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STRUCTURAL FEATURES IN ERNST SCHRÖDER'S WORK. Part I

Abstract. In this paper articulated in two parts we propose a structural interpretation of Schröder's work, pointing out his insistence on the priority of a whole in comparison with its parts. The examples are taken from the diverse areas in which Schröder was active, with a particular interest in his project of an *absolute algebra*. I am regretting for the *bad* quality of my English, hoping that notwithstanding the reader can grasp at least the fundamental tracts of my reasoning.*

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1. Ouverture

Only the whole is real[1]

If one reads a Schröder's text, he or she will note a typical way of writing. Dual theorems are put side by side with a vertical stroke dividing them. If a theorem admits more than two forms, for example the *original*, its *dual*, its *conjugate* and its *dual-conjugate*, the formulas are arranged in the text to form a sort of square. In the top left corner lays the original form, in the top right its dual, in the bottom left ist conjugate and in the bottom right ist dual conjugate, with a vertical stroke

* I thank very much prof. A. Pietruszczak and prof. M. Nasieniewski to have accepted my paper in spite of my deep linguistic deficiencies. Their journal is in no way responsible for my errors. Finally I thank an anonymous referee for his valuable suggestions which brought me to reformulate my ideas.

[1] [Heg07, p. 901]. All translations are mine, if not otherwise indicated.

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separating the right formulas from the left ones (see [Bon11b], [Bon11c] and [Bon11a]). This is an example from Schröder's third volume of the *Vorlesungen*:[2]

$$\begin{array}{c|c} R \circ S \subseteq (R \bullet S) \bullet \mathsf{Di} & R \circ (S \bullet \mathsf{Id}) \subseteq R \bullet S \\ R \circ S \subseteq \mathsf{Di} \circ (R \bullet S) & (\mathsf{Id} \bullet R) \circ S \subseteq R \bullet S \end{array}$$

One could object that such arrangement is useless. Once we know how to obtain from a theorem its dual and its conjugate, it makes no sense repeating any time a new theorem is stated also its three other possible forms. Notwithstanding the visual matter becomes important, if we aim to stress the *relationships* among the elements of our theory. Schröder had not an *atomic* vision of mathematics (or logic): a theory is not built up grouping together various *single* concepts. On the contrary, at the basis we have a structure,[3] a totality in which and only in which any concept acquires its meaning. In other words, the meaning of a concept is the result of the relations obtaining amongs it and the other concepts. We have a conceptual web in which the concepts are interwoven each other.

In fact, for Schröder the *same* symbolic language [Zeichensprache] was susceptible of various interpretations. Today, we could say that Schröder distinguished among a unique syntactical language and its many possible semantics.[4] In this situation, where the elements have no meaning, only a theory in its totality can give sense to them.

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[2] [Sch66b, p. 524]. R and S are *binary relations*, $\circ$ stands for the *relative composition*, $\bullet$ for the *relative sum*, Id is the *diagonal* and Di the relation of *diversity*. It is *inessential* that the reader grasps the meaning of the examples from Schröder's papers proposed in this text. It is not important their content, but their layout.

[3] I use the word *structure* throughout this paper in a not technical way. A structure is simply an ordered whole; i.e. a whole whose elements are arranged according to some law, relation, or set of relation. I refer the reader to the latin *structura* which means, for example, an *ordered disposition of the words in a sentence*. This noun derives from the verb *struo*, i.e. *to put in order* [copias struere ⟨to put in order the troops for the battle⟩, Caesar], or also *to build up a composite word*. In any case, we have not only a whole, but an *ordered* whole. This is what I mean for *structure* and this is its commonest meaning. It makes no sense using our modern notion of [algebraic] *structure* in a context in which this notion is still fuzzy. Our model-theoretic notion of *structure* arises from Schröder's investigations, but it is not present in Schröder's work in a precise and modern fashion. For this reason, I believe that in this paper it is not possible digging in the philosophical facets of such concept. That I propose is simply an interpretation of Schröder philosophy which be consistent and which reflects Schröder's investigations. The concept of *structure* which we will use is too *vague* to permit a serious philosophical analysis.

[4] See below at page 346 the excerpt from [Sch74b]. That Schröder was really

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We could say that there are not *firstly* the single concepts and *then* a totality embracing them, but there is first a theory which step by step enlarges itself in its entirety. A concept, a theorem say, derives its meaning from the web of relations in which it is inserted and this web enlarging continuously itself, enlarges in the same time also the relational net; so the theorem with the time aquires new senses. This way, the meaning of a concept is not fixed for ever, but its identity is *liquid*, in a continuous metamorphosis resulting from its interaction with other (new) concepts.

In the next section we put forth some examples which path the way to a structural interpretation of mathematics. We will see that the context in which a concept is embedded is fundamental in order to determine its meaning. We exemplify this point of view pondering on the different meanings of the so called δ-function and of the fixed point theorems. Then we will show as this line of thought implies a step by step generalization and abstraction from the context of the scientific results. The formal statement of a scientific concept is necessary in order that it embraces diverse possible meanings according to the various situations. This effort of generalization will exemplified by Banach's work on functional analysis.

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conscious of the distinction between syntax and semantics is a matter of fact. At most, we can question if Schröder had in mind a *formal and axiomatized system* on one side, and a *model* on the other. Of course this is to be denied. Schröder had no a clear understanding of the meanings of *axiom*, *definition* or *postulate*. A theorem is confused sometimes with a law of derivation. Schröder refers to its language always as a *Zeichensprache*, a language by signs without a preassigned reference. For Schröder the language is build up by ink signs on the paper which are susceptible of many interpretations. For this reason, I don't agree with Roger D. Maddux who adfirms: *I think Schröder had no syntax, no variables, no predicates, no connectives, and no language. Specialists in PDEs talk about equations, of course, but they do not use or need any formal language. Ditto for Schröder and Löwenheim* [personal conversation]. I don't understand Maddux's position on this point, because elsewhere he writes: *The Peirce–Schröder calculus of relations may be defined as Boolean combination of equations between terms* ***denoting*** *relations* [Mad10, p. 49]. The emboldening is mine. So he is compelled to admit a difference between *terms* (syntax) and *relations* (semantics). At any rate, such sort of confusion is present also in a recent review of John T. Baldwin, where it is asserted: [...] *the source of confusion* [in Löwenheim's argument] *is that the distinction between syntax and semantics that is fundamental for the model theoretical advances* [...] *were not available to Löwenheim but arose in the context of his work* [Bal10, p. 182]. False. This distinction is crucial to understand Schröder's work. As we have seen, such distinction is present already in [Sch74b], forty years before the publication of Löwenheim' seminal paper [Löw15].

Indeed this section is crucial to grasp Schröder's *naive philosophy*, because his work culminates in a formal theory in which the usual concepts of mathematics and logic are recast in the most possible general way, in order to receive different meanings in different situations and in different branches of sciences. For this reason, we must read these example always keeping in mind their rôle in understanding Schröder's philosophy.

Schröder's philosophy arises from his work. There is no place in Schröder's literature in which we can find a precise statement of his philosophy. Furthermore, no contemporary left a record on him. For these grounds, in order to understand Schröder's thought we must make use of metaphors, comparisons and other rhetorical tools.

2. Background

2.1. Dirac's δ-function

Paul M. Dirac introduced its celebrated function as a *generalization* of Kronecker δ-operator [Dir58, p. 63]:

> We can develop the theory on closely parallel lines for the discrete and continuous case. For the discrete case we have [...]:
>
> $$\sum_{\xi'} |\xi'\rangle \delta_{\xi'\xi''} = |\xi''\rangle,$$
>
> the sum being taken over all eigenvalues. [...] Similarly, for the continuous case we have [...]:
>
> $$\int |\xi'\rangle d\xi' \delta(\xi' - \xi'') = |\xi''\rangle.$$

Incidentally, Dirac was not much clear on the deep nature of this new symbol. He wrote that:

> $\delta(x)$ is not a function of x according to the usual mathematical definition of a function [...] we may call [it] an *improper function* [...].
> [Dir58, p. 58]

Indeed, it is not a function but a *functional*:

> Of course, also in Analysis there are linear Functionals; for example, the integral is a linear Functional on the space of $[a, b]$ Riemann-integrable functions:
>
> $$\int_a^b : R([a, b]) \to \mathbb{R}, \qquad f \mapsto \int_a^b f.$$

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> The *Distributions* (for example, the so called δ-function, which we mentioned previously, is **not** a function) are (continuous) functionals on $C_0^\infty(\mathbb{R})$ [...].[5] [Rai09a, p. 426]

Compare the preceding quotation from [Dir58] with the following one in which is introduced the *same* concept but from another point of view:

> We denote by $L^p(\Omega)$ $(1 \leq p \leq \infty)$ the space $L^p(X, \mu)$, where (X, μ) is the Lebesgue measure space corresponding to an open set Ω in R^n. Let $u \in L^p(\Omega)$, and define
>
> $$(J_\epsilon u)(x) = \epsilon^{-n} \int_\Omega \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy \qquad (\epsilon > 0).$$
>
> We call $J_\epsilon u$ a *mollifier* of u [...].[6]
> 1. $J_\epsilon u$ is in $C^\infty(R^n)$.
> 2. If u vanishes outside a subset A of Ω, then $J_\epsilon u$ vanishes outside an ϵ-neighbourhood of A [that is, outside the set $\{x; \rho(x, A) < \epsilon\}$].[7]
> 3. If K is a closed subset of Ω with $\rho(K, R^n - \Omega) \geq \delta > 0$, then
>
> $$J_\epsilon u = \epsilon^{-n} \int_{|y-x|<\epsilon} \rho\left(\frac{x-y}{\epsilon}\right) u(y)dy = \int_{|z|<1} \rho(z) u(x - \epsilon z) dz$$
>
> for any $x \in K$, provided $\epsilon < \delta$. [Fri82, p. 103].

Where is the difference between Friedman's approach and Dirac's one? While Dirac was interested in generalizing Kronecker δ-operator to the case of continuum, Friedman was interested in the possibility to *smooth* a function. That is, Friedman focused on the concept of *derivation*. As a matter of fact, the function u under the action of the

[5] The emboldening is mine. I observe (with fun, of course) that von Neumann's famous statement is so correct: *The region of* $\delta(q)$ [i.e. between the graph of $\delta(q)$ and the abscissa] *is indeed infinitely extended and infinitely high having in* $q = 0$ *a top of area* 1. *This is almost the limit value of the function* $\sqrt{\frac{a}{\pi}}e^{-aq^2}$ *for* $a \longrightarrow +\infty$. *In spite of this, it* [i.e. the function $\delta(q)$] *is impossible* [vN96, p. 240, footnote 32]. In fact it is not a function, but a functional (a function of functions). In any case, von Neumann described rightly the behaviour of this *strange* mathematical object. He noted also that *it* [i.e. $\delta(q)$] *is beyond the realm of the general and usual mathematical methods, and we will describe the Quantum Theory with help of only these methods* [vN96, p. 15]. I give the pardon of the reader for not having used the English translation of [vN96] by the Princeton University Press. Unfortunately, being only an independent scholar without any sort of support, I am unable to buy all books which I need.

[6] Please note that u is a function.

[7] The square brackets are in the original text. I refer to the previous footnote.

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mollifier becomes infinitely differentiable and continue.[8] So the same mathematical object can be introduced with different accents. In the first case, the context is provided by the continuity while in the second case, the context is provided by the derivability.

Of course we can envisage other environments in which introducing (and *carving*) the mollifier. We could be also interested in the relation between the meta-linguistic mollifier and the function to which it is applied. This relation is a relation of *convolution*. In a paragraph entitled *Convolution and Physical Linear Systems* Bramanti *et al.* state:

> The answer of a linear system to an input whatever $e(t)$ is a *convolution* with the answer to a *Dirac's impulse*. [BPS08, p. 312].

In this case the accent is on the relation of convolution between two functions and on the physical interpretation of the δ-function. It is not without reason that Bramanti et al. call the mollifier, *Dirac impulse*.

Continuity, derivability and convolution are three possible models which give a precise meaning to the formal concept of mollifier. In these example, there is no mollifier *in itself and for itself*, but a mollifier embodied in a precise mathematical context. This means that the mollifier has meaning only in relations to other mathematical concepts provided by the context. In itself the mollifier has no meaning at all. It is a simple string of signs which we can interpret differently according to our purposes. As one can easily understand, we can grasp formally the mollifier only by a process of abstraction, i.e. leaving aside its various possible interpretations.

2.2. Fixed Point Theorems

Another example could be the search for the fixed points of a function. This bring us to a theorem which states the conditions which a function must fulfil in order to have a fixed point. This is the well known Picard-Banach theorem:

> Let **A** be a contraction mapping of a complete metric space M into itself. Then **A** has a unique fixed point. [Shi96, p. 161]

This theorem is inserted by the Russian mathematician Shilov in a chapter devoted to the solution of differential equations. In this sense, the Picard-Banach is only a *possible tool* among others, useful to find a

[8] $J_\epsilon(u) \in C^\infty(R^n)$.

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unique solution for a system of differential equations. We could assert that a similar theorem with the same result would have the same effect in this context. The accent is on the solution of a system of differential equation. The main goal is to solve this system; it is a secondary matter if we attain this solution by a fixed point theorem or by another result. Using Gianni Gilardi's[9] words:

> From my point of view [...] the results attained with fixed points theorems [...] are merely tools to investigate the PDEs.[10]

This opinion underpins that of Shilov, but, notwithstanding, there is an important difference. The Picard-Banach Theorem is called by Gilardi *Theorem on the Contractions* and is formulated in this way:

> Be C a closed set belonging to $\mathbb{R}^n$ and $f : C \to C$ a mapping satisfying the following condition: it exists an $\alpha \in [0, 1)$ such that
>
> $$|f(x) - f(y)| \leq \alpha|x - y| \quad \text{for every} \quad x, y \in C.$$
>
> Then, there is a unique $x \in C$ such that $f(x) = x$. [Gil11, p. 67]

The theorem is inserted in a part of the book devoted to numerical series. So, the focus is on the concept of *Lipschtzean* with on the background the convergence of particular sequences. No word is made on the solution of differential equations, also if this theorem is introduced for this purpose. Furthermore, Shilov states the Picard-Banach Theorem in the context of Banach (metric) spaces. If for Shilov the context for a fixed point theorem is given by the Banach spaces, for Gilardi the context is that of numerical series.

Let us go on. Take Rainer Wüst's work on mathematical physics [Rai09a]. At the beginning of the first volume he introduces the Lipschitzean in a similar situation to that in [Gil11]:

> Be f a mapping, $a \in \mathcal{D}(f), \gamma$ a positive number. If there is a $c > 0$ such that
>
> $$|f(x) - f(a)| \leq c|x - a| \qquad (x \in \mathcal{D}(f) \cup (a - \gamma, a + \gamma)),$$
>
> then f is continuous at the point a. (The mapping f is at the point a *Lipschitz-continuous*). [Rai09a, p. 80]

[9] He is author of a nice book on the calculus [Gil11], most diffused in Italian universities.

[10] Personal conversation.

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Wüst has the necessary tools to prove a fixed point theorem, as Gilardi, but he doesn't it. Cleary, the interest in this precise point of the book for Wüst doesn't lay neither in the solution of differential equations, nor in the sequences, but in the diverse concepts of continuity. So in this case to be crucial is the concept of *Lipschtzean*. Let go further. If we open the second volume [Rai09b] at page 1028, that is near the end, we encounter this theorem:

> Be $\langle X, \|\cdot\|\rangle$ a Banach space and $T\colon X \to X$ a mapping. If T is a contraction, then T a unique fixed point. [Rai09b, p. 1028]

Why does Rainer Wüst state so late a fixed point theorem? Because he will formulate it in terms of metric spaces and as a tool to solve the Cauchy Problem. Now, the interest is shifted from the concept of Lipschtzean to the solution of Cauchy Problems in a physical context. Note also that Wüst occupies himself with the solution of differential equations yet in the first volume and at the start of the second. But he waits to formulate a fixed point theorem until he could define a Banach space and a Cauchy Problem.

Let us change scenery. In a very introductory book, the Russian mathematician N. Ya. Vilenkin formulates geometrically and informally a fixed point theorem in a work devoted to the *approximation* of a solution. He introduces the method of iterating a function and then, exploiting this tool he arrives to a fixed point. The title of the chapter where there is stated a fixed point theorem is *The Geometric Interpretation of the Method of Iteration* (see [Vil64, p. 51 and ff.]). In this case, a fixed point theorem is regarded under the study of functional iteration:

> One of the most powerful methods of approximate solution of such equations [i.e. differential equations] is [...] the method of successive approximation (iteration). [Vil64, p. 68]

The same argument applies for Shashkin [Sha91, p. 40]:

> There is a number α, such that $0 < \alpha < 1$ and for any points x_1 and x_2 of the closed interval $[a, b]$ one has
>
> $$|f(x_1) - f(x_2)| \leq \alpha |x_1 - x_2|.$$ [11]

There is however a big difference with Vilenkin. For Vilenkin the context is given by the method of approximation and by the functional iteration;

[11] f is a contraction.

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for Shashkin the main context is given by Brouwer's Fixed Point Theorem and this result is only secondary. For Shashkin the accent is not on the solution of differential equations but on the review of many results on fixed points.

The next example introduces a very different background:

> Let $\mathcal{D}$ be a topological space and assume that F is a self-mapping of $\mathcal{D}$, i.e., $F(\mathcal{D}) \subseteq \mathcal{D}$. [...] A point $z \in \mathcal{D}$ is called a *fixed point* of the mapping F if $F(z) = z$. [RS05, p. 107]

In this definition Reich and Shoikhet defines a fixed point in the context of a topological space and in their book they formulates many fixed points results in the milieu of the unitary disk in $\mathbb{C}$. In particular, the main fixed point theorem of the book is the following:

> Let Δ be the open unit disk in the complex plane $\mathbb{C}$. If $F \in \mathrm{Hol}(\Delta)$[12] is not the identity and is not an automorphism of Δ with exactly one fixed point in Δ, then there is a unique point a in the closed unit disk $\bar{\Delta}$ such that the iterates $\{F^n\}_{n=1}^{\infty}$ of F converge to a, uniformly on compact subsets of Δ.[13] [RS05, p. 119]

In this case we have again a fixed point theorem in the context of functional iterations as in [Vil64] and [Sha91], but the vocabulary and the mathematical environment is totally different. Reich and Shoikhet investigates fixed points theorems using different metric spaces (not only Banach spaces), different norms, different geometries (also non-euclidean) and topological tools. This time the interest is not more on the utility of a fixed point theorem to solve a problem, but on the fixed point theorems themselves. It is a sort of theme and variations on the concept of *fixed point theorem.*

We cannot not quote Schröder's Fixed Point Theorem, now fundamental for Complex Dynamics [Sch70b, p. 322]:

> If $f(z)$ is a injective mapping in a neighbourhood of a point z, which satisfies the conditions (13) [i.e. $f(z) = z$; $f(z)$ has a fixed point z] and (15) [i.e. $|D_1 f(z)| < 1$], then one can always choose an arbitrary point z_1 in the neighbourhood of z such that the equation (11) [i.e. $\lim_{n\to\infty} f^n(z_1) = z$] is true; in other words, for any point z_1 belong-

[12] $F \in \mathrm{Hol}(\Delta)$ means that F is holomorphic on Δ.

[13] This is the *Denjoy–Wolff Fixed Point Theorem.*

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> ing to the neighbourhood of z, the n-iteration of the function $f(z_1)$ converges to the root z of the equation $f(z) = z$ as $n \longrightarrow \infty$.[14]

In other words, if a function f has a fixed point z and it is such that the module of its first derivative is < 1,[15] then we can choice a z_1 in the neighbourhood of z such that $\lim_{n\to\infty} f^n(z_1) = z$.[16]

Schröder is working in a functional analytical setting (although he had not a precise definition of a functional space) using iterations of mappings with value in Δ. The scope of [Sch70b] (and of [Sch70a]) is to find an algorithm to solve any algebraic and transcendental equation:

> The investigations [described in this text] refer not only to algebraic equations, but also to transcendental ones with *one* variable.[17]
>
> [Sch70b, p. 317]

Schröder Fixed Point Theorem acquires his meaning in the algebraic perspective of the solution problem. As we will show, Schröder's main goal in [Sch73], in [Sch66a] and in [Sch66b] is to solve any equation of the calculus in issue. Kolmogorov rightly speaks of Schröder's contribute to logic stressing the centrality of the solution problem:

> Schröder, like other mathematicians in this discipline [i.e. the logic], regarded the solution of logical equations to be one of the central problem of logical algebra. [KY01, p. 29]

The solution problem was a *Leitmotiv* in Schröder's investigations. As regard the matter of this section, the solution problem is the context in which Schröder formulated his Fixed Point Theorem.

Summing up, the same concept of *fixed point theorem* acquires a diverse meaning according to the context: Shilov, Gilardi, Wüst, Vilenkin, Shashkin, Reich and Shoikhet, and Schröder formulate a fixed point theorem but with different nuances. This shows that a fixed point the-

[14] This translation differs heavily from Alexander's one [Ale94, p. 6].

[15] The first derivative of f is strictly contained in the *open* disk Δ.

[16] For lack of our modern topological tools, Schröder could not express adequately the difference between an *open* set, a *closed* set and the *closure* of a set. Probably, Schröder thought that a distinction between open and closed intervals could be translated in some manner to apply also to the unit disk. He couldn't state, for example, that $D_1 f(z) \in \partial\Delta$. In his theorem he limits himself to state that the first derivative of $f(z)$ cannot belong to the boundary, with *boundary* meant in a vague sense. For Schröder the *boundary* could be only a sort of *perimeter*.

[17] This paper is the first part of a work on functional iterations; the second part is [Sch70a].

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orem has a meaning not in itself but only in a web of relations with other entities. This theorem can be put in relation with the continuity or with the differential calculus or with functional spaces or with other similar theorems. In any case it is the structure, the totality in which the theorem in issue is formulated, to found[18] ontologically the theorem and to give a meaning to it.

These examples show that a single mathematical concept can be interpreted differently according to the structure in which it is inserted. Now, the landscape can be seen also by another point of view: we have different structures and we are in search of the things common to all them. It is what Schröder did searching a theory which could unify various mathematical and logical branches. By a process of generalization he found a formal milieu in which putting the various objects of mathematics and logic. This milieu was the *Universal Algebra*, called by Schröder *Absolute Algebra* or sometimes *Theory of Connections*. Such theory is *formal* because, without any context (a semantic), its elements are spoiled of any meaning. It is a sort of skeleton made up by the relations obtaining among its elements.

We have two procedures pointing to the same result: the first one is a process of particularization of formal concepts which we have seen at work in the previous examples; the second is a process of generalization, by which the mathematical or logical concepts are spoiled of their meaning. In the first case, we end with a contextualization of the scientific concepts in which they are in relation with other concepts. In the second case, we terminate with a theory of purely formal elements without meaning. In this theory to survive are only the *relations*, the *connections* [Verknüpfungen] between the various elements. It is what we do when we introduce, say, the *distributivity law* focusing on the operations in issue and disregarding the nature of the involved objects.

I stress the fact that both in the first and in second case we contemplate a structure, a web of relations. In any of these cases the relational web is given from the context. The next section exemplifies the ascent to universal with Banach's work on functional spaces.

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[18] For the concept of *foundation* I refer to Husserl's *Third Logical Investigation*: *If a law of essence means that an A cannot as such exist except in a more comprehensive unity which connects it with an M, we say that an A* as such requires foundation by an M *or also that* an A as such needs to be supplemented by an M [Hus70, p. 25]. I refer a German reader to [Hus09, p. 267]. At any rate I will use the word *foundation* with a certain degree of freedom.

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2.3. Stefan Banach

This process of progressive abstraction, from particular to universal (and formal) could be exemplified by Banach's seminal work in functional analysis. Banach, introducing in his PhD-thesis the concept of *Space B*, today called *Banach Space*, generalized the notion of functional space of which the Hilbert space with operators was only a particular case. It was a great step ahead, indeed:

> Let us remember that during the final decades of the 19th century and the beginnning of the 20th century, in Mathematics there appeared sets that had as their elements sequences, series, functions and similar objects [...]. Such sets, having distinguishable structures, had interesting properties and were named *function spaces*. They were studied by Vito Volterra, Hilbert, Frigyes, Riesz and others. But they looked at these *spaces* **one by one**.[19] What was missing was a general definition that could accommodate all those *function spaces* as a single notion, in order to investigate just one single *space* instead of what had hitherto been many. And that was the task that Banach took up, introducing in his doctoral thesis the notion of a *type B* space, which encompassed all the known function spaces.[20] [Dud10b, p. 42]

This path towards the generalization is witnessed by Banach's own words:

> Aim of this work is to formulate some theorems valid in different functional spaces which I am to define [...] I consider in a general way the sets of theorems and then I prove that the adopted postulates are true for any particular functional space. [Ban20, p. 134]

Duda, in a first German version of his previous cited paper, quotes Köthe:

> The functional analysis replaces the concept of number which is fundamental for the calculus with a more general concept, which one today identifies in thousand contributes with the concept of *point in a Banach Space*. The generalization of mathematical analysis attained in this way, which is called functional analysis, allows to deal with **at first sight independent and diverse** problems of mathematical analysis in a simple and unitary way, and to solve a lot of problems which harassed before the mathematicians in vain.[21] [Dud10a, p. 10]

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[19] The emboldening is mine.

[20] This a new version of a paper by Duda published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* [Dud10a].

[21] The emboldening is mine.

It was just what Schröder did, comparing the similar rôle of mathematical and logical objects in different contexts. His universal algebra allows to embed various mathematical and logical concepts in a general theory in which they could be tackled in an uniform manner. If for Banach the matter was to unify many functional spaces, for Schröder was to isolate the common tracts of various mathematical and logical disciplines and to recast them in a theory of pure relations (the Universal Algebra). This way Schröder could apply the same procedure in diverse contexts. I think to his combinatorial approach and to the solution problem, which was investigated in algebra, in functional analysis and in logic.

Schröder's Theory of Relations. Albeit Schröder is mentioned often (and only) for his work in logic, logic was for him only a possible semantics for a formal and general theory. Obviously, as said above, in such formal theory its elements have not a meaning; there is only a web of relations which gives a structure to this totality of objects. Is this web of relation to give a meaning to the objects putting them in a structured totality. At this point the focus is on the relations among these formal elements and not more on their meaning, because there is no particular context to support the reference of them.

Logic becomes important at this stage because it permits to investigate the concept of *relation*. We can agree that to survive in this formal theory are only the relations, but what is precisely a relation? The third volume of Schröder's *Vorlesungen* is devoted just to this topic. It is not only the conclusion of a more general work on logic but the apex of Schröder's logical investigations. It is functional to this idea that there is a more general and structural theory than mathematics and logic. This volume would be written in vain if not contextualized in the search for a general theory.

Let us look more closely to the structure of [Sch66b]. The volume is a set of twelve lectures on the calculus of binary relatives, which we can group in five parts:

First part: { Lectures I–IV; introduction of the concept of *relation*, of the operations between relations and of the quantification

Second part: { Lectures V–VIII, XI; *solution problem*

Third part: Lecture IX: the *Theory of Chains* in the calculus of relatives and the concept of *number*

Fourth part: Lecture X: the concept of *individuum*

Fifth part: Lecture XII: the concepts of *function* and *set*

Note that after the introduction of the main objects of the calculus of relatives (first part), Schröder devotes a large space to the solution problem, then he analyses the concept of *number*, *function* and *set*. Because Dedekind's theory of chains allows the definition of a set of number, Schröder introduce it to formulate the concept of *number* relying only on a lattice[22] of relations. The parts three, four and five embed some mathematical objects in a larger structure (of relations). In this volume we can also see that the logical concept of *relation* is an interpretation of an abstract concept of *connection*. From this point of view, the context of this book is the project of finding a theory of connections, i.e. to build an universal algebra.

The dream of a structural theory arises also from Schröder's apparently pure logical investigations. This and the fact that the greater part of [Sch66b] focuses on mathematical entities testifies Schröder's mathematical, and precisely, algebraic attitude. For this reason logic doesn't deserve a particular place in Schröder's work, being only functional to dig in the concept of *relation*, essential for a theory which pretends to be abstract.

Furthermore, once formulated such structural theory, we need only to formulate in its language at least the main concepts of logic and mathematics. This explain the end of [Sch66b] and the successive papers. In fact, Schröder's last papers concern set-theory and well-ordening (see below Subsection 4.7 (second part).[23]

[22] I use this term in a non technical fashion.

[23] I stress also that Dedekind's work was useful in the context of [Sch66b] not only for defining the concept of number, but also as a source of inspiration for Schröder's investigations on the concept of *set*, and *function*. Finally, Dedekind's book on the theory of chain [Ded96b] is present in the background in the *Schröder-Cantor-Bernstein Theorem* with his definition of *finite set* [Ded96b, p. 806 and ff.].

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2.4. From Hegel to Leśniewski

At any rate, Schröder's structural approach in mathematics and logic has also a great *philosophical* tradition.[24] As a matter of fact Schröder belongs to the German–Austrian (and Polish) tradition of *Mereology*, according to which the totality is more important than its elements. We can think to Hegel, to Marx, etc. arriving to Husserl's phenomenology and to *Gestaltpsychologie*, not neglecting topology and Leśniewski's own Mereology. In all these cases, the whole has a sort of precedence over its pieces. For example, Leśniewski in his paper of 1916, *Foundations of the General Theory of Sets* [Leś92], in order to avoid *Russel Paradox* envisaged a theory alternative to *ZF* in which it is absent the set-theoretical notion of *belonging* [$\in$]. On its place Leśniewski introduces the concept of *ingredient*:

> I use the expression *ingredient of object* P to denote the same object P and every part of that object. [Leś92, p. 132]

In this situation we cannot speak of *elements* but of *parts*. An object is not element of another greater object, but it is part of this object with the possibility to coincide with it. A *part* is not something precise in a mathematical sense, but it is qualitative in its essence. It has not a *definite* boundary. In any case, we note the at the basis of Leśnieski's mereology there is the concept of *whole* (object). An individual can be thought of only as a limit entity. An individual is the result of the infinite splitting up of a whole, an *infinitesimal* piece. Being the individual the conclusion of an infinite process, the individual is only *possible* but *de facto* unattainable. In nature we encounter never an individuum, but something greater, a *neighbourhood* say, an infinitesimal *piece* of space–time.

Henri Lebesgue. A way to illustrate this splitting up of a whole is the definition of the *Lebesgue set of measure* 0. Such set is not empty, but more empty than other possible set. In fact, in the definition we have a *tacit* quantification over ϵ and ϵ is strictly greater than 0 [Rai09a, p. 340]:

> Let $A \subset \mathbb{R}$.
> A is a *Lebesgue set of measure zero*
> :$\Leftrightarrow$ for [any] $\epsilon > 0$ there is a sequence $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ of open intervals

[24] I speak of *philosophy* with care in this context, inasmuch Schröder had not a sufficient philosophical background. His knowledge of the history of logic in 1877 is limited to 60 pages about Leibiniz [Tre67, 1–63].

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such that
$$A \subset \bigcup_{k\in\mathbb{N}} (a_k, b_k) \quad \text{and} \quad \sum_{k\in\mathbb{N}} (b_k - a_k) < \epsilon.$$
From this definition it is manifest that the rôle of Leśniewski's concept of *object* is here exemplified by that of an *open interval*.

Again on Leśniewski. Going back to Leśniewski's mereology, a possible rebuke towards such type of wholes is justified, if we desire that a totality be something *quantitative*. Nevertheless, topology taught us that we must not fear *qualitative* objects. That a totality can be qualitative in its essence and notwithstanding be harmless we will prove with an example from [vN96]. I insist on the fact that a totality can be qualitative and be well structured at the same time. I think to the contexts exhibited above. They have not definite boundaries. A theory is always a work in progress, because the mathematicians or the logicians introduce new concepts, dilating its boundary.

I note also that, albeit the truth of a sentence is not in discussion, a theorem preserving its truth value for ever, the context in which is formulated changes its meaning.

2.5. John von Neumann

In [WZ83], John von Neumann wrote:

> [...] we must always divide the world into two parts, the one being the observed system, the other the observer. [...] The boundary between the two is arbitrary to a very large extent.[25] [WZ83, p. 622]

von Neumann is speaking about the measurement of a quantum-particle. What is a measurement? For him it is a relation between an *observer* on one side and an *observable* on the other side. That is, on one side we have a *subject* and on the other an *object*. von Neumann gives for granted the independent existence of a subject and of an object; then he puts them in a relation (the experiment). The difficulty lies in the fact that we cannot divide sharply the subject from the object. So we are faced with two opposite possibilities: 1) we push the boundary subject/object in the direction of the object; 2) we push the boundary subject/object in the direction of the subject. In the first case, we have a sort of

[25] I discussed this sentence in my unpublished paper [Bon10a], where it is argumented Bell's position and it is found a feasible way out.

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brutal *idealism* in which there is no room for the object and the subject occupies the totality of the real. So the context of experiment (of a possible experience) would be the entire universe. This is a position extremely subjective. The subject is everything. On the contrary, in the second case there is no room for the subject and the situation is more intricate.

We can circumvent the obstacle, asserting that the experiment is an indivisible totality in which anything is *entangled*. The process of individuation of one element is a process of abstraction which results in a figure by fuzzy borders, this fuzzyness being harmless for many purposes.[26] In the previous examples, we have seen the importance of the context in defining the meanings of some mathematical objects. One more I insist that the truth or falsity of a sentence is not in issue; from an algorithmic point of view, there is no danger in assuming a *liquid* setting. For this reason we can say that the qualitative essence of a structure encompassing scientific concepts is harmless for a merely computation purpose.

The liquidity of the context becomes important, on the contrary, in defining the *meanings* of the involved entities being something more than a truth value. Therefore, the contextual vagueness is not in contradiction with the requested rigour of the definitions, theorems, and so on. They are two sides of the same matter. The algorithm works independently from the semantic nuances associated to the objects that it puts in connection.

I stress also the fact that, following this line of thought, the various situations which structure the entities are highly objective. Is there something more objective than a *phenomenon*? The phenomenon is not objective in a crude positivistic sense, not satisfying a quantitative way of thought, but it is the only thing that a man can share. In any case, it is more understandable the concept of *whole*, of an *object inserted in a phenomenon*, than a limit entity as an *individuum* or a *point*. This doesn't mean that we advocate for a *nominalistic* philosophy, notwithstanding many possible overlappings. We maintain a structural approach to reality in which any element has meaning only in connection with all the other elements. Our approach is diverse from nominalism inasmuch we insist on the structural character of the wholes, which in their growing

[26] Bell said that it was harmless *For All Practical Purposes*.

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modify the meanings of the parts. This confers to the parts a context-dependent meaning.

Going back to the problem of measurement, we must observe that our possible way out is not new, having its roots in Husserl's phenomenology.

Transition. Keeping in mind what said in the previous section, we can pass to the main object of our research: to verify a structural philosophy in Ernst Schröder's work. The previous arguments must be kept in the background in the following parts. Albeit not always manifest, they remain in the shadow, to support Schröder's words.

3. Ernst Schröder

3.1. Formalism

The idea that mathematics is structural in its essence is so not disturbing. It doesn't affect the pragmatic job of the mathematician.[27] On the contrary, it casts light on many mathematical and logical concepts, providing a context and a relational web in which embedding them. Furthermore, it permits to find connections between elements at first sight not related. There was not a Schröder mathematician, a Schröder logician, a Schröder set-theoric, a Schröder analyst, etc., but only one Schröder declined in diverse ways according to the topic in issue. There was not in particular a different approach to mathematics and to logic. Both were approached in a combinatorial and abstract way and both had their main goal in the *Solution Problem*.

That Schröder gave such importance to the solution problem [Auflösungsproblem] is not surprising, because in a theory structured as a formal calculus it is the only possible algorithm to make deductions possible in form of solution of a system of equations.[28] As we saw above, Schröder formulated his fixed point theorem to solve a system of algebraic or transcendental equations. The signifiance of the solution problem in logic is the same. In fact, in the most intricate lecture of [Sch66b] we read [Sch66b, p. 163]:

[27] In this sense we can state with Bell that such structural philosophy, albeit qualitative, is FAPP.

[28] Obviously, there is a big difference between *deduction* and *solution*, the first requiring a semantic. We cannot deny that Schröder's was limited by this confusion. I refer the reader to [Bon10b, pp. 112–116 and p. 124].

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> [...] one can always formulate the general solution σ [of the equation $F(x) = 0$] both from a theoretical and a practical point of view in such a form that it fulfils the following *(first) additional condition*:
>
> $$\{F(x) = 0\} \leftrightarrow \{\sigma(x) = x\} \quad [\dots].$$

In other words, if an equation $[F(x)]$ is solvable, then its general solution can be expressed as a function with a fixed point. Of course, if the solution is a constant, say a, then we consider it as a 0-ary function, i.e. a^0. In this case the function coincides with its fixed point.

I stress that this quotation is from a logical book. Notwithstanding, we find concepts belonging to functional analysis. If we pay attention to the previous equation, we note that there is no hint referring to a logical world. A reader who didn't read the *Vorlesungen* could tag this equation with a mathematical label. As a matter of fact, we can rephrase some logical sentence in mathematical terms and *vice-versa*. For example [Sch90a, p. 602]:

> [...] I became to be interested on the geometrical and combinatorial problems [...] which can be built up with [only] three concepts or classes A, B, C with the help of the particles *and*, *or* and *not* [...].[29]

Compare it with the following excerpt from [Sch73]:

> If three number $[a, b, c]$ are linked progressively by two operations of first grade [i.e. sum and subtraction], then these operations are both additions $[(a+b)+c]$ or they are the result of the connection [Verknüpfung] of an addition and a subtraction in a order whatever $[(a+b)-c = a-(b+c)]$, or they are the result of the connection of two subtractions $[(a-b)-c]$. [Sch73, p. 187].

Replacing in the second quotation *number* with *concept* or *class*, the operations $+$ and $-$ with the connectives *and* and *not* we obtain the first quotation.[30] In both cases Schröder is putting the same question: *given three elements* a, b, c *(whatever they are), how ways are there to connect them with a binary operation* $\sqcup$ *and* $\neg$*?*[31] This is an unique combinatorial problem tackled in the first case in a logical context, in the second in an algebraic one. It is not important, and in any case

---

[29] In this quotation *concept* is the intensional side of a class.

[30] I remember that *or* can be defined with *and* and *not*.

[31] $\sqcup$ is the lattice *join* and $\neg$ the lattice *ortho-complementation*.

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not in issue, the nature of the objects which are put in relation as the following quotation shows:

> Be assumed that it is given an unlimited manifold of objects (of a sort whatever) which are clearly different each from other – by a distinguishing mark or a boundary [Grenze]. Generic elements of this manifold are indicated with the letters $a, b, c \ldots$.[32] [Sch74b, p. 1]

The unique request on the universe of discourse is that its elements be in some way different; there is no other condition to satisfy. These objects have no meaning in itself, but in relation with each other:

> I put forward as examples of such objects belonging to a manifold [...]: proper nouns, concepts,[33] judgements, algorithms, numbers, symbols of quantities or of operations, points and sets of points, or some geometrical constructions [Gebilde], quantities of substances, etc. [Sch74b, p. 1]

Because the elements of a manifold are without meaning, we can contextualize them differently. In other words, this abstract manifold admits a variety of different semantics.[34] To underpin this possibility Schröder states:

> As **particular** examples of such operations which are subject to the laws of O_1,[35] one knows until now:
> 1. the *logical addition* of *concepts* (or of individuals)[36]
> 2. [the logical addition] of *judgements*[37] (or of algorithms)
> 3. the *numerical addition* of *generic complex numbers*[38] and analogous
> 4. the *geometrical addition* with the *points belonging to the number plane*[39]

32 Schröder is suggesting that the elements of the manifold are disjoint.

33 In logical sense.

34 Today we would speak of *models*. Obviously, Schröder could not have this concept in mind, but nevertheless he foresaw albeit vaguely a structured semantic.

35 O_1 is the formal system of ordinary algebra. See Figure 1.

36 I.e. *predicative* logic.

37 I.e. *propositional* logic.

38 I.e. $(x + iy) + (u + iv) = (x + u) + i(y + v)$ [Bea05, p. 32]. I adduced this definition, because it is possible to express complex numbers with the help of *polar coordinates* [BN10, p. 8] or using *De Moivre Formula* [D'A10, pp. 42–43]. Nevertheless the word *numerical* suggests an *algebraic* formulation of complex numbers.

39 I refer the reader to my [Bon10b, p. 151] and to [Sch77, p. 484] where Schröder introduced a possible geometrical interpretation of the *logical* product: *As you can*

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5. the *addition* of von Staudt's *dices*

$$O_1 = \begin{cases} \text{I.} \quad a(bc) = b(ca) = c(ab) = (ca)b = (ab)c = (bc)a \\ \text{II.} \quad a\frac{b}{c} = \frac{ab}{c} = \frac{a}{(\frac{c}{b})} = \frac{a}{c:b} = \frac{a}{c}b = \\ \quad a(b:c) = b : \frac{c}{a} = b : (c:a) = (ab) : c = (a:c)b \\ \text{III.} \quad \frac{a}{bc} = \frac{(\frac{a}{c})}{b} = \frac{a:c}{b} = \frac{a}{b} : c = (a:b) : c = a : (bc) \end{cases}$$

Figure 1. The three laws of O_1 [Sch74b, p. 24].

> furthermore [...] the *multiplication* on the same domains [Gebieten], which stays often with the corresponding addition in the relation expressed by the distributivity law. [Sch74b, p. 25]

In other words, these examples are true also for the operation of product which often distributes over the sum. It deservers our attention, before going on, the last example, because Schröder considered von Staudt's theory from an abstract point of view.

3.2. von Staudt

At the beginning of [Sch76], we read:

> From two years I am investigating the formal connections [Beziehungen][40] of the simplest type, which can obtain between an operation and its inverses, in their mutual logical independence, and I wrote yet for a particular occasion a preliminary communication[41] on the results of these researches. [Sch76, p. 289]

The formal and combinatoric investigations which were until now exemplified in an algebraic context in [Sch73] or in a geometrical one, are now exemplified in a more formal setting:

see, for example in a text by Otto Boeddicker, one may express the measure $A(\cdot)B$ of the area [Gebiet] *which is common to two uni- or multi-dimensional surfaces A and B by multiple integrals extended partly over the last areas themselves, partly over their boundaries* [...]. Schröder is saying that the *measure* of the overlapping of two areas A, B equals the product of the surface integrals of A and B. The implicit reference must be [Boe76, p. 22]. In any case, we can consider a point as a *vector* and a sum of points a *vectorial* sum (according to the usual *Parallelogram Law* [Bea05, p. 53]).

[40] I translate *Beziehung* with *connection* to avoid confusion. The expression *relation* would be misleading in this context.

[41] [Sch74b]. The footnote is by Schröder himslef. The *particular occasion* refers to the fact that this book was written as a contribute for the school program (years 1873/1874) of the Gymnasium in Baden-Baden.

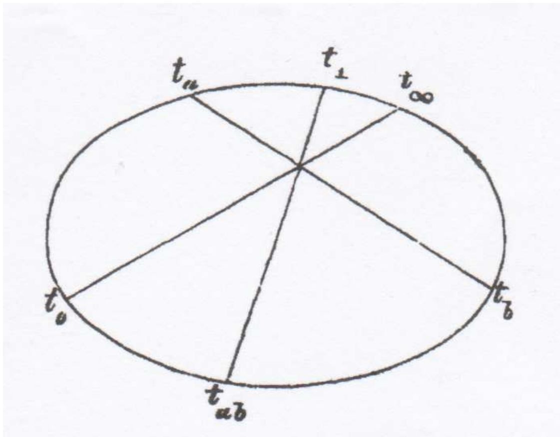

Figure 2. I am regretting for the bad quality of the scannerization. Proceeding clockwise, we have t_1, t_∞, t_b, t_{ab}, t_0, t_a.

> The multiplication and the addition of dices with their inverse operations, which were considered [until now] only as geometrical operations from the point of view of the associativity, commutativity and the distributivity, [...] provide analogous [...] an analytical example of one operational sphere [Operationskreis] corresponding to the four species [of operations]. [Sch76, p. 290]

Being a *von Staudt Dice-point* defined by four *fundamental points* t_0, t_1, t_∞, t, we state the product of two Dice-points in the following way:

> I will indicate the number corresponding to the von Staudt-*product* of two arbitrary Dice-points t_a and t_b with t_{ab} or with the **symbolic** product $t_c = t_a(\times)t_b$.[42] [Sch76, p. 291]

Furthermore:

> von Staudt' *sum* of both the Dice-points t_a and t_b will be expressed, accordingly, with t_{a+b} or $t_a(+)t_b$. We can treat the *symbolic addition* [...] *as a special case of the previous mentioned multiplication* [...]. Therefore, it is sufficient to solve the problems only for the multiplication which we can express by t_c, t_a and t_b. [Sch76, p. 291]

Without entering in the details of [Sch76], it is clear from these excerpts that Schröder is adopting a formalistic point of view. In any case,

[42] The emboldening is mine. See Figure 2.

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these quotations explain the meaning of the sum of Dice-points to which Schröder refers in [Sch74b] as a possible semantic for an abstract theory. It is interesting also the relation between an operation and its possible inverses, because it express the search for a formal symmetry between the elements of a theory.

[Sch74b] is so the pivot and the more clear statement of Schröder's philosophy. All his work finds his reason d'être in this booklet: the absolute algebra is a formal theory which generalizes the concepts and operations of many fields. The question, now is: what does remain in an abstract structure of the objects under consideration? Nothing but the *relationships* among them.[43] In fact Schröder himself in his last paper defined the *absolute algebra* a theory of the *Verknüpfung*:

> [The investigations in arithmetic and algebra] [. . .] path the way to the *absolute algebra*, i.e. to a general theory [. . .] of connection [Verknüpfung]. Of such works, which represent Schröder **very own** [ureigenest][44] field of research, it were only a little part published.[45]
>
> [Eck01, without number of page; the second page of Schröder contribute]

To speak is Schröder in 1901, that is thirty year after the publication of [Sch73] or of [Sch74b], to prove as the project of a formal theory of connections was the main goal of Schröder's work.

---

[43] The relations are what survives in a theory, once we have spoiled its elements of their possible meaning.

[44] I translated *ureigenest* with *very own*. This word is a combination of the prefix *ur-* (reinforcing the following adjective) and the superlative of *eigen* (proper, own) [Dud07, pp. 1783–1784]. With such expression Schröder stressed that the absolute algebra was *really* the matter in which he was most interested.

[45] I must thank prof. Volker Peckhaus in Padeborn for these pages. I own a copy of this book, but Schröder is not mentioned. The volume is a collection of profiles of artists, scientists, musicians, etc. There is no criterion in the choice of the persons embraced in this text. Perhaps, Eckstein collected some notes about important people he knew personally. Incidentally, not any person quoted there is German; E. Grieg, E. Ibsen, L. Duse, etc. were not German. Eckstein himself supported the costs of the publication which ought circulate only privately. So it is possible that there are more versions of the same book. Mine is from the *Bibliothek der Hansestadt–Lübeck*. For any person enumerate in the book there is a picture probably of the same Eckstein, because he was also a photographer. The question is: *why did Eckstein choose Schröder, a little known mathematician?* Eckstein was an historian focused on the history of the Hebrews in the same places where Schröder lived and wrote a monograph on the Hebrew history in the Baden-Baden. Perhaps he became acquainted with Schröder during one of his researches.

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Repertorium. Furthermore, note this quotation taken not from the original paper of Schröder [Sch76], but from a *mathematical Repertorium* where this paper is abridged:

> In my *Handbook of Arithmetic and Algebra* [Sch73], I spoke for the first time of a discipline, of which the usual algebra and analysis appear to be only a *special case* [specieller Fall], and in which they are included as a [...] yarn in a wide dress.[46] [Sch79, p. 81]

I think that Schröder could not be more clear. Universal algebra is a formal theory of which the various mathematical branches are only possible semantics. Obviously, lacking of a precise understanding of the concept of *axiomatic system*, Schröder could not think as a modern model theorist, but this does not imply that Schröder didn't recognize a formal structure on one side and its possible interpretations on the other side.

Note also in this quotation the metaphor of the yarn and of the dress. A yarn is included in a dress, like algebra or analysis in a formal theory of connections. A dress is not an uniform and vague totality, but it is a structured whole in which the yarns are in relation with each other, forming a lattice of relations, a *warp-yarn* or a *web*. Schröder is saying that the various and different logical and mathematical branches are inserted in a more formal theory in a structural way. It is such structure to be the common feature of particular contextualized theories.

3.3. Structure

I rephrase an example yet proposed in [Bon01, p. 34] from an husserlean point of view. Let us think a chessboard as a structured whole. For an accident we have lost a black tower. We could then bring a little wood piece and agree on the fact that from now on this little piece replaces the lost tower. It is not important that this wood piece be similar or not to the lost black tower. We could brought a thing whatever. The essential is that this new object has the *same rôle* in the play of the preceding tower. For this reason in *the context in issue* our wood piece is not distinguishable from the black tower inasmuch it has the same meaning. In other words, the meaning of a tower derives from the relational web in

[46] Incidentally, Schröder in [Sch79] refers to his *Lehrbuch*, but in the original paper of 1876 [Sch76] he referred to the *Abriss*. These two volumes are often confused in the literature. [Sch74a] is a short version for students of the *Lehrbuch*.

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which it is inserted. Fulfilling some relationships and not others a tower *is* a tower.

It is the totality of the laws of chess which founds the meaning of any single piece of the chessboard. The particular shape of the pieces is not so relevant; they can be made of different material and carved in different shapes. They must only obey to some laws and not to others. The pieces are in a structural web in a particular position and not in another. The structural web of chess determines that a tower to be a tower must be in a determinate place of this relational lattice.

Language by signs. For Schröder the objects of a theory are nothing but pure *ink spots on paper*:

> [. . .] the deduction turns out to be in his highest form a computation, a calculus, in which *the order and position*[47] *of the little pictures of the letters* or of the ciphers constitutes from now on the **unique object of observation**.[48] [Sch90b, p. 14]

If the real objects of a theory are only the concrete signs on papers we could doubt that these signs are always the same, especially during our absence as observers. For this purpose Schröder yet in 1873 formulated an axiom on the persistence of signs:

> [. . .] in our deductions and consequences the signs [Zeichen] are fixed in our memory – and much more on paper [. . .]. [Sch73, pp. 16–17].

Schröder puts in comparison the signs of the calculus with the mushrooms in a wood:

> [. . .] they [i.e. the signs] cannot grow from paper as they were mushrooms, neither they can vanish by themselves [. . .]. [Sch73, p. 17]

I would stress the *physical* character of the signs. They are *things* [Dinge], something which we can touch, see with our eyes, modify with our calligraphy. The deduction is not an operation executed with concepts symbolized with signs, but it is an operation on the signs themselves:

> The letters [Buchstaben] are always identical to themselves while we operate with them; an a can never transform itself in a b [. . .]. [Sch73, p. 17]

[47] In the original *Taxis und Thesis*.

[48] The emboldening is mine. I translated *Ziffern* with *ciphers* and not with *numbers*, because Schröder is speaking of any possible sign on paper.

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If the objects of our possible theory are only signs on paper, then they can receive their meaning only in relation to other signs. A sign has no meaning in itself: ∂ can mean the *infimum* of a set, a commutative operation, Euler's constant, and so on. No pre-assigned meaning is possible for Schröder. Only the context can give meaning to a sign, and a denotation is only a possibility among others. For this reason it is necessary to be consistent with the use of signs and postulate that they cannot change by themselves.

This explains Schröder's interest in the relations. Relying on a structural philosophy in the background, Schröder understood that the outmost generality of mathematical concepts could be obtained only in a formal theory. The elements of this theory are meaningless and then there are only the relations between the elements to survive. At this point, it is natural asking: *what is a relation?* The answer to this question is in [Sch66b].

3.4. The concept of relation

To be honest Schröder was ambiguous in defining a *relation*; in some place he introduces it as a class of ordered couples, but in other ones he considers it as a totality from which to derive the concept of *set*, *function* and *individual*. I think that in this context is interesting focusing on the second way of defining a relation. In this case, a relation is a whole which ontologically founds the elements which satisfy them. A binary relation founds the being of two individuals and it establishes an arrow from one to the other; that is, it states an order from one individual (the *relate*) to another one (the *correlate*).

From this point of view, a relation is as an experiment à la Bohr which founds its various elements. The relation is a gestaltic situation in which everything is entangled and where there is an overlapping between the various objects. What is the meaning of these diverse elements? Their meaning depends on the the meaning of the whole (the relation) founding them. The individuals have a meaning only as founded entities of a totality which has a meaning and not another. Let us make a little example. We consider the relation *is lover of* and two *objects* Alice and Bob. The relation considered implies that one object be a lover and one object be a loved. In this way, the relation *is lover of* gives a meaning to Alice and Bob and established an order from the first to the second. In other words, only in a relation of *being lover of* Alice can have the meaning of *lover* and only loving a guy. So, Alice acquires his meaning

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only in the context of a large structure in which she is embedded and only in relation to the other components of this structure.[49]

Obviously, Alice can changes meaning with the passing of time or inasmuch she is founded by another relation. In fact, Alice can be a lover with Bob and a loved with Alfred. It is the context to give meaning to Alice. One possible context is given from the relation exemplified, but Alice can be also in a relation of *serving*, *helping*, *reading*, etc. In any case, it is the structural context to determine the reference of Alice.[50]

So, albeit his ambiguities, Schröder's analysis of the concept of *relation* reveals as part of his thought had a structural source. It is not by chance that Schröder tried to *algebrize* [condensieren] any formula of the calculus of relatives, because in this manner is possible to eliminate the concept of individual. Using Löwenheim's words:

> To *condense* a relative expression means to transform it so that no $\exists$ or $\forall$ occurs any longer. For example, $\exists y(xRy \wedge ySz)$, when condensed yelds $R \circ S$. [Löw67, p. 233]

Distributivity. In a meeting of the *British Association for the Advancement of Science* hold in the September 1883 [Sch84] Schröder's contribute focused on the *distributivity* which he affirmed being a logical principle discovered by the algebra of logic. This is highly interesting. For Schröder the most important principle discovered in logic from the ancient Greek thought is a *formal law*. In fact, Schröder formulated in an abstract way the distributivity.

3.5. Combinatorics

It is not only Schröder's formalism to suggest a structural approach, but also his combinatoric vein. Let us remind a well known combinatorial problem:

> In the plane there are six points h_1, h_2, h_3 (houses) and f_1, f_2, f_3 (fountains). Is it possible drawing in the plane a path from any house to any fountain, such that diverse paths do not intersect? [BE86, p. 29].

For the solution of the problem, it is indifferent the nature of the objects which are connected with a path. They can be houses, fountains, or

[49] In this example Bob.

[50] I remember Bohr's reply to EPR: it is the experiment to *cause* the non-locality of the two particles.

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simply points in the plane. What it is important are the *relations* among these elements. If you prefer, it is the totality of the situation to be problematic, not its parts. The meaning of the six elements is determined by the structure of the path. The path founds the elements, inserting them in an appropriate context.

This way a combinatorial problem evokes a structure–problem. It is this last to be solved. Step by step we erase any specific feature of the elements involved to obtain only formal elements. At this point we can focus ourself on the *true* problem. This process of abstraction is helpful in determining what is essential to the solution of the problem and what is not. In the previous example, it is inessential speaking of *houses* and *fountains*, being relevant only the form of the path.

For this reason I cited [Sch84]. In the definition of *distributivity* is irrelevant not only the meaning of the elements to which the dual operations apply, but also the specific nature of these operations. The distributivity law states that an operation distribute over its dual. Two arbitrary operations dual each other are sufficient.

So, when we ask *how many ways are there to combine some elements?*, we are putting a combinatorial question which points to the relations between the elements of a totality.

Let us quote again from *Über die formalen Elemente*:

> If one connects [verknüpft] progressively three numbers a, b, c with two of the three fundamental operations [i.e. time and division], eighteen elementary expressions arise which can be ordered in three groups of six elements according to our principles of permutation [Vertauschungsprincipien] [...] in such way that in [any group] the **letters** can be permuted again.[51]
> [Sch74b, pp. 12–13]

As one can easily see, the focus is on the *grouping* of some elements sharing a common property. This is a purely combinatorial problem which leaves the nature of the grouped and permuted elements out of consideration. To be in the foreground are the *fundamental operations*[52] and the principles of permutation. In any case, relations.

[51] The emboldening is mine. I refer the reader above to page 345 where we quoted Schröder tackling a similar combinatorial problem in [Sch90a] and [Sch73].

[52] A fundamental operation with its inverse. Schröder could choose instead of *time* and *division*, *addition* and *subtraction*, *raising to power* and *root*, or *exponentiation* and *logarithm* [Sch74a, p. 12].

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Coda. As noted above, formalism implies that an *object* has more interpretations. As a matter of fact, Schröder applied the operations not on objects, but on signs, on the letters of the calculus in issue. This is a crude formalism. In the same way Schröder considers the operations or the tools employed in the various situations. In the case of distributivity, are not in question the meanings of the operations distributing each on other. All is secondary or marginal *but* the meta-relationship of duality between these operations.

For this reason the same formal law of distributivity can be applied in different fields and with different meanings. Like the distributivity there is a lot of formal operations, instruments or concepts which can be contextualized differently. One of these formal concepts is the *solution problem*. Unhesitatingly we can say that all Schröder's work revolves around the solution problem, showing Schröder's algebraic *vocation*.

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