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ON RELATIONAL AND FUNCTIONAL LANGUAGES

Abstract. We prove two theorems concerning expressive power of relational and functional languages. The theorems have interesting consequences for the history of philosophy and logic.

Keywords: relational language, functional language, substantive metaphysics

1. Introduction

It is a quite common view that the transition to modern logic, made at the turn of the XX century, was associated with the increase in the expressive power of language through the inclusion of, among other things, judgments about relationships. It is believed that the transition to the language and the logic of relations was a natural step, since it allows an opportunity to give a more complete description of the world around us. It enriched not only the logic itself, but also the overall development of science.

This view was shared by R. Carnap, when he wrote in [1] that the restriction only to predicative sentences fatally affected areas lying outside the sphere of logic, that Russell was right explaining some of the errors of metaphysics by shortcomings of logic, and that any substantive metaphysics can be explained as based on this error, which allegedly caused a long delay in the development of physics, giving rise to substantive view of matter.

We show that, contrary to the received view, the language of properties (one-place relations) and functions is sufficient for the expression of the same mathematical and physical ideas, which are usually presented

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in terms of multi-place relations. We must conclude that, from a logical point of view, there was no need to abandon the subject-predicate language and substantive metaphysics.

2. Basic notions

If S is a theory in a language L, then $A \in L$ means that the formula A belongs to L, and $S \vdash A$ means, that the formula A is provable in the theory S.

We say that a theory S_1 is *subtheory* of S_2 (in short: $S_1 \subseteq S_2$) if and only if $L_1 \subseteq L_2$ and from $S_1 \vdash A$ it follows $S_2 \vdash A$.

Let S_1 and S_2 be two theories in languages L_1 and L_2, respectively. A recursive function $\varphi\colon L_1 \to L_2$ is called *operation, which embeds S_1 into S_2*, if and only if φ satisfies the following condition for any $A \in L_1$:

$$S_1 \vdash A \Longleftrightarrow S_2 \vdash \varphi(A).$$

We say that a theory S_1 is *embeddable* in a theory S_2 if and only if there is an operation which embeds S_1 in S_2. Theories S_1 and S_2 are *mutually embeddable* if and only if S_1 is embeddable in S_2, and S_2 is embeddable into S_1. Relation of being mutually embeddable is reflexive, symmetric and transitive [4].

LEMMA 1. *Let $S_1 \subseteq S_2$ and there exists a recursive function $\phi\colon L_2 \to L_1$ which satisfies the following conditions*:

(i) *if $A \in L_2$ and $S_2 \vdash A$, then $S_1 \vdash \phi(A)$,*
(ii) *if $A \in L_1$ and $S_1 \vdash \phi(A)$, then $S_1 \vdash A$,*
(iii) *if $A \in L_2$ and $S_2 \vdash \phi(A)$, then $S_2 \vdash A$.*

Then

1. *the identity function $\iota(A) = A$ embeds S_1 in S_2,*
2. *the function ϕ embeds S_2 in S_1,*

Thus, the theories S_1 and S_2 are mutually embeddable.

PROOF. 1. For any $A \in L_1$: if $S_1 \vdash A$, i.e. $S_1 \vdash \iota(A)$, then $S_2 \vdash \iota(A)$, by $S_1 \subseteq S_2$. Reversely, if $S_2 \vdash \iota(A)$, i.e. $S_2 \vdash A$, then $S_1 \vdash \phi(A)$, by (i). So $S_1 \vdash A$, by (ii).

2. For any $A \in L_2$: if $S_2 \vdash A$, then $S_1 \vdash \phi(A)$, by (i). Reversely, if $S_1 \vdash \phi(A)$, then $S_2 \vdash \phi(A)$, $S_1 \subseteq S_2$. Hence $S_2 \vdash A$, by (iii). $\dashv$

3. The main result

Let T_1 be an axiomatic first-order theory in a language L_1 with equality such that $T_1 \vdash \neg\, a = b$, for some closed terms a and b of L_1. Then for any n-ary predicate symbol P of L_1, the following formulas are provable in T_1:

$$\exists y((P(\boldsymbol{x}) \wedge y = a) \vee (\neg P(\boldsymbol{x}) \wedge y = b))$$

$$[((P(\boldsymbol{x}) \wedge y = a) \vee (\neg P(\boldsymbol{x}) \wedge y = b)) \wedge ((P(\boldsymbol{x}) \wedge z = a) \\ \vee (\neg P(\boldsymbol{x}) \wedge z = b))] \supset y = z$$

where $\boldsymbol{x}$ is an n-tuple of pairwise distinct variables, and different from the variable y.

The above facts allow us to construct the theory T_2, obtained by expanding the language L_1 to L_2 by adding for each n-placed predicate symbol P a new functional symbol f_P and a new axiom

$$(P(x_1, ..., x_n) \wedge f_P(x_1, ..., x_n) = a) \vee (\neg P(x_1, ..., x_n) \wedge f_P(x_1, ..., x_n) = b)$$

The reason for the introduction of new functional symbols is Theorem 2.28 from Mendelson [3]. For each functional symbol f_P the following formulas are provable in T_2:

$$f_P(x_1, ..., x_n) = a \vee f_P(x_1, ..., x_n) = b \tag{†}$$

$$P(x_1, ..., x_n) \equiv f_P(x_1, ..., x_n) = a \tag{‡}$$

From Theorem 2.28 of Mendelson [3] and Theorem 42 of Kleene [2] we obtain the following two facts.

Lemma 2. *$T_1 \subseteq T_2$ and T_2 is a conservative extension of T_1. So for any $A \in L_1$: $T_1 \vdash A$ iff $T_2 \vdash A$.*

Lemma 3. *There exists a recursive function $\phi\colon L_2 \to L_1$ such that:*

(1) *if $A \in L_2$ then $T_2 \vdash A \equiv \phi(A)$,*
(2) *if $A \in L_2$ and $T_2 \vdash A$, then $T_1 \vdash \phi(A)$.*

Lemma 4. *Theories T_1 and T_2 are mutually embeddable.*

Proof. Since $T_1 \subseteq T_2$, it suffices to find a recursive function satisfying the conditions of Lemma 1. We show that the mapping ϕ from Lemma 3 possesses the required properties.

Indeed, by Lemma 3(2), we obtain the condition (i) of Lemma 1.

For (ii): if $A \in L_1$ and $T_1 \vdash \phi(A)$, then $T_2 \vdash \phi(A)$, by $T_1 \subseteq T_2$. Hence $T_2 \vdash A$, by Lemma 3(1). Thus $T_1 \vdash A$, by Lemma 2.

For (iii): if $A \in L_2$ and $T_2 \vdash \phi(A)$, then $T_2 \vdash A$, by Lemma 3(2). $\dashv$

Define the language L_3, which is obtained from the language L_2 by deleting all predicate symbols other than the symbol of equality. Next, we define the following function $\alpha \colon L_2 \to L_3$ such that:

- $\alpha(t_1 = t_2) = \ulcorner t_1 = t_2 \urcorner$,
- $\alpha(P(t_1, ..., t_n)) = \ulcorner f_P(t_1, ..., t_n) = a \urcorner$, and P is different from '$=$',
- $\alpha(\neg A) = \ulcorner \neg\alpha(A) \urcorner$,
- $\alpha(A \circ B) = \ulcorner (\alpha(A) \circ \alpha(B)) \urcorner$, where $\circ \in \{\wedge, \vee, \supset, \equiv\}$
- $\alpha(Qx\, B) = \ulcorner Qx\, \alpha(B) \urcorner$, where $Q \in \{\forall, \exists\}$.

We define the theory T_3 in the language L_3 by the following condition:

> A is a non-logical axiom of the theory T_3 if and only if
> $A = \alpha(B)$ for some a non-logical axiom B of T_2.

Lemma 5. $T_3 \subseteq T_2$.

Proof. By the definitions, $L_3 \subseteq L_2$ and for any non-logical axiom A of T_3 there is a non-logical axiom B of T_2 such that $A = \alpha(B)$. Moreover, since $T_2 \vdash$ (‡), by the definition of α and the Equivalence Theorem we have that

$$T_2 \vdash B \equiv \alpha(B) \quad \text{and} \quad T_2 \vdash \alpha(B).$$

From this fact it follows that any proof in the theory T_3 at the same time is a proof in the theory T_2 and consequently $T_3 \subseteq T_2$. $\dashv$

Lemma 6. (1) *The identity function $\iota(A) = A$ embeds T_3 in T_2.*
(2) *The function α embeds T_2 in T_3.*
Thus, the theories T_2 and T_3 are mutually embeddable.

Proof. Since $T_3 \subseteq T_2$, we need to show that α is a recursive function satisfying the conditions of Lemma 1.

For (i): By induction on the construction of a proof of a formula A in T_2, we demonstrate that formula $\alpha(A)$ is provable in the theory T_3.

If A is a logical axiom of T_2, then $\alpha(A)$ is also a logical axiom and it is provable in T_3. If A is a non-logical axiom of T_2, then $\alpha(A)$ is a non-logical axiom of T_3.

Suppose that a formula A is obtained by the rule of modus ponens from two previous formulas B and $\ulcorner B \supset A \urcorner$. By assumption, formulas $\alpha(B)$ and $\alpha(B \supset A)$ are provable in T_3. Hence the formula $\alpha(A)$ is also provable in T_3.

Suppose that a formula $A = \ulcorner \forall x\, B \urcorner$ and A is obtained by the rule of generalization from B. By assumption, the formula $\alpha(B)$ is provable in T_3. Hence, the same holds good for $\ulcorner \forall x\, \alpha(B) \urcorner$. Since $\ulcorner \forall x\, \alpha(B) \urcorner = \ulcorner \alpha(\forall x\, B) \urcorner$ it follows that the formula $\ulcorner \alpha(\forall x B) \urcorner$ is provable in T_3.

Thus, we have shown that from $T_2 \vdash A$ it follows $T_3 \vdash \alpha(A)$.

For (ii): If $A \in L_3$ and $T_3 \vdash \alpha(A)$, then $T_3 \vdash A$, since $\alpha(A) = A$, by the definition of α.

For (iii): Suppose that $A \in L_2$ and $T_2 \vdash \alpha(A)$. Since $T_2 \vdash (\ddagger)$, by the Equivalence Theorem, we have that $T_2 \vdash A \equiv \alpha(A)$. Hence $T_2 \vdash A$. $\dashv$

The next theorem follows from lemmas 4 and 6.

THEOREM 1. *Let T be a first-order theory with equality such that $T_1 \vdash \neg\ a = b$, for some closed terms a and b. Then there exists a first-order theory T' such that T and T' are mutually embeddable, and T' is formulated in a language with functional symbols, and only the equality predicate.*

We can go further and show that the logical predicate of equality is not necessary and can be replaced by a special one-place predicate.

Let the theory T_4 be obtained by expanding the language of the theory T_3 with the new monadic predicate symbol 'H' and addition of the following new axiom

$$H(x) \ \equiv\ x = a$$

Notice that, by the definition, $T_3 \subseteq T_4$.

LEMMA 7. *The following formula*

$$H(f_=(x_1, x_2)) \ \equiv\ x_1 = x_2$$

is provable in the theory T_4.

PROOF. Since $T_2 \vdash (\ddagger)$, so $T_2 \vdash x_1 = x_2 \ \equiv f_=(x_1, x_2) = a$. By the definition of α and Lemma 6(2), we have that $T_3 \vdash x_1 = x_2 \ \equiv f_=(x_1, x_2) = a$. Hence also $T_4 \vdash x_1 = x_2 \ \equiv\ f_=(x_1, x_2) = a$, since $T_3 \subseteq T_4$. Now notice that $T_4 \vdash H(f_=(x_1, x_2)) \ \equiv\ f_=(x_1, x_2) = a$, because $\forall x(H(x) \equiv x = a)$ is an axiom of T_4. Thus, $T_4 \vdash H(f_=(x_1, x_2)) \ \equiv\ x_1 = x_2$. $\dashv$

LEMMA 8. (1) *The identity function* $\iota(A) = A$ *embeds* T_3 *in* T_4.
(2) *There is a function* $\psi\colon L_4 \to L_3$ *which embeds* T_4 *in* T_3.
Thus, the theories T_3 *and* T_4 *are mutually embeddable.*

PROOF. Since $T_3 \subseteq T_4$, it is enough to find a recursive function $\psi\colon L_4 \to L_3$, satisfying the conditions of Lemma 1. We define ψ as follows:

- $\psi(H(t)) = \ulcorner t = a \urcorner$,
- $\psi(t_1 = t_2) = \ulcorner t_1 = t_2 \urcorner$,
- $\psi(\neg A) = \ulcorner \neg\psi(A) \urcorner$,
- $\psi(A \circ B) = \ulcorner (\psi(A) \circ \psi(B)) \urcorner$, where $\circ \in \{\wedge, \vee, \supset, \equiv\}$,
- $\psi(Qx\, B) = \ulcorner Qx\, \psi(B) \urcorner$, where $Q \in \{\forall, \exists\}$.

For (i): By induction on the construction of a proof of a formula A in T_4 we demonstrate that formula $\psi(A)$ is provable in the theory T_3.

If A is a logical axiom of T_4, then $\psi(A)$ is also a logical axiom and it is provable in T_3.

If A is the axiom '$H(x) \equiv x = a$', then $\psi(A)$ is '$x = a \equiv x = a$', so $T_3 \vdash \psi(A)$.

If A is a non-logical axiom of T_4 which is a non-logical axiom of T_3, then $\psi(A) = A$, because A contains no occurrences of 'H'. Therefore $T_3 \vdash \psi(A)$.

For the rule of modus ponens and the rule of generalization we apply in the standard way the definition of the function ψ. Thus, we have shown that from $T_4 \vdash A$ it follows $T_3 \vdash \psi(A)$.

For (ii): Suppose that $A \in L_3$ and $T_3 \vdash \psi(A)$. Then $\psi(A) = A$, because A contains no occurrences of 'H'. Therefore, $T_3 \vdash A$.

For (iii): Suppose that $A \in L_4$ and $T_4 \vdash \psi(A)$. Since '$\forall x(H(x) \equiv x = a)$' is an axiom of T_4, so $T_4 \vdash \psi(A) \equiv A$, from the Equivalence Theorem. Thus, $T_4 \vdash A$. $\dashv$

Let the language L_5 be obtained by deleting symbol of equality from the language L_4. We define the following function $\beta\colon L_4 \to L_5$:

- $\beta(H(t)) = \ulcorner H(t) \urcorner$,
- $\beta(t_1 = t_2) = \ulcorner H(f_{=}(t_1, t_2)) \urcorner$,
- $\beta(\neg A) = \ulcorner \neg\beta(A) \urcorner$,
- $\beta(A \circ B) = \ulcorner (\beta(A) \circ \beta(B)) \urcorner$, where $\circ \in \{\wedge, \vee, \supset, \equiv\}$,
- $\beta(Qx\, B) = \ulcorner Qx\, \beta(B)) \urcorner$, where $Q \in \{\forall, \exists\}$.

We define the theory T_5 as follows:

> A is a non-logical axiom of the theory T_5 if and only if $A = \alpha(B)$ for some B which either is an axiom of equality or is a non-logical axiom of T_4.

Lemma 9. $T_5 \subseteq T_4$.

Proof. If A is an axiom of T_5, then $A = \beta(B)$, for some B which is an axiom of equality or is a non-logical axiom of T_4. By Lemma 7, the definition of β and the Equivalence Theorem, we have that $T_4 \vdash B \equiv \beta(B)$ and $T_4 \vdash \beta(B)$. Therefore any proof in the theory T_5 at the same time is the proof in T_4 and consequently $T_5 \subseteq T_4$. $\dashv$

Lemma 10. (1) *The identity function $\iota(A) = A$ embeds T_5 in T_4.*
(2) *The function $\beta\colon L_4 \to L_5$ embeds T_4 in T_5.*
Thus, the theories T_4 and T_5 are mutually embeddable.

Proof. Since $T_5 \subseteq T_4$, we need to show thet β is a recursive function satisfying the conditions of Lemma 1.

For (i): By induction on the construction of a proof of a formula A in T_4 we demonstrate that $T_5 \vdash \beta(A)$.

If A is a propositional or quantifier axiom of T_4, then $\beta(A)$ is a logical axiom of T_5, and therefore $T_5 \vdash \beta(A)$.

If A is an axiom of equality of T_4 or A is a non-logical axiom of T_4, then $\beta(A)$ is a non-logical axiom of T_5 and therefore $T_5 \vdash \beta(A)$.

As in the case of ψ, for the rule of modus ponens and the rule of generalization we apply in the standard way the definition of the function β. Thus, we have shown that from $T_4 \vdash A$ follows $T_5 \vdash \beta(A)$.

For (ii): If $A \in L_5$ and $T_5 \vdash \beta(A)$, to $T_5 \vdash A$, since $\beta(A) = A$, by the definition of β.

For (iii): Suppose that $A \in L_4$ and $T_4 \vdash \beta(A)$. By Lemma 7 and the Equivalence Theorem, we have that $T_4 \vdash A \equiv \beta(A)$. So $T_4 \vdash A$. $\dashv$

The following theorem is a consequence of Theorem 1 and lemmas 8 and 10.

Theorem 2. *Let T be a first-order theory with equality such that $T_1 \vdash \neg\, a = b$, for some closed terms a and b. Then there exists a first-order theory T' such that T and T' are mutually embeddable and the language of T' contains only functional symbols, and a single one-place predicate.*

4. Concluding remarks

The proofs of the theorems in Section 3 are not complicated, and because of that, their content seems to be trivial. However, from a philosophical point of view, they are interesting, because they refute some widespread views. The ontology of things, properties, and functions is not worse than the ontology of things and relations, which *de facto* has become a model for the presentation of scientific ideas.

Moreover, it appears that we can also delete properties from our model. The ontology of objects and functions is also universal in its expressive power. The relation of equality, which is included in the language, is purely logical and says if two terms denote the same object or not.

For the philosophically oriented logician it is obvious that the replacement of one ontology by another is not a purely formal trick, but brings a completely different view of the surrounding world, leads us to use new heuristics in the construction of scientific theories.

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