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TWO WEAK LAMBEK-STYLE CALCULI: DNL AND DNL^{-}*

Abstract. The calculus **DNL** results from the non-associative Lambek calculus **NL** by splitting the product functor into the right (▷) and left (◁) product interacting respectively with the right (/) and left (\) residuation. Unlike **NL**, sequent antecedents in the Gentzen-style axiomatics of **DNL** are not phrase structures (i.e., bracketed strings) but *functor-argument structures*. **DNL**$^{-}$ is a weaker variant of **DNL** restricted to fa-structures of order $\leqslant 1$. When axiomatized by means of introduction/elimination rules for / and \, it shows a perfect analogy to **NL** which **DNL** lacks.

Keywords: Lambek calculus, Gentzen formalism, axiomatization.

1. Introduction

The syntactic type calculus of J. Lambek was presented by him in [7] (the associative variant **L**) and [8] (the non-associative variant **NL**). For both **L** and **NL**, Lambek provides Gentzen-style axiomatics. **L** and **NL** have the same axioms and (roughly) the same rules. Sequent succedents are always single types (which resembles intuitionistic logic); however, sequent antecedents of **L** are plain strings of types, with no internal structure, while those of **NL** are *bracketed* strings. This slight difference has far-reaching consequences: some type transformation laws of **L**, e.g., the associativity laws

$$(x \cdot y) \cdot z \to x \cdot (y \cdot z) \qquad x \cdot (y \cdot z) \to (x \cdot y) \cdot z \qquad \text{(assoc)}$$

fail in **NL**. The converse does not hold: **NL** is weaker than **L**.

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The theory was subsequently developed in several ways. One of them was that by augmenting the calculus with (some or all) of Gentzen's structural rules. When J.-Y. Girard [5] came out with his linear logic, **L** was shown to be its fragment (non-commutative implicational intuitionistic linear logic, possibly with the Lambekian product $\cdot$ playing the part of "multiplicative" conjunction). As a consequence of non-commutativity, there are two implications, to be identified with the two Lambekian residuations $/$ and $\backslash$. If the rule of Permutation is admitted, **L** changes into the commutative **LP** in which both implications coincide. **LP** gained some popularity (see [2]) because of its connections with the calculus of typized λ-terms. Some other extensions of **L** by structural rules, e.g., **LPC** with Permutation and Contraction, have also been considered.

The lack of structural rules in **L** is due to the fact that it is the *logic of concatenation*. Unlike the classical (or intuitionistic) conjunction, the concatenation of expressions is neither commutative nor idempotent and therefore it does not obey the rules of, respectively, Permutation and Contraction. To be sure, concatenation is associative, but associativity is inherent in the notation and this is evidently why Gentzen did not need to introduce it explicitly in his sequential systems of logic. In contrast to the *structural* properties of commutativity, idempotence, and monotonicity, associativity is, so to speak, a *substructural* one.

However, passing from **L** to **NL** is not the only step to be made in the substructural direction opposite to that which leads from **L** to **LP** and **LPC**. There is another substructural property, more fundamental than associativity, which may be called "non-directionality". That concatenation is non-directional means that, no matter whether one writes down x to the left of y or y to the right of x, the result is the same. In other words: one does not distinguish between the action of the *functor* $x\,\cdot$ on the *argument* y and the action of the functor $\cdot\, y$ on the argument x. The system **DNL** of this article is the **D**irectional **N**on-associative **L**ambek calculus. It was presented by the author in [11] an then studied more profoundly in [12]. The unique product $\cdot$ splits in it into the right-searching $(x \triangleright y)$ and left-searching $(x \triangleleft y)$ functional application (similarly, the unique residuation splits into $/$ and $\backslash$ when passing from **LP** to **L**).

In both **L** and **NL**, the following equivalences hold:

$$x \to z/y \equiv x \cdot y \to z \quad \text{and} \quad x \cdot y \to z \equiv y \to x\backslash z. \tag{1}$$

(That means, if one side of $\equiv$ is derivable, so is the other side). In a sense to be precised in Section 3, these equivalences together with the identity law (id) and a restricted form (cut) of the cut rule (for **L**, also with (assoc)) provide alternative axiomatics for both calculi. In **DNL**, the analogues of (1) are

$$x \to z/y \equiv x \triangleright y \to z \quad \text{and} \quad x \triangleleft y \to z \equiv y \to x\backslash z; \tag{2}$$

so one might expect that (2), (id), and (cut) are enough to axiomatize **DNL**. This is, however, not true: the resulting system **DNL**$^-$ is essentially weaker than **DNL**. To compare both systems, it is convenient to present **DNL**$^-$ in sequential form. Now, **DNL**$^-$ turns out to be the calculus of fa-structures of order $\leqslant 1$, the order being defined as in [3].

Sections 2 and 3 of the present article are devoted respectively to **DNL** and **DNL**$^-$. Proofs are outlined or even omitted; readers interested in technical details may find them in [12].

2. The calculus DNL

We start with reminding the calculi **L** and **NL**. Let a denumerable set of symbols called *primitive types* be given. **Sequents** of **L** and **NL** are defined as follows:

- primitive types are *types*;
- if x and y are types, so are (x/y), $(x\backslash y)$, and $(x \cdot y)$ (we usually omit the outermost parentheses);
- types are *terms*;
- if X and Y are terms, so is XY (for **L**) and $[XY]$ (for **NL**; we usually omit the outermost brackets);
- if X is a term and y is a type, then $X \to y$ is a *sequent*.

Axioms of **L** and **NL** are all sequents $s \to s$ with s primitive.

Rules of **L** and **NL** are the following:

$$(\to /)\ \frac{T\,y \to x}{T \to x/y} \qquad (/ \to)\ \frac{T \to y \quad U[x] \to z}{U[x/y\,T] \to z}$$

$$(\to \backslash)\ \frac{y\,T \to x}{T \to y\backslash x} \qquad (\backslash \to)\ \frac{T \to y \quad U[x] \to z}{U[T\,y\backslash x] \to z}$$

$$(\to \cdot)\ \frac{P \to x \quad Q \to y}{P\,Q \to x \cdot y} \qquad (\cdot \to)\ \frac{U[x\,y] \to z}{U[x \cdot y] \to z.}$$

Here, lower-case (resp. capital) letters denote types (resp. terms), and $U[Y]$ results from the term $U[X]$ by substitution of Y for a single occurrence of X.

THEOREM 1 (Lambek [7], [8]). **L** *and* **NL** *are closed under the* cut rule

$$\frac{T \to x \quad U[x] \to y}{U[T] \to y} \tag{CUT}$$

Sequents of **DNL** are like those of **L** and **NL**, except that

- if x and y are types, so are (x/y), $(x\backslash y)$, $(x \triangleright y)$ and $(x \triangleleft y)$;
- if X and Y are terms, so are $[X(Y)]$ and $[(Y)X]$.

Axioms of **DNL** are those of **L** and **NL**.

Rules of **DNL** are the following:

$$(\to /)\quad \frac{T(y) \to x}{T \to x/y} \qquad\qquad (/ \to)\quad \frac{T \to y \quad U[x] \to z}{U[x/y(T)] \to z}$$

$$(\to \backslash)\quad \frac{(y)T \to x}{T \to y\backslash x} \qquad\qquad (\backslash \to)\quad \frac{T \to y \quad U[x] \to z}{U[(T)y\backslash x] \to z}$$

$$(\to \triangleright)\quad \frac{P \to x \quad Q \to y}{P(Q) \to x \triangleright y} \qquad (\triangleright \to)\quad \frac{U[x(y)] \to z}{U[x \triangleright y] \to z}$$

$$(\to \triangleleft)\quad \frac{P \to x \quad Q \to y}{(Q)P \to y \triangleleft x} \qquad (\triangleleft \to)\quad \frac{U[(y)x] \to z}{U[y \triangleleft x] \to z.}$$

The rules of **NL** may be obtained therefrom by replacing everywhere $X(Y)$ and $(X)Y$ by XY, as well as $x \triangleright y$ and $x \triangleleft y$ by $x \cdot y$.

THEOREM 2. **DNL** *is closed under the cut rule.*

PROOF. Similar to that of Theorem 1. ⊣

A sequent of **DNL** is *product-free* if it involves neither $\triangleright$ nor $\triangleleft$. The *product-free fragment* of **DNL** is its restriction to the first four rules. Only product-free sequents are derivable in it because of the subtype property (types in premises are subtypes of those in the conclusion) and the fact that axioms are product-free.

Denote by **AB** the system whose only rule is (CUT) and whose axioms are all product-free sequents of the form $x \to x$ (with x not necessarily primitive!) as well as of the form

$$x/y(y) \to x \tag{A0$'$}$$

$$(y)y\backslash x \to x \tag{A0$''$}$$

Let $\boldsymbol{R}$ be any calculus of sequents. We write $X \vdash_{\boldsymbol{R}} y$ instead of "$X \to y$ is derivable in $\boldsymbol{R}$".

LEMMA 1. *If* $x \vdash_{\mathbf{AB}} y$, *then* $x = y$.

LEMMA 2. **AB** *is closed under the rules* $(\to /)$ *and* $(\to \backslash)$.

PROOF. By induction on derivations in **AB**, using Lemma 1. ⊣

THEOREM 3. **AB** *is equivalent to the product-free fragment of* **DNL**.

PROOF. Axioms of **AB** may be derived in **DNL** without the product introduction rules:

$$\dfrac{\dfrac{y \to y \quad x \to x}{x/y(y) \to x}}{x/y \to x/y} \qquad\qquad \dfrac{\dfrac{y \to y \quad x \to x}{(y)y\backslash x \to x}}{y\backslash x \to y\backslash x;}$$

thus **AB** is a product-free subsystem of **DNL** by cut elimination theorem.

On the other hand, axioms of **DNL** are axioms of **AB**. By Lemma 2, $(\to /)$ and $(\to \backslash)$ do not lead out of **AB**. The rules $(/ \to)$ and $(\backslash \to)$ may be derived in **AB** as follows:

$$\dfrac{\dfrac{T \to y \quad x/y(y) \to x}{x/y(T) \to x} \qquad U[x] \to z}{U[x/y(T)] \to z}$$

and similarly for $(\backslash \to)$. ⊣

Example. The *type raising laws*

$$y \to x/(y\backslash x) \qquad \text{and} \qquad y \to (x/y)\backslash x$$

of **L** and **NL** fail in **DNL**. In fact, they are not **AB**-derivable by Lemma 1 and thus not **DNL**-derivable by Theorem 3.

3. The calculus DNL$^-$

A sequent $X \to y$ is said to be *simple* if X is a type. Denote by $\mathbf{NL_s}$ the calculus of simple sequents whose axioms and rules are

$$x \to x \qquad \text{(id)}$$

$$\frac{x \cdot y \to z}{x \to z/y} \qquad \frac{y \cdot x \to z}{x \to y\backslash z} \tag{intr}$$

$$\frac{x \to z/y}{x \cdot y \to z} \qquad \frac{x \to y\backslash z}{y \cdot x \to z} \tag{elim}$$

$$\frac{x \to y \quad y \to z}{x \to z} \tag{cut}$$

and let $\mathbf{L_s}$ be $\mathbf{NL_s}$ with (assoc) added as new axiom schemata.

To every term X of $\mathbf{L}$ or $\mathbf{NL}$, we associate a type $\overline{X}$ by induction. If X is a type, then $\overline{X} = X$. If X is a term YZ of $\mathbf{NL}$, then $\overline{X} = \overline{Y} \cdot \overline{Z}$. If a term X of $\mathbf{L}$ is not a type, it must have the form Yz for some type z and then $\overline{X} = \overline{Y} \cdot z$.

Theorem 4 (Lambek [7], [8]). *$X \vdash_{\mathbf{L}} y$ (resp. $X \vdash_{\mathbf{NL}} y$) iff $\overline{X} \vdash_{\mathbf{L_s}} y$ (resp. $\overline{X} \vdash_{\mathbf{NL_s}} y$).*

Theorem 4 gives rise to the identification of $\mathbf{L}$ with $\mathbf{L_s}$ and $\mathbf{NL}$ with $\mathbf{NL_s}$ (in fact, they coincide pairwise in the scope of simple sequents). It seems natural to look for a similar simple-sequent axiomatics of **DNL**. Now, the **DNL**-analogues of (intr) and (elim) are

$$\frac{x \triangleright y \to z}{x \to z/y} \qquad \frac{y \triangleleft x \to z}{x \to y\backslash z} \tag{intr$'$}$$

$$\frac{x \to z/y}{x \triangleright y \to z} \qquad \frac{x \to y\backslash z}{y \triangleleft x \to z} \tag{elim$'$}$$

This strongly suggests that it suffices to add (id) and (cut) to them in order to obtain what is needed. However, a simple model-theoretical argument due to W. Buszkowski [4] disproves this conjecture. In the set of natural numbers, define $x \to y$ iff $x \leqslant y$, $x/y = y\backslash x = x + y$, $x \triangleright y = y \triangleleft x = \max(0, x - y)$. We get a structure in which the **DNL**-derivable sequent $z/((x \triangleright y)/y) \to z/x$ is not valid (it is false whenever $x < y$) but which is a model of (id)+(intr′)+(elim′)+(cut). Since the latter system is not the desired $\mathbf{DNL_s}$, let us denote it by $\mathbf{DNL_s^-}$ and try to adjust an adequate Gentzen-style calculus $\mathbf{DNL^-}$ to it.

Let types and axioms of $\mathbf{DNL^-}$ be those of **DNL**. **Sequents** of $\mathbf{DNL^-}$ (to be called *1-sequents*) are defined as follows:

- types are *1-terms*;
- if X is a 1-term and y is a type, then $[X(y)]$ and $[(y)X]$ are 1-terms (the outermost brackets are to be omitted);
- under the same assumptions, $X \to y$ is a 1-sequent.

Every 1-term is its own *subterm*. All subterms of X are also subterms of $X(y)$ and $(y)X$. There are no other subterms. In particular, the occurrence of y in parentheses is *not* a subterm of $X(y)$ or $(y)X$. It follows that one 1-term may occur at most once as a subterm of another. We denote by $U[Y]$ the result of substitution of Y for *the* subterm X of $U[X]$. **Rules** of $\mathbf{DNL}^-$ are the following:

$$(\to /)\ \frac{T(y) \to x}{T \to x/y} \qquad (/ \to)\ \frac{U[x] \to z}{U[x/y(y)] \to z}$$

$$(\to \backslash)\ \frac{(y)T \to x}{T \to y\backslash x} \qquad (\backslash \to)\ \frac{U[x] \to z}{U[(y)y\backslash x] \to z}$$

$$(\to \triangleright)\ \frac{T \to x}{T(y) \to x \triangleright y} \qquad (\triangleright \to)\ \frac{U[x(y)] \to z}{U[x \triangleright y] \to z}$$

$$(\to \triangleleft)\ \frac{T \to x}{(y)T \to y \triangleleft x} \qquad (\triangleleft \to)\ \frac{U[(y)x] \to z}{U[y \triangleleft x] \to z.}$$

Theorem 5. $\mathbf{DNL}^-$ *is closed under the cut rule*

$$\frac{T \to x \quad U[x] \to y}{U[T] \to y} \tag{CUT}$$

Proof. Induction on the *degree* of (CUT) which is defined to be the total number of occurrences of $/$, $\backslash$, $\triangleright$ and $\triangleleft$ in $U[T]$, x and y. ⊣

We write $x \vdash y$ for "$x \vdash_{\mathbf{DNL_s^-}} y$" and $X \vdash_{\mathrm{G}} y$ for "$X \vdash_{\mathbf{DNL}^-} y$"

Theorem 6. *If* $x \vdash y$, *then* $x \vdash_{\mathrm{G}} y$.

Proof. Since (cut) is a particular case of (CUT), it holds in $\mathbf{DNL}^-$ by Theorem 5. Next, we have the $\mathbf{DNL}^-$-derivations

$$\frac{\dfrac{x \to x}{x/y(y) \to x}}{x/y \to x/y,} \qquad \frac{\dfrac{x \to x}{x(y) \to x \triangleright y}}{x \triangleright y \to x \triangleright y,}$$

and similarly for $\backslash$ and $\triangleleft$. Since $s \to s$ is an axiom of $\mathbf{DNL}^-$ for s primitive, we get (id) by induction on the complexity of x. Using (id), (cut) and the rules of $\mathbf{DNL}^-$, it is easy to derive (intr′) and (elim′). ⊣

The *length* of a 1-term is the number of its subterms. For any 1-term X, we define the type $\overline{X}$ by induction on the length of X as follows: $\overline{x} = x$; $\overline{X(y)} = \overline{X} \triangleright y$; $\overline{(y)X} = y \triangleleft \overline{X}$.

Lemma 3. *If $\overline{X} \vdash y$, then $\overline{U[X]} \vdash \overline{U[y]}$.*

Proof. Induction on the length of $U[y]$. ⊣

Theorem 7. *If $X \vdash_{\mathrm{G}} y$, then $\overline{X} \vdash y$.*

Proof. In a $\mathbf{DNL}^{-}$-derivation of $X \to y$, replace every sequent $Z \to z$ by $\overline{Z} \to z$. Thus, axioms of $\mathbf{DNL}^{-}$ remain axioms of $\mathbf{DNL_s}$. The rules $(\to /)$ and $(\to \backslash)$ become (intr′). The rules $(\triangleright \to)$ and $(\triangleleft \to)$ become the identity rule. The remaining rules assume now the form

$$(\to \triangleright)\quad \frac{\overline{T} \to x}{\overline{T} \triangleright y \to x \triangleright y} \qquad (/ \to)\quad \frac{\overline{U[x]} \to z}{\overline{U[x/y(y)]} \to z}$$

$$(\to \triangleleft)\quad \frac{\overline{T} \to x}{y \triangleleft \overline{T} \to y \triangleleft x} \qquad (\backslash \to)\quad \frac{\overline{U[x]} \to z}{\overline{U[(y)y\backslash x]} \to z}$$

and may be easily derived in $\mathbf{DNL_s}$ (for $(/ \to)$ and $(\backslash \to)$, we apply Lemma 3). ⊣

Theorems 6 and 7 give rise to the identification of $\mathbf{DNL}^{-}$ and $\mathbf{DNL_s}$.

We shall now show that the product-free part of $\mathbf{DNL}^{-}$ is essentially weaker than **AB**. Let X be a term. The *order* $o(X)$ of X is defined inductively as follows: $o(x) = 0$; $o([X(Y)]) = o([(Y)X]) = \max(o(X), o(Y) + 1)$. Clearly, $o(X) \leqslant 1$ iff X is a 1-term.

Theorem 8. *A* **DNL**-*derivable product-free sequent is* $\mathbf{DNL}^{-}$-*derivable iff it is a 1-sequent.*

Proof. $(\Rightarrow)$ It is easy to see that the rules of $\mathbf{DNL}^{-}$ do not lead out of 1-sequents.

$(\Leftarrow)$ Observe that the product-free rules of **DNL** yield sequents of order $\geqslant 2$ whenever any of the premises has order $\geqslant 2$. Consequently, in a **DNL**-derivation of a 1-sequent only 1-sequents may occur. In particular, for every application of $(/ \to)$ or $(\backslash \to)$ we have $o(T) = 0$, i.e. T is a type. Since $T \to y$ is **DNL**-derivable by assumption, we get $T = y$ by Theorem 3 and Lemma 1; thus the conclusion follows from the premise $U[x] \to z$ respectively by $(/ \to)$ or $(\backslash \to)$ of $\mathbf{DNL}^{-}$. ⊣

Example. $[(x/y)/z(z)](y) \vdash_{\mathrm{G}} x$ but $x/y([y/z(z)]) \nvdash_{\mathrm{G}} x$. In fact, we have

$$\cfrac{\cfrac{x \to x}{x/y(y) \to x}}{[(x/y)/z(z)](y) \to x.}$$

On the other hand, $x/y([y/z(z)]) \to x$ is not a 1-term; so $x/y([y/z(z)]) \nvdash_{\mathrm{G}} x$, by Theorem 8.

4. Summary. Logical and linguistic motivation

One might write a whole book — if not a library — about logical questions underlying theory of syntactic calculi. Those who touch on this topic are confronted with the problem of matter selection. For the purposes of the present article, let us concentrate on ideas motivating the author's interest in the subject.

It would be unsuccessful to expect any new deeper insight into syntactic phenomena of natural languages by means of **DNL** and **DNL**$^{-}$. In fact, product types do not play any part in linguistic applications and Theorem 3 establishes the equivalence of **DNL** and the well-known **AB** in the scope of product-free sequents. As for **DNL**$^{-}$, matters stand still worse: terms of order 1 are not sufficient for natural language syntax; even as a simple term as

([poor (John)]) works $\qquad\qquad$ $([n/n\,(n)])\,n\backslash s$

(example taken from [7]) has order 2. Thus, **AB** seems to supply the bare minimum necessary for syntactic description.

This being so, what are the reasons of the author's interest in both calculi?

The principal reason is the idea of Gentzenian sequents being the most appropriate formalism for syntactic calculi. It seems that all the differences between reasonable — actual or potential — variants of the Lambek calculus may be expressed in terms of algebraic properties of the *structural* (in the sense of Belnap [1], i.e., term-forming, not type-forming) product: commutativity, associativity, idempotence, existence of the neutral element (here: the empty term) etc.

Consider the axiom system for the product-free **L** given in [10] which consists of (id), (CUT) and

(A1′) $x/y\, y \to x$ (A1″) $y\, y\backslash x \to x$
(A2′) $y \to x/(y\backslash x)$ (A2″) $y \to (x/y)\backslash x$
(A3′) $x/y \to (x/z)/(y/z)$ (A3″) $y\backslash x \to (z\backslash y)\backslash(z\backslash x)$

(R1′) $\dfrac{x \to y}{x/z \to y/z}$ (R1″) $\dfrac{x \to y}{z\backslash x \to z\backslash y}$

(R2′) $\dfrac{x \to y}{z/y \to z/x}$ (R2″) $\dfrac{x \to y}{y\backslash z \to x\backslash z.}$

In view of (R1) and (R2), (id) may be restricted to primitive x. Also, (R1) and (R2) may be applied to axioms only.

By removing (A3), we get **NL** (cf. [6]). Thus, (A3) are some kind of associativity axioms. And what happens if we remove both (A2) and (A3)? The remainder is evidently **AB**: rules (R1) and (R2) are now unable to produce anything but (id).

What property of the structural product (analogous to the associativity expressed by (A3)) do (A2) express? This is just what we call *non-directionality*: the lack of distinction between the functor and the argument. By restoring this distinction, we turn **NL** into **DNL** and its product-free part into **AB**. In such a way, **AB** may be placed in a uniform Gentzenian perspective with the product-free parts of **L** and **NL**, as a result of elimination of a constraint (viz., non-directionality) imposed on the non-associative (in **NL**) term-forming product, rather than as an axiomatic weakening of **NL**.

There is another argument in favour of **DNL** which, in fact, reduces to the same. Recall that expressions of a language may be viewed in three ways, viz., as

- *strings*: finite sequences of atoms;
- *phrase structures (p-structures)*: strings with a (preferably binary) tree structure defined on them by means of bracketing;
- *functor-argument structures (fa-structures)*: p-structures where, for every internal node in a tree, one of its successors is the *functor*, the other(s) being its *argument(s)*.

This threefold approach is thoroughly discussed in [3]. It transfers to structures built of types assigned to atoms by a categorial grammar;

these structures, in turn, form sequent antecedents of the Lambek calculus: **L** and **NL** are the calculi of, respectively, strings and p-structures. Now, **DNL** was devised to occupy the vacant position of the calculus of fa-structures.

It becomes now clear that, contrarily to a widespread opinion,[1] this is **DNL**, not **NL**, which is the very bottom of the hierarchy of syntactic calculi. In fact, it is nothing more but a product version of **AB**.

Having sufficiently motivated the need for **DNL**, we turn to $\mathbf{DNL}^-$. As argued in the preceding paragraphs, **L**, **NL** and **DNL** form some kind of triad of *really substructural* type logics (i.e., those which do not involve *any* structural rules). Surely, one may look for a calculus which is directional and associative but this is apparently an approach of little interest, non-directionality being somehow a more elementary property than associativity, to such a degree that it remained unnoticed for a long time. For similar reasons, the commutative but not associative Lambek calculus **NLP** never gained as much popularity as **L**.

Consequently, it seems natural to ask whether **DNL** shares some nice properties of **L** and **NL**. One of them is the possibility of being axiomatized as a calculus of simple sequents, by means of (id), (cut) and some kind of reversible rules of product/slash introduction/elimination analogous to (intr) and (elim). In fact, this is exactly in that form that Lambek introduces both his calculi; Gentzen-style axiomatization comes afterwards, as a tool for decidability proofs. Now, the most natural attempt at giving a similar form to **DNL** fails: its plausible equivalent $\mathbf{DNL}^-$ proves to be too weak. But, according to our *idée fixe*, what is natural, should be Gentzen-style axiomatizable. And indeed: $\mathbf{DNL}^-$ has a sequential form which admits cut elimination. For this purpose, the cut rule (or rather the notion of subterm which underlies it) must be appropriately modified. This having been done, it becomes clear how $\mathbf{DNL}^-$ is situated inside **DNL**: surprisingly enough, it is the calculus of fa-structures of order $\leqslant 1$; and let this surprise be one more argument for our interest in Gentzen techniques, particularly when applied to syntactic type calculi.

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[1] Let us quote, e.g., Moortgat [9]: "The most rudimentary type system that qualifies as a logic in the sense of the above attributes to the structure building operator '$\circ$' *no* properties at all beyond what is required for residuation. We obtain the non-associative type calculus of Lambek (1961)."

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