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WHAT IS A GENUINE INTUITIONISTIC NOTION OF FALSITY?

What is it to be false? Shall we call a sentence false if and only if it has a true negation? Or if and only if it isn't true?

David Lewis, "Logic for Equivocators"

Abstract. I highlight the importance of the notion of falsity for a semantical consideration of intuitionistic logic. One can find two principal (and non-equivalent) versions of such a notion in the literature, namely, falsity as non-truth and falsity as truth of a negative proposition. I argue in favor of the first version as the genuine intuitionistic notion of falsity.

Keywords: Intuitionism, truth, falsity, negation.

1. Preliminaries

Since Gottlob Frege substantiated his view on logic as the science of "the most general laws of being true" [4, p. 39], the notion of *truth* has generally been considered to be central notion for logical semantics. Accordingly, many other logical notions, *validity* and *entailment* among them, are usually introduced through the one of truth. In Fregean philosophy of logic truth is represented by the corresponding semantical value, a specific (abstract) object serving as a denotation for sentences, their *truth value* (see, e.g. [26]).

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Now, one can ask, whether we need other logical value(s) distinct from truth, and if we do, what the use of such extra value(s) should be. Frege did not mention any "laws of being false", after all. A little reflection, however, reveals a destructive effect of taking truth as a sole (universal) logical value filling the whole semantic domain (of truth values). In such a construction all the sentences would be true, denoting one and the same truth value, and the resulting "mono-valued" logic would be absolutely inconsistent.

Thus, any non-trivial logic has to be no less than *two-valued*, and hence, we cannot do with truth alone — at least one more logical value is needed. The standard semantics for classical logic is paradigmatic in this respect. One takes here *falsity* as such extra value, treating it simply as absence of truth, i.e., as *non-truth*: sentence A is (classically) false if and only if it is not (classically) true. Falsity so introduced plays an important *restrictive* role, and as such it is an indispensable component of the whole semantic construction even though it is not explicitly involved in defining the key logical notions (validity and entailment).

Moreover, since classical falsity and classical truth are complementary notions, it is not difficult to formulate an adequate semantics for classical (propositional) logic purely in terms of falsity. One only needs to specify *falsity conditions* instead of *truth conditions* for propositional connectives and define validity and entailment by means of falsity rather than truth. By such a "dualization" truth and falsity will reverse their roles with truth playing the role of restriction for falsity to prevent its universalization.

But in non-classical logics the situation may not be so transparent, and explication of the notion of falsity in various non-classical systems and its interrelation to the given notion of truth can be rather complicated. In particular, one can consider *intuitionistic logic* and ask what kind of falsity is (or should be) used there. In the present paper I am going to compare two possible ways of understanding falsity in intuitionism and adduce arguments in favor of one of them, which I regard as a *genuine intuitionistic notion of falsity*.

It should be clear from the very outset that truth and falsity are taken in this paper as categories of logical semantics. Thus, under "intuitionistic falsity" I understand here a semantic notion from a metatheory of intuitionistic logic, which in its turn presents certain embodiment of a broader conception adopted in intuitionism as a philosophical program in foundations of mathematics and anti-realistic epistemology.

2. Intuitionistic conception of truth and its semantic implementation

Intuitionistic logic belongs to the family of *constructive logics*, and as such it adopts a *constructive conception of truth*. Generally speaking, this means that

> the notion of truth of a proposition should be explained in terms of the notion of proof, or verification, rather than as correspondence with some sort of mind-independent realm of mathematical objects.
>
> [18, p. 131]

This is indeed a very general view, which needs further specification. First, one should observe that "proof" is understood here as a *constructive* proof, i.e., some kind of construction that proves the proposition in question. Second, there is a point of discrepancy here as to whether we should speak of *actually presented* proofs or rather of some *potential* provability. This controversy has been explicitly indicated by M. Dummett, see e.g. [2, pp. 18-19], and touched upon by some authors, D. Pravitz [17], P. Martin-Löf [14] and W. Rabinowicz [19] among them (see also 1994 issue of *Topoi* [29] devoted to intuitionistic truth, especially the contribution by E. Martino and G. Usberti [15]). The opposition of "actualist" and "possibilist" approaches to intuitionistic truth has been also considered in [25, pp. 766 ff.], and recently examined in some detail by P. Raatikainen in [18].

According to the first view, "A is true" should be understood as "A has been (actually) proved", whereas the second view interprets this as "A is (in principle) provable (A can be proved)". In the first case truth is essentially of a temporal character. It is not uncommon that for some sentence no actual proof was available yesterday, but today we are lucky enough to obtain such a proof (recall *Fermat's Last Theorem* or *Poincaré Conjecture*). In the second case we deal with a tenseless notion — *a possibility* of a proof of a true sentence must always be present even if nobody knows its actual proof so far (otherwise the very notion of possibility would be destroyed).

Nowadays the possibilist (atemporal) approach to intuitionistic truth is rather wide-spread in the literature. This view goes hand in hand with some allegedly forceful arguments against the actualist approach, which can be summarized in the following two points: (1) actualist truth seems to be not closed under the logical consequence; (2) taking the actualist interpretation it would be impossible to suppose (and to claim) truth of a sentence, which is not yet proved. Rabinowicz [19, p. 192] finds these objections "compelling" enough to abandon the actualist understanding of intuitionistic truth right off the bat, and to take the possibilist side.

However, the problem with these counterarguments is that they look very much like an attempt to "smuggle" under the guise of intuitionistic truth the essentially classical (realistic) conception, only slightly decorated with intuitionistic terminology. One repudiates the actualist understanding of the truth of single sentences, but accepts instead the actualist treatment of intuitionistic theories as a whole.

Indeed, by assuming that logical consequence should ("customary") be defined "in terms of necessary truth preservation" [19, p. 191], and by taking theories to be *actually closed* under the consequence so defined, one commits oneself, explicitly or implicitly, to *the abstraction of actual infinity*, which is an anathema to any true intuitionist. Real intuitionistic theories are never closed under the intuitionistic consequence actually, but only *potentially*. And if we understand the deductive closure of a proved sentence not as an actual presenting (here and now) the whole infinity of its consequences, but as a potential possibility to perform a desired proof whenever it is needed (as it should be in intuitionism), then the actualist understanding of the proofs themselves seems to be unproblematic.

One might remark that it is generally accepted by semantic considerations to operate with intuitionistic theories as actually closed unities. Well, it only means that in these cases one performs what can be called a *classical analysis of intuitionistic logic*, i.e, one considers intuitionistic theories from the classical perspective. Such kind of analysis is most often employed by formulating semantics for intuitionistic logic in classical metalanguage, which can be quite useful for establishing some important meta-properties, such as completeness. One should only be aware that by this analysis one just forces intuitionistic theories to behave in a classical manner, and not to claim such a behavior to be authentic from intrinsic intuitionistic standpoint.

The same holds if one wishes "to talk about truths waiting to be discovered" [19, p. 192], or if one needs "a notion of truth where, without falling into absurdities, we may say, e.g., that there are many truths that are not known today" [17, p. 9]. This is, of course, an extremely classical (realistic) view on truth, which has little in common with the original intuitionistic understanding.

The main trouble with the possibilist account on truth is that it almost inevitably leads to acceptance of an objective realm of preexisting proofs independent of anybody's hitting them. In fact, Prawitz had explicitly postulated such a realm, what was criticized by Dummett as driving fatally into the realist position with a justification of bivalence, see [18, p. 139-141]. Martino and Usberti characterize "potential intuitionism" as a realistic position, which may be of interest by itself "as a way of reconciling, to some extent, classical and intuitionistic mathematics" [15, p. 88]. They further explain:

> However [...] the philosophical position of the potential intuitionist is very far from Brouwer and Heyting's view of mathematics. The essential difference between orthodox intuitionism and classical mathematics turns out to lie in the opposition between temporal and atemporal provability rather than in the opposition between transcendent truth and provability. [...] [I]f one is interested in intuitionism as an antirealistic philosophy of mathematics and in intuitionistic logic as the basis of antirealistic reasoning, he must reject atemporal provability and resist the temptation of introducing into intuitionism any surrogate of classical truth. He must rather accept certain consequences of his antirealistic view, even when they are not in agreement with the naive, realistic, attitude of the working mathematician. Temporal truth, the notion according to which a proposition becomes true only when it is proved, is certainly counterintuitive insofar as it conflicts with the usual naive notion of proof as recognition of preexistent facts. But that is as it should be, since the antirealist must, of course, refuse any intuition of a realistic nature. [15, p. 88-89]

It has been repeatedly observed in the literature (see, e.g., [15, 18]) that an orthodox intuitionism, as conceived and advocated by Brouwer and Heyting, expressly asserts the temporal (and thus, actualist) approach to the main mathematical and logical notions, and this temporality reflects the very essence of intuitionist philosophy of mathematics[1].

The primary purpose of the present paper is to explore some authentic intuitionistic notions rather than to reconcile intuitionism with classical logic by finding some compromise formulations of intuitionistic

[1] It is possible to differentiate between "strict actualism" and "liberalized actualism", see [18], and to demonstrate that, e.g., Brouwer's views have been sometimes oscillated between strict and liberalized versions of actualism, but we disregard such subtleties here. On all occasions it would be inadequate to interpret Brouwer's views on provability as a possibilist position.

principles to make them acceptable for those who tend to adhere to realist views. Therefore in what follows I adopt the orthodox intuitionistic understanding and consider a sentence to be *constructively true* if and only if it is *constructively proved* (or, as Heyting put it, "*it is known how to prove p*", see [9, p. 959]).

This understanding can be nicely modeled by means of standard Kripke-semantics for intuitionistic propositional logic. A *Kripke-model* for intuitionistic logic is defined as a triple $\langle W, \leq, \Vdash_T \rangle$, where W is some set, $\leq$ is a reflexive and transitive relation (a pre-order) defined on W, and $\Vdash_T$ is a constructive "forcing relation" between elements of W and sentences of our language. The relation $\Vdash_T$ is defined by the model in such a way that for any $a, b \in W$ and for every atomic sentence p_i the following "hereditary condition" holds:

$$a \Vdash_T p_i \text{ and } a \leq b \Rightarrow b \Vdash_T p_i. \tag{1}$$

The following definitions extend $\Vdash_T$ to compound sentences:

$$a \Vdash_T A \wedge B \text{ iff } a \Vdash_T A \text{ and } a \Vdash_T B; \tag{2}$$

$$a \Vdash_T A \vee B \text{ iff } a \Vdash_T A \text{ or } a \Vdash_T B; \tag{3}$$

$$a \Vdash_T A \to B \text{ iff } \forall b(a \leq b \Rightarrow (b \nVdash_T A \text{ or } b \Vdash_T B)); \tag{4}$$

$$a \Vdash_T {\sim}A \text{ iff } \forall b(a \leq b \Rightarrow b \nVdash_T A). \tag{5}$$

A standard induction procedure demonstrates that the hereditary condition holds for all the sentences of our language. The validity of a sentence ($\models A$) and the entailment relation between two sentences ($A \models B$) are defined as usual:

$$\models A \text{ iff } \forall W \forall a \in W(a \Vdash_T A). \tag{6}$$

$$A \models B \text{ iff } \forall W \forall a \in W(a \Vdash_T A \Rightarrow a \Vdash_T B). \tag{7}$$

Intuitively this construction fits very well the interpretation of intuitionistic truth in terms of constructive proofs. The set W represents some intuitionistic theory in the course of its development. Every element $a \in W$ can be understood as a state of this theory at a certain stage of development, and $\leq$ is a possible time-relation between states of the theory represented by W (a "later or simultaneously" relation). That is, "$a \leq b$" means that having a state of theory a we can reach (on some later stage of the theory development) a state of theory b.

Relation $\Vdash_T$ represents the notion of intuitionistic truth, and expression "$a \Vdash_T A$" can correspondingly be read as "the state of theory a forces the constructive truth of sentence A", i.e., "sentence A is constructively proved at the state of theory a".

The hereditary condition, being interpreted as a principle of preservation for proved sentences, plays an important role in expressing the constructive character of intuitionistic truth on the semantic level. In accordance with this condition, a sentence once proved remains such for ever (in all the next states of the given theory), i.e., the set of intuitionistic truths can only grow. Taking into account that we deal with constructive mathematical proofs, this property looks quite natural.

Consider clause (5) and its possible intuitive interpretation. In accordance with the above understanding, this clause says that $\sim A$ is constructively proved if and only if A will not be proved in any later state of our theory. In other words, it is established that A will never be proved. The latter means simply that *it is impossible to prove* A, and as soon as one really succeeds in demonstrating such impossibility the sentence A can be considered *intuitionistically refuted*. In intuitionism one usually demonstrates such an impossibility by showing that taking the sentence as proved leads to a contradiction. That is, intuitionistic refutation expressed by a negated sentence amounts to a kind of an indirect proof — a derivation of contradiction from the sentence itself.

3. Two ways of defining falsity in intuitionistic logic

The semantic model formulated above makes no explicit use of the notion of falsity. Still, truth is here of a non-universal character since it is quite possible that some state of a theory does not force this or that sentence. Such a possibility is clearly reserved, e.g., by definitions of forcing relation for implication (4) and negation (5). Hence, falsity has to be present (at least implicitly) in this domain devoid of truth, and the question naturally arises of how to explicate this notion and to elucidate its characteristic features.

Generally, there are two basic ways of introducing falsity into a semantic construction. One way is to interpret falsity as a complementary notion to the one of truth: "A is false" means nothing else but "A is not true". According to another view falsity ought to be interpreted as a direct semantic representation of the syntactic notion of negation, i.e., "A is false" is understood as an abbreviation of the expression "$\sim A$ is true". The main difference between these approaches is that in the first case falsity is defined exclusively in semantic terms (in a metalanguage), and as such is treated as a purely semantic notion; whereas in the second case falsity is introduced by means of some object language terms (namely, the connective of negation), and becomes thus of a mixed semantic-syntactic character.

As it has already been mentioned, in classical logic both ways are equivalent through the standard definition of truth conditions for classical negation: "$\sim A$ is true if and only if A is not true". But in intuitionistic logic this equivalence fails — one can observe that $a \nVdash_T A \Leftrightarrow a \Vdash_T \sim A$ *does not* hold in Kripke models in general[2]. Hence, the question arises, which way should one take to deal with intuitionistic falsehood.

There is a tradition to present intuitionistic falsity of a sentence as truth of its (intuitionistic) negation. E.g., Heinrich Wansing in [30] claims that in intuitionistic logic *both* the conception of truth and the conception of falsity are inferentialist, anti-realistic, and intuitionistic falsity is indicated by intuitionistic negation. Dummett also equates expressions "A is false" and "not-A is true" (where A is a proposition of intuitionistic logic) when he remarks:

> On [...] interpretation of "A is true" [...] as meaning "A has been proved", the statement "A is false", *that is* "$\neg A$ is true", is much stronger than "A is not true". (italics mine) [2, p. 19]

Brouwer and Heyting seem not to have any special conception of falsity as a philosophical (or semantic) notion. Whenever they occasionally speak of falsity, they simply mean intuitionistic negation. Thus, Brouwer usually refers to negation of a sentence as to its "absurdity", sometimes identifying the latter with "falsity", see, e.g. [1, p. 114]. Whereas Heyting in [9, p. 18] implicitly discriminates between object language and metalanguage speaking of "mathematical assertions" on the one side, and "statements about the states of our knowledge" on the other side, he

[2] By the way, this observation brings another counter-argument against the possibilist account of intuitionistic truth. Indeed, if one interprets "A is true" as "A is provable", then "A is *not* true" is to be read as "A is unprovable". But this means exactly that A can never be proved, and hence, that $\sim A$ is true. Thus, interpreting intuitionistic truth as a potential provability forces us to equate "A is not true" and "$\sim A$ is true" (cf. [2], p. 19), which we consider to be a rather unwanted outcome of such an interpretation.

reckons "true" and "false" among the terms of object language and takes "not true" and "false" to be synonyms both standing for intuitionistic negation:

> In mathematical assertions no ambiguity can arise: "not" has always the strict meaning. "The proposition p is not true", or "the proposition p is false" means "If we suppose the truth of p, we are led to a contradiction". [9, p. 18]

At the same time Heyting considers not only mathematical (i.e., object language), but also *factual* negation, which belongs to a metalanguage and is supposed to describe states of our knowledge:

> Where there is some danger of ambiguity, we express the mathematical negation by such expressions as "it is impossible that", "it is false that", "it cannot be", etc., while the factual negation is expressed by "we have no right to assert that", "nobody knows that", etc. [9, p. 18]

Here, again, "it is false that" correlates with mathematical negation. One can call this kind of falsity "indirect", and introduce a specific forcing relation $\Vdash_\perp$ for it:

$$a \Vdash_\perp A \text{ iff } a \Vdash_T {\sim}A. \tag{8}$$

Evidently, $\Vdash_\perp$ is based on the notion of constructive proof, demonstrating thus close relatedness between indirect falsity and intuitionistic truth. To establish the indirect falsity of sentence A one has to present some *conditional proof*, namely, a derivation of contradiction from this sentence. That is, one has to prove some invariably unacceptable sentence (such as $1 = 2$) upon condition that A is true. Consequently, indirect falsity turns out to be of a complex character: it never presents elementary statements about states of our knowledge, even in case of atomic sentences.

It would be inappropriate to consider $\Vdash_\perp$ a straightforward constructive counterpart of $\Vdash_T$. Such a counterpart of constructive truth (which could be labeled by $\Vdash_F$, cf. [25]) would be the notion of constructive ("constructible") falsity introduced by Nelson in [16]. Constructive falsity is to be understood not as reducing a sentence to an absurd, but as its direct constructive refutation or disproof by some construction refuting this sentence. However, such constructive falsity does not belong to *intuitionistic* conceptual framework.

By contrast, in intuitionistic logic it is possible to consider a notion of falsity which represents a metalanguage ("factual", in Heyting's terminology) negation, and to introduce another forcing relation $\Vdash_f$ as follows:

$$a \Vdash_f A \text{ iff } a \nVdash_T A. \tag{9}$$

It is worthy of note that Kripke by formulating his semantic model for intuitionistic logic makes an explicit use of the notion analogous to $\Vdash_f$, see [12, p. 94]. Namely, he considers a set of truth values $\{\mathbf{T}, \mathbf{F}\}$ and introduces a truth value function $\phi(A, a)$ (notation adjusted), which for every propositional formula A, and for every possible world a takes one of the two truth values. Moreover, Kripke defines truth conditions for complex formulas as follows:

$$\phi(A \wedge B, a) = \mathbf{T} \text{ iff } \phi(A, a) = \mathbf{T} \text{ and } \phi(B, a) = \mathbf{T};$$
$$\text{otherwise, } \phi(A \wedge B, a) = \mathbf{F},$$

and likewise for other connectives and quantifiers. This means that in his original semantic construction for intuitionistic logic Kripke interprets falsity as a simple absence of (constructive) truth.

Expression $a \Vdash_f A$ can be understood as "A is not proved in a", or "sentence A is *rejectable* at the state of theory a". That is, at the state a we cannot accept the sentence, since we do not have enough reasons for including it in our theory. If we cannot accept the sentence, we can reject it *so far*, and the sentence can be considered to be *rejectable* within the state of theory a, although this does not exclude the possibility of including it in our theory on some later stage.

Clearly, this notion of falsity is not constructive, it simply records absence of proof in our theory and the possibility of rejection of some sentence, but it says nothing about accomplishing any construction. It cannot be interpreted as intuitionistic refutation: if we do not have a proof of a sentence, this does not necessarily mean that this sentence is refuted. Therefore $a \Vdash_f A$ is not subject to the hereditary condition — a sentence may lack proof at some moment, but this proof can be found later. Still, the non-constructive falsity is of temporal character, but this temporality goes in inverse direction. That is, for $\Vdash_f$ a "backward heredity" naturally holds: if a sentence is not proved in some state of a theory, it has not been proved at any previous state.

Now, the following questions arise. First, taking into account definitions (8) and (9), would it be justified to assume a simultaneous existence of two notions of falsity in a semantics for intuitionistic logic? Second, what could (and should) be the role of falsity in logical semantics, and why do we need a separate (autonomous) notion of falsity in intuitionistic logic? Third, if any coexistence of two notions of falsity in intuitionistic semantics is inappropriate, which of the above notions should be taken as adequate for intuitionistic logic?

4. Falsity in intuitionistic logic: how many and what for?

In what sense can we speak of *two* different falsity notions within *one* logic? Consider a semantic construction for some many-valued logic with a set of designated values interpreted as an analogue of the classical notion of truth. The set of designated values may contain more than one element representing, thus, various *degrees of truth*. Therefore, in many-valued logics one sometimes speaks not of "truth values" but of "truth degrees", see [6, p. 4]. It can be stated that the set of designated values represents a general notion of truth in the given logic, where truth is of a gradual (manifold) character, and each designated value expresses some particular aspect of this general notion. Likewise, one can distinguish a set of *anti-designated values* among the values of a many-valued logic, see, e.g., [32]. Again, if this set contains more than one element, it represents a general notion of falsity in the given logic, whereas individual elements may stand for various kinds of this notion.

However, an attempt to incorporate *both* $\Vdash_{\perp}$ and $\Vdash_{f}$ into a semantical framework for intuitionistic logic on the basis of a many-valued construction faces some serious difficulties. Here it is impossible just to do with $\Vdash_{T}$, $\Vdash_{\perp}$ and $\Vdash_{f}$, and to conceive a kind of an "intuitionistic three-valued semantics", since, according to a well-known result by Gödel [5], intuitionistic logic does not allow any (adequate) *finite-valued* semantic realization. Yet, one can construct a many-valued semantics for intuitionistic logic on the basis of *infinite matrices*. Such an infinite matrix-construction has been proposed by Stanisław Jaśkowski [11], although, as Siegfried Gottwald [7] remarks, "the truth degrees of this matrix do not have a nice and simple intuitive interpretation". Remarkably, Jaśkowski-matrix has *one* designated element. This is as it should be: taking into account the proof-interpretation of intuitionistic truth, it would hardly be justifiable to assume any "gradations" of constructive proofs. Now, if we have one designated value and infinitely many others (non-designated), it seems rather difficult to find *natural* grounds for distinguishing among them just two elements as anti-designated.

Therefore, it is more appropriate to rank intuitionistic logic among the *two-valued logical systems*, which precludes any multiplication of the falsity notion. Hence, if we need falsity as an explicit element of an intuitionistic semantic construction, we have to decide, which of the notions introduced in the previous section should be taken for an adequate expression of the intuitionistic idea of falsity. To make the right choice we should determine cases in which we can need "falsity" as a full fledged semantical value.

One may wish to consider two such cases. *First*, one can employ falsity for a "dualization" of usual verificationistic semantics by reformulating it in such a way that it takes falsity as a primitive (and the main) semantic notion. *Second*, falsity is needed if it is impossible to construct a desired semantics solely in terms of truth, and both values should act as "peer to peer" components of a general semantic framework[3].

My next task is to consider both these aspects in the context of a Kripke-style semantical analysis of intuitionistic logic, and to demonstrate that whereas relation $\Vdash_f$ allows an adequate semantic realization of the indicated cases, $\Vdash_\perp$ remains deficient in this respect.

5. A "falsificationistic semantics" for intuitionistic logic

It is not difficult to see how the Kripke-semantics from Section 2 can be reformulated by using relation $\Vdash_f$ as a primitive. Let a *falsificationistic Kripke model for intuitionistic logic* be a triple $\langle W, \leq, \Vdash_f\rangle$, where W and $\leq$ are defined as in usual Kriple models, and $\Vdash_f$ is a (falsity) forcing relation between elements of W and sentences of the language. For atomic sentences, $\Vdash_f$ is defined in such a way that the following "backward hereditary condition" holds:

$$a \Vdash_f p_i \text{ and } b \leq a \Rightarrow b \Vdash_f p_i. \tag{10}$$

For compound sentences, $\Vdash_f$ is determined by the following definitions:

$$a \Vdash_f A \wedge B \text{ iff } a \Vdash_f A \text{ or } a \Vdash_f B; \tag{11}$$

[3] As it is the case, e.g., in semantical models for Nelson's logic of constructible falsity, where truth and falsity are taken as autonomous values from the outset, moreover, truth conditions and falsity conditions for logical constants are formulated separately, see [28].

$$a \Vdash_f A \vee B \text{ iff } a \Vdash_f A \text{ and } a \Vdash_f B; \tag{12}$$

$$a \Vdash_f A \to B \text{ iff } \exists b (a \leq b \text{ and } b \nVdash_f A \text{ and } b \Vdash_f B); \tag{13}$$

$$a \Vdash_f {\sim}A \text{ iff } \exists b (a \leq b \text{ and } b \nVdash_f A). \tag{14}$$

Again, by induction condition (10) can be easily extended to all the sentences of language. The notions of intuitionistic validity ($\models A$) and intuitionistic entailment ($A \models B$) can be defined through relation $\Vdash_f$ as follows:

$$\models A \text{ iff } \forall W \forall a \in W (a \nVdash_f A); \tag{15}$$

$$A \models B \text{ iff } \forall W \forall a \in W (a \Vdash_f B \Rightarrow a \Vdash_f A). \tag{16}$$

The falsificationistic Kripke-models so formulated are adequate for intuitionistic logic. Soundness and completeness are straightforwardly obtainable from the corresponding proofs for the usual Kripke-models by a sort of "dualization procedure". The formulation of an adequate semantics for intuitionistic logic exclusively in terms of falsity displays a possibility of interpreting this logic as a logic of *potential falsifiability*.

If we try to formulate an analogous semantics with relation $\Vdash_\perp$ as a primitive, then some serious problems seem unavoidable. For one thing, the definitions of falsity conditions for compound sentences turn out to be rather artificial. Consider conjunction. Taking into account the main idea which underlies definition (8), we have to build on the following equivalence:

$$a \Vdash_\perp A \wedge B \text{ iff } a \Vdash_T {\sim}(A \wedge B). \tag{17}$$

By employing definitions (2) and (5), we get:

$$a \Vdash_\perp A \wedge B \text{ iff } \forall b (a \leq b \Rightarrow (b \nVdash_T A \text{ or } b \nVdash_T B)). \tag{18}$$

And likewise for other connectives:

$$a \Vdash_\perp A \vee B \text{ iff } \forall b (a \leq b \Rightarrow (b \nVdash_T A \text{ and } b \nVdash_T B)); \tag{19}$$

$$a \Vdash_\perp A \to B \text{ iff } \forall b (a \leq b \Rightarrow \exists c (b \leq c \text{ and } c \Vdash_T A \text{ and } c \nVdash_T B)); \tag{20}$$

$$a \Vdash_\perp {\sim}A \text{ iff } \forall b (a \leq b \Rightarrow \exists c (b \leq c \text{ and } c \Vdash_T A)). \tag{21}$$

An insuperable problem with definitions (18)–(20) is that their right-hand side contains relation $\Vdash_T$ (even if only negated), which cannot be eliminated. It would be incorrect to replace expression $\nVdash_T$ by $\Vdash_\perp$ and expression $\Vdash_T$ by $\nVdash_\perp$ in these formulations, since such a replacement would demonstrate an obvious incoherency in treatment of $\Vdash_{\perp}$, virtually identifying it with $\Vdash_{f}$. Hence, the falsity conditions defined through $\Vdash_{\perp}$ turn out not to be formulated exclusively in terms of falsity, but also to employ the notion of truth. But if $\Vdash_T$ should obligatorily be employed in a semantic construction, $\Vdash_{\perp}$ turns out to be superficial since it can always be introduced by definition (8).

Moreover, taking into account the equivalence $a \Vdash_T {\sim}A \Leftrightarrow \forall b(a \leq b \Rightarrow b \nVdash_T A)$, any definition of the relation $\Vdash_{\perp}$ (for atomic sentences) should be subject to the following condition:

$$a \Vdash_{\perp} p_i \text{ iff } \forall b(a \leq b \Rightarrow b \nVdash_T p_i). \tag{22}$$

Again, we see here expression $\nVdash_T$ which cannot be replaced by $\Vdash_{\perp}$, first, for the reason indicated above, and second, since such a replacement would lead us into a *vicious circle*.

To sum up: in contradistinction to $\Vdash_f$, relation $\Vdash_{\perp}$ is not self-sufficient, it should be accompanied with $\Vdash_T$. Relation $\Vdash_{\perp}$ cannot be taken as a primitive semantic notion proper, and one cannot formulate a *purely falsificationistic* semantics on basis of $\Vdash_{\perp}$ alone.

Relation $\Vdash_{\perp}$ is also useless for the definition of the notion of validity. By considering the only possible candidate for such a definition in terms of $\Vdash_{\perp}$:

$$\models A \text{ iff } \forall W \forall a \in W(a \nVdash_{\perp} A), \tag{23}$$

one has to conclude that this definition is inadequate. Indeed, since relations $\Vdash_T$ and $\Vdash_{\perp}$ are not complementations of each other, there should be such W, $a \in W$, and A for which neither $a \Vdash_T A$ nor $a \Vdash_{\perp} A$ holds. In this case A evidently falls within the scope of definition (23), however it cannot be recognized as intuitionistically valid, because it is not constructively proved.

Thus, whereas it is throughout possible to construct an adequate *falsificationistic analog* of intuitionistic semantics by means of relation $\Vdash_f$, the relation $\Vdash_{\perp}$ turns out to be absolutely inapplicable in this respect. From this perspective relation $\Vdash_f$ is the most authentic representative of the falsity notion in intuitionism.

6. The notion of falsity and the entailment relation in intuitionistic logic

As we have seen, the relation $\Vdash_\perp$ does a poor job of modeling intuitionistic falsity as the *basic* semantic value. But, perhaps, it can be effectively employed by a simultaneous usage of *both* truth and falsity as autonomous values equally needed for a semantic construction? Indeed, notwithstanding that intuitionistic logic as such has no need in the parallel functioning of truth and falsity, some problems of traditional intuitionistic semantics could be resolved by employing this kind of construction.

In particular, one can try to eliminate on this way the so-called "paradoxes of logical entailment" which hold not only in classical, but also in intuitionistic logic. The following principles are valid in any Kripke-model:

$$A \models B; \tag{24}$$

$$\sim B \models A, \tag{25}$$

where A is any sentence, and B is any logically valid sentence. In accordance with this principles, a logical law is entailed by *any* sentence *whatsoever*, and logically false sentence entails *anything*. This is often considered to be counterintuitive, even "paradoxical", as it is desirable that premises and conclusion of a "good" logical inference should be *relevant* to each other.

Relevance logic attempts to eliminate the "paradoxes of entailment". One of the most effective eliminative strategies, which essentially is due to J. Michael Dunn [3], is sometimes dubbed as "American plan", see, e.g., [20]. The crux of the strategy consists in treating truth and falsity *on a par*, as mutually independent semantic values, and thus to allow under-determined ("gaps") and over-determined ("gluts") valuations. An entailment relation defined on this basis is not subject to the "paradoxical" principles (24) and (25).

It is interesting to observe that this strategy can be successfully applied to intuitionistic entailment by using relation $\Vdash_f$. Define *relevant intuitionistic model* as a quadruple $\langle W, \leq, \Vdash_T, \Vdash_f \rangle$, where W and $\leq$ are as in standard intuitionistic models, and for $\Vdash_T$ and $\Vdash_f$ the corresponding hereditary conditions (1) and (10) hold. For conjunction and disjunction definitions (2), (11), (3) and (12) are taken. To ensure a required interrelation between truth and falsity, definitions for implication and negation have to be slightly modified:

$$a \Vdash_T A \to B \text{ iff } \forall b(a \leq b \Rightarrow (b \Vdash_f A \text{ or } b \Vdash_T B)); \tag{26}$$
$$a \Vdash_f A \to B \text{ iff } \exists b(a \leq b \text{ and } b \Vdash_T A \text{ and } b \Vdash_f B); \tag{27}$$
$$a \Vdash_T {\sim}A \text{ iff } \forall b(a \leq b \Rightarrow b \Vdash_f A); \tag{28}$$
$$a \Vdash_f {\sim}A \text{ iff } \exists b(a \leq b \text{ and } b \Vdash_T A). \tag{29}$$

An important point of these modifications is that here definition (9) is *not* generally taken (although it is meant as an underlying background), and relations $\Vdash_T$ and $\Vdash_f$ are introduced separately as primitive notions. As a result, the following conditions do *not* generally hold:

$$a \Vdash_T A \text{ or } a \Vdash_f A; \tag{30}$$
$$a \nVdash_T A \text{ or } a \nVdash_f A. \tag{31}$$

That is, speaking informally, a sentence may well be neither true nor false, or true and false simultaneously. A definition for entailment relation remains literally the same. Moreover one can take any of the two definitions (7) or (16) at choice — it can be demonstrated that these definitions turn out to be equivalent with respect to any relevant intuitionistic model, see [24].

Principles (24) and (25) do not hold in relevant intuitionistic models. In this way we achieve a "relevantization" of the first-degree entailment for intuitionistic logic, see in more detail [21, 22, 23, 24].

Remarkably, any attempt to perform an analogous procedure with relation $\Vdash_\perp$ (rather than $\Vdash_f$) fails. Indeed, consider a *strong intuitionistic model* virtually introduced in the previous section as a quadruple $\langle W, \leq, \Vdash_T, \Vdash_\perp\rangle$, where W and $\leq$ are again as in standard intuitionistic models, $\Vdash_T$ and $\Vdash_\perp$ are subject to conditions (1) and (22) correspondingly, and for compound sentences definitions (2)–(5) and (18)–(21) are taken. Note, however, that it is impossible to modify definitions for implication and negation in the way the corresponding definitions have been modified in relevant intuitionistic models above, since such a modification would imply an identification of $\Vdash_\perp$ with $\nVdash_T$ (and hence, with $\Vdash_f$), which is inadmissible.

The following lemma is easy to establish:

LEMMA 6.1. *$\forall W \forall a \in W \forall A\,((a \Vdash_\perp A$ and $a \leq b) \Rightarrow b \Vdash_\perp A)$.*

But what can be a definition of logical entailment? As distinct from the relevant intuitionistic models described above, it makes no sense to use definition (7) here, since now it does not determine any new relation (in comparison with the standard intuitionistic logic), turning thus $\Vdash_\perp$ into a useless redundancy.

It might seem natural to consider the following definition:

$$A \models B \text{ iff } \forall W \forall a \in W(a \Vdash_\perp B \Rightarrow b \Vdash_\perp A)). \tag{32}$$

A simple check shows, however, that this definition not only validates the "paradoxical" principles (24) and (25), but also makes valid some intuitionistically unacceptable principles, so that $\sim\sim A \models A$. That is, the entailment relation so defined is not only irrelevant, but also non-intuitionistic. Thus, employing the pair $\Vdash_T$ and $\Vdash_\perp$ as a basis for semantic construction, and accompanying it with definition (32) do not lead us to any logic of intuitionistic type.

7. Intuitionistic truth and intuitionistic falsity: monism or dualism?

In this paper some arguments have been put forward in favor of the claim that intuitionistic falsity has to be treated as non-truth, where truth is understood as a here-and-now availability of constructive proof[4]. On the basis of such a notion one can construct a full-fledged falsificationistic semantics for intuitionistic logic, and to achieve a "relevantization" of intuitionistic entailment. At the same time, an alternative understanding of falsity as intuitionistic truth of negated sentence turns out to be non-competitive in this regard. Properly speaking the latter notion is semantically superfluous, and cannot play any weighty role by semantical considerations in intuitionism.

For the matter of that, intuitionistic logic is to be considered a typical representative of logics which can be characterized as *strongly two-valued*,

[4] Let me stress once again that truth and falsity in intuitionistic logic are *not* symmetric. As compared with truth, falsity is here a rather weak notion presenting a simple absence of proof. With this understanding it may well happen that a sentence once false can change its semantic value to truth, as soon as one manages to find its proof. But this is as it should be, since intuitionistic logic belongs to a *verificationistic* paradigm, with verification (proof) as the principal notion. An intuitionistic theory accumulates proved sentences, and whenever we have got a constructive proof it remains strongly fixated, whereas simple intuitionistic falsehoods are only for a time.

and so to say "monistic". The two-valuedness of such logics rests upon monism of a certain truth value (called "designated") explicitly used for introducing the most essential logical notions. In such a case, one usually proceeds from the monism of truth, and what is not true is just taken to be false. It is the way of formulating the standard semantics for classical and intuitionistic logics[5].

As pointed out above, such a construction can be "dualized" in the sense that one can replace a semantics based on the monism of truth by a semantics (adequate to the logic in question) based on the monism of falsity. The procedure of such a dualization provides for further transition to a semantics which assumes a "dualism" of truth and falsity. The latter semantics is an amalgamation of two monistic semantics dual to each other, whereas truth and falsity are treated as primitive and autonomous (not inter-definable) notions. This approach allows for relevantization of a number of logics, classical and intuitionistic among them.

8. Concluding remarks

In conclusion, let me summarize the exposition of the present paper as follows:

1. The notion of falsity is no less important semantic notion than the notion of truth.
2. A genuine intuitionistic notion of falsity represents an absence of a constructive proof, which indicates a possibility for a sentence to be rejected (its potential rejectability). Intuitionistic falsification is a provisional rejection.
3. Such a notion allows one to build a full-fledged falsification (rejectability) semantics for intuitionistic propositional logic.
4. It also opens a possibility to treat intuitionistic truth and intuitionistic falsity *on a par* (as independent notions subsisting in their own rights) and to eliminate the paradoxes of intuitionistic entailment in this way.
5. Intuitionistic negation embodies the notion of intuitionistic refutation which is a particular case of (conditional) proof, and thus, represents a variation of intuitionistic truth.

[5] According to the much-debated *Suszko's Thesis* semantics for *any* logic can be equivalently reduced to a semantics which is monistic in the indicated sense. This claim has been criticized, however, and several constructions have been proposed which can be considered to be counterexamples to Suszko's Thesis, see in [31].

6. Any attempt to introduce the notion of falsity as constructive truth of negated proposition (sentence) into intuitionistic semantics, and to formulate full-fledged semantic analogues to items 3 and 4 on this basis fails, turning out to be a typical case of "multiplication of entities".

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