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REFUTATION SYSTEMS FOR A SYSTEM OF NONSENSE-LOGIC*

Abstract. In the paper rejection systems for a system of nonsense-logic are investigated. The first rejection system consists of four rejected axioms and only one rejection rule — the rule of rejection by detachment. The second one consists of one rejected axiom and two rejection rules: the rule of rejection by detachment and the rule of rejection by substitution. The aim of the paper is to present also a proof of Ł-decidability for the considered systems.

Keywords: refutation systems, rejected axioms, Ł-decidability

1. Introduction

The paper concentrates on the notion of Ł-decidability of a logical system. The notion of an Ł-decidable system, under the name of a saturated system, was introduced by Łukasiewicz in his paper [2]. Łukasiewicz posed there the following problem: given a logical system, provide a list of rejected axioms on the basis of which, using the rules of rejection: by detachment and by substitution, one can reject all the formulae which are not theses of the considered logic. Some results concerning the notion of Ł-decidability of logical systems can be found in [1, 4]. In the monograph [4], refutation systems were given for selected many-valued logics, in which the rule of rejection by substitution was eliminated. The construction of relevant rejected axioms given there, in some cases, proved

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most difficult due to the lack of certain property of sentential calculi called definitionally completeness. Finn's nonsense-logic was considered there, among others, for which a certain system of rejected axioms was given, yet it is more complicated than that proposed in the present work. Consequently, the refutation system proposed here, with only the rule of rejection by detachment is much simpler and is based on rejected axioms which are formed of unary functors and which - in a more direct way - penetrate the structure of a formula being rejected. The present work considers also a refutation system for Finn's nonsense-logic calculus with both rules of rejection (this problem was not analyzed in [4]). It seems also interesting (for methodological reasons) to compare both refutation systems for Finn's calculus. A refutation system with two rules of rejection is far simpler, whereas the other of the systems turns more complicated both as regards the form of rejected axioms and in terms of applied techniques of proving.

First, we shall provide some comments on sentential calculi. Any deductive system has the biaspectual axiomatic method of its characterization. By the tuples $\langle S, A, R\rangle$ and $\langle S, A^*, R^*\rangle$ we mean the asserted system for any deductive system and the refutation system for any deductive system respectively, where S is the set of all well–formed formulae of a system, A the set of its axioms, R the set of its primitive inference rules, A^* the set of its rejected axioms and R^* the set of its primitive rejection rules. The tuple $\langle S, A, R\rangle$ determines the set T of all theses and the tuple $\langle S, A^*, R^*\rangle$ determines the set T^* of all rejected formulae. So, we have:

$C_R(X)$	the least set of formulae containing $A \cup X$ and closed with respect to the rules from the set R (in our case RU_1 and RU_2),
$C_{R^*}(X)$	the least set of formulae containing $A^* \cup X$ and closed with respect to the rules from the set R^* (in our case RO_1 and RO_2).

It follows from the above definitions that $T = C_R(A)$ and $T^* = C_{R^*}(A^*)$.

We adopt the following rejection rules:

$$RO_1 : \frac{\vdash \alpha \Rightarrow \beta \qquad \dashv \beta}{\dashv \alpha} \qquad\qquad RO_2 : \frac{\beta \in Sb(\{\alpha\}) \qquad \dashv \beta}{\dashv \alpha}$$

These rules we read as follows:

RO_1 if the implication $\ulcorner\alpha \Rightarrow \beta\urcorner$ is asserted, but its consequent β is rejected, then its antecedent α must be rejected, too;

RO_2 if β is a substitution instance of α, and β is rejected, then α must be rejected, too.

In addition to the rules of rejection we adopt also the well known rules of assertion: the rule of detachment (modus ponens) and the rule of substitution:

$$RU_1 : \frac{\begin{array}{c}\vdash \alpha \Rightarrow \beta \\ \vdash \alpha\end{array}}{\vdash \beta} \qquad RU_2 : \frac{\begin{array}{c}\beta \in Sb(\{\alpha\}) \\ \vdash \alpha\end{array}}{\vdash \beta}$$

Let the tuple $\langle S, Ax, R\rangle$ be a deductive system based on a language $J = (S, \Im)$, where $Ax \subseteq S$ and $R \subseteq 2^S \times S$ is an arbitrary finite set of rules of inference. By the symbol $Sb(X)$ we mean a set of the form $\{e(\alpha) : \alpha \in X$ and e is an endomorphism of the language $J\}$. If $Sb(X) = X$, then the set X is called invariant. Moreover, if the condition $(X, \alpha) \in R \Rightarrow (e(X), e(\alpha)) \in R$ holds for any endomorphism e of the language J, then the rule $(X, \alpha) \in R$ is called structural. The sentential calculus $\langle S, Ax, R\rangle$ is called invariant, if $e(Ax) \subseteq Ax$ for any endomorphism e of J and R is the set of structural rules of inference. Recall that the rule RU_2 does not satisfy the condition of structurality, since if $(p, q) \in RU_2$ and $e(p) = Cpq$, $e(q) = s$, then $(Cpq, s) \notin RU_2$. The case of the rule RO_2 is similar.

In the further considerations we will use the notion of logical matrix. Assume that for a sentential calculus $\langle S, Ax, R\rangle$, with the language $J = (S, \Im)$, there exists a logical matrix $M = (U, V, f)$, where $\emptyset \neq V \subset U$ and the algebra (U, f) is similar to $J = (S, \Im)$. The content of the matrix $M = (U, V, f)$ is the set $E(M)$ defined as follows:

$$E(M) := \{\alpha : \forall_{h \in \mathrm{Hom}}\, h(\alpha) \in V\},$$

where Hom denotes the set of homomorphisms from the algebra of the language into the algebra of the matrix. Consequences of logical matrices are defined as follows:

$$C_M(X) := \{\alpha : \forall_{h \in \mathrm{Hom}}[h(X) \subseteq V \Rightarrow h(\alpha) \in V]\}.$$

It is easy to see that $C_M(\emptyset) = E(M)$.

We assume that $C_M(\emptyset) = C_R(\emptyset)$, i.e. $E(M) = T$. It means that the matrix $M = (U, V, f)$ is weakly adequate for the logic $\langle S, Ax, R\rangle$. This assumption will play a very important role in the futher considerations.

Definition 1. A deductive system is Ł-decidable (i.e. decidable in the sense of Łukasiewicz) if and only if

$$T \cap T^* = \emptyset, \tag{1}$$

$$T \cup T^* = S, \tag{2}$$

where T is the set of all theses and T^* is the set of all rejected formulae (see [1, 4]). The condition (1) is called the *consistency condition* and the condition (2) is called the *completeness condition*.

2. Two refutation systems for Finn nonsense-logic

Now we consider the nonsense-logic system constructed by W. K. Finn. This system is discussed in more detail in [3]. Let the symbol **FN** denotes the system built by an axiomatic method. The primitive terms of this logic are: $\neg$, $\cap$, $\Rightarrow$. The adequate matrix $M_{\mathbf{FN}}$ has the form $M_{\mathbf{FN}} = (\{1, 0, \frac{1}{2}\}, \{1\}, n, k, c)$, where the functions n, k, c correspond to the functors $\neg$, $\cap$, $\Rightarrow$, respectively. The set of theses of the logic **FN** is the content of the matrix $M_{\mathbf{FN}}$. The values 1, 0 i $\frac{1}{2}$ are called truth, false and nonsense. The functions of the matrix $M_{\mathbf{FN}}$ are interpreted as:

x	$n(x)$
1	0
0	1
$\frac{1}{2}$	$\frac{1}{2}$

$k(x,y)$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	0	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$c(x,y)$	1	0	$\frac{1}{2}$
1	1	0	0
0	1	1	1
$\frac{1}{2}$	1	1	1

We consider two cases depending on the rules of rejection, which we will use.

2.1. A invariant refutation system

In order to prove Ł-decidability of the systemu **FN** in the invariant version, we will construct some new functors.

$$G_0(\alpha, \beta) = \ulcorner(\alpha \cap \neg\beta) \Rightarrow (\alpha \cap \beta)\urcorner.$$

To this functor there corresponds in the matrix $M_{\mathbf{FN}}$ the following function g_0:

$g_0(x,y)$	1	0	$\frac{1}{2}$
1	1	0	1
0	1	1	1
$\frac{1}{2}$	1	1	1

Now, we will define an important functor M.

$$V(\alpha,\beta) = \ulcorner G_0(\alpha,\beta) \cap G_0(\beta,\alpha)\urcorner,$$
$$M(\alpha,\beta) = \ulcorner V(\alpha,\beta) \Rightarrow (\alpha \cap \beta)\urcorner.$$

To this functor there corresponds in the matrix $M_{\mathbf{FN}}$ the function m of the form:

$m(x,y)$	1	0	$\frac{1}{2}$
1	1	1	0
0	1	0	0
$\frac{1}{2}$	0	0	0

As can be seen, the function m has the property: $m(x,y) = \max\{x,y\}$ for $x,y \in \{0,1\}$. We adopt the convention that $M(\alpha) = \alpha$, $M(\alpha_1,\dots\alpha_n) = M(M(\alpha_1,\dots,\alpha_{n-1}),\alpha_n)$ for, $n \geqslant 2$.

In the invariant system **FN** the following formulae will play the role of rejected axioms:

$$F_0(\alpha) = \ulcorner\neg\alpha \Rightarrow \alpha\urcorner,$$
$$F_{\frac{1}{2}}(\alpha) = \ulcorner(\neg\alpha \Rightarrow \alpha) \Rightarrow \alpha\urcorner,$$
$$F_1(\alpha) = \ulcorner\alpha \Rightarrow \neg\alpha\urcorner$$

To this functors there correspond in the matrix $M_{\mathbf{FN}}$ the following functions f_i:

x	$f_0(x)$	$f_{\frac{1}{2}}(x)$	$f_1(x)$
1	1	1	0
0	0	1	1
$\frac{1}{2}$	1	0	1

In invariant systems schemes of axioms are considered. So, the set of rejected axioms will consist of formulae $F_i(\alpha)$, $i \in \{0,1,\frac{1}{2}\}$ and $M(F_{i_1}(\alpha_1), F_{i_2}(\alpha_2), \dots, F_{i_n}(\alpha_n))$, where $i_1, i_2, \dots, i_n \in \{0,1,\frac{1}{2}\}$ (in the paper [4] the other rejected axioms have more complicated form).

Theorem 1. *For any formula* $\alpha : \alpha \notin T$ *iff* $\alpha \in T^*$.

Proof. "$\Rightarrow$" Suppose that $\alpha \notin T$ and α contains variables p_1, p_2, $\ldots$, p_n. Since $T = E(M_{\mathbf{FN}})$, this means that there is a valuation $\phi_0 \colon \{p_1, p_2, ..., p_n\} \to \{0, \frac{1}{2}, 1\}$ such that $\phi_0(p_1) = w_1$, $\ldots$, $\phi_0(p_n) = w_n$ and $h^{\phi_0}(\alpha) \in \{0, \frac{1}{2}\}$, where h^{ϕ_0} is the standard homomorphic extension of ϕ_0 to the set of all formulae. In order to reject the formula α we consider the following rejected axiom:

$$M(F_{w_1}(p_1), F_{w_2}(p_2), \ldots, F_{w_n}(p_n)).$$

It is easy to see that $h^{\phi_0}(M(F_{w_1}(p_1), F_{w_2}(p_2), \ldots, F_{w_n}(p_n))) = 0$. Moreover the formula $\ulcorner \alpha \Rightarrow M(F_{w_1}(p_1), F_{w_2}(p_2), ..., F_{w_n}(p_n)) \urcorner \in E(M_{\mathbf{FN}}) = T$. Thus $\alpha \in T^*$ by the rejection rule RO_1.

"$\Leftarrow$" This case is trivial (it is easy to prove by induction on the length of a proof). $\square$

The conditions for Ł-decidability immediately follow from this theorem for the nonsense-logic system **FN**.

2.2. A non-invariant refutation system

Now, we will use also the rejection rule by substitution (see the rule RO_2), which is not structural. In this case, in order to reject a formula which is not a thesis of the system **FN**, we use both rejection rules and only one rejected axiom. The only rejected axiom may be a formula with one variable p, which has the following property:

$$\mu(\alpha) = \begin{cases} \frac{1}{2}, & \text{if } \mu(p) = \frac{1}{2}, \\ 1, & \text{if } \mu(p) \neq \frac{1}{2}. \end{cases}$$

For example, such formulae are:

$$\neg(p \cap \neg p) \quad \text{or} \quad \neg[\neg p \cap (\neg p \Rightarrow p)] \quad \text{or} \quad \neg[\neg p \cap (p \Rightarrow \neg p)].$$

Theorem 2. *For any formula* α*: if* $\alpha \notin T$*, then* $\alpha \in T^*$.

Proof. Consider a formula α with variables $p_1, p_2, \ldots, p_n$, which is not a thesis. By the theorem of completeness, there exists a falsifying valuation μ_1. Let $\alpha(p)$ denote the formula obtained from α by substituting for each of its variables p_i, $i \in \{1, 2, \ldots, n\}$ the following formulae:

- $p \Rightarrow p$, if $\mu_1(p_i) = 1$,
- $\neg(p \Rightarrow p)$, if $\mu_1(p_i) = 0$
- $\neg(\neg p \cap p)$, if $\mu_1(p_i) = \frac{1}{2}$.

One can see that for every valuation $\mu\colon \{p_1, p_2, ..., p_n\} \to \{0, \frac{1}{2}, 1\}$ the following condition holds: if $\mu(p) = \frac{1}{2}$, then $h^{\mu}(\alpha(p)) \in \{0, \frac{1}{2}\}$. Thus for every valuation $\mu\colon p \to \{0, \frac{1}{2}, 1\}$ we obtain $\mu(\alpha(p) \Rightarrow \neg(\neg p \cap p)) = 1$. Treating the formula $\neg(\neg p \cap p)$ (any formula of (6) may be treated as a rejected axiom) as the only rejected axiom and using the rejection rule by detachment, we obtain $\dashv \alpha(p)$. Since the formula $\alpha(p)$ is the substitution of the formula α, then using the rejection rule by substituting we conclude that $\dashv \alpha$. $\square$

The above theorem implies also the Ł-decidability of the system **FN** in the non-invariant version. The condition (2) follows directly from Theorem 2, and the condition (1) is obvious.

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