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ON AXIOMATIZATION OF ŁUKASIEWICZ'S FOUR-VALUED MODAL LOGIC*

Abstract. Formal aspects of various ways of description of Jan Łukasiewicz's four-valued modal logic £ are discussed. The original Łukasiewicz's description by means of the accepted and rejected theorems, together with the four-valued matrix, is presented. Then the improved E. J. Lemmon's description based upon three specific axioms, together with the relational semantics, is presented as well. It is proved that Lemmon's axiomatics is not independent: one axiom is derivable on the base of the remanent two. Several axiomatizations, based on three, two or one single axiom are provided and discussed, including S. Kripke's axiomatics. It is claimed that (a) all substitutions of classical theorems, (b) the rule of modus ponens, (c) the definition of "$\Diamond$" and (d) the single specific axiom schema: $\ulcorner \Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B} \urcorner$, called the *jumping necessity* axiom, constitute an elegant axiomatics of the system £.

Keywords: Łukasiewicz's modal logic, axiomatization, modal logic.

1. Introduction

Search after the most simple axiomatization of any deductive system has always been one of dominant characteristics of the polish school of logic. In this paper we focus on Jan Łukasiewicz's system of four-valued modal logic. To facilitate our exposition we refer to this logic as £. The system £ is a conservative extension of the Classical Propositional Logic, to which we refer as CL. Although the system £ has never belonged to the

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mainstream of modal logic, it definitely deserves profound examination of both mathematical and philosophical nature. In the case of £ philosophical discussion may be even more desirable. In this paper, however, we put all the philosophical questions — temporarily — aside and indulge ourselves to quest of the best axiomatization of the calculus in question.

2. The System £ from Łukasiewicz to Lemmon

In this section we familiarize ourselves with two important accounts of the system £: the original Łukasiewicz's account and the momentous account by E. J. Lemmon. Łukasiewicz constructed the system £ as a system of both accepted and rejected formulas. The accepted formulas are preceded with the sign "⊢", while the rejected ones are preceded with the sign "⊣". Lemmon provided a very transparent reconstruction of £ as a typical axiomatic system of strictly regular modal logic.

The language of modal logic. The object language of the system £ is just the typical language of modal propositional logic. In metalanguage we refer to any formulas of the object language with help of such variables as "$\mathcal{A}$" and "$\mathcal{B}$".

The alphabet consists with (a) countably many schematic sentence letters: "p", "q", "r" etc., which represent any sentences, (b) five connectives adopted from CL, specifically the connective of negation "$\neg$", the connective of conjunction "$\wedge$", the connective of disjunction "$\vee$", the connective of (material) implication "$\to$" and the connective of equivalence "$\equiv$", (c) two modal connectives, specifically the connective of necessity "$\Box$" and the connective of possibility "$\Diamond$", and finally (d) parentheses serving as punctuation marks.

The set of object language formulas is the smallest collecton containing all schematic sentence letters and closed under following operations: (a) if $\mathcal{A}$ is a formula, then $\ulcorner(\neg\mathcal{A})\urcorner$, $\ulcorner(\Box\mathcal{A})\urcorner$ and $\ulcorner(\Diamond\mathcal{A})\urcorner$ are also formulas, (b) if $\mathcal{A}$ and $\mathcal{B}$ are formulas, then $\ulcorner(\mathcal{A}\wedge\mathcal{B})\urcorner$, $\ulcorner(\mathcal{A}\vee\mathcal{B})\urcorner$, $\ulcorner(\mathcal{A}\to\mathcal{B})\urcorner$ and $\ulcorner(\mathcal{A}\equiv\mathcal{B})\urcorner$ are also formulas. The formulas mentioned in the points (a) and (b) are to be read in order of appearance: it is not the case that $\mathcal{A}$, it is necessary that $\mathcal{A}$, it is possible that $\mathcal{A}$, $\mathcal{A}$ and $\mathcal{B}$, (either) $\mathcal{A}$ or $\mathcal{B}$, if $\mathcal{A}$, then $\mathcal{B}$, $\mathcal{A}$ if and only if $\mathcal{B}$. For the sake of simplicity we allow to omit external parentheses, and those default, when scopes of connectives in order "$\Box$", "$\Diamond$", "$\neg$", "$\wedge$", "$\vee$", "$\to$", "$\equiv$" become longer and longer.

Furthermore Łukasiewicz makes use of a *variable connective* "δ", which actually extends the language described thus for. The variable connective is to be added to the alphabet, and in the definition of a formula (a) there is another recursive condition required: if $\mathcal{A}$ is a formula, then so is $\ulcorner(\delta\mathcal{A})\urcorner$. The variable "$\delta$" may be substituted by any aggregate of elements of the alfabet, provided the aggregate forms a formula together with a single item of the formula $\mathcal{A}$ [4, § 2–3]. We deal with the variable "δ" only when referring directly to the original work [5] by Łukasiewicz.

Basic modal logic. Łukasiewicz was first considering necessary conditions for any logical theory to be regarded as modal logic. Those conditions constitute what Łukasiewicz called *basic modal logic* [5, § 1–2]. According to Łukasiewicz axiomatization of any modal logic is to be based upon CL. This clearly means that all substitutions of theorems of CL are theorems of modal logic, provided they are formulas of the logic. Subsequently four specific axioms are also assumed:

$$\vdash p \to \Diamond p \tag{1}$$

$$\dashv \Diamond p \to p \tag{2}$$

$$\dashv \Diamond p \tag{3}$$

$$\vdash \Diamond p \equiv \Diamond\neg\neg p \tag{4}$$

The fifth specific axiom defines the connective of necessity "$\Box$":

$$\vdash \Box p \equiv \neg\Diamond\neg p \tag{5}$$

Four primitive transformation rules are assumed: the Rule of Modus Ponens and the Rule of Uniform Substitution for theorems and their counterparts for rejected formulas. According to the Rule of Modus Ponens $\mathcal{B}$ is accepted, provided so are $\ulcorner\mathcal{A} \to \mathcal{B}\urcorner$ and $\mathcal{A}$:

$$\frac{\vdash \mathcal{A} \to \mathcal{B} \qquad \vdash \mathcal{A}}{\vdash \mathcal{B}} \tag{6}$$

According to the rejective counterpart of Modus Ponens $\mathcal{A}$ is rejected, provided $\ulcorner\mathcal{A} \to \mathcal{B}\urcorner$ is accepted but $\mathcal{B}$ is rejected:

$$\frac{\vdash \mathcal{A} \to \mathcal{B} \qquad \dashv \mathcal{B}}{\dashv \mathcal{A}} \tag{7}$$

Two rules of uniform substitution are also required. Assuming that $e\mathcal{A}$ is a uniform substitution of $\mathcal{A}$, according to the Rules of Uniform Substitution (a) if $\mathcal{A}$ is accepted, then so is $e\mathcal{A}$, (b) if $e\mathcal{A}$ is rejected, then so is $\mathcal{A}$. Of course, e is any arbitrary mapping of the set of schematic letters into the set of all object language formulas extended to endomorphism in a usual way.

There is another axiomatization of the basic modal logic: the connective of possibility "$\Diamond$" is here defined by means of the connective of necessity "$\Box$". In such case, instead of the specific axioms (1)–(5), the following specific axioms should be assumed:

$$\vdash \Box p \to p \tag{8}$$

$$\dashv p \to \Box p \tag{9}$$

$$\dashv \neg\Box p \tag{10}$$

$$\vdash \Box p \equiv \Box\neg\neg p \tag{11}$$

and the definition:

$$\vdash \Diamond p \equiv \neg\Box\neg p \tag{12}$$

The axioms adopted from CL and the transformation rules remain unchanged. Both axiomatizations give exactly the same set of theorems.

All the assumptions of the basic modal logic are based upon quite fundamental philosophical ideas concerning *alethic* modalities. Four famous scholastic bywords are reflected in axioms (1), (2), (8) and (9), i.e. respectively: *ab esse ad posse valet consequentia formalis* ("it is the case that" entails "it is possible that"), *a posse ad esse non valet consequentia formalis* ("it is possible that" does not entail "it is the case that"), *a necesse ad esse valet consequentia formalis* ("it is necessary that" entails "it is the case that") and *ab esse ad necesse non valet consequentia formalis* ("it is the case that" does not entail "it is necessary that"). They express the general concept of an alethic modality: necessity is something more than mere occurence (mere being or truth), while possibility is something less than mere occurence (mere being or truth). By these assumptions some non alethic modal concepts, like epistemic, moral or legal modalities, are excluded.

By axioms (3) and (10) collapsing of modal logic into CL is excluded. By the axiom (3) the connective "$\Diamond$" is not to be identified with the connective of tautology: not all formulas are possible (cf. f_5, Table 2).

By the axiom (10) the connective "$\Box$" is not to be identified with the connective of antilogy (cf. f_6, Table 2). Note that there is no need of the opposite restrictions. The connective "$\Diamond$" is not identifiable with the connective of antilogy by the axiom (1) and the connective "$\Box$" is not identifiable with the connective of tautology by the axiom (8) — provided the system is consistent.

So, one can say, necessity is something stronger than mere truth, but something weaker than antilogy. Whereas possibility is something weaker than mere truth, while still something stronger than tautology.

According to Łukasiewicz the axioms (3) and (10) have not been under any consideration in antiquity. It is not quite accurate. There was a great controversy between Peripatetic School on one side and Megarian and Stoic ones on the other over the distinction between modalities and mere truths.

Definitions (5) and (12) are claimed by Łukasiewicz to be obvious. In fact they were considered so by most logicians from antiquity to the present day. It should be, however, emphasised, that the definitions in question express one of many ways of relating modal concepts. Axioms (4) and (11) are deductively equivalent to definitions (12) and (5) respectively within the basic modal logic. Łukasiewicz decided just to use only one modal connective as primitive and define the other. He just did not want derivative terms to appear within the proper specific axioms. For both sets of axioms presented Łukasiewicz delivered proofs that specific axioms of the basic modal logic are independent from one another. He also prooved that the system in question is consistent.

C. A. Meredith's axiom. To complete the description of his logic, Łukasiewicz assumes one more axiom. Łukasiewicz intends to extend the well known principles of extensionality for the connectives of CL over all connectives, including the modal ones [5, §1, §4]:

$$\vdash (p \equiv q) \to (\Diamond p \to \Diamond q) \tag{13}$$

$$\vdash (p \equiv q) \to (\Box p \to \Box q) \tag{14}$$

The consequents of these formulas may be easily strengthened to equivalences. This point of Łukasiewicz's work — unlike his costruction of the basic modal logic — is philosophically highly controversial, but in this paper we put it aside. To achieve that objective Łukasiewicz assumes protothetic C. A. Meredith's axiom:

$$\vdash \delta(p) \to (\delta(\neg p) \to \delta(q)), \tag{15}$$

which is a kind of extension of the Duns Scotus' principle: if anything is true of a formula and of its negation, then the same is true of any formula at all. The formula (15) is sufficient as a single axiom of CL and Łukasiewicz extends the scope of the variable "δ" over of the formulas of the system Ł. To the effect of that the system Ł may be described by means of axioms: (15), (1), (2) and (3), both Rules of Modus Ponens and both Rules of Substitution, and the definition (5). The required theses of extensionality are theorems and so are all substitutions of theorems of CL [5, §5].

Four-valued matrix. There are several semantic descriptions of the system Ł. Łukasiewicz [5, § 6] himself provided a four-valued matrix, which is the product of two different submatrices of the classical matrix (both are definitionally complete). Let us first recall the tableaux defining seven functions f_1–f_7 of the classical matrix. On Table 1 we present binary functions and on Table 2 unary functions. Of course, f_1 is the

f_1	1	0
1	1	0
0	0	0

f_2	1	0
1	1	1
0	1	0

f_3	1	0
1	1	0
0	1	1

f_4	1	0
1	1	0
0	0	1

Table 1. Tableaux of classical binary functions

x	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
1	1	0	1	0
0	1	0	0	1

Table 2. Tableaux of classical unary functions

function of conjunction, f_2 is the function of disjunction, f_3 is the function od implication, f_4 is the function of equivalence, f_5 is the function of tautology, f_6 is the function of antilogy, f_7 is the function of affirmation and f_8 is the function of negation. Now, consider two similar matrices:

$$\mathfrak{C}_1 = \langle \{1, 0\}, \{1\}, f_1, f_2, f_3, f_4, f_7, f_7, f_8 \rangle,$$
$$\mathfrak{C}_2 = \langle \{1, 0\}, \{1\}, f_1, f_2, f_3, f_4, f_5, f_6, f_8 \rangle.$$

In both cases 1 is the designated value. The product $\mathfrak{C}_1 \times \mathfrak{C}_2$ of those two matrices is the four-valued matrix

$$\mathfrak{L} = \langle \{11, 10, 01, 00\}, \{11\}, f_{11}, f_{22}, f_{33}, f_{44}, f_{75}, f_{76}, f_{88} \rangle.$$

Referring to the values of the matrix $\mathfrak{L}$ instead of (x, y) we say simple xy. The designated value of $\mathfrak{L}$ is 11. In the language £ f_{11} is the function of conjunction, f_{22} is the function of disjunction, f_{33} is the function of implication, f_{44} is the function of equivalence, f_{75} is the function of possibility, f_{76} is the function of necessity and f_{88} is the function of negation. The result functions are presented on tableaux 3 and 4.

f_{11}	11	10	01	00
11	11	10	01	00
10	10	10	00	00
01	01	00	01	00
00	00	00	00	00

f_{22}	11	10	01	00
11	11	11	11	11
10	11	10	11	10
01	11	11	01	01
00	11	10	01	00

f_{33}	11	10	01	00
11	11	10	01	00
10	11	11	01	01
01	11	10	11	10
00	11	11	11	11

f_{44}	11	10	01	00
11	11	10	01	00
10	10	11	00	01
01	01	00	11	10
00	00	01	10	11

Table 3. Binary functions of the matrix $\mathfrak{L}$

x	$f_{75}(x)$	$f_{76}(x)$	$f_{88}(x)$
11	11	10	00
10	11	10	01
01	01	00	10
00	01	00	11

Table 4. Unary functions of the matrix $\mathfrak{L}$

To prove that $\mathfrak{L}$ is an adequate matrix of £ Łukasiewicz provided an argument, which is not quite clear [5, § 6]. This argument has been regimented and improved by T. J. Smiley [6] and Lemmon [3]. Anyway, $\mathfrak{L}$ is actually an adequate matrix of £.

Decidability via interpretation. The matrix $\mathfrak{L}$ provides a decidability procedure for £. A version of the procedure is combined interpretation in CL. It is easy to observe on tables 1–4, that functions f_{11}, f_{22}, f_{33}, f_{44} and f_{88} are just second powers of the respective classical functions f_1, f_2, f_3, f_4 and f_8. So the connectives of conjunction, disjunction, implication and equivalence in £ are just classical, truth-functional connectives.

Whereas modal connectives are reduced to classical in a more complicated way. A possibility formula has the desiganted value if and only if so do both respective formulas with affirmation f_7 and with tautology f_5. And a necessity formula has the desiganted value if and only if so do both respective formulas with affirmation f_7 and with antilogy f_6. This observation is often attributed to Arthur N. Prior [1, p. 170]. Actually it has been claimed by Łukasiewicz himself in his original paper on Ł, although Łukasiewicz emphasised this fact wit respect to the connective of possibility [5, § 7]. Nevertheless the general result is quite easy to extend by (IP) having the first claim established.

Lemmon's axiomatics. A very elegant and, so to say, contemporarily standard, formalization of Ł thus far, comes from Lemmon. He put aside the rejected formulas and described Ł as typical axiomatic system, containing only accepted theorems. Lemmon's axiomatization is also invariant, so one has to talk about metalanguage structural schemata beside object language formulas in the strict sense of the word. However, sometimes, having ensured misunderstanding be not imminent, we say simply of axioms or theorems instead of schemata of axioms or theorems.

The system Ł is a conservative extension of the system CL, which means that all substitutions of theorems of CL, being formulas of Ł, are assumed as adopted axioms of the latter systems. So, let $\mathcal{A}$ be any formula of Ł:

$$\text{if } \mathcal{A} \text{ is a substitution of any theorem of CL, then } \mathcal{A} \text{ is an axiom of Ł.} \tag{CL}$$

Besides adopted axioms one has to assume the specific axioms. So, let $\mathcal{A}$ and $\mathcal{B}$ ba any formulas of Ł, all formulas:

$$\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}) \tag{K}$$

$$\Box\mathcal{A} \to \mathcal{A} \tag{T}$$

$$\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B}) \tag{L}$$

are specific axioms of the system Ł. The Rule of Modus Ponens is assumed as a sole rule of inference: if formulas $\ulcorner\mathcal{A} \to \mathcal{B}\urcorner$ and $\mathcal{A}$ are both theorems of the system Ł, then so is the formula $\mathcal{B}$, schematically:

$$\frac{\begin{array}{l}\mathcal{A} \to \mathcal{B}\\ \mathcal{A}\end{array}}{\mathcal{B}} \tag{MP}$$

Moreover modal connectives are interchangable in a usual manner:

$$\ulcorner\Diamond\mathcal{A}\urcorner \stackrel{\text{df}}{=} \ulcorner\neg\Box\neg\mathcal{A}\urcorner. \tag{IP}$$

Adopted axioms (CL), specific axioms (K), (T) and (L), the rule of inference (MP) and the rule of replacement (IP) constitute the description of the system £ provided by Lemmon in his work [3]. We will shortly endeavour to show Lemmon's axiomatics be not independent and open to improvement.

Problem of independence of Lemmon's axiomatics. One can observe that the set of specific axiom schemata in Lemmon's axiomatization is not independent. Particularly the schema (K) is derivable from schemata (T) and (L). Therefore these two specific axioms are sufficient description of the system £. Firstly, we prove that from those two axioms we obtain the following schema:

$$(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}) \tag{M}$$

This schema was a matter of special focus of Łukasiewicz, who attributed it to Aristotle.

1. $(\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B}))$	(L)
2. $\Box\mathcal{A} \to \mathcal{A}$	(T)
3. $1. \to (\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B}))$	(CL)
4. $\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B})$	3, 1 × (MP)
5. $2. \to (\neg\mathcal{A} \to (\Box\mathcal{A} \to \Box\mathcal{B}))$	(CL)
6. $\neg\mathcal{A} \to (\Box\mathcal{A} \to \Box\mathcal{B})$	5, 2 × (MP)
7. $(\neg\mathcal{A} \to (\Box\mathcal{A} \to \Box\mathcal{B})) \to ((\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B})) \to$ (M))	(CL)
8. $(\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B})) \to ((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}))$	7, 6 × (MP)
9. $(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}))$	8, 4 × (MP)

Secondly, we prove the schema (K) from (T), (M) and (CL):

1. $\Box(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{B})$	(T)
2. $(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$	(M)
3. $1. \to (2. \to (\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})))$	(CL)
4. $2. \to (\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}))$	3, 1 × (MP)
5. $\Box(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$	4, 2 × (MP)

This ends the proof of our claim: schemata (T) and (L) describe the system £ even in the absence of the schema (K) in the set of axioms.

Lemmon's axiomatics and Łukasiewicz's axiomatics. To become convinced that Lemmon's description of the system Ł is in fact inferentially equivalent to the original Łukasiewicz's one, observe the following derivation of the schema (L) from invariantly taken set {(T), (14)}:

1. $\Box\mathcal{A} \to \mathcal{A}$ (T)
2. $(\mathcal{A} \equiv \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$ (14)
3. $(\Box\mathcal{A} \to \mathcal{A}) \to (\Box\mathcal{A} \to (\mathcal{B} \to (\mathcal{A} \equiv \mathcal{B})))$ (CL)
4. $\Box\mathcal{A} \to (\mathcal{B} \to (\mathcal{A} \equiv \mathcal{B}))$ 1, 3 × (MP)
5. 2. → (4. → $(\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B})))$ (CL)
6. 4. → $(\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B}))$ 5, 2 × (MP)
7. $\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B})$ 6, 4 × (MP)

The derived schema is, of course, exemplified by the formula (8). The schema (K) has already been derived from schemata (T) and (L). On the other hand the formula (12) is obviously derivable from (IP) and (CL), the formulas (11) and (14) are obviously derivable from (M) and (CL), and finally the formula (13) is obviously derivable from (M), (CL) and (IP).

Regularity of the system Ł. It may be easily observed, the system Ł be a regular modal logic. To become convinced of that notice that the following rule of monotonicity:

$$\frac{\mathcal{A} \to \mathcal{B}}{\Box\mathcal{A} \to \Box\mathcal{B}} \tag{RM}$$

is immidiately derivable in the logic Ł by means of the schema (M) and the rule (MP). And it is a well known fact that the following rule of regularity:

$$\frac{(\mathcal{A} \land \mathcal{B}) \to \mathcal{C}}{(\Box\mathcal{A} \land \Box\mathcal{B}) \to \Box\mathcal{C}} \tag{RR}$$

is derivable in the presence of the schema (K) and the rule (RM). This makes the system Ł regular.

Relational semantics. Having axiomatized the system Ł and proven the matrix $\mathfrak{L}$ to be adequate for it, Lemmon [3] provided also relational semantics for the system Ł as a special case of the general schema of regular modal logic semantics.

Let us define a frame $\mathfrak{F} = \langle W, Q, R\rangle$ and a model $\mathfrak{M} = \langle W, Q, R, V\rangle$ on the frame $\mathfrak{F}$. Here W and Q are sets, R is a relation and V is a function. We assume that $W \neq \varnothing$, $Q \subseteq W$, $R \subseteq W \times W$ and V maps the set of formulas into 2^W (the power set of W). The set W is usually counted as a set of *possible worlds* and its subset Q as a set of *queer* possible worlds. In Lemmon's construction in a queer world anything (including contradiction) is possible and nothing is necessary [3, p. 57]. Possible worlds that are not queer, are called *normal.* R is counted as an accessibility relation between worlds and V is taken to attribute a world to a formula exactly when the formula is true in the world. That $x \in V(\mathcal{A})$ is understood as $\mathcal{A}$'s being true in the world x. If $\mathcal{A}$ is a senence letter, $V(\mathcal{A})$ may be arbitrary, as long as $V(\mathcal{A}) \subseteq W$. For molecular formulas values of V meet the conditions:

$x \in V(\neg\mathcal{A})$ if and only if $x \notin V(\mathcal{A})$,
$x \in V(\mathcal{A} \wedge \mathcal{B})$ if and only if $x \in V(\mathcal{A})$ and $x \in V(\mathcal{B})$,
$x \in V(\mathcal{A} \vee \mathcal{B})$ if and only if $x \in V(\mathcal{A})$ or $x \in V(\mathcal{B})$,
$x \in V(\mathcal{A} \to \mathcal{B})$ if and only if $x \notin V(\mathcal{A})$ or $x \in V(\mathcal{B})$,
$x \in V(\mathcal{A} \equiv \mathcal{B})$ if and only if $x \in V(\mathcal{A}) \cap V(\mathcal{B})$ or $x \notin V(\mathcal{A}) \cup V(\mathcal{B})$,
$x \in V(\Box\mathcal{A})$ if and only if $x \notin Q$ and for all y, if xRy then $y \in V(\mathcal{A})$,
$x \in V(\Diamond\mathcal{A})$ if and only if $x \in Q$ or for some y, xRy and $y \in V(\mathcal{A})$.

One says that a formula $\mathcal{A}$ is true in a model $\mathfrak{M}$ ($\mathfrak{M} \models \mathcal{A}$) if and only if $\mathcal{A}$ is true in all worlds of this model (i.e. $V(\mathcal{A}) = W$). One says that a formula $\mathcal{A}$ is true in a frame $\mathfrak{F}$ ($\mathfrak{F} \models \mathcal{A}$) if and only if $\mathcal{A}$ is true in all models on the frame $\mathfrak{F}$, i.e. is true for any function V, having established W, Q and R. A formula of modal logic is considered as *valid* if and only if it is true in all frames of a particular kind, ascribed to a particular system of modal logic.

Relational semantics related to the system £ is constituted by the set of those frames $\mathfrak{F}$ meeting two following conditions:

$$xRy \to x = y, \tag{16}$$

$$x \notin Q \to xRx. \tag{17}$$

According to the first condition R is a subrelation of identity: no world can see anything except itself. According to the other any world can see itself, unless it is queer. If R meets the condition $\forall x\, xRx$, we call it *reflexive.* So R is to be here reflexive in the set $W \setminus Q$ (*normal worlds*). According to the above formulated conditions, in a frame being considered any normal world can see exactly itself and any queer world can see at most itself. A formula of £ is regarded as valid if it is true in all frames of the above described kind. Lemmon proved the system £ to be sound and complete with respect to the set of such frames.

Some theorems of £. In the system £ there is a lot of laws of distribution of modal connectives over classical ones. All these schemata are provable in the system £ [5, §8–9]:

$$\Box(\mathcal{A} \wedge \mathcal{B}) \equiv (\Box\mathcal{A} \wedge \Box\mathcal{B}) \tag{18}$$

$$\Box(\mathcal{A} \vee \mathcal{B}) \equiv (\Box\mathcal{A} \vee \Box\mathcal{B}) \tag{19}$$

$$\Diamond(\mathcal{A} \wedge \mathcal{B}) \equiv (\Diamond\mathcal{A} \wedge \Diamond\mathcal{B}) \tag{20}$$

$$\Diamond(\mathcal{A} \vee \mathcal{B}) \equiv (\Diamond\mathcal{A} \vee \Diamond\mathcal{B}) \tag{21}$$

$$\Diamond(\mathcal{A} \to \mathcal{B}) \equiv (\Diamond\mathcal{A} \to \Diamond\mathcal{B}) \tag{22}$$

Only two of those schemata, namely (18) and (21), are provable in the most typical modal calculi. On the other hand, the schema

$$\Box(\mathcal{A} \to \mathcal{B}) \equiv (\Box\mathcal{A} \to \Box\mathcal{B})$$

is not provable in £, for it is false in any queer world x: notice, that $x \notin V(\Box\mathcal{A})$, so $x \in V(\Box\mathcal{A} \to \Box\mathcal{B})$, whereas $x \notin V(\Box(\mathcal{A} \to \mathcal{B}))$, as long as $x \in Q$.

In £ there are theorems of the form of disjunction $\ulcorner\mathcal{A} \vee \mathcal{B}\urcorner$, such that the disjuncts $\mathcal{A}$ and $\mathcal{B}$ do not share any sentence letters and neither of them is itself a theorem of £. By (L), (IP) and (CL), the following formula

$$\Diamond q \vee (\Diamond p \to p) \tag{L$'$}$$

may well serve as an example, for it is a theorem of £, while neither "$\Diamond q$" nor "$\Diamond p \to p$" is a theorem.

This fact may be easily explained by means of double interpretation in CL, described above (cf. page 221). Due to the presence of the classical connectives all formulas

$$\begin{aligned} &\mathcal{A} \to (\mathcal{A} \vee \mathcal{B}) \\ &\mathcal{B} \to (\mathcal{A} \vee \mathcal{B}) \end{aligned} \tag{23}$$

are theorems of £. So, for any disjunction, if one or the other disjunct is a theorem, so is the disjunction. Let us now choose any formula $\mathcal{A}$ of £ which is a theorem of CL, if "$\Box$" and "$\Diamond$" are interpreted as the connective of affirmation f_7, but not if "$\Box$" is interpreted as the connective of antilogy f_6 and "$\Diamond$" is interpreted as the connective of tautology f_5. Of course $\mathcal{A}$ is not a theorem of $\pounds$. Let us now choose any formula $\mathcal{B}$ of $\pounds$ which is not a theorem of CL, if "$\Box$" and "$\Diamond$" are interpreted as the connective of affirmation f_7, but is a theorem of CL if "$\Box$" is interpreted as the connective of antilogy f_6 and "$\Diamond$" is interpreted as the connective of tautology f_5. But $\mathcal{B}$ is not a theorem of $\pounds$ as well. However, due to theorems (23) — of course, by means of (MP) — the disjunction $\ulcorner \mathcal{A} \vee \mathcal{B} \urcorner$ must be a theorem of CL in both considered interpretations, and so is a theorem of $\pounds$. As an example let us take two formulas "$\Diamond p \to p$" and "$\Diamond q$" explicitly rejected by Łukasiewicz (see (2) and (3), respectively).

If "$\Diamond$" was the connective of affirmation f_7, then (2) would be a theorem of CL, but neither would (3). However the disjunction (L′) of those two formulas must have been a theorem of CL, as its latter disjunct would be a theorem of CL. Now, if "$\Diamond$" was the connective of tautology f_5, then (3′) would be a theorem of CL, but neither would (2). And again, the disjunction (L′) must have been a theorem of CL, since its former disjunct would be a theorem of CL. It follows that the disjunction (L′) is a theorem of $\pounds$, although neither $\mathcal{A}$ nor $\mathcal{B}$ is a theorem of $\pounds$. In fact, if both letters "p" and "q" have the value 00, than the formula "$\Diamond p \to p$" has the value 10 and the formula "$\Diamond q$" has the value 01. So neither of formulas (2), (3′) turns out to be a theorem of the system $\pounds$, whereas the formula (L′) is a tautology of $\mathfrak{L}$.

Parallel explanation may be delivered on the base of relational semantics. As we have said in all frames $\mathfrak{F}$ related to the system $\pounds$ the accessibility relation R meets two conditions: (16) and (17). The formula (2) is true in all qeer worlds, but false in some normal ones, so is not valid. On the other hand the formula (3′) is true in all normal worlds but false in some queer worlds. It is true in all normal worlds due to the condition (16) and (17): any normal world can see exactly itself, so any formula $\mathcal{A}$ is true in that world, provided it is true in any world accessible. However, the formula (3′) is false in any queer world x, such that $x \notin V(p)$. Because x is queer, $x \in V(\Diamond p)$. So, again, (3′) is not valid. However, in the frame $\mathfrak{F}$ any world is either normal or queer. Therefore for any world x, either $x \in V(\Diamond q)$, or $x \in V(\Diamond p \to p)$. Therefore the disjunction (L′) is true in all worlds of any model, on any frame related to $\pounds$, and so is valid.

3. Alternative Axiomatizations of £

We are about to recomend some alternative ways of derscription of the system £, having Lemmon's axiomatics as a starting point. There are some elements that remain unchanged in all axiomatizations we discuss: the set (CL) of adopted axioms, the rule (MP) and (IP). So we modify the schemata of specific axioms of £.

Variations on the Theme of Kripke's Axiomatics. Independently of Lemmon, Saul Kripke provided an axiomatics of the system £ [2]. That axiomatics consists of two specific schemata: (M) and (T). Kripke has not examined the relationship of the axiomatics to any other description of the system £.

For the schema (L) looks weaker than the schema (M), it may be interesting to show Kripke's axiomatics being equivalent to the collection {(L), (T)} of schemata (and by that to the Lemmon's axiomatics as well). Firstly, on p. 223 we have already proven the schema (M) on the base {(T), (L)}. Notice that derivation of the schema (M) from {(L), (K)} also exists. Secondly, the following derivation proves (L) from (M):

1. $\mathcal{B} \to (\mathcal{A} \to \mathcal{B})$	(CL)
2. $(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})$	(M)
3. $1. \to (2. \to (\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B})))$	(CL)
4. $2. \to (\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B}))$	3, 1 × (MP)
5. $\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B})$	4, 2 × (MP)
6. $(\mathcal{B} \to (\Box\mathcal{A} \to \Box\mathcal{B})) \to (\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B}))$	(CL)
7. $\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B})$	6, 5 × (MP)

This ends the proof of equivalence of both presented sets of axioms. Nota bene, on the page 223 we have shown a derivation of the schema (K) from the set {(T), (M)}.

It may also be of sam interest that both, in the axiomatics {(T), (L)} (resp. {(K), (T), (L)}) and in the axiomatics {(T), (M)}, the schema (T) may be equivalently replaced with the following schema:

$$\Box\mathcal{A} \to \Diamond\mathcal{A} \qquad \text{(D)}$$

which — as it is well known — is in turn equivalent to the schema:

$$\Diamond(\mathcal{A} \to \mathcal{A}) \qquad \text{(P)}$$

within the confines of any regular modal logic. Obviously any schema $\ulcorner\Diamond\top\urcorner$, with $\top$ being an arbitrary theorem, may be used instead of (P). It is interesting because in the field of typical modal calculi the schema (D) is weaker then (T).

Most derivations required here are absolutely commonly known and so may be omitted. It is enough to show (T) be derivable. Observe, that the schema (L), being an axiom in the latter case, has been already proved on the base of the axiom (M) and the adopted axioms alone (without any use of the axiom (T)). So, it is enough to derive the schema (T) on the base of (L) and (D):

1. $(\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\neg\Box\neg\mathcal{A} \to \neg\neg\mathcal{A})$ (CL)
2. $(\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \neg\neg\mathcal{A})$ 1 × (IP)
3. $\neg\neg\mathcal{A} \to \mathcal{A}$ (CL)
4. $(\neg\neg\mathcal{A} \to \mathcal{A}) \to ((\Diamond\mathcal{A} \to \neg\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A}))$ (CL)
5. $(\Diamond\mathcal{A} \to \neg\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A})$ 4, 3 × (MP)
6. $((\Diamond\mathcal{A} \to \neg\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A})) \to (((\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to$ $\to (\Diamond\mathcal{A} \to \neg\neg\mathcal{A})) \to ((\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A})))$ (CL)
7. $((\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \neg\neg\mathcal{A})) \to$ $\to ((\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A}))$ 6, 5 × (MP)
8. $(\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A})$ 7, 1 × (MP)
9. $((\neg\mathcal{A} \to \Box\neg\mathcal{A}) \to (\Diamond\mathcal{A} \to \mathcal{A})) \to ((\Box\mathcal{A} \to (\neg\mathcal{A} \to \Box\neg\mathcal{A})) \to$ $\to (\Box\mathcal{A} \to (\Diamond\mathcal{A} \to \mathcal{A})))$ (CL)
10. $(\Box\mathcal{A} \to (\neg\mathcal{A} \to \Box\neg\mathcal{A})) \to (\Box\mathcal{A} \to (\Diamond\mathcal{A} \to \mathcal{A}))$ 9, 8 × (MP)
11. $\Box\mathcal{A} \to (\neg\mathcal{A} \to \Box\neg\mathcal{A})$ (L)
12. $\Box\mathcal{A} \to (\Diamond\mathcal{A} \to \mathcal{A})$ 10, 11 × (MP)
13. $(\Box\mathcal{A} \to (\Diamond\mathcal{A} \to \mathcal{A})) \to ((\Box\mathcal{A} \to \Diamond\mathcal{A}) \to (\Box\mathcal{A} \to \mathcal{A}))$ (CL)
14. $(\Box\mathcal{A} \to \Diamond\mathcal{A}) \to (\Box\mathcal{A} \to \mathcal{A})$ 13, 12 × (MP)
15. $\Box\mathcal{A} \to \Diamond\mathcal{A}$ (D),
16. $\Box\mathcal{A} \to \mathcal{A}$ 14, 15 × (MP)

So, from {(D), (L)} we obtain both (T) and (M). Thus, we have at least two new descriptions of £: {(D), (L)} and {(D), (M)}.

The system £ based on a single axiom. However, having axiomatics of the system £ simplyfied thus far, a natural question arises of a single, simple, specific axiom of the logic in question. Of course, one can just assume conjunction of any two presented axioms. But there exist a little more simple and more elegant solutions of the problem. Having proven schemata (T) and (L) to describe the system £, one gets immediately the single axiom of the system:

$$\Box\mathcal{A} \to (\mathcal{A} \wedge (\mathcal{B} \to \Box\mathcal{B})) \tag{24}$$

The schema (24)'s being equivalent to schemata (T) and (L) is easily provable by means of the axiom (CL):

$$((\Box\mathcal{A} \to \mathcal{A}) \wedge (\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B}))) \equiv (\Box\mathcal{A} \to (\mathcal{A} \wedge (\mathcal{B} \to \Box\mathcal{B})))$$

only. This exempts us from providing here a strict proof. But one can also observe another simple candidate for the single axiom of £:

$$(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{B}) \tag{25}$$

for all $\mathcal{A}$ and $\mathcal{B}$ being formulas, can serve as the single axiom of the system £. Derivability of the axiom (M) from (25) is quite obvious and rests fully upon the axiom (CL):

$$((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{B})) \to ((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B}))$$

Here is the derivation of the schema (D):

1. $(\mathcal{A} \to \mathcal{A}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{A})$ (25)
2. $\mathcal{A} \to \mathcal{A}$ (CL)
3. $\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{A}$ 2, 1 × (MP)
4. $(\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{A}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A})$ (CL)
5. $\Box\mathcal{A} \to \Diamond\mathcal{A}$ 4, 3 × (MP)

To prove the schema (25) be a theorem of the system £, we use our axiomatics (M), (D):

1. $(\Box\mathcal{A} \to \Diamond\mathcal{A}) \to ((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A}))$ (CL)
2. $(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A})$ 1, (D) × (MP)
3. $((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A})) \to (((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})) \to$ $\to ((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{B})))$ (CL)
4. $((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Box\mathcal{B})) \to$ $\to ((\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{B}))$ 3, 2 × (MP)
5. $(\mathcal{A} \to \mathcal{B}) \to (\Box\mathcal{A} \to \Diamond\mathcal{A} \wedge \Box\mathcal{B})$ 4, (M) × (MP)

This is sufficient to prove our claim. So, simple schemata may be found to serve as single specific axioms of the system £. There is one more schema to draw our attention as a single axiom of £.

The jumping necessity axiom. The most interesting and simple single specific axiom of Ł we know thus far is the schema

$$\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B} \tag{26}$$

It might be called a *jumping necessity axiom*. According to (26), for any formulas $\mathcal{A}$, $\mathcal{B}$, if $\mathcal{A}$ is necessary and $\mathcal{B}$ is true, then $\mathcal{A}$ is also true and $\mathcal{B}$ is necessary. The proof:

1. $(\Box\mathcal{A} \to \mathcal{A}) \to (\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A})$ (CL)
2. $\Box\mathcal{A} \to \mathcal{A}$ (T)
3. $\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A}$ 1, 2 × (MP)
4. $(\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B})) \to (\Box\mathcal{A} \wedge \mathcal{B} \to \Box\mathcal{B})$ (CL)
5. $\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B})$ (L)
6. $\Box\mathcal{A} \wedge \mathcal{B} \to \Box\mathcal{B}$ 4, 5 × (MP)
7. $(\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A}) \to ((\Box\mathcal{A} \wedge \mathcal{B} \to \Box\mathcal{B}) \to$ $\to (\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B}))$ (CL)
8. $(\Box\mathcal{A} \wedge \mathcal{B} \to \Box\mathcal{B}) \to (\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B})$ 7, 3 × (MP)
9. $\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B}$ 8, 6 × (MP)

So, the schema (26) is derivable in the system Ł.

Let us now show the schemata (T) and (L) be derivable from (26). Actually those two derivations are quite obvious. Here is the derivation of the schema (T) on the base of the schema (26):

1. $\Box\mathcal{A} \wedge \Box\mathcal{A} \to \mathcal{A} \wedge \Box\Box\mathcal{A}$ (26)
2. $(\Box\mathcal{A} \wedge \Box\mathcal{A} \to \mathcal{A} \wedge \Box\Box\mathcal{A}) \to (\Box\mathcal{A} \to \mathcal{A})$ (CL)
3. $\Box\mathcal{A} \to \mathcal{A}$ 2, 1 × (MP)

and the derivation of the schema (L) on the same base:

1. $\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B}$ (26)
2. $(\Box\mathcal{A} \wedge \mathcal{B} \to \mathcal{A} \wedge \Box\mathcal{B}) \to (\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B}))$ (CL)
3. $\Box\mathcal{A} \to (\mathcal{B} \to \Box\mathcal{B})$ 2, 1 × (MP)

So, actually, adopted axioms (CL), the single specific axiom (26), the rules (MP) and (IP) constitute the description of the system Ł. That would be our main claim in this paper.

References

[1] Font, J. M., and P. Hájek, "On Łukasiewicz's four-valued modal logic", *Studia Logica* 70, 2 (2002): 157–182.
[2] Kripke, S., "Semantical analysis of modal logic II", pages 206–220 in: *The Theory of Models. Proc. of 1963 International Symposium at Berkeley*, J. W. L. Henkin and A. Tarski (eds.), Nord-Holland, Amsterdem, 1965.
[3] Lemmon, E. J., "Algebraic semantics for modal logics", *The Journal of Symbolic Logic* 31 (1966), part I: p. 46–65; part II: 191–218.
[4] Łukasiewicz, J., "On variable functors of propositional arguments", *Proceedings of the Royal Irish Academy* 54 (1951), A 2: 25–35.
[5] Łukasiewicz, J., "A system of modal logic", pages 352–390 in: J. Łukasiewicz, *Selected Works*, L. Borkowski (ed.), transl. from Polish by O. Wojtasiewicz, North-Holland, Amsterdam 1970.
[6] Smiley, T. J., "On Łukasiewicz's Ł-modal System", *Notre Dame Journal of Formal Logic* 2 (1961): 149–153.

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