case of connectives, contributes to determine its meaning. A *minimalist* thinks that not every such element makes such a semantic contribution.[2] The other elements would give shape to a logic, but not meaning to the logical notions. Thus, classical features result from truth conditions plus other restrictions, concerning frames and truth values and their respective structures. More generally, a logic L can be seen as a result from general truth conditions, common to several logics, and particular desiderata concerning frames and truth values and their respective structures. By the *structure of truth values* I will mean, roughly, the exact number of truth values, how do they relate to each other as well as how do they relate to propositions. Similarly, the *structure of frames* consists of the nature of indexes of evaluation (whether they are worlds or times or epistemic states, etc.) and their exact number, how do they relate to each other as well as how do they relate to propositions and truth values.

Minimalism for the connectives suffices to make room for possibilism and trivialism in the realm of logic. Nonetheless, I will try to advance also minimalism for truth values, that is, I will suggest that truth values can be characterized in such a way that it is compatible with, for example, the collectively exhaustiveness or the mutually exclusiveness of truth and falsity, but does not imply them.

Minimalism can be seen as supporting a distinction between investigating what are the most general features of logical notions and investigating what are *our* logical notions (the right logic, if any, for studying

[2] I take the labels from [5], and build upon his characterization of minimalism. For my present purposes, I have given model-theoretic characterizations of minimalism and maximalism, even though Hjortland was originally interested in proof-theoretic semantics.

the modal properties of our world, or for using our language(s), or for doing the *mainstream* mathematics, etc.). I do not know whether our world is a trivial one or not, whether it is consistent or not. In that sense, possibilism might not be the case, in the sense that our modal space do not correspond to that described in possibilism. However, it does not dismiss it as a logical theory. Thus, it should be emphasized that more than attempting a case for possibilism and trivialism, I want rather to contribute to a better understanding and formulation of them.

The plan of the paper is as follows. In section 2 I introduce the basic features of the philosophical ideas of possibilism and trivialism. As to the first, Mortensen characterizes it as the idea that necessitarianism is false, i.e. there are no necessary truths and everything is possible. In section 3 I use the apparatus of relational structures used in the semantics for modal logics to engineer some models of possibilism and trivialism. More than one may miss the philosophical bases of the analyses in section 3, so in section 4 I discuss a philosophical stance about logic, truth values and the meaning of connectives such that possibilism and trivialism find place (more or less) easily in the realm of logic. The models are decidedly straightforward and the philosophical discussion is as deep as the current understanding of trivialism allows, but I hope that what I present here is still useful.

At last, let me make some disclaimers. For brevity, only zero-order logics will be considered here. I will assume the reader is familiar with basic modal logic and some acquaintance with non-classical logics like *FDE*, K_3 or Priest's *LP*, though not necessary, would prove useful. About these logics, including basic modal logic, one can consult [14]. The technical apparatus presented here overlaps in some respects with that of non-normal modal logics, but there are also several important differences ([14] is also a good start point on non-normal modal logics). One of the most significant is that I do not require "non-normal" worlds to have special truth conditions for connectives in order to accommodate either trivialism or possibilism. In order to avoid making of this an unnecessarily lengthy text, I reserve a more exhaustive comparison both formal and philosophical for further work. The truth conditions I use here are not necessarily what I would regard as the "true" truth conditions or meanings of connectives. These are working truth conditions for a specific purpose: to show that the connectives of some families of logics share the same truth conditions, and hence those truth conditions are compatible with trivialism and different versions of anti-trivialism, including, say, the consistentist cases of classical logic or K_3 and the paraconsistent cases of *FDE* or *LP*. I must say that what will be discussed here would require only a slight modification in order to discuss instead logical nihilism, the idea that no proposition is true or, more generally, that no proposition has a designated value. However, for reasons of space and presentation I will not discuss logical nihilism explicitly. I will give some hints here and there about how to do that, but I am sure that the reader can figure out completely, or at least significant part of, the required modifications.

2. Brief descriptions of possibilism and trivialism

2.1. Mortensen's possibilism

The most detailed defense of possibilism, the idea that anything is possible, has been given by Mortensen (see [9], [10]). Mortensen's possibilism is also non-necessitarianist, i.e. also defends that nothing is necessary. The point of view that there are necessary truths concerning the empirical world has always been a problematic position, but many thinkers found solid bases of necessary truths in the realms of mathematics and logic. And even in face of the "crises" in mathematics in the turn from XIX century to XX century, logic was thought of as having a special and secure status. Mortensen's arguments for "weakening the necessitarian intuitions and strengthening the possibilist intuitions" are mainly epistemological and model-theoretic. Let me focus on the model theoretic side. Contemporary model theory has far gone beyond the models of classical logic. Several truths regarded as necessary in classical logic have been doubted. From semantic principles (the existence of two and only two truth-values, truth functionality for zero-order logic, etc.) to the validity of once sacred formulas ($A \supset A$, $A \vee \neg A$, $\neg(A \wedge \neg A)$, etc.), through axioms and rules of inference, there have had a proliferation of logics in which some of such features are absent. For example, intuitionists restrict some principles only to finite domains; paraconsistentists consider that classical logic is inapplicable in inconsistent domains, and so on. Now virtually every logical truth or valid inference has been thrown out, in the sense that there are models thought of as "situations" where they no longer hold, and now nearly every structure resulting from dropping such logical truths or valid inferences is accepted as a logic.

The following excerpt from Mortensen's [9] summarizes well the possibilist attack against model-theoretic necessitarianism (italics in the original):

> The semantics of classical first order logic is very much a special case of a much wider semantical framework, many of the details of which have only become apparent recently. I do not maintain the thesis that the *mere* existence of alternative models *suffices* to demonstrate that propositions refuted therein are not really necessary truths. I am making the weaker point that a model-theoretic necessitarian is in no position to say that various theses are logically true *solely* on the grounds that they hold in all models. Therefore, the model-theoretic necessitarian is in the position of having to argue that certain models have a *preferred* status over others. But it is not easy to see how this is to be done short of metaphysical dogmatism. The model theory *by itself* does not provide this.

2.2. Trivialism

Consistentism is the idea that no formula of the form $A \wedge \neg A$ has a designated value; dialetheism is the idea that some contradictions, but not all, have a designated value. Trivialism is the thesis that every proposition has a designated value, and so consistentism and dialetheism should be allied against it.[3] Both consistentism and dialetheism can be presented as modal theses. For example, consistentism would say that contradictions necessarily fail to have a designated value, or that are impossible, whereas dialetheism (in Priest's version) is the thesis that in the actual world there are some true contradictions (and hence, some contradictions are possible). But then "Every proposition has a designated value" could mean several things in a modal context, not all them blatantly implausible. Let me clear this formulating the following modal varieties of anti-trivialism and then some varities of trivialism:[4]

(AT0) In the actual world some propositions fail to have a designated value. Call this *actualist minimal anti-trivialism*.

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[3] One can also find the formulation of trivialism as the claim that every proposition is both true and false (cf. [19]). For reasons that will become clear later, I do not think this formulation is correct.

[4] Form more generality, quantified worlds below can be taken as relative or accessible to a given world.

(AT1) In the actual world all propositions fail to have a designated value. Call this *actualist absolute anti-trivialism*

(AT2) In some worlds some propositions fail to have a designated value. Call this *minimal anti-trivialism.*

(AT3) In some worlds every proposition fails to have a designated value. Call this *pointed anti-trivialism* or *minimal logical nihilism.*

(AT4) In every world some propositions fail to have a designated value. Call this *distributed anti-trivialism.*

(AT5) Some propositions fail to have a designated value in every world. Call this *strong anti-trivialism.*

(AT6) Every proposition fails to have a designated value at some world. Call this *super anti-trivialism* or *moderate logical nihilism.*

(AT7) All propositions fail to have a designated value in every world. This can be called *absolute anti-trivialism* or *maximal logical nihilism.*

(AT7) implies all other anti-trivialisms but none of them implies it, while (AT0) is implied by all other anti-trivialisms and implies none of them. The complete relations of logical dependence between these anti-trivialisms can be figured out by the reader.

Both consistentism and dialetheism are necessitarianist: Both accept some necessary truths, whether the same or not. For example, consistentism endorse (AT5) and is strongly anti-trivialist: Suppose A is necessary for the consistentist, i.e. has a designated value in every world. Then $\neg A$ is impossible, for it fails to have a designated value at every world. Dialetheists are in general at most minimally anti-trivialists, but nonetheless usually take the arithmetical statement $\neg(0 = 1)$ as an example of proposition that has a designated value at every world, so it is necessary.

To the above taxonomy of anti-trivialisms corresponds a taxonomy of trivialisms:

(T0) Every proposition has a designated value at some world. Call this *minimal trivialism.*

(T1) In some worlds every proposition has a designated value. Call this *pluralist trivialism.*

(T2) In the actual world every proposition has a designated value. Call this *actualist trivialism.*

(T3) Every proposition has a designated value at every world. Call this *absolute trivialism.*

These varieties of trivialism are of increasing strength. Some of these varieties are far more plausible than others as descriptions of our "modal space", e.g. (T0) seems much more appealing than (T2) or (T3). Moreover, possibilism and trivialism seem to be closely tied: if possibilism is the case, then trivialism is possible; if trivialism is the case then (trivially!) possibilism is the case.[5] Trivialism is typically thought to be discarded on purely logical grounds, but as I hope to have made clear with the modal formulation, some trivialisms have certain degree of plausibility or attractiveness. This is not enough, though. In the next section I extract models of possibilism from some very general yet familiar characterizations of connectives and truth values.

3. Some models of possibilism

3.1. Technical preliminaries

Semantically, a zero-order logic can be characterized by a structure $S_L = \langle F,\ \mathcal{V},\ D^+,\ \{f_c : c \in C\},\ \mathfrak{v}\rangle$.[6] F stands for a collection of formulas, C is a collection of connectives and for each connective c, f_c is the truth function it denotes. $\mathcal{V}$ is a non-empty collection of truth values. It is standard to assume that $\mathcal{V}$ comes with an ordering, $\leq$ (which may be a partial ordering). I will assume in what follows that this is so.[7] I will also assume that every subcollection of $\mathcal{V}$ has a greatest lower bound (Glb) and least upper bound (Lub) in the ordering. Let us suppose that there is an $x \in \mathcal{V}$ such that for every $y \in \mathcal{V}$, $y \leq x$. Let us call *true* such x and denote it '$\top$'. An interpretation, $\mathfrak{v}$, assigns values in $\mathcal{V}$ to propositional parameters; the values of all formulas can then be computed using the f_c. $D^+ \subseteq \mathcal{V}$ is a collection of *designated values* if it satisfies the following conditions:

(a^+) $\top \in D^+$;
(b^+) for every $x, y \in \mathcal{V}$, if $x \in D^+$ and $y \notin D^+$ then $x \nless y$;

[5] See [10]. Kabay (cf. [7, ch. 3]) has argued that possibilism implies not only the possibility of trivialism but its truth. An interesting version of Kabay's argument is studied in [6].

[6] In this section I have built closely upon Priest's characterization of many-valued modal logics in [15].

[7] Every partial order induces a strict order, defined as $x < y = (x \leq y \text{ and } x \neq y)$.

(c^+) an inference is D^+-valid in L, denoted $A \Vdash_L^{D^+} B$, if and only if whenever $\mathfrak{v}(A) \in D^+$, $\mathfrak{v}(B) \in D^+$ too.

A collection $D^- \subseteq \mathcal{V}$ is going to be called collection of *antidesignated values* if it satisfies the following conditions:

(a^-) There is an $x \in \mathcal{V}$ such that for every $y \in \mathcal{V}$, $x \leq y$. Let us call *false* such x and denote it '$\perp$'. $\perp \in D^-$;
(b^-) for every $x, y \in \mathcal{V}$, if $x \in D^-$ and $y \notin D^-$ then $y \not< x$;
(c^-) an inference is D^--valid in L, denoted $A \Vdash_L^{D^-} B$, if and only if whenever $\mathfrak{v}(B) \in D^-$, $\mathfrak{v}(A) \in D^-$ too.

As a loose but useful (at least for the purposes of this paper) working characterization of a modal logic, let me say that it is a logic containing connectives indicating a "mode" in which propositions have their truth values. Two of the most well-known are the alethic modalities "necessarily" and "possibly", usually denoted '$\Box$' and '$\Diamond$'.

A *model* of a modal logic is a structure $\mathfrak{M} = \langle W, O, R, S_L, \mathfrak{v} \rangle$, where W is a non-empty collection of indexes of evaluation and R an accessibility relation between those indexes; $O \subseteq W$ is a collection of indexes such that $W - O$ is the collection peculiar indexes; S_L is a structure for a logic, L, $\mathfrak{v}$ is a mapping from $F \times W$ to $\mathcal{V}$ such that for each zero-order parameter, p, and index, w, $\mathfrak{v}$ assigns the parameter a value, $\mathfrak{v}_w(p)$, in $\mathcal{V}$.[8] D^+-validity and D^--validity can be redefined accordingly taken into account the relevant quantification over indexes of evaluation. In what follows, I will use p, q, ... for propositional parameters and A, B, ... for arbitrary sentences.

The truth conditions for the connectives at a world w simply deploy the functions f_c. Thus, if c is an n-ary connective, $\mathfrak{v}_w(c(A_1, \ldots, A_n)) = f_c(\mathfrak{v}_w(A_1), \ldots, \mathfrak{v}_w(A_n))$. For the sake of definiteness I will use the following truth conditions:

(i) For every $w \in W$:
$\mathfrak{v}_w(A) \in \mathcal{V}$,
$\mathfrak{v}_w(\neg A) = f_\neg(\mathfrak{v}_w(A)) = \top$ if and only if $\mathfrak{v}_w(A) = \perp$, otherwise $f_\neg(\mathfrak{v}_w(A)) = \mathfrak{v}_w(A)$,
$\mathfrak{v}_w(A \wedge B) = f_\wedge(\mathfrak{v}_w(A), \mathfrak{v}_w(B)) = \mathrm{Glb}(\mathfrak{v}_w(A), \mathfrak{v}_w(B))$,

[8] Strictly, the mapping $F \times W \longrightarrow \mathcal{V}$ considered here is an extension of the mapping $\mathfrak{v} : F \longrightarrow \mathcal{V}$ introduced above. I make an abuse of notation and will also use '$\mathfrak{v}$' for the extended mapping.

$\mathfrak{v}_w(A \vee B) = f_\vee(\mathfrak{v}_w(A), \mathfrak{v}_w(B)) = \mathrm{Lub}(\mathfrak{v}_w(A), \mathfrak{v}_w(B))$,
$\mathfrak{v}_w(\Box A) = f_\Box(A) = \mathrm{Glb}\{\mathfrak{v}_{w'}(A) : wRw'\}$,
$\mathfrak{v}_w(\Diamond A) = f_\Diamond(A) = \mathrm{Lub}\{\mathfrak{v}_{w'}(A) : wRw'\}$.

(ii) If $w \in W - O$ there might be further truth conditions, with the only proviso that they do not imply the violation of truth conditions in (i).

The conditional can be defined as $A \supset B =_{\text{def}} \neg A \vee B$.

A is *designated at an index* w if $\mathfrak{v}_w(A) \in D^+$ and is *designated in a model* $\mathfrak{M}$, denoted $\mathfrak{M} \Vdash A$, if A is designated at every w in $\mathfrak{M}$. A formula A is *valid in a class* $\mathcal{C}$ of models, denoted $\mathcal{C} \Vdash A$ or sometimes simply $\Vdash A$ if the context prevents confusion, when it is designated in every model of the class.

A logic satisfying the following conditions is going to be called *proto K-like logic*, denoted *proto* K_L:

(K1) It contains the zero-order logical truths of a base logic L;
(K2) it satisfies the necessitation rule (RN): if $\Vdash A$ then $\Vdash \Box A$ too.

The family of logics just delineated are proto K-like. For the validity of (RN), suppose $\nVdash \Box A$. Then there is a $\mathfrak{v}_w$ such that $\mathfrak{v}_w(\Box A) \notin D^+$. Thus, for some w' such that wRw', $\mathfrak{v}_{w'}(A) \notin D^+$. Hence $\nVdash A$.

A *quasi K-like logic*, denoted *quasi* K_L, is a proto K-like logic satisfying the following further condition:

(K3) it satisfies the modal axiom (K): Being © the conditional of the given logic, $\Box(A © B) © (\Box A © \Box B)$ is valid.

A *pseudo K-like logic*, denoted *pseudo* K_L, is a proto K-like logic satisfying the following condition:

(K4) $\mathcal{L}$ is closed under the detachment rule: Being © the conditional of the given logic, from $A © B$ and A, infer B.

A *K-like logic*, denoted K_L, is a logic which is both quasi K-like and pseudo K-like.

Let me consider a particular group of proto K-like logics.[9] Let $\mathcal{V} = \mathcal{P}\{\bot, \top\} = \{\varnothing, \bot, \top, \{\bot, \top\}\}$ with $\bot$ as the least element, $\top$ as the

[9] Compare with [15]. Priest calls there 'K logicsÂİ a number of logics which do not satisfy what is proper to K, namely the axiom (K). I have introduced more labels in order to facilitate comparison.

greatest one and $\varnothing$ and $\{\perp, \top\}$ as incomparable. $D^+ = \{\{\perp, \top\}, \top\}$.[10] These requirements, taken together with the above truth conditions for connectives, constitute the logic *FDE* and, in the modal case, one can get at most *proto* K_{FDE}, since one cannot obtain (K) and the rule of detachment is not valid for *FDE*. If one ignores the value $\varnothing$ in the non-modal case (i. e. if $\mathcal{V} = \{\perp, \{\perp, \top\}, \top\}$, the logic obtained is Priest's logic of paradox *LP*. In the modal case, one obtains at most *quasi* K_{LP}, for detachment is not valid for *LP*. If one ignores the value $\{\perp, \top\}$, one gets Kleene's logic K_3. In the modal case, at most one gets *pseudo* K_{K_3}, for there are no logical truths in K_3 (and this does not change when modal particles are added) although detachment holds in it. Finally, if in the non-modal case one ignores both $\varnothing$ and $\{\perp, \top\}$, one gets classical logic. In the modal case, one can get K_{CL}, for it satisfies $(K1) - (K4)$.[11]

3.2. Models of possibilism

In order to get a model of possibilism, the intended interpretation of the elements of W would be as "worlds", i.e. as indexes of evaluation of alethic modes: necessarily, possibly, etc.[12] According to the informal

[10] Intuitively, the values $\varnothing, \perp, \{\perp, \top\}, \top$ stand for *neither true nor false*, *(just) false*, *both true and false* and *(just) true*, respectively.

[11] Some authors have proposed to modify *FDE* or *LP* introducing suitable conditionals for formal reasons, say, to ensure the validity of detachment or the validity of certain formulas. There are also philosophical reasons, for example:
- It is desirable to consider the conditional as a connective conceptually different from the others; interdefinability between connectives must appear after the fact, not due to their meanings.
- Some valuations for $\neg A \vee B$ look odd as valuations of a conditional. For example, $\mathfrak{v}\ (A \supset B) = \top$ when $\mathfrak{v}(A) = \{\perp, \top\}$ (both true and false) and $\mathfrak{v}(B) = \varnothing$ (neither true nor false), or $\mathfrak{v}(A \supset B) = \{\perp, \top\}$ when $\mathfrak{v}(A) = \{\perp, \top\}$ and $\mathfrak{v}(B) = \perp$. In both cases, the conditional has a designated value even though the antecedent is designated and the consequent is not.

There are several ways of trying to accomplish these formal and philosophical desiderata. For more on this see see [14, chapters 7 and 8].

[12] I avoid the label "possible worlds" because it is potentially misleading, and it is even more misleading talking about "impossible worlds". The notions of possibility and impossibility are mainly logic-relative. For example, a contradiction is impossible in both classical and intuitionistic logic, as well as in Kleene's (strong) logic K_3, but it is not in other logics like Priest's *LP*. In the case of K_3, if $\Diamond A$ were defined as "there is a world w in which $\mathfrak{v}_w(A) \neq \perp$", contradictions could be possible in it, although they would remain impossible in classical and intuitionistic logics. $p \vee \neg p$ failing to be a logical truth is impossible according to classical logic and *LP*, but not

exposition of possibilism in section 2, we are looking for models in which $\Diamond A$ has a designated value.

It could be thought that a model of $\Diamond A$ is one where $\mathcal{V} = \{\bot, \top\}$, $O = \varnothing$ (that is, there are only peculiar worlds) and truth conditions are as in (i), except for the modal connectives, whose truth conditions are instead those usually used for peculiar or "non-normal" worlds w^* in the semantics of non-normal modal logics:

$\mathfrak{v}_{w^*}(\Box A) = \bot$;
$\mathfrak{v}_{w^*}(\Diamond A) = \top$

This works. In a very broad sense, it is a model of possibilism, but at best it is *ad hoc*; at worst it introduces undesirable arbitrariness. Since the non-modal base for non-normal modal logics could be classical logic, it would be rather odd that formulas whose valuation at a world is never $\top$ (i.e. at every world are not true) are nonetheless possible. It is equally odd that formulas whose valuation at a world is always $\top$ (i.e. are true at every world) are nonetheless not necessary. Those truth conditions does not seem to be capturing the meaning of "necessarily" and "possibly". Said otherwise, the truth of possibilism in this model is by decree, not by a special interplay between worlds and their nature, on one hand, and truth values and truth conditions, on the other.

That is why I have given the clause (ii). Roughly, (ii) says that the meaning of connectives are fixed by (i), so whatever is peculiar in the propositions being designated at peculiar worlds must be due not to a change in the meaning of connectives, but to the peculiarity of other components of the model. Making $\vee$ to mean by decree what $\wedge$ means is not very interesting; in contrast, it would be very interesting that there were worlds where, without changing their respective truth conditions, the true formulas containing $\vee$ were such that one rather would consider them formulas containing $\wedge$.

Thus, in my narrower but, I hope, more accurate sense, this model of possibilism is not a model at all.

Intuitively, $\Diamond A$ holds in a world w if it is related to another world w' where A holds. Now, for $\Diamond A$ to hold in a model it must be the case for

according to intuitionistic logic or K_3. The notion of an index of evaluation of alethic modes, that is, a *world*, is not tied to any particular logic, so it is not necessary to add the logic-relative adjectives "possible" or "impossible". I have applied similar considerations to avoid other biased and potentially pejorative labels such as "non-normal", "abnormal", "queer" for the worlds in O. For a similar stance see [17].

every w and every A of the model ($\Diamond A$ as holding in a class of models can be defined *mutatis mutandis*). More formally, a model of possibilism is a model in which R satisfies the following condition:

(POS) For every A and w, there is w' such that wRw' and $\mathfrak{v}_{w'}(A) \in D^+$.

(POS) thus has two components: the seriality of R (for every w there is a w' such that wRw') and the condition that every formula is satisfied at some world (related to another given world). Let call this accessibility relation "possibilist seriality". Clearly, (POS) plus the satisfaction condition for $\Diamond A$ above imply that in a model $\mathfrak{M}$ where (POS) holds, $\mathfrak{M} \Vdash \Diamond A$, for every A. R, possibilist seriality, may be equivalent to other kinds of relations or it can be strengthened, all this depending either on the structure of W or $\mathcal{V}$, or in the way in which $\mathfrak{v}$ is extended in a given model of possibilism.

Note that the logical notions defined in the previous section are pretty standard, yet they are compatible with models where the following hold:

(∗1) For every A and world w, $\mathfrak{v}_w(A) \in D^+$
(∗1.1) For every A and world w, $\mathfrak{v}_w(A) = \top$
(∗2) For every A, $\mathfrak{M} \Vdash A$
(∗3) $\top = \bot$

Thus, some models of possibilism are those which equates it with trivialism. One of the simplest models of trivialism is one where W consists of a unique w such that every formula has a designated value in it, or where every formula is a logical truth or every inference is valid. I will use 't' to denote such a world. How can one get a trivial world? The notions defined in 3.1 allow several constructions:

Example 1. One of the simplest is considering a model $\mathfrak{M}$ where $W = \{w\}$ and $\mathcal{V} = \{\top\}$. In that case it is easily proved that R being reflexive suffices for being possibilistly serial. Moreover, for every A, not only $\mathfrak{M} \Vdash \Diamond A$, but also $\mathfrak{M} \Vdash A$ and $\mathfrak{M} \Vdash \Box A$.

Example 2. Yet another model contains a special world $t \in W - O$ and the valuation $\mathfrak{v}$ is defined as to contain the clause *If* $w = t$*, then* $\mathfrak{v}(A) \in D^+$ *for every* A.

When there are two or more worlds in W, the construction in Example 2 will be the way to get a trivial world.

Example 3. There is also a model $\mathfrak{M}$ with exactly two worlds, w and w' and one of them is t. In this case it suffices that wRt and tRt to satisfy (POS).

But one does not need a trivial world in W in order to get a model of possibilism. A more interesting case seems to be that where there are denumerable many worlds, whether with a t or not. Indeed, the more interesting case is a model without such a t. Again, the easiest way would be requiring that $\mathcal{V}$ has some special features.

Example 4. Let W contain no trivial worlds and $\mathcal{V}$ as in *FDE*. In this logic there is no A such that for every $\mathfrak{v}$, $\mathfrak{v}(A) \notin D^+$. Thus, for every A, some $\mathfrak{v}$ is such that $\mathfrak{v}(A) \in D^+$. This can be used to construct a model $\mathfrak{M}$ where (POS) holds.[13] But this implies that in $\mathfrak{M}$ $\mathrm{Lub}\{\mathfrak{v}_{w'}(A) : wRw'\} \in D^+$. Hence, for every w, $\mathfrak{v}_w(\Diamond A) \in D^+$, so $\mathfrak{M} \Vdash \Diamond A$.

Recall that if one ignores the value $\varnothing$ in the non-modal case (i. e. if $\mathcal{V} = \{\bot, \{\bot, \top\}, \top\}$, the logic obtained is *LP*. In the modal case, one obtains quasi K_{LP}. If one ignores the value $\{\bot, \top\}$, one gets K_3 and, in the modal case, one gets pseudo K_{K_3}. Finally, if in the non-modal case one ignores both $\varnothing$ and $\{\bot, \top\}$, one gets classical logic and, in the modal case, one gets K_{CL}. They all are at least proto K-like logics, as seen also in the previous section, but they are indeed very different modal logics nonetheless:

Proto K_{FDE}: For every A, $\Diamond A$ is a logical truth but $\neg\Diamond A$ is not, and $\Box A$ is logical truth only if A has the form $\Diamond B$. Intuitively, in it there are no impossibilities, only possibilities, and the only necessities are those of the form $\Diamond A$.

Quasi K_{LP}: For every A, $\Diamond A$ is a theorem; it contains no theorems of the form $\neg\Diamond A$ but it does contain some of the form $\Box A$. Intuitively, in it there are no impossibilities, anything is possible but there are also some necessary truths.

[13] As a simple illustration, consider in particular a clearly toy model with three worlds w, w' and w'' such that wRw', wRw'', $w'Rw$, $w'Rw'$, $w''Rw'$, and three propositions A, B, C such that

$$\begin{array}{lll} \mathfrak{v}_w(A) = \bot; & \mathfrak{v}_w(B) = \{\bot, \top\}; & \mathfrak{v}_w(C) = \bot \\ \mathfrak{v}_{w'}(A) = \{\bot, \top\}; & \mathfrak{v}_{w'}(B) = \varnothing; & \mathfrak{v}_w(C) = \{\bot, \top\} \\ \mathfrak{v}_{w''}(A) = \top; & \mathfrak{v}_{w''}(B) = \{\bot, \top\}; & \mathfrak{v}_{w''}(C) = \varnothing. \end{array}$$

None of the worlds is trivial and it can be easily verified that $\Diamond A$ for every A and every world.

Pseudo $K_{K_3}(= K_{K_3})$: $\Diamond A$ is not a theorem for every A; it contains some theorems of the form $\neg\Diamond A$ and $\Box A$ is a theorem only if A is a theorem (and the only theorems are some formulas of the form $\Diamond B$). Intuitively, in it not everything is possible and the only necessary truths are those concerning some possibilities; also, some formulas are impossibilities.[14]

K_{CL}: $\Diamond A$ is not a theorem for every A; it contains some theorems of the form $\Box A$ and also some of the form $\neg\Diamond A$, regardless of whether A contains modal vocabulary or not. Intuitively, there are some necessities, some impossibilities and not everything is possible.

3.3. Possibilism and conditions on frames

It seems that there is problem if we combine the proper axiom of possibilism, $\Diamond A$, with other well-known and well-regard axiom, namely

$$\Diamond\Box A \supset A \tag{B}$$

Consider the following proof:

1.	$\Diamond A$	Possibilist axiom
2.	$\Diamond\Box A \supset A$	Axiom (B)
3.	$\Diamond\Box A$	From 1 by uniform substitution
4.	A	From 2 and 3 by detachment

This holds for every A and one can even obtain $\Box A$ with (RN). For more drama, let A be the proposition *Every proposition is true*: Not only our world would be trivial, but also every other world.

Consider now the following two axiom schemes:

$$\Diamond\Box A \supset \Box A \tag{5}$$

$$\Box A \supset A \tag{T}$$

Now one can display the following proof:

1.	$\Diamond A$	Possibilist axiom
2.	$\Diamond\Box A \supset \Box A$	Axiom (5)

[14] As I have said in a previous footnote, there would be no impossibilities in K_{K_3} if the truth condition of $\Diamond A$ were defined as "$\mathfrak{v}(\Diamond A) = \top$ if and only if there is a world w in which $\mathfrak{v}_w(A) \neq \bot$". The other logics considered would not be affected by the change and still K_{K_3} would be different from them. For example, defined in the way here suggested, K_{K_3} would be a possibilist yet non-dialetheist modal logic in that it could have $\mathfrak{v}(\Diamond(A \wedge \neg A)) = \top$ without $\mathfrak{v}(A) \in D^+$ nor $\mathfrak{v}(\neg A) \in D^+$.

3. $\Box A \supset A$ Axiom (T)
4. $\Diamond\Box A$ From 1 by uniform substitution
5. $\Box A$ From 2 and 4 by detachment
6. A From 3 and 5 by detachment

In [6] Humberstone has studied the arguments above with respect to a model based on a classical modal logic enriched with a trivial world t and where there are pretty standard characterizations of absolute and relative modalities. More in detail, in the model he considers, Humberstone proves the equivalence between absolute necessity (truth in every world, which he denotes $\Box^{+}$) and relative necessity (truth in every world other than t, denoted $\Box$ as usual). As it is based on a classical normal modal logic, there are truths of the form $\Box\Phi$ in the model. This has the effect that absolute necessity and absolute possibility (denoted $\Diamond^{+}$) could not be interdefinable. Suppose there is a truth $\Box^{+}\Phi$. Given that $\Diamond^{+}\neg\Phi$ for every Φ, if $\Box^{+}\Phi$ implied $\neg\Diamond^{+}\neg\Phi$, the collection of valid formulas would be inconsistent, which is not. Thus $\Box^{+}\Phi$ does not imply $\neg\Diamond^{+}\neg\Phi$. Thus, the derivations above, put in terms of absolute modalities, are invalid, because (B) does not hold for absolute modalities in the model: $\Diamond^{+}\Box^{+}A$ is true as $\Box^{+}A$ is true at least in t, where everything is true, but A might be false (in a world other than t). Similar considerations apply as to the invalidity of (5) when it is about absolute modalities. (T), in turn, holds even for absolute modalities.

But what Humberstone has proved is only that the derivations above are unsound in the model considered. It leaves open the possibility that there are models where they are sound. Humberstone's treatment leaves open the question about the existence of models where, say, both $\Diamond A$ and (B) are valid without leading to *absolute trivialism*. Besides, Humberstone considers a variety of possibilism where there are truths of the form $\Box\Phi$, which is not central to the spirit of possibilism, and for some it might be even contrary to its spirit (for example, for the supporters of proto K_{FDE} or pseudo K_{K3}). Remember also that detachment is in general not available for some of these modal logics, so possibilism does not necessarily leads to absolute trivialism. On the other hand, given the possibly non-classical behaviour of the conditional and that $\Box$ and $\Diamond$ are in general not interdefinable, one has to be careful with the versions of modal axiom schemes one is using in the derivations of absolute trivialism from possibilism. For example, $\Diamond\Box A \supset A$ and $A \supset \Box\Diamond A$ could be not equivalent. These open questions arise because the presence of a trivial world is by no means the only way of having a model in which $\Diamond A$ is valid, and the carrier logic does not need be the classical logic.

In models where the accessibility relation is symmetric, (B) holds. In models where R is Euclidean (for every w, w' and w'', if wRw' and wRw'' then $w'Rw''$) and reflexive, (5) and (T) hold. Symmetry is equivalent to Euclideaness plus reflexivity, so there seems to be a close connection between models of absolute trivialism and models where both serial possibility and symmetry hold. This connection should be treated in a separate work.

4. Some philosophical remarks

The philosophical foundations of the constructions in the previous section might be not neat enough, so now I will try to make explicit the stance on logic, connectives and truth values underlying them. One of the most pressing problems for possibilism so far seems to be its compatibiliy with trivialism since, as noted above, if possibilism is the case, then trivialism is possible. Since trivialism is regarded as the worst logical scenario, there seems to be an easy *reductio* of possibilism. This is why in what follows I work to make trivialism not look so bad from a purely logical point of view.

Recall that by "structure of truth values" I mean, roughly, the exact number of truth values, how do they relate to each other as well as how do they relate to propositions. Recall also that I have characterized $\mathcal{V}$ as a non-empty partially ordered collection of truth values. $\top$ was defined as an $x \in \mathcal{V}$ such that for every $y \in \mathcal{V}$, $y \leq x$. *Designated values*, D^+, satisfy the following conditions:

(a^+) $\top \in D^+$;
(b^+) for every $x, y \in \mathcal{V}$, if $x \in D^+$ and $y \notin D^+$ then $x \not< y$;
(c^+) an inference is D^+-valid in L, denoted $A \Vdash_L^{D^+} B$, if and only if whenever $\mathfrak{v}(A) \in D^+$, $\mathfrak{v}(B) \in D^+$ too.

Antidesignated values, D^-, satisfy the following conditions:

(a^-) there is an $x \in \mathcal{V}$ such that for every $y \in \mathcal{V}$, $x \leq y$. Let us call *false* such x and denote it '$\perp$'. $\perp \in D^-$;
(b^-) for every $x, y \in \mathcal{V}$, if $x \in D^-$ and $y \notin D^-$ then $y \not< x$;
(c^-) an inference is D^--valid in L, denoted $A \Vdash_L^{D^-} B$, if and only if whenever $\mathfrak{v}(B) \in D^-$, $\mathfrak{v}(A) \in D^-$ too.

An interpretation, $\mathfrak{v}$, was characterized as a function between propositions (at a world) and truth values. Given the functionality of $\mathfrak{v}$ and the notions whose characterization we have just recalled, the determining factor of the structure of truth values is their exact number. In the classical case where exactly two truth values are given, we have as consequence that $D^+ = \top$ and $D^+ = \bot$ (so that $\neg\top = \bot$ and $\neg\bot = \top$). In classical logic D^+ and D^- satisfy thus the following equalities:

(d) $D^+ \cup D^- = \mathcal{V}$,
(e) $D^+ \cap D^- = \varnothing$.

When more values are given, one has to choose by hand, as it were, what the elements of D^+ are. It has been standard to assume (d)-(e) (equivalently, the coextensionality of D^+-validity and D^- validity) even for the case where there are more than two elements in $\mathcal{V}$. Thus, an exact, definite number of truth values just varies the collections of logical truths or of valid inferences but the properties of truth values and designated and antidesignated values described in (a^+)–(c^+), (a^-)–(c^-) are not changed and even the properties (d)–(e) may still hold.

Note that the initial characterizations of truth values and designated and antidesignated values in (a^+)–(c^+) and (a^-)–(c^-), which are pretty standard and are compatible with nice equalities like (d)–(e), only assume the non-emptiness of $\mathcal{V}$. Thus, in the limit case when there is only one element in $\mathcal{V}$, one gets the trivialist property

(TP1) for all A, $\mathfrak{v}(A) = \top$,

and since $\top \in D^+$,

(TP2) for all A, $\mathfrak{v}(A) \in D^+$.

Moreover,

(TP3) $\top = \bot$

and hence

(TP3*) $D^+ = D^-$,
(TP1*) for all A, $\mathfrak{v}(A) = \bot$,
(TP2*) for all A, $\mathfrak{v}(A) \in D^-$,
(TP4) for all A, $\mathfrak{v}(A) \in D^+$ and $\mathfrak{v}(A) \in D^-$,

and for more drama

(TP5) for all A, $\mathfrak{v}(A) = \top$ and $\mathfrak{v}(A) = \bot$.

These results refer us back to the etymology of 'trivial'. This word derives from the Latin *trivium*, literally "(junction of) three ways". Hence, by extension, *trivialis* was used to designate something that may be found everywhere, common, commonplace, vulgar, ordinary. *Trivium* was also the set of grammar, rhetoric and dialectic or logic, the three propaedeutic subjects for the more advanced ones, the *quadrivium* (arithmetic, geometry, music and astronomy). Thus, over time, 'trivial' was used to suggest that something was "inceptive", "introductory" or "simple". In mathematics, 'trivial' is used just to describe, among a group of objects, those with the simplest structure (such as a degenerated categories among toposes in general, the empty set among sets in general, or the trivial group between all groups), or an uninteresting or lacking of interest option or possibility, but that should be mentioned for the sake of conceptual completeness. Such use is present in many current logics, like when it is said that a contradiction trivializes a theory. That a theory is trivialized means, then, that it becomes as simple as possible. This is regarded as a defect because in general the phenomena to be described by the theory are complex, not adequately represented by the simplicity of a trivialized theory. In fact, the trivialist thesis should be expressed as holding that what there is is in the simplest possible way.

It is noteworthy that many of Aristotle's arguments in *Metaphysics* Γ in favor of the principle of non contradiction are rather arguments against trivialism. In particular, there is a family of arguments between 1008^a26 and 1007^b12 of the form "If trivialism is right, then X is the case, but if X is the case then all things are one. But it is impossible that all things are one, so trivialism is impossible." Seemingly, these Aristotelian considerations are the seeds of virtually all subsequent suspicions against trivialism: Trivialism has to be rejected because it identifies what should not be identified, and is undesirable from a logical point of view because it identifies what is not identical, namely, truth and falsehood.[15]

But those considerations are also the seed of what has been unsuccessful about the suspicions. Indeed, Aristotle is right in saying that if there are several different things (as in fact there are), then it is true that these different things are not one and the same. Perhaps it is even necessary that these different things are not one and the same. However,

[15] A nice discussion of Aristotle's arguments against trivialism in *Metaphysics* can be found in [11]. To be fair, Aristotle also makes a case against metaphysical monism, not in *Metaphysics* (see [2]) but in the first book of *Physics* ([1]).

it does not follow that it is impossible to have only one thing or that all what is predicated of different things cannot be predicated truthfully of that only one thing. The limits of the possible have been constrained by an excessive focus on how things actually are. The best arguments against trivialism just manage to be (quasi-)transcendental, as they come to establish at most that rational agents like us could not be trivialist, but most of them are also defective in this more modest aim (cf. [11], [12], [7]). However, even if successful, these transcendental arguments are far from showing that trivialism is either logically or metaphysically impossible, in the same way that if there were no rational agents like us in a certain kind of geometrical space that would not mean that such a space is impossible.

One interpretation of the results is that our logical notions are compatible with trivialism, just as they are compatible with several logics, like classical logic, LP, K_3 and FDE. Another interpretation of the results above is that there must be something wrong with such an abstract account in somehow being compatible with trivialism. According to the latter interpretation, trivialism can be rejected on purely logical grounds and a proposal which does not reflect that is *eo ipso* wrong. I prefer the former interpretation and given that trivialism has not been widely discussed, much less in the present terms, in order to provide support for it I will try to do my best effort to figure out what those logical grounds to reject trivialism could be and show why I think that they do not succeed. I shall not attempt a comprehensive examination of criticisms and replies, but I hope that the selection is large enough as to persuade the reader at least that trivialism deserves a chance among logics *tout court*.

The objections I will examine here include:

- A trivial theory is empirically false
- Trivialism is logically impossible, either because
  - Our best characterizations of certain logical notions, for example, of the notion of logical consequence, the connectives or truth values, imply the impossibility of trivialism, or because
  - Trivialism is self-refuting.
- Trivialism is just technical curiosity but ultimately unimportant.

The most common charge against a trivial theory is that it is useless because it is false. But this can be understood in at least two ways: Either a trivial theory may be merely empirically false, or it may be logically impossible. Let me discuss the first option. Consider a physical theory according to which every proposition (of physics) is true. Even if according to this theory it is true that not all bodies move at the speed of light, it would also be true that all bodies move at the speed of light. Then I would not have to leave home an hour earlier to get to school because, if I walk at the speed of light, I just need a few milliseconds to arrive (although it really would take forever to get to school, for it is also true that I do not walk at the speed of light; in particular it is true that I move at a speed of one picometre per year). A theory telling that it takes me any amount of time whatsoever to get to school is surely useless to plan my trip. This is perhaps the weakest objection against trivialism, though. Being wrong, false or empirically inadequate does not disqualify a theory as a theory. This objection is at most valuable evidence that trivialism is an incorrect theory, but that does not make trivialism the worst of theories.

The objector may still reply that, unlike other incorrect theories, trivialism could never have been a useful theory because it could never have been correct. In that sense trivialism could be the worst of theories and should not even count as a theory. But now we come to a different field, namely the attribution of impossibilities. I will assume that what the objector is saying is that our best characterization of certain logical notions, such as those of logical consequence, truth values or the connectives should entail the impossibility of trivialism, because it is somehow self-refuting. I do not know if we have available or even if there is something like *the best characterizations* of these logical notions, but the characterizations offered here are fairly standard and, as has been already noticed, they do not imply the impossibility of trivialism, which is regarded as a special case of the structure of truth values. One could take here one of the following options: Either independent arguments for rejecting trivialism must be given, and hence the reasonableness of the above characterizations, or it must be accepted that such characterizations are indeed reasonable and a philosophical effort has to be made to show that triviality is not so bad. What I will do here is to defend an indirectly the characterizations I have given, and thus the case of a collection of truth values with only one element. I will outline some objections to trivialism directed to reject the aforementioned characterizations of logical notions that are compatible with it and I will show their lack of support.

Recall that the notions of validity (D^+-validity and (D^--validity) are pretty standard. Take D^+-validity, for example. This does not exclude by itself that every proposition A is such that $v(A) \in D^+$, and hence every inference is D^{+}-valid. The "contrapositive" form, D^{-}-validity, does not exclude by itself that $D^{+} = D^{-}$, and hence that every inference is D^{-}-valid. In short, when there is only one element in $\mathcal{V}$, i.e. $\top$, no proposition can have a different value than $\top$, so the notions of validity are satisfied.

If the notions of validity do not provide the desired logical exclusion of trivialism, maybe something in the characterization of truth values might be helpful. mistake in defining *false* ($\perp$) when $\mathcal{V}$ is a singleton and having already defined *true* ($\top$). In this case it makes no sense to say *true* = *false* ($\top = \perp$). It is like saying of a unique person in a domain of discourse that he is the tallest and the shortest, the youngest and the oldest, etc. These properties are simply ill-defined and are inapplicable when talking about a single object. There should be different definitions of truth values (and perhaps also of connectives) appropriate to deal with singletons.

This objection makes a wrong assumption: Logical notions are defined for nearly every domain of truth values with a certain kind of (universal) properties. Besides demanding non-emptiness, cardinality is not one of those universal properties. Hence, trivialism appears after the fact. However, the objection makes an interesting point. Let me state it differently. Skipping for the moment the case of negation, *false* is not needed to characterize logical notions, hence its introduction in (a^{-}) is unnecessary. Trivialism only commit us with the existence of at least one value, *true*.[16] Minimalism rests on a mistake trying to make us believe that there could be at least two truth values, *true* and *false*, which somehow become one and the same in certain situations. What is actually happening is that we are viciously giving different names to one and the same truth value. It is like viciously calling a single person in a domain "the tallest" and "the shortest" because there is no one taller than him and there is no one shorter than him: It is so because there are no other people to make that kind of comparison and hence applying those adjectives makes no sense. Note that it is no problem in defining an order when a single object is involved, but it simply makes no sense to use labels with such opposite connotations if it is not needed and if we are not ruling out from the beginning the possibility of collections of values with only one element.

[16] The following could be changed accordingly to make a plea for logical nihilism if the unique assumed value were *false*.

In this form the criticism seems legitimate. A problem remains, though. Even in the amended version of minimalism, if we regard *true* = *false* (and in general (TP3)-(TP5) and (TP1*)–(TP3*)) as ill-formed, (TP1) and (TP2) would not necessarily be ill-formed. Said otherwise, cases where all propositions are *true* are still rightful, namely those where *true* is the only element in $\mathcal{V}$. The critic may say that an expression like "All propositions are (assigned the value) *true*" has a perverse rhetorical twist, suggesting again that we start with several propositions which somehow become *true* in certain cases. But this objection goes further: The objection in the preceding paragraph was rightly stressing an abusive practice of giving unnecessary names, and now we are told that certain uses of quantifiers may appear as misleading. But in this case the minimalist cannot keep company the critic. In a domain with only one person, it is perfectly right to say that all the persons of the domain are Martians. There might be a psychological impact with the use of plurals, but it is done for allowing the possibility that there is more than one person in the domain. But such a defense is not even required. There is a way in English to express universal quantification without suggesting multitudes: "For every x belonging to the domain, x has the property F" or "Anyone belonging to the domain has the property F". No plurals. No collapse of the many into one. And "For every A and some or all $\mathfrak{v}$ in all or some worlds w, $\mathfrak{v}_w(A) = \top$(or $= D^+$)" is exactly what a trivialist would say.

However, one does not need to grant that there is a bad practice of labeling here. What the special cases are saying is that sometimes truth and falsity coincide. But now the obvious reply is that the characterization given here is wrong because it is part of the notion of truth and falsity that they are different truth values. It is not an extra-logical issue. We have been led astray by the mere consideration of structural properties and logic goes beyond that. The moral of the story is clearly that all this is wrong: There is more to truth than its universal structural properties as implied by the characterization offered here.

My reply is that it is a clash of views: Maximalism against minimalism. According to minimalism, it is not part of the notions of truth and falsity that they are different. Maybe we are again buying a by-product of certain logics, like when we believed that the validity of certain formulas was a necessary core of the meaning of certain connectives. As to descriptions of the logical landscape, both in theory and practice, minimalism possesses certain advantages over maximalism. For example, minimalism can explain rather easily what is that "family resemblance" which make us to count as logics so many different things, departing in several ways of the paradigmatic case of classical logic. Now, minimalism can be seen as supporting a distinction between investigating what are the most general features of logical notions and investigating what are *our* logical notions (the right logic, if any, for studying the modal properties of our world, for using our language(s), for doing the *mainstream* mathematics, etc.). Trivialism is one of those interesting limit cases common in logic and mathematics. If philosophers or whoever are happy thinking that our logic is not trivialist, that is ok, the minimalist can keep company to him. Things are different when the philosopher or whoever maintains that nothing else besides our logical notions (or notions that we could imagine as playing a similar role) should be counted as logical.

Once it was used to think that truth and falsity were the only truth values and that they were incompatible. There have been impressive cases for both the non-disjointness and non-exhaustiveness of truth and falsity, which sometimes take the form of cases for the existence of truth values beyond those two. Nonetheless, Suszko's theorem asserts that every Tarskian logic has a (generally non truth-functional) bivalent semantics, insofar as each of those extra values are either *designated* or *antidesignated*. Designated and antidesignated are the only two values necessary to define Tarskian validity, so they are the only truth values which deserve the appellative 'logical'. Suszko's theorem plus the premise that Tarskian validity is the only notion of validity which deserves to be called logical imply Suszko's thesis: Every logic is bivalent or, said otherwise, many-valued logics do not exist at all. In fact, the properties (d)-(e) look pretty "classical": They imply that every proposition is either designated or antidesignated, and that no proposition is both designated and antidesignated. But when other notions of validity (indistinguishable from the traditional one under the constraints imposed on truth values by classical logic!) enter the scene, designated and antidesignated do not need longer to behave "classically" as in (d)-(e), the limits of Suszko's theorem become evident and Suszko's thesis loses a great ally.[17]

None of the characterizations offered here excludes themost feared situation in logic, namely to obtain falsity from truth, $\top \Vdash \bot$, and thus,

[17] For a nice overview of Suszko's theorem, Suszko's thesis and non-Tarskian notions of consequence, see [18].

$\top \Vdash A$ and ultimately $\Vdash A$ for every A. It can be thought that there is nothing more mistaken than this (cf. [16]). However, that $\top \Vdash A$ and $\Vdash A$ for every A hold in the above models of trivialism is harmless. In terms of the order relation, what one wants to avoid is that both $\bot < \top$ and $\bot \leq \top$. But this does not occur in the models considered. Note that in this case one does not have both $\bot < \top$ and $\bot \leq \top$, but only the latter option, and in fact only $\bot = \top$, which allows $\top \Vdash \bot$. However, this option is safe because $\bot$ is only a definitional variant of $\top$. Not even in a trivial world, at least as those considered here, happens that $\bot$ is a "bad" value (lesser in the order than $\top$) that is derived from $\top$. No A is such that $\mathfrak{v}(A) = \bot$ is a theorem or derived from $\top$, if by '$\bot$' is meant a value different from $\top$.

But this is far from have alleviated the doubt, and in fact led us to an apparently worse situation. If in a model of trivialism A is such that $\mathfrak{v}(A) = \top$ and $\top = \bot$, then every A is such that $\mathfrak{v}(A) = \top$ and $\mathfrak{v}(A) = \bot$. It appears to be very difficult to get an intuition of how this could be the case, and it is (cf. [16]).

Objections to $\top \Vdash \bot$ and $\top = \bot$ arise because it is assumed that the defining property of *false* has to be *false* $<$ *true*, or *true* $\nVdash$ *false* if the order relation is interpreted directly as an inferential relation. However, for $\bot < \top$ it is necessary that $\mathcal{V}$ has at least two elements, but this might not be the case (in Example 2, $\mathcal{V}$ may have more than two elements, but the valuation for t behaves as if for that world had only one value). The defining properties of *false* and D^- stated above, that is, $(a^-) - (c^-)$, have fewer assumptions and covers more cases. Of course, if there are at least two distinct elements *true* and *false* it is desirable that $\bot < \top$, and hence *true* $\nVdash$ *false*. But this is, what is usually expected, is obtained as a result of $(a^+) - (c^-)$ provided that, as noted above, $\mathcal{V}$ has at least two different elements. This suggests that the analysis presented above distinguishes truth from falsehood when it is possible to distinguish them, and includes cases where this is not possible. Those "anomalous" cases should be treated like any other limit cases in mathematics, that is, they are tolerated because they follow from certain definitions that work extremely well for the rest of cases.

This discussion is helpful for pointing out how to block a *consequentia mirabilis* argument against trivialism. A *mirabilis* argument tries to conclude that not everything is true from the assumption that everything is true. In a model of trivialism, $(A \supset \neg A) \supset \neg A$ is valid, as is everything else. It seems that one can use it against trivialism in the following way:

If trivialism is true, then trivialism is false. But then trivialism is false. However, the argument fails in several respects. The argument would have some force only if "false" is intended to mean a value different from *true*. Otherwise "*false*" might be a mere definitional variant of "*true*" and one could put "true" instead of "false" in every relevant part of the *mirabilis* argument. Related to this, the proponent of the argument might be taking for granted that A and $\neg A$ have different truth values which, as we have seen, is not guaranteed by the truth conditions of $\neg$. On the contrary, the *consequentia* is not so *mirabilis* when one plays fairly to the trivialist. Let A designate the trivialist claim *Everything is true*. Now, $(A \supset \neg A) \supset \neg A$ should be read as follows: If *Everything is true* is true, then *Not everything is true* is also true. Then *Not everything is true* is also true. Indeed. But that is what trivialism is saying: Everything is true, in particular *Everything is true* and *Not everything is true*. Summarizing, the original argument of the mirabilist does not work in general because it might not even begin. In general one cannot use "What trivialism says does not have a designated value", because there might be no such truth values as different from designated ones, even though one could use *names* which in most situations would refer to values different from designated ones. Remember that when there is only one truth value, *true*, both A and $\neg A$ are true (as is everything else). If A is read "The thesis endorsed by a trivialist is true", $\neg A$ cannot be read "The thesis endorsed by a trivialist is false" (trying to mean "different from true" with "false") but only "The negation of the thesis endorsed by a trivialist is true", which is more true to the symbolism '$\neg A$'.

Now, on the case of negation, someone might object that the characterization of its truth conditions does require at least two distinct values. But the truth condition

$$f_{\neg}(\mathfrak{v}_w(A)) = \top \text{ if and only if } \mathfrak{v}_w(A) = \bot, \text{ otherwise } f_{\neg}(\mathfrak{v}_w(A)) = \mathfrak{v}_w(A)$$

is just a way of stating

$$f_{\neg}(\mathfrak{v}_w(A)) = x \text{ if and only if } \mathfrak{v}_w(A) = x', \text{ otherwise } f_{\neg}(\mathfrak{v}_w(A)) = \mathfrak{v}_w(A),$$

where $x \in \mathcal{V}$ is such that for every $y \in \mathcal{V}$, $y \leq x$ and $x' \in \mathcal{V}$ is such that for every $y \in \mathcal{V}$, $x' \leq y$. It does not rule out by itself that $x = x'$, so the truth condition for negation does not demand at least two truth values. It is an illusion caused by the traditional connotations of '$\top$' (*true*) and '$\bot$' (*false*).

It is very unlikely that avoiding the name "true" in models where there is only one element in $\mathcal{V}$ helps. First, that name is invited by the definitions of connectives. Saying that a conjunction has the value x if and only if its components have each the value x seems to be a very elaborate and unmotivated way of speaking of the usual truth condition speaking of *true*. Secondly, it introduces an asymmetry difficultly explainable besides of trying to avoid models of trivialism. There should be a general argument intended to show that when $\mathcal{V}$ has certain cardinality some standard names cannot be assigned to the elements of $\mathcal{V}$, and that $\mathcal{V}$ being a singleton is one of those cases. I am not saying that such an argument cannot be given, but accepting trivialism as a limit case of logic makes perfect sense without introducing further complications.

The objection above can be generalized. Elements of $\mathcal{V}$ form a partial order, but we should resist the move to try to read also a validity relation in that order. We cannot use then the rejoinder that connectives seem to be using the notions of *true* and *false*, for what appear to be connectives are not really connectives: The entire analysis is flawed. This suggestion looks unfeasible, though. In magnificent Quinean prose we could say that we are scratching where it does not itch. In more arid terms already used, the objector is suggesting, contrary to well motivated practices in adopting theories, to reject all perfectly "normal" (non-trivial) cases in order to reject some cases where one gets trivialism, trivialism that not necessarily expands to the rest of worlds and which may receive nice technical treatments as outlined above, where there are abnormalities but not necessarily incoherencies.

Philosophy should be especially wary of the obvious, although it is very difficult not to take as an indisputable truth something that holds for a prodigious number of cases. For example, nothing is more obvious that the whole is greater than its parts, or that if n is a number, $n+1$ is greater than n. Long time ago, the violation of these truisms was used as an argument to show the incoherence of the notion of (actual) infinity, but later they became part of their defining features. Similarly, nothing is more obvious that not everything is true and the preceding analysis agrees, with the sole exception of a case that is simple enough that the distinction between true and false cannot be made, but complex enough to have something recognizable as logic. This is precisely the case where there is only one element in $\mathcal{V}$ (perhaps appropriate for a rare world where there is but one fact). Given that this case is still conservative enough as to not allow the terrible situation in which both *false* $<$ *true* and *false* $\leq$ *true* hold and there is, I think, a reasonable explanation of why nonetheless it may be admitted that *false* $\Vdash$ *true*, we should consider the truth of trivialism, the equivalence of all propositions, the equality of truth and falsehood, not as incoherencies, but as part of the defining properties of worlds where there is only one truth value. Since Aristotle, arguments against triviality are usually based on what Wittgenstein would call a diet poor in examples. There may be other ways to get trivial situations that have not been covered in logic, perhaps simply because they do not resemble the actual world with its many objects and where not everything is true.

In summary, in a model of trivialism as the above there may not be false propositions, if by *false* is meant a value different from *true*. That there are no propositions other than true ones ("false" ones as usually understood) is not only possible, but seems to be a natural and quite acceptable consequence of the definitions of *true* and other logical notions. That *true* $=$ *false* certainly looks strange, but it is a case compatible with the characterizations of both values and the only cases where you get it is when $\mathcal{V}$ has a single element or a valuation behaves as if $\mathcal{V}$ were so. In any other case, the definitions of these values imply that *false* $<$ *true*. Finally, *false* $\Vdash$ *true* might be obtained, but it is not necessarily harmful because *false* $\Vdash$ *true* obtains only when *false* is a definitional variant of *true*. The same answers can be given to one who objects that a proposition cannot be both true and false. Thus far there are only oddities, but not impossibilities.

A final objection trying to prove the logical impossibility of trivialism is that, if what saves the idea of a trivial world of being nonsense is that *true* $\nVdash$ *false* (*false* $<$ *true*) false and *true* $\Vdash$ *false* (*false* $\leq$ *true*) do not hold together but only the latter, then the logic of this world should not be considered trivial. In an authentically trivial situation one would have that both *It is not the case that true* $\nVdash$ *false* and *It is the case that true* $\Vdash$ *false* hold together. But this objection loses sight of the different nature of the propositions involved. In one sense, *It is not the case that true* $\nVdash$ *false* and *It is the case that true* $\Vdash$ *false* are not propositions in F, but are metalogical statements about $\mathcal{V}$ and its elements. Of course it is possible to represent *false* $<$ *true* and *false* $\leq$ *true* as propositions in F, but this does not change the things: As propositions of F, *false* $<$ *true* and *false* $\leq$ *true* are both evaluated as true (it is a trivial world, after all), but that *false* $<$ *true* (*true* $\nVdash$ *false*) is not true from an external point of view is not as harmful to trivialism, since the justification is that from an external point of view there is no such value *false* distinguishable from *true* to be located at a different place in the order.

The final objection is that the admission of trivialism as presented here is just a mere curious technical spandrel on logic, but in the long run it lacks any interest, particularly philosophical. The proponent of this objection could even say, sarcastically, that the objection does justice to the two senses of 'trivial' as used in mathematics. My models of trivial cases can be accepted as legitimate, as a collection of truth values with a single element is, not counting the empty case, the simplest one structurally speaking, and must be included for the sake of conceptual completeness. However, they are trivial also in the sense that their very simplicity plays down their significance, especially philosophical.

The last seven pages or so are part of a reply to this objection. The discussion of an issue that seriously challenges very entrenched ideas on some core logical notions is not a waste. To quote a famous advocate of implausible theses, "I still find it hard to see how any half-way competent attack on an orthodoxy that is some two and a half thousand years old, and scarcely defended during that period, can fail to have some 'fruitful and interesting' results. Even if it the attack is wrong, discovering why this is so cannot but help deepen our understanding of the orthodoxy." ([13, xviii]) I hope that the proposal presented here is at least half-way competent and that will help to give a better understanding of the anti-trivialist orthodoxy.

5. Conclusions

I have presented here relational models of possibilism and trivialism and have discussed the underlying characterization of truth values and connectives as to make clear their philosophical bases. I have not been able to discuss all the relevant objections against trivialism, but apart from that there remains a lot of work to be done. For example, I was far from being able to study all the interesting varieties of possibilism and trivialism allowed by the abstract characterizations of truth values and the connectives, but I hope to have at least laid down the bases for further work on these issues. To the extent that they contain (RN) and hence some theorems of the form $\Box A$, none of the particular logics considered here fully satisfy Mortensen's version of possibilism, but in principle there is no obstacle to get a logic closer to it. It would also be interesting to study whether and how trivialism can be shown compatible with other kinds of logics giving a central role to consistency, like intuitionistic logic and intermediate logics. It would be also worth studying the resemblances and the differences, both formal and philosophical, between the models here presented and more traditional studies of "non-normal" modalities. Logical nihilism also deserves a philosophical discussion on its own and not as a mere formal dual of trivialism.

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