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A CLASS OF SIMPLER LOGICAL MATRICES FOR THE VARIABLE-SHARING PROPERTY

Abstract. In our paper "A general characterization of the variable-sharing property by means of logical matrices", a general class of so-called "Relevant logical matrices", RMLs, is defined. The aim of this paper is to define a class of simpler Relevant logical matrices RMLs′ serving the same purpose that RMLs, to wit: any logic verified by an RML′ has the variable-sharing property and related properties predicable of the logic of entailment E and of the logic of relevance R.

Keywords: logical matrices, variable-sharing property, relevant logics

1. Introduction

As it is well-known, according to Anderson and Belnap, the *variable-sharing property* (vsp) is a necessary property of any relevant logic S (see [1]). The vsp reads as follows:

Definition 1 (Variable-sharing property –vsp). A logic S has the vsp iff in any theorem of S of the form $A \to B$, A and B share at least one propositional variable.

In [4], a general class of so-called "Relevant logical matrices", RLMs, is defined. RLMs have the following property: if a logic S is verified (cf. §2) by an RLM, then, in addition to the vsp, S has exactly the same properties predicable of the logic of entailment E and of the logic of relevance R (cf. [1], §22.1.3).

The aim of this paper is to define a class of simpler Relevant logical matrices, which serve the same purposes that RLMs.

We shall begin with some preliminary definitions.

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2. Logical matrices. Preliminary definitions

We shall consider propositional languages with a set of denumerable propositional variables and the following connectives: $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), the biconditional ($\leftrightarrow$) being defined in the customary way. The set of wff is also defined in the usual way. Then, the notion of logical matrix is defined as follows:

Definition 2. A logical matrix M is a structure $(K, T, F, f_\to, f_\wedge, f_\vee, f_\neg)$ where:

1. K is a set.
2. T and F are non-empty subsets of K such that $T \cup F = K$ and $T \cap F = \varphi$,
3. $f_\to$, $f_\wedge$, $f_\vee$ are binary functions (distinct of each other) on K, and $f_\neg$ is a unary function on K.

It is said that K is the *set of elements* of M; T is the *set of designated elements*, and F is the *set of non-designated elements*. The functions $f_\to$, $f_\wedge$, $f_\vee$ and $f_\neg$ *interpret* in M the conditional, conjunction, disjunction and negation, respectively. In some cases one or more of these functions may not be defined.

Now, let L be a propositional language, $A_1, ..., A_n$, B be any wff of L and S be a logic defined on L. On the other hand, let M be a logical matrix and v_m an assignment of elements of M to the propositional variables of B. That B is assigned the element j of K is expressed as follows: $v_m(B) = j$.

Then, we set:

Definition 3. Let M be a logical matrix. M *verifies* B iff for any assignment of elements of K to the propositional variables of B, v_m, $v_m(B) \in T$.

Definition 4. Let M be a logical matrix. M *falsifies* B iff for some assignment of elements of K to the propositional variables of B, v_m, $v_m(B) \in F$.

Definition 5. Let $A_1, ..., A_n \vdash_S B$ be a rule of derivation of S, and M be a logical matrix. Then, M *verifies* $A_1, ..., A_n \vdash_S B$ iff for any assignment of elements of K to the variables of $A_1, \ldots, A_n$ and B, v_m, if $v_m(A_1) \in T, ..., v_m(A_n) \in T$, then, $v_m(B) \in T$.

Finally,

DEFINITION 6. Let M be a logical matrix. M *verifies* S iff M verifies all axioms and rules of derivation of S.

3. Simplified Relevant logical matrices

DEFINITION 7 (Simplified Relevant logical matrix). Let M be a logical matrix in which a_T, a_F, a_i, a_r are elements of K distinct of each other. And let us designate by K_i and K_r the subsets of K $\{a_i\}$ and $\{a_r\}$, respectively. In addition, the following conditions are fulfilled:

1. $a_T \in T$,
2. $a_F \in F$.
3. (a) $f_\wedge(a_T, a_T) = f_\vee(a_T, a_T) = f_\rightarrow(a_F, a_T) = f_\neg(a_F) = a_T$,
   (b) $f_\wedge(a_F, a_F) = f_\vee(a_F, a_F) = f_\rightarrow(a_T, a_F) = f_\neg(a_T) = a_F$.
4. $\forall x \in K_i \cup K_r$
   (a) $f_\wedge(a_F, x) = f_\wedge(x, a_F) = f_\rightarrow(a_T, x) = f_\rightarrow(x, a_F) = a_F$,
   (b) $f_\vee(a_T, x) = f_\vee(x, a_T) = f_\rightarrow(a_F, x) = f_\rightarrow(x, a_T) = a_T$,
   (c) $f_\wedge(a_T, x) = f_\wedge(x, a_T) = f_\vee(a_F, x) = f_\vee(x, a_F) = x$.
5. (a) $f_\wedge(a_i, a_i) = f_\vee(a_i, a_i) = f_\rightarrow(a_i, a_i) = f_\neg a_i = a_i$,
   (b) $f_\wedge(a_r, a_r) = f_\vee(a_r, a_r) = f_\rightarrow(a_r, a_r) = f_\neg(a_r) = a_r$.
6. $f_\rightarrow(a_i, a_r) = a_F$.

Then, it is said that M is a *(simplified) relevant (logical) matrix*, *relevant matrix*, for short.

Remark 1. 1. Notice that it is not necessary to stipulate the following:
   (a) To which subset, T or F, belong the elements of $K_i \cup K_r$.
   (b) The elements of K assigned to $f_\rightarrow(a_T, a_T)$, $f_\rightarrow(a_F, a_F)$, $f_\wedge(a_F, a_T)$, $f_\wedge(a_T, a_F)$, $f_\vee(a_F, a_T)$ and $f_\vee(a_T, a_F)$.
   (c) The elements of K assigned to $f_\wedge(a_i, a_r)$, $f_\wedge(a_r, a_i)$, $f_\vee(a_i, a_r)$, $f_\vee(a_r, a_i)$ and $f_\rightarrow(a_r, a_i)$.
2. Remark that condition 5 establishes that K_i and K_r are closed under $f_\rightarrow$, $f_\wedge$, $f_\vee$ and $f_\neg$. This fact, i.e., the fact that K_i and K_r are singletons closed under the logical operations, together with the standardization of some open values in the functions, is the simplification here presented in respect of the Relevant matrices defined in [4].

4. Relevant matrices and the variable-sharing property

We prove that if a logic S is verified by a relevant matrix M, then S has the properties that Anderson and Belnap prove in [1], §22.1.3 as predicable of the logic of entailment E and of the logic of relevance R.

In order to prove that this is the case, we define *antecedent part* ("ap") and *consequent part* ("cp") of wff inductively as follows (see [1], p. 240).

Definition 8 (Antecedent parts and consequent parts). Let A be a wff. Then,

1. A is a cp of A.
2. If $B \wedge C$ is a cp (ap) of A, then both B and C are cps (aps) of A.
3. If $B \vee C$ is a cp (ap) of A, then both B and C are cps (aps) of A.
4. If $B \to C$ is a cp (ap) of A, then B is an ap (cp) of A and C is a cp (ap) of A.
5. If $\neg B$ is a cp (ap) of A, then B is an ap (cp) of A.

The properties referred to above are expressed in the following theorems (cf. [1], §22.1.3):

Theorem 1. *If $A \to B$ is provable in S, then some variable occurs as an ap of both A and B, or else as a cp of both A and B.*

Theorem 2. *If A is provable in S and A contains no conjunctions as aps and no disjunctions as cps, then every variable in A occurs at least once as ap and at least once as cp.*

Let us proceed to prove Theorem 1. Suppose that $A \to B$ is a wff in which no variable occurs as an ap of both A and B or as a cp of both A and B. Then, each variable p occurring in $A \to B$ has to appear in A and/or in B in one of the six situations tabulated below:

	A	B
$p:$	cp	-
	ap	-
	-	ap
	-	cp
	cp	ap
	ap	cp

The first row is read “p occurs as a cp in A, but does not occur in B”, and the rest of the rows are read similarly.

Now, let M be a relevant matrix that verifies S. According to these possibilities, the following assignment v_m of elements of M is defined for each variable p in $A \to B$:

	A	B	$v_m(p)$
$p:$	cp	-	a_i
	ap	-	a_i
	-	ap	a_r
	-	cp	a_r
	cp	ap	a_T
	ap	cp	a_F

Then, Theorem 1 follows immediately from the following lemmas:

LEMMA 1. *For every ap C of A, $v_m(C) \in \{a_i, a_F\}$; and for every cp C of A, $v_m(C) \in \{a_i, a_T\}$.*

LEMMA 2. *For every ap C of B, $v_m(C) \in \{a_r, a_T\}$; and for every cp C of A, $v_m(C) \in \{a_r, a_F\}$.*

The proofs of lemmas 1 and 2 are by induction on the length of C, and they are easy given that M is a relevant matrix. Let us prove as way of an example a couple of cases. By 3a, 4b etc. we refer to the clauses in Definition 7. First, the conditional case in Lemma 1:

PROOF. Suppose that C is of the form $D \to E$. 1. C is an ap. Then, D is a cp and E is an ap by Definition 8. By hypothesis of induction (H.I), $v_m(D) \in \{a_i, a_T\}$ and $v_m(E) \in \{a_i, a_F\}$, whence by 3b, 4a and 5a, $v_m(D \to E) \in \{a_i, a_F\}$ as it was to be proved.

2. C is a cp. Then, D is an ap and E is a cp by Definition 8. By H.I, $v_m(D) \in \{a_i, a_F\}$ and $v_m(E) \in \{a_i, a_T\}$, whence by 3a, 4b and 5a, $v_m(D \to E) \in \{a_i, a_T\}$, as it was to be proved. ⊣

Next, the negation case in Lemma 2:

PROOF. Suppose that C is of the form $\neg D$:

1. C is an ap. Then, D is a cp by Definition 8. By H.I, $v_m(D) \in \{a_r, a_F\}$. Then, $v_m(\neg D) \in \{a_r, a_T\}$ by 3a and 5b.

2. C is a cp. The proof is similar to the previous one by 3b and 5b. ⊣

Then, the proof of Theorem 1 is immediate. As each formula is a cp of itself, by lemmas 1 and 2, $v_m(A) \in \{a_i, a_T\}$ and $v_m(B) \in \{a_r, a_F\}$ whence $v_m(A \to B) = a_F$ by 3b, 4a and 6 in Definition 7; that is, $A \to B$ is not a theorem of S. Therefore, if $A \to B$ is a theorem of S, then some variable occurs either as an ap or else as a cp of both A and B.

An immediate consequence of Theorem 1 is the following:

COROLLARY 1. *S has the vsp.*

Next, we proceed to the proof of Theorem 2.

Suppose that A is a wff in which some variable, say p, occurs only as cp. Then, we set the following assignment under M: $v_m(p) = a_F$, and each variable (distinct of p) in A is assigned the element a_i of K_1 by v_m. Then, it is proved:

LEMMA 3. *If B is any part of A in which p does not occur, then $v_m(B) \in \{a_i\}$.*

PROOF. Induction on the length of B. As it was the case with Lemma 1 and Lemma 2, it is easy. Notice, in this sense, that K_i and K_r are closed under $\to$, $\wedge$, $\vee$ and $\neg$. ⊣

LEMMA 4. *If B is any part of A in which p does occur, then:*

1. *If B is an ap of A, then $v_m(B) = a_T$.*
2. *If B is a cp of A, then $v_m(B) = a_F$.*

PROOF. Induction on the length of B. If B is a propositional variable, then B is p and $v_m(B) = a_F$. (Recall that p occurs only as a cp). Regarding complex formulas, we prove the conditional case, and leave the rest of the cases to the reader.

B is of the form $C \to D$:

1. B is an ap. Then, C is a cp and D is an ap.
   (a) p occurs in C and D: by H.I, $v_m(C) = a_F$ and $v_m(D) = a_T$. So, $v_m(C \to D) = a_T$ by 3a.
   (b) p occurs in C but not in D: by H.I, $v_m(C) = a_F$. By Lemma 3, $v_m(D) \in \{a_i\}$. So, $v_m(C \to D) = a_T$ by 4b.
   (c) p occurs in D but not in C: by Lemma 3, $v_m(C) \in \{a_i\}$; by H.I, $v_m(D) = a_T$. Then, $v_m(C \to D) = a_T$ by 4b.
2. B is a cp. Then, C is an ap and D is a cp.
   (a) p occurs in C and D: by H.I, $v_m(C) = a_T$ and $v_m(D) = a_F$. Then, $v_m(C \to D) = a_F$ by 3b.

(b) p occurs in C but not in D: by H.I, $v_m(C) = a_T$. By Lemma 3, $v_m(D) \in \{a_i\}$. Then, $v_m(C \to D) = a_F$ by 4a.

(c) p occurs in D but not in C: by Lemma 3, $v_m(C) \in \{a_i\}$; by H.I, and $v_m(D) = a_F$. Then, $v_m(C \to D) = a_F$ by 4a. $\dashv$

The proof of the conditional case is now finished. The proof of the conjunction, disjunction and negation cases is similar (recall that conjunctions can only appear as cps and disjunctions as aps).

The proof of Theorem 2 is now immediate. As each formula is a cp of itself, from Lemma 4 it follows that $v_m(A) = a_F$. That is, A is not a theorem of S. Consequently, if A is a theorem of S without conjunctions as aps and disjunctions as cps, then every variable in A occurs at least once as ap and at least once as cp.

We end the paper with an example.

5. Example

We set the following example:

DEFINITION 9. Consider the matrix $\mathrm{M}_{\mathrm{DF9}} = (K, T, F, f_{\to}, f_{\wedge}, f_{\vee}, f_{\neg})$ where:

- $K = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- $F = \{0\}$
- The functions $f_{\to}$, $f_{\wedge}$, $f_{\vee}$, $f_{\neg}$ are defined as shown in the tables below.

$\to$	0	1	2	3	4	5	6	7	8	9	$\neg$
0	9	9	9	9	9	9	9	9	9	9	9
1	0	1	2	3	4	5	6	7	8	9	8
2	0	0	2	3	0	5	6	7	8	9	7
3	0	0	0	3	0	0	0	7	8	9	3
4	0	0	0	0	4	5	6	7	8	9	6
5	0	0	0	0	0	5	6	7	8	9	5
6	0	0	0	0	0	0	6	0	8	9	4
7	0	0	0	0	0	0	0	7	8	9	2
8	0	0	0	0	0	0	0	0	8	9	1
9	0	0	0	0	0	0	0	0	0	9	0

$\wedge$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1
2	0	1	2	2	1	2	2	2	2	2
3	0	1	2	3	1	2	2	3	3	3
4	0	1	1	1	4	4	4	4	4	4
5	0	1	2	2	4	5	5	5	5	5
6	0	1	2	2	4	5	6	5	6	6
7	0	1	2	3	4	5	5	7	7	7
8	0	1	2	3	4	5	6	7	8	8
9	0	1	2	3	4	5	6	7	8	9

$\vee$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	1	2	3	4	5	6	7	8	9
2	2	2	2	3	5	5	6	7	8	9
3	3	3	3	3	7	7	8	7	8	9
4	4	4	5	7	4	5	6	7	8	9
5	5	5	5	7	5	5	6	7	8	9
6	6	6	6	8	6	6	6	8	8	9
7	7	7	7	7	7	7	8	7	8	9
8	8	8	8	8	8	8	8	8	8	9
9	9	9	9	9	9	9	9	9	9	9

Moreover,

- $K_i = \{5\}$
- $K_r = \{3\}$
- $a_T = 9$
- $a_F = 0$

It is proved:

PROPOSITION 1. *Matrix M_{DF9} is a relevant matrix.*

PROOF. It is a matter of checking that conditions 1-6 in Definition 7 are fulfilled by $\mathrm{M_{DF9}}$. ⊣

Consequently, by Proposition 1, we have:

PROPOSITION 2. *Any logic S verified by M_{DF9} has the vsp. Moreover, it has Theorem 1 and Theorem 2 as properties.*

Remark 2. 1. Matrix $\mathrm{M}_{\mathrm{DF9}}$ could have been defined with $K_i = \{3\}$ and $K_r = \{5\}$.

2. The logic RMO defined in [2] (cf. also [3]) is verified by $\mathrm{M}_{\mathrm{DF9}}$. RMO is the result of adding to the positive fragment of relevance logic R, R_+ (cf. [1]) the axiom mingle $A \to (A \to A)$, the axioms of double negation $A \to \neg\neg A$ and $\neg\neg A \to A$, the axiom of specialized reductio $(A \to \neg A) \to \neg A$ and the rule of contraposition, if $A \to B$ is a theorem, then $\neg B \to \neg A$ is a theorem. By means of a (non-simplified) relevant matrix, in [2], it is proved that the vsp and the properties stated in Theorem 1 and Theorem 2 are predicable of RMO.

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