Logic and **L**ogical **P**hilosophy
Volume 19 (2010), 195–242
DOI: 10.12775/LLP.2010.008

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CAN CONCEPTS BE DEFINED IN TERMS OF SETS?

Abstract. The goal of this paper is a philosophical explication and logical rectification of the notion of concept. We take into account only those contexts that are relevant from the logical point of view. It means that we are not interested in contexts characteristic of cognitive sciences, particularly of psychology, where concepts are conceived of as some kind of mental objects or representations. After a brief recapitulation of various theories of concept, in particular Frege's and Church's ones, we propose our own theory based on procedural semantics of Transparent Intensional Logic (TIL) and explicate concept in terms of the key notion of TIL, namely *construction* viewed as an abstract, algorithmically structured procedure.

Keywords: Procedural semantics, Transparent Intensional Logic, concepts, structured meanings.

Introduction

The term 'concept' is frequently used in various contexts but hardly well understood. The intuitions connected with its use are vague, and thus an *explication* is needed. The goal of this paper is a philosophical explication and logical rectification of the notion of concept. We will take into account only those contexts that are relevant from the *logical* point of view. It means that we are not interested in contexts characteristic of cognitive sciences, particularly of psychology, where concepts are conceived

Received July 14, 2010; *Revised* August 18, 2010

ISSN: 1425-3305

of as some kind of mental objects or representations. Hence concepts conceived as *mental* entities, as for instance, Fodor characterizes them in [1998, p. 23]

> Concepts are mental particulars; specifically, they satisfy whatever ontological conditions have to be met by things that function as mental causes and effects.

are not a subject of our scrutiny here. We aim at *logical* explication and conceive concepts as *objective*, extra-mental entities. Thus Aristotle's theory of definitions, Bolzano's *Begriffe*, Kauppi's theory of conceptual systems based on Aristotle and Frege, Bealer's inspiring [1982] and, of course, Frege's [1891] and [1892] theory are examples of studies relevant for our explication.

When comparing contemporary and traditional textbooks, one is likely to come away with the impression that contemporary logic is no more interested in studying concepts. No wonder; a chapter dealing with concepts in the traditional textbooks (mostly by German authors like Ziehen or Prantl) has been usually so much influenced by mentalistic (psychologistic) conceptions that it is of no interest for modern philosophers and logicians. True, already in 1837 the psychologistic tradition of construing concepts as a sort of mental objects (and thus of nil interest to logic) was dealt a serious blow by Bolzano, who worked out, in his *Wissenschaftslehre*, a systematic realist theory of concepts. In Bolzano concepts are construed as objective entities endowed with structure. But his ingenious work was not well-known at the time when modern logic was founded by Frege and Russell. Thus the first theory of concepts that was recognized as being compatible with modern, entirely anti-psychologistic logic was Frege's [1891] and [1892]. A notable conception of concepts has been presented by Church [1956] who tries to adhere to Frege's principles of semantics, but comes to realize that Frege's explication of the notion of concept is untenable.

Thus we can ask what shape would a modern theory of concepts be of if it were free from any psychologistic features. To sum up, Aristotle's theory seems to be too remote from the ideas that underlie modern logic. Bolzano is very inspiring but his language is specific and difficult to understand. Kauppi can hardly be understood if Frege's theory is not known. Setting aside some more or less specific theories like Bealer's or Peacocke's, our starting point to elaborate a modern theory of concepts might be Frege's [1891] and [1892]. Yet we are going to show that Frege's attempt yields a misconception, because Frege's notion of concept is a notion of class and is thus superfluous. Moreover, Frege's concepts are untenable due to their set-theoretical character, because concepts should be structured entities that can be used to identify other entities while sets lack a structure. In particular, sets cannot be *executed* to obtain an entity. In this paper we are mostly inspired by Bolzano and propose a *procedural* conception of structured concepts based on Transparent intensional logic (TIL). Moreover, in the Appendix we show that Gödel's uneasiness concerning concepts in [1944] can be most naturally explained from the viewpoint of procedural theory of concepts.

1. Frege on concepts[1]

Frege's theory, as presented in [1891] and [1892] (see also Frege [1952] and [1971]) construes concepts as total monadic *functions* whose arguments are objects (*Gegenstände*) and whose values are truth-values. At first sight this definition seems to be plausible. The concept of *dog* could be such a function: for such objects that are dogs the function takes the truth value **T**, for all other objects it takes **F**. However, this conception is vulnerable to several objections that we will list below. The positive feature of this conception is the fact that the so defined Frege's concepts comply with the principle of extensionality [1971, 25]:

> dass nämlich, unbeschadet der Wahrheit, in jedem Satze Begriffswörter einander vertreten können, wenn ihnen derselbe Begriffsumfang entspricht, [. . .].

If concepts are Fregean functions, i.e., 'unsaturated entities', then the expression ('Begriffswort') that denotes a concept should never stand in the position of grammatical subject. A grammatical subject should stand for an *object* ('Gegenstand'), whereas concepts are functions; hence, a concept is no object from the viewpoint of Frege's dichotomy between *Gegenstand* and *Funktion* (*Begriff*). This solution is possible as soon as we distinguish between Frege's notion of function as an unsaturated entity and his notion of *Wertverlauf*. The former is far from being clear

[1]Here we do not want to multiply numerous texts concerning Frege's philosophy and, in particular, his theory of concepts. We provide only a brief recapitulation of those features that are relevant from our point of view.

but *might* be taken to be a notion of function-in-intension, as suggested by Church in [1941, pp. 2–3], whereas the latter seems to correspond to the notion of *function as a mapping* or function-in-extension.[2]

In his famous polemic with Kerry in [1952] Frege defends his view that in sentences of the form

The concept X is . . .

the first three words make up a name of an *object*, not of a concept so that the sentence

The concept of horse is not a concept

is a meaningful and even true sentence. The core of his argument consists in the claim that a concept, being a function, is 'unsaturated' ('predicative') whereas if *the concept X* is itself an object of predication then it cannot be completed by an argument and is thus a name of an *object*.

Yet the intuition that a concept remains a concept whether used to identify an object or *mentioned* as being itself an object of predication (to use our terminology) has much to be said for it, since it seems that Frege's concept vacillates between being a concept and being an object only relative to a flawed theory of concepts.

Another notable consequence of Frege's definition of concepts can be found in [1952, pp. 51–52], where Frege distinguishes between *properties* of an object and *marks* (*Merkmale*) of a concept.[3] Briefly: Let Φ, X, Ψ be properties of an object Γ, and let their sum be denoted Ω. Then Φ, X, Ψ are marks of Ω. This issue is related to the traditional doctrine of concepts as known from textbooks based on simplified Aristotelian theory of definitions. The same holds of Frege's distinguishing between

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[2]To anticipate a possible misunderstanding, note that in the semantics of mathematics, the terms 'function-in-intension' and 'function-in-extension' are used in this sense: function-in-extension corresponds to the modern notion of function as a mapping, and function-in-intension could arguably correspond to our notion of *construction* of a function, see below. Thus function-in-intension is a structured *way* or a *rule* how to obtain the function-in-extension. However, since the notion of function-in-intension is a vague one, and obviously dependent on the formal system in which the meaning of the correspondence rule is captured, we will *not* use the term 'function-in-intension'.

[3]See similar considerations in Bolzano [1837].

content (*Inhalt*) and extension (*Umfang*) of a concept.[4] It follows from Frege's scheme that any 'concept word' (*Begriffswort*) possesses its 'reference' (or 'denotation' in Church's terminology) as well as its 'sense'. The reference is the respective concept but being the reference it must be obviously represented by its Wertverlauf, which is an object (*Gegenstand*) according to Frege. Now there are two questions:

(a) What are the content and the extension of a concept?

(b) What is the sense of a concept word?

It is far from clear what answer could Frege propose to the question (b). After all, no genuine definition of sense can be found in Frege's work.[5] As for the question (a), it is obviously Wertverlauf what can be called *extension*. So it seems that it is the sense of the concept word what can be conceived as the content of a concept. This is well compatible with Frege's criticism of "Inhaltslogiker" in [1971, pp. 31–32].

In his [1956] Church tries to adhere to Frege's principles of semantics, but comes to realize that Frege's explication of the notion of concept is untenable. Concepts should be located on the level of Fregean *sense* in fact, as Church maintains, the *sense* of an expression E should be a *concept* of what E denotes. Consequently, concepts should be associated not only with predicates (as was the case of Frege), but also with definite descriptions, and in general with any kind of expression, since all (meaningful) expressions are associated with a sense. Even sentences express concepts; in the case of empirical sentences the concepts are concepts of propositions ('proposition' as understood by Church, as a concept of a truth-value, and not as understood in this article, as a function from possible worlds to (functions from times to) truth-values).

The degree to which 'intensional' entities, and so concepts, should be fine-grained was of the utmost importance to Church.[6] When summarising Church's heralded Alternatives of constraining intensional entities,

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[4] Sometimes we can read the English translation *intension* of a concept. Since this term is more frequently used as naming intensions in the sense of, for example, Possible-World Semantics, we will use the term *content*.

[5] As for a detailed analysis of the problems with sense in Frege see Tichý [1988], in particular chapters 2 and 3.

[6] Now we are using Church's terminology; in TIL concepts are hyperintensional entities.

Anderson [1998, p. 162] canvasses three options considered by Church. Senses are identical if the respective expressions are (A0) 'synonymously isomorphic', (A1) mutually λ-convertible, (A2) logically equivalent. (A2), the weakest criterion, was refuted already by Carnap in his [1947], and would not be acceptable to Church, anyway. (A1) is surely more fine-grained. However, partiality throws a spanner in the works: β-reduction is not guaranteed to be an equivalent transformation as soon as partial functions are involved. The alternative (0) arose from Church's criticism of Carnap's notion of intensional isomorphism and is discussed in Anderson [1980]. Carnap proposed intensional isomorphism as a criterion of the identity of belief. Roughly, two expressions are intensionally isomorphic if they are composed from expressions denoting the same intensions *in the same way.*

Church, in [1954], constructs an example of expressions that are intensionally isomorphic according to Carnap's definition (i.e., expressions that share the same structure and whose parts are necessarily equivalent), but which fail to satisfy the principle of substitutability.[7] The problem Church tackled is made possible by Carnap's principle of tolerance (which itself is plausible). We are free to introduce into a language *syntactically simple* expressions which denote the same intension in *different ways* and thus fail to be synonymous. Yet they are intensionally isomorphic according to Carnap's definition. Church used as an example of such expressions two predicates P and Q, defined as follows: $P(n) = n < 3$, $Q(n) = \exists xyz(x^n + y^n = z^n)$, where x, y, z, n are positive integers. P and Q are necessarily equivalent, because for all n it holds that $P(n)$ if and only if $Q(n)$. For this reason P and Q are intensionally isomorphic, and so are the expressions '$\exists n(Q(n) \wedge \neg P(n))$' and '$\exists n(P(n) \wedge \neg P(n))$'. Still one can easily believe that $\exists n(Q(n) \wedge \neg P(n))$ without believing that $\exists n(P(n) \wedge \neg P(n))$.[8]

Church's Alternative (1) characterizes synonymous expressions as those that are λ-convertible.[9] But, Church's λ-convertability includes also β-conversion, which goes too far due to partiality; β-reduction is not

[7]See also Materna [2007].

[8]Criticism of Carnap's intensional isomorphism can be also found in Tichý [1988, pp. 8–9], where Tichý points out that the notion of intensional isomorphism is too dependent on the particular choice of notation.

[9] See Church [1993, p 143].

guaranteed to be an equivalent transformation as soon as partial functions are involved. Church also considered Alternative ($1'$) that includes η-conversion. Thus ($1'$) without β-conversion is the closest alternative to our definition of synonymy based on the procedural isomorphism that we are going to define below.

Summarising Church's conception, we have: Concept is a *way* to the denotation rather than a special kind of denotation. Thus concepts should be situated at the level of *sense*. There are not only general concepts but also singular concepts, concepts of propositions, etc. More concepts can identify one and the same object. Now what would we, as realists, say about the connection between *sense* and *concept*? Accepting, as we do, Church's version as an intuitive one, we claim that

senses are concepts.

Can we, however, claim the converse? This would be:

concepts are senses.

A full identification of senses with concepts would presuppose that every concept were the meaning of some expression. But then we could hardly explain the phenomenon of historical evolution of language, first and foremost the fact that new expressions are introduced into a language and other expressions vanish from it. Thus with the advent of a new $\langle$expression, meaning$\rangle$-pair a new concept would have come into being. Yet this is unacceptable for a realist: concepts, *qua* logical entities, are *abstract* entities and, therefore, cannot come into being or vanish. Therefore, concepts outnumber expressions; some concepts are yet to be discovered and encoded in a particular language while others sink into oblivion and disappear from language, which is not to say that they would be going out of existence. For instance, before inventing computers and introducing the noun 'computer' into our language(s), the procedure that von Neumann made explicit was already around. The fact that in the 19^{th} century we did not use (electronic) computers, and did not have a term for them in our language, does not mean that the concept (*qua* procedure) did not exist. In the dispute over whether concepts are discovered or invented we come down on the side of discovery.

Hence in order to assign concept to an expression as its sense, we first have to define and examine concepts independently of a language, which we are going to do in the next paragraphs. Needless to say that our starting point will be Church's rather than Frege's conception of concepts. Yet to put our arguments on a more solid ground, in the next section we are going to summarise objections raised against Frege's theory.

2. Criticism of the Fregean notion of concept

Briefly, the objections raised against Frege's theory are: (1) Frege's concept viewed as a class is redundant, (2) Frege's concept is only a concept of universals, (3) Frege's concept is implausible in the empirical case and (4) Frege's concept situated at the level of denotation cannot serve as a *way* to the object denoted.

2.1. Redundancy of Frege's notion of concept

The way Frege defines function is *ambiguous*. Frege oscillates between function as a mapping (*Wertverlauf*) and function as a rule defining the mapping.[10] As mentioned above, this oscillation is not unlike the distinction between Church's functions-in-extension and functions-in-intension. Function-in-extension corresponds to the modern notion of function as a mapping, and function-in-intension could arguably correspond to the *mode of presentation* of the mapping. This is to say that Frege confuses a mapping with the mode of presentation of the mapping. Yet these are two entirely distinct entities. While mapping is a simple set of tuples, (abstract) mode of presentation is a complex and structured entity consisting of particular step constituents. The former interpretation is open to the additional objection that the concept of any class C would be identical with C. However, one and the same class can obviously be specified in many different ways. This is an important issue; intuitively, concepts are *ways to specify an object* rather than the object itself. For a time-honoured example, the same set of geometrical figures can be equally well conceptualized as a set of *triangular figures* (*triangles*) or a set of *trilateral figures*. Put differently, concepts are reasonably expected to be mode of presentations, *conceptualizations* of objects rather than the

[10] This fact has been noticed by Tichý in [1988], where Frege's oscillation is demonstrated.

conceptualized objects themselves. But Frege in his well-known schema consistently locates concepts at the level of *Bedeutung* rather than the level of *Sinn*.[11]

Thus the way Frege uses the term 'concept' justifies the conception of concept being a class. Yet Frege does distinguish the traditional dichotomy of the extension of a concept (*Wertverlauf*) and its content (*Inhalt*). Interesting enough, Frege's content of a concept should be the sense of the definiens that determines the concept. In general, this is the traditional view; *the content of a concept is the sum of its marks* (*Merkmale*). Frege says (commenting his example of a definition):

> Wir sehen hier einen Begriff [. . .] zusammengesetzt aus Teilbegriffen [. . .]. Diese nennen wir Merkmale jenes zusammengesetzten Begriffes. [1971, p. 132]

To the best of our knowledge, the only logician before Frege and long after Frege who warned against confusing concepts and their contents was Bolzano in [1837, p. 244]. Bolzano characterized a concept as *the way the components of the content are composed*. Yet a *sum* of Merkmale (subconcepts of a given concept) lacks a glue to hold them together into one whole. Though Frege maintained that the sense of a complex expression is composed of the senses of its subexpressions, he situated concepts at the level of *Bedeutung*. Thus Frege's concept is just the characteristic function of a class, which is not a structured entity.

Remark. It is interesting to note that the traditional theory of concepts which is based on the Aristotelian theory of definition is in some respects more adequate than Frege's theory and actually any other set-theoretical theories. Anticipating our later partial identification of concepts with senses (or *meanings*) we can state that the attempts "to define senses by means of the notion of synonymy or analytical identity of expressions" (Tichý [1968, 2004, p. 81]) have been mistaken. Comparing the traditional ('classical') theory with these attempts Tichý says:

> True, the classical idea of sense being a simple family of features or qualities is inadequate as is the idea that all the simple sentences are

[11] As mentioned above, Church anticipated this objection and put forward a different proposal to adjust Frege's definition of concept. Another problem arises from the fact that n-adic functions for $n > 1$ are obviously not conceived as concepts, see [1914] and [1971, p. 134]. This problem is perhaps not essential being just a terminological problem.

> of the form S-P. However, the opinion that the notion of intension[12] logically precedes the notions of truth, analyticity and synonymy, and not vice versa, is in our opinion quite justified, [...].
>
> Tichý [1968, 2004, p. 81]

By the way, the fact that the trend to define senses in terms of synonymy and analyticity proved to be untenable (which Tichý convincingly shows in the above mentioned paper) has led Quine to his attack on semantics in, e.g., [1953]. Yet another solution is at hand. We first exactly define the sense of an expression and only then synonymy of expressions as the relation of sharing the same sense.

Thus the first objection against Frege's concept can be summarized as follows: *If whatever can be said about Frege's concept can be said about class then Occam's Razor should be applied: using Frege's notion of concept we are not in a position to solve any problem whose solution would not be possible using the notion of class instead. Frege's notion of concept is redundant and thus not needed.*

2.2. Frege's concept does not cover non-universals

Frege's concept represents only universals. Yet obviously there are *non-general* concepts as well. Such singular concepts like 'THE RICHEST MAN', 'THE HIGHEST MOUNTAIN', 'THE PRESIDENT OF THE USA', 'THE SUM OF 3 AND 5', 'THE SUCCESSOR FUNCTION, etc., etc.[13] cannot be represented in Frege's theory, since each of them has to be replaced by the respective singleton, so we would get sets, the only member of which would be the respective object. But to claim that the richest man is married is not to claim that the respective set is married.

According to Frege whichever expression that is neither universal nor a proper name cannot denote a concept. This elimination of important meaningful expressions from the class of concept words is not plausible. Besides definite descriptions that undeniably represent concepts, we should also take into account expressions of other types, for instance adverbial modifiers like 'quickly', attitude verbs like 'knowing', 'believing', 'seeking', attributes like 'father of', 'murderer of', etc.

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[12] The term 'intension' is meant here as a structured explication of sense.

[13] Capitals indicate that the expression represents a concept (concepts being whatever).

About half a century before Frege, Bolzano built up in [1837] a remarkable theory of concepts. Bolzano came very close to a most general theory of structured meanings understood as concepts that are not set-theoretical mappings. For Bolzano any kind of an expression represents a concept, with the only exception of sentences. In this respect Bolzano's notion of concept is not superfluous and is also more general than the notion defined and used by Frege.

By way of summary, *Frege's notion of concept is unnecessarily restricted. Bolzano's notion and in some respects even the traditional Aristotle notion are much closer to our intuitions.*

2.3. The Fregean notion is not applicable to empirical concepts

The Fregean definition of concepts is not applicable to *empirical* cases. Characteristic function of a set of individuals is just this: characteristic function of one and the same set. No modally and temporally sensitive intensionality is present here. Thus, for instance, as soon as some dog dies, the population of dogs changes. As a result, during the development of the populations of dogs there are as many distinct characteristic functions as there are distinct populations. Yet the concept of dog remains the same independently of contingent facts like dogs' dying or being born. But since the Fregean concept is identified with the characteristic function, there are infinitely many Fregean concepts of dogs. This cannot be right, for concepts ought not to be susceptible to empirical vicissitudes.

Frege as many other semanticists has not taken into account the important distinction between empirical and non-empirical expressions. This is well understandable. The main interest of Frege was logic and mathematics scrutinized under the idea of logicism. After all, even most contemporary (mainly mathematical) logicians are obviously convinced that the problem of *logically* analyzing empirical expressions is, properly speaking, a pseudoproblem, since empirical expressions are logically inaccessible. True, Montague, Kripke and other intensional logicians have demonstrated that it is not a justified hypothesis, and even in Bolzano's work the distinction is respected, but sceptical views are still very strong.

Frege seems not to ignore the distinction; rather, he is presumably not aware of the problem. Therefore he neglects his own principle expressed in [1884, p. 60]:

> Überhaupt ist es nicht möglich von einem Gegenstand zu sprechen, ohne ihn irgendwie zu bezeichnen oder benennen.

and believes that 'morning star' denotes Venus although Venus is not mentioned by this expression.

This flaw is closely connected to the previous one. Since Frege works only with concept words and names (and sentences, of course), 'morning star' must be a name for him. He is not aware of a most important consequence thereof, namely that the problem of informativeness of the true sentences of the form $a = b$, is not actually a problem, if a and b are individual names, because then the semantic status of '$a = b$' is analytical; it is trivially true by claiming that one and the same individual is referred to here. Indeed, if 'morning star', 'evening star' are simple names, then they both must denote Venus. But then the sentence 'Morning star is evening star' is not empirically informative at all. In such a case it is a linguistic banality not needed to be verified by *astronomers*. The revival of the problem is possible only if 'morning star' is a semantically complex expression with the sense like *the brightest celestial body in the morning heaven* (similarly for 'evening star'), which is a typical definite description. However, since Frege does not consider singular definite descriptions to refer to concepts, he has no means to solve the problem.

Frege obviously did not see the difference between the semantics of empirical and mathematical expressions. Thus the semantics of 'primes' and 'dog' is, in principle, the same. The former is a class of natural numbers possessing the property of being a prime, the latter is a class of individuals possessing the property of being a dog. As we have seen in Subsection 2.1, in both cases the notion of concept is redundant, because we can simply speak about a class. Moreover, in the latter case, another objection is applicable: *which* class of dogs is denoted by the concept word 'dog'? Since the population of dogs is gradually changing, each such a change would, as a consequence, mean that the concept word 'dog' denoted another class. Thus denotation would be dependent on empirical facts. On the other hand, Frege obviously presupposes that denotation should be unambiguously determined by a sense.

For Frege, the sense of a concept word is most likely the sum of particular marks forming the content of the concept. Here a further problem arises: these marks of a concept are totally independent of empirical facts (such as which individuals form the population of dogs). Thus the sense cannot unambiguously determine variable denotations, despite Frege's assumption. That dogs are animals, four-legged, carnivores, etc., are facts that do not vary according to variable populations. Hence these marks, that is the elements of the sense of the concept word 'dog', do not determine any Fregean denotation of 'dog'.

In general, *Frege cannot distinguish classes and properties, similarly as he cannot distinguish truth-values and propositions. More generally, he cannot tell intensions from extensions.*[14]

2.4. Concepts as "Bedeutungen"? Church's criticism

According to Frege, whereas in fiction the denotation is not too important, it is most important in Science. Thus Frege situated concept at the level of denotation (*Bedeutung*). This conception is vulnerable to a severe objection. Concepts should *identify* entities denoted by expressions rather than being these denoted entities themselves. But sets or classes cannot be executed, cannot serve as an "intellectual journey" (as Tichý sometimes expressed his intuition) to the object denoted. For Frege concept is the end destination of the journey rather than the journey itself. Following Carnap's principle of explication we should explicate the notion of *concept* in accordance with the philosophical desiderata stated above. This means that concept should be situated at the level of *Sinn* rather than *Bedeutung*. Note that Aristotelian tradition corresponds closer to these intuitions than Frege's explication. Traditionally, we start with some marks or features and obtain an object that satisfies these features. Alonzo Church also noticed the problems connected with Frege's definition of concept and proposed an essential shift in Frege's scheme:

> Of the sense we say that it *determines* the denotation, or *is a concept* of the denotation. [1956, p. 6]

Hence Church situated concept at the level of Frege's Sinn, and we have

The sense (meaning) of an expression E is a concept of the object denoted by E.

[14] This is not to be understood as a severe criticism of Frege himself, who was undeniably a great logician. In the time when Frege was developing his logical framework, his system was a great achievement, for sure. Possible-word semantics with its distinguishing intensions and extensions came much later.

Church's proposal indicates his ingenious insight into the character of concepts. Observe: An unambiguous expression must have just one sense, so Church speaks about *the* sense, while there are more concepts of the object denoted by a given expression; therefore *a* concept.

Compare two expressions:

(a) 'a natural number greater than 1 divisible just by itself and 1',

(b) 'a natural number possessing just two factors'.

One would surely agree that (a) as well as (b) are unambiguous definitions of the set of primes and have each just one sense. Yet the sense of (a) is different from the sense of (b). Since both (a) and (b) denote the class of prime numbers, they are two different concepts of this class. Actually, there are infinitely many concepts of this class, each of them being *the* sense of some expression.

Summarizing: *Frege's notion of concept as the object denoted by the concept word is strongly counterintuitive. We have to respect the way the term 'concept' is used. Thus concepts are means of obtaining denotation rather than simple universals (classes).*

2.5. Intensionalization of the Fregean concept does not solve the problems

If we want to adjust Frege's schema and shift the concept at the level of sense, then a question arises what kind of entity a concept is. In case of empirical concepts it might seem that concepts can be defined as PWS intensions.[15] Regardless that some of the above objections are applicable in this case as well, in particular the redundancy objection, we must ask in which way can PWS intensions play the role of the mode of presentation (*Art des Gegebenseins*) of the denoted object and which entity the denoted object is. Presumably the denoted object could be the value of the respective intension in the actual world. But then such a denotation is only *contingently* determined by the sense conceived as PWS intension, because it is a contingent fact that this or that intension has this or that value in the actual world now. For instance, it is only contingently so that the Queen of the United Kingdom is Elizabeth II and the

[15] PWS standing here for possible-world semantics. Hence PWS intensions are set-theoretical mappings with the domain of possible worlds.

office of King of France is vacant now. Put in another way, if the meaning of an empirical concept word is an intension then the meaning does not only in itself determine the denotation. An empirical investigation is needed, and as already Carnap knew, such an empirical investigation is, of course, out of scope of logical semantics. It is beyond the capacities of human beings to obtain the actual value of an intension without empirical investigation. It would amount for empirical omniscience.[16]

In the case of mathematical concepts no intenzionalization is needed, because in mathematics possible worlds are out of any importance. Thus if concepts are situated at the level of sense, it is rather enigmatic how they determine mathematical objects. Moreover, the question about the character of a concept (i.e. sense) remains open.

In order to answer these questions, we first answer the question raised in the title of the next section.

3. What is wrong with the Fregean notion of concepts?

3.1. Aristotle vs. Frege; structured meanings

As mentioned above, the traditional, essentially Aristotelian theory of definition is more plausible for explication of the character of concepts than the Fregean theory of concepts. Tichý also evaluated the traditional theory more positively in this respect than that of Frege. Here is his brief characterization of the traditional approach:[17]

> The sense of the term (in classical terminology rather the "content of concept" of a term) is understood as a collection or a family of features, i.e. properties, which is something that does not logically depend on any semantic notion, in particular on the notion of truth. Just the opposite, the notion of truth and analytical truth logically depends on the notion of sense: Let S and P be terms, and $s(S)$ and $s(P)$ their respective senses. Then the sentence SaP is true if each object that has all the features from $s(S)$ has also all the

[16]This is a consequence of defining possible worlds as maximal consistent collections of possible facts. That such a conception of possible worlds is not necessarily circular is explained in Tichý [1988, Ch. 11].

[17]This conception, according to which the sense (meaning) has to be defined independently of analyticity and/or synonymy, could not be accepted by Quine, for whom *meaning* was an obscure entity. See Materna [2007] for a substantial criticism of Quine.

> features from $s(P)$. The sentence SaP is analytically true, and thus it can be a meaning postulate, if all the features from $s(P)$ belong to $s(S)$. [2004, p. 81]

So what are the positives of the Aristotelian approach to concepts?

Recall the characteristic of the sense of a concept word (that is of the content of a concept) as the sum of the marks of the concept. Setting aside vagueness of the notion 'sum', we can view such an enumeration of particular features forming the content of a concept as a specification of *criteria* that an object falling under the concept must satisfy. This is very distinct from Frege's set-theoretical definition. Frege's class is a *product* of applying particular criteria; yet in the product, that is the class, there is no trace of the criteria that have been used to produce the class. Since when using a concept we follow the 'itinerary' yielding the product, the way of providing criteria *via* a definition is much more natural than the notion of concept in terms of Frege's simple universal where those criteria are irretrievably lost.

Now we must reconsider the notion of the *sum* and content of a concept. As mentioned above, Bolzano followed the sharp differentiation between the content of a concept and the concept itself. Concept is *not a sum* of its marks. Cresswell's proposal replaces the *sum* by an *ordered tuple*, so instead of a sum of criteria he introduces a *list* of criteria. Yet even this proposal has been subjected to a severe criticism (see, e.g., Jespersen [2003]). Briefly, the tuple conception of meanings suffers these flaws: (a) a tuple is not a procedure that could be executed in order to obtain a product; (b) even if one includes a function that should be applied to the other elements of the tuple, an *operation* of application is still missing; (c) tuples cannot serve to specify functions: there is no gap for an argument in a tuple. Simply, tuples being sets cannot play the role of *structured meanings*.

Generalizing we can say that the bifurcation between structured meaning and a simple object is not unlike the distinction between an itinerary and its destination. The need for structured meaning is now broadly recognized by the philosophers of language and logicians. Since Frege's [1892] pioneer paper the advocates of denotational semantics like Carnap [1947], Montague [1970], Cresswell [1975, 1985] and others strive at defining 'structured meanings' which would comply with the principle of Compositionality and universal Transparency. Various adjustments of

Frege's semantic schema have been proposed, shifting the denoted entity named by an expression from the extensional level of atomic (physical/abstract) objects to the intensional level of molecular abstract objects such as sets or functions/mappings. Yet the denoted entity, be it a molecular mapping, cannot serve as a sense, because molecular mappings are not structured; they are just sets of tuples. We need to shift the sense up to *hyperintensional* level. In the rest of this paper we introduce *procedural explication of hyperintensions* and demonstrate that procedures are structured entities that can be assigned to expressions as their senses.

3.2. Sets vs. constructions of sets

Let us return to our example from Subsection 2.4:

(a) 'a natural number greater than 1 divisible just by itself and 1',

(b) 'a natural number possessing just two factors'.

The Fregean concept would in both cases be just the class of prime numbers. This class does not contain any of the features specified by the definition (a) or (b) like *being greater than 1*, *divisible by itself*, *possessing two factors*, ... It is a simple, unstructured entity. Thus the Fregean concept cannot play the role of Frege's sense that should be a bridge between an expression and its denotation. Being situated at the level of denotation, Fregean concept cannot play the role a concept should play. In particular, a class cannot be *executed* to obtain a product, because the class itself is the result of some lost itinerary yielding it.

Now consider two options:

(i) There is just one concept connected with (expressed by, denoted by, or whatever) the expressions (a) and (b); hence these expressions express/denote one and the same concept in two distinct ways.

(ii) The expression (a) expresses a concept of the class of prime numbers and the expression (b) expresses another concept of the same class of numbers.

Needless to say that we vote for the option (ii). If (i) were the case, then the objections enumerated in Section 2 would be applicable. The positive reason for our option is that *concepts expressed by (a) and (b)* *can now be viewed as two different ways to the same class of primes.* If we manage to define concepts in such a way, then the objection 2.1 from the previous section is no more applicable.

The general idea of the explication is clear. Obviously, (a) and (b) differ in i) containing different constituents and ii) the way these constituents are composed together. The way to obtain the class of primes is governed by other criteria in the case (a) than in the case (b). Moreover, as mentioned above, neither the set nor the list of the criteria does the job of obtaining the respective denoted object, because sets cannot be *executed*. Even a *function* (*mapping*) that would map particular criteria to the denoted object cannot serve the purpose. The *operation* of *application* of the function is missing. Hence no set-theoretical object (like a function/mapping) is plausible for the explication of a concept.

So what do we need? There are entities that are not reducible to sets, namely *complexes*. Yet complexes have been neglected by philosophers for the most part of the last century. Tichý characterizes this standpoint as a 'metaphysical purge' and says:

> It is not as if complexes had been singled out as special targets for ontological cleansing. They were forgotten as a part [of] a general shift of philosophical interests from things to words. The fact is, however, that now that the linguistic turn is hopefully behind us and it is once again respectable to discuss things as distinct from words, there is still no discussion of complexes, because the notion has simply disappeared from the philosopher's conceptual armoury. [1995, 2004, p. 874]

In what follows we are going to introduce our program of *procedural semantics*, and show that the role of complexes is, to our best knowledge, most plausibly played by *procedures*. Hence our position is a plea for a realist *procedural semantics*, which is at variance with set-theoretic semantics such as model theory and pragmatic semantics such as inferentialism. Language expressions represent or rather encode their structured meanings, which are abstract entities of Platonic realm. The subject matter of logical, *a priori* semantics is to study these entities independently of their encoding in a particular language. But these abstract entities that are assigned to expressions as their meanings are neither extensional atomic objects nor intensional set-theoretical mappings. Rather, they are hyper-intensional, algorithmically structured *procedures* producing extensional/intensional entities or lower-order procedures as their products. This approach, which could be characterized as an *algorithmic* or *procedural turn*, has been advocated for by Moschovakis in his [1994]. Yet much earlier in the early 70-ties Tichý introduced his notion of *construction* and developed the system of Transparent Intensional Logic, TIL (see Tichý [2004]). We argue for a robust concept of semantic structure as an extra-linguistic, abstract *procedure* (a generalized *algorithm* known as *TIL construction*), because procedures are inherently structured. They consist of one or multiple steps that have to be executed in order to arrive at the product produced by the respective procedure.

We will show that *concepts* can be well defined in terms of TIL constructions, and that this definition does not suffer from the defects characteristic of explications based on set-theoretical objects.

4. General features of TIL

In general, TIL is a fine-grained logical semantic theory of meaning. It is a logical framework, within which particular formal theories of something (like a theory of attitudes, logic of intensions, etc.) can be specified. But TIL is not defined as a *formal system* though particular calculi can be specified within TIL.[18] For instance, Tichý [1982] specified the TIL calculus for the simple theory of types. From the formal point of view TIL can be viewed as a hyperintensional, partial, typed lambda calculus. Hyperintensional, because the terms of the 'language of constructions' in which constructions are encoded are not interpreted as the functions denoted by the terms. Rather, they directly encode particular procedures the products of which are functions produced by them. Partial, because we work with partial functions; and typed, because all the entities within TIL ontology receive a type.

Why do we need a robust, hyperintensional semantics? The reason is that many paradoxes or puzzles stem from an inadequate, coarse-grained analysis of premises. First-order predicate logic is standardly used to analyse empirical sentences. This practice creates a mismatch between the analytic tool and what is to be analysed. The analyses in the first-order predicate logic are too coarse-grained, as well as being ambiguous.

[18] After all note that neither predicate logic can be identified with a particular formal system.

These difficulties would be neglectable if we could always infer the correct consequences from the premises. Unfortunately, we cannot. An up-dated puzzle of old shows why:

$$\frac{\begin{array}{c}\text{Necessarily, 8 is greater than 5}\\ \text{The number of planets equals 8}\end{array}}{\text{Necessarily, the number of planets is greater than 5}}$$

We just used Leibniz's law of substitution of identicals to infer from true premises a false conclusion. Modal logic sorts out the fallacy, though:

$$\frac{\begin{array}{c}\Box G(8,5)\\ n(p)=8\end{array}}{\Box G(n(p),5)}$$

The conclusion is not derivable, just as we desired. '$G(8,5)$' occurs within the scope of a modal operator, and we must not substitute co-extensional terms into contexts governed by a modal operator. But we are left in the dark as to *why* not. A rule is required that suspends the applicability of Leibniz's Law in precisely circumscribed cases. Without such a rule available to us, blocking an argument such as this remains *ad hoc*. As with solutions *ad hoc* in general, while they may succeed in alerting us to the fact *that* there is a problem, they fail to show *how* to solve the problem. Little logical insight can be garnered from a mere ban on substituting into modal contexts.

Another problem concerning this solution is what the *meaning* of the modal operator '$\Box$' is. Obviously, it is not a property of the truth-value **T**, though '$(8 > 5)$' denotes **T**. One may grant that the 'language' of modal logic is handy shorthand and still suspect that it hardly provides a transparent analysis. Furthermore, many other fallacies cannot be solved by modal logic, like this one:

$$\frac{\begin{array}{c}\text{John McCain wanted to become the President of the USA}\\ \text{Barack Obama is the President of the USA}\end{array}}{\text{John McCain wanted to become Barack Obama}}$$

We have to switch to a system of some intensional logic in order to render the fact that 'to become' establishes intensional contexts that are not to be substituted into. If B is an attitudinal operator, the shared analysis is

$$\frac{\begin{array}{c}B(a,f(b))\\ c=f(b)\end{array}}{B(a,c)}$$

Again, the undesirable substitution is said to be blocked, because the substitution of 'c' for '$f(b)$' in a context preceded by B is banned. But *why and how*? What is the meaning of the operator B? Obviously, B does not stand for a relation between two individuals; an individual cannot become another individual, unless it would somehow bizarrely alter its identity. Yet $f(b)$ does denote an individual.

In general, a ban on substitution will cure the symptom, but not the disease. Addressing the underlying problem requires formulating a non-circular, independently motivated rule to regulate substitution in intensional contexts.

Attitudes are another notorious troublemaker. They force us to switch to some epistemic, doxastic, etc., logic. Here is an example.

$$\frac{\text{Tom knows that Prague is greater than Brno}}{\begin{array}{l}\text{Tom knows that (Prague is greater than Brno} \\ \text{and no bachelor is married)}\end{array}}$$

It may be the case that the first sentence is true whereas the second is false. Yet the standard possible-world semantics of epistemic logic yields the result that the second sentence must be true as well, 'Prague is greater than Brno' and 'Prague is greater than Brno and no bachelor is married' denoting the same proposition. This is due to the fact that the proposition that no bachelor is married is the necessary proposition TRUE, which takes the truth-value **T** for all possible worlds and times. Provided (as we are supposing) that we understand the meaning of 'is a bachelor' and 'is married' as these predicates are used in current English, if an individual is known to be a bachelor, we need not (empirically) examine the state of the world in order to get to know that the individual is not married. Qualms about substitution within attitude contexts motivate the need to ascend from intensional logic to *hyperintensional* logic. Here is an example in which it is indisputable that hyperintensional attitude complements are called for.

$$\frac{\begin{array}{c}3+5=6+2 \\ \text{Tom calculates } 3+5\end{array}}{\text{Tom calculates } 6+2}$$

It is no option to relate Tom to possible-world intensions. Their granularity is far too crude for them to figure as complements in mathematical attitudes. Thus, Tom would be related to a constant function from possible worlds and instants of time to a number. This grossly misrepresents what the activity of calculating is all about, which is to apply arithmetic operations to numbers. Finer granularity that would block the undesirable derivation would relate Tom to the expression '$3 + 5$'. Yet Tom cannot be related to a piece of mathematical *notation*. The argument does not say what syntactic transformation Tom performs in order to calculate the sum of 3 and 5. In the case at hand Tom calculates $3 + 5$ by applying the addition function to the pair of numbers $(3, 5)$. When calculating, Tom is related to this very procedure rather than to the number 8; he aims at finding the product of the procedure.

For these reasons we vote for a hyperintensional semantics, where abstract procedures defined as TIL constructions are assigned to expressions as their structured meanings. TIL is a *realistic* logic. Logical objects as well as any *abstract* objects like constructions are objective: they are not excogitated but discovered. Logic does not study (formal) languages; these are just "a mere shorthand facilitating discussion of extra-linguistic entities" (Tichý [1988, viii]). This point is important for understanding what constructions are (abstract procedures) and what they are *not* (expressions of some artificial language).

The *epistemic framework* is based on the notion of (partial) *functions*. Functions serve as surrogates for intuitively, pre-theoretically given objects. Thus functions from possible worlds to chronologies of classes of individuals are surrogates for what we intuitively call *properties of individuals*. The purpose of introducing the category of surrogates is that the resulting system of explications makes it possible to represent the relations between intuitively given objects "by the mathematically rigorous relationships between the functional surrogates" (Tichý in [1988), p. 195]).

4.1. TIL constructions

Constructions are based on a robust concept of semantic structure as an extra-linguistic, abstract procedure (a generalized algorithm).[19] Because procedures are inherently structured, they consist of one or multiple constituent subprocedures that are to be executed in order to arrive at the

[19] Portions of this section draw on material presented in Duží, Jespersen and Materna [2010].

product (if any) produced by the respective procedure. TIL agrees with Moschovakis's conception of Frege's sense as "an (abstract, idealized, not necessarily implementable) algorithm which computes the denotation of [a term]" [2006, p. 27].

To anticipate a common misapprehension, we wish to emphasize that the procedures we have in mind are not syntactic objects. Just like one and the same algorithm can be encoded by different programs possibly written in different programming languages, so one and the same procedure can be encoded by different pieces of syntactic items belonging to different languages. Our procedural conception of hyperintensionality is not a syntactic conception, and TIL constructions are not syntactic structures. They are objectual procedures consisting of sub-procedures. Thus an answer to Russell's question, "What binds the constituents of propositions together?" can be offered.[20] Propositional unity is established by the very procedure that generates a compound whole from its individual constituents. The meaning of an expression E is not a *list* of the meanings of the sub-expressions of E. Rather it is the *procedure* detailing in what particular ways its sub-procedures are combined.

A most important feature of our procedural semantics is that to exercise linguistic competence with respect to an expression is to know its sense, i.e. the procedure encoded by the expression, rather than the entity that this procedure produces. For instance, to master the mathematical constant 'π' is not to know what real number it denotes. Obviously, no finitely limited agent, such as a human being, can know the infinite sequence of digits 3.14159.... Rather, being linguistically competent with respect to 'π' is tantamount to knowing a procedure for, e.g., computing *the ratio of the circumference of a circle to its diameter*. Similarly, to master the empirical predicate 'is a whale' is neither to know what individuals or set of individuals it refers to, nor is it to know the property it denotes.[21] Rather it is to know a procedure which for any state of affairs enables the language-user to determine whether a given individual is a whale.[22] Moreover, some expressions do not denote anything, yet are anything but meaningless. For instance, mathemati-

[20] See also King [2001].

[21] Empirical properties of individuals are construed as functions from logical space to chronologies of sets of individuals.

[22] Fuzziness and vagueness aside.

cians needed to understand the meaning of 'the greatest prime' prior to proving that there is no greatest prime. They had to master the procedure expressed by this expression in order to show that the procedure fails to produce a product.

To adduce an example, the expressions '$3^2 - 2^2$' and '$(3 + 2) \times (3 - 2)$' do not have the same sense. They encode two different ways of constructing the number 5 in terms of two other numbers, two and three. The sense of '$3^2 - 2^2$' is the procedure that consists in the application of the square function to the number 3, application of the square function to the number 2, and subtraction of the result of the latter from the result of the former. On the other hand, the sense of '$(3 + 2) \times (3 - 2)$' is a completely different procedure, of course.

The fundamental primitive objects of the ontology of TIL are functions rather than relations or sets. Thus the formal language in which TIL constructions are encoded is inspired by the (typed) λ-calculi. Yet *a function is not a procedure*. We view functions as set-theoretical mappings, and one and the same mapping can be produced by infinitely many procedures. The terms of the language in which TIL constructions are encoded are viewed procedurally. Abstraction ('Closure' in TIL) is the very procedure of forming a function (and not the resulting function), and application ('Composition' in TIL) is the very procedure of applying a function to an argument (and not the resulting value). The functional dependencies underlying compositionality are technically accommodated by means of the interplay between abstraction and application. Note that we strictly distinguish between procedures and their products, and between functions and their values.[23]

Constructions are abstract procedures, i.e., they are not spatially or temporally localizable. No deep metaphysics is here. This can easily be explained. Consider two copies of a record of a (computer) program. The records are well localizable and they differ by their position, so we have *two distinct* objects here. Yet there is just one abstract algorithm encoded by these copies. Constructions are procedures (generalized algorithms), and thus they are structured just as algorithms are; they consist of *constituents* (or *instructions*) to be executed. By generalized

[23]The contrast between functions and constructions of functions is not unlike the contrast between functions-in-extension and functions-in-intension. But we are hesitant to push the parallel, since function-in-intension remains a poorly-understood notion. See Church [1956, pp. 2–3].

algorithm we mean that not every construction is an algorithm. The point is that *algorithms* are effectively executable procedures, while constructions may contain constituents that are not effectively executable. For example, the application of any function at any argument is admissible, but the function itself may be non-recursive, or the argument may be infinite, etc. Tichý has characterized constructions as *ideal* procedures.

TIL constructions are assigned to expressions as their context-invariant, algorithmically structured meanings. When claiming that constructions are algorithmically structured, we mean the following. Constructions consist of constituents. Yet the objects a construction operates on are not constituents of the construction. Just like the constituents of a computer program are its sub-programs, so the constituents of a construction are its sub-constructions. Thus on the lowest level of non-constructions, the objects that constructions work on have to be supplied by other (albeit trivial) constructions. This is in principle achieved by using atomic constructions. A construction C is atomic if it does not contain any other constituent but C. There are two atomic constructions: *Variables* and *Trivializations*. They supply objects (of any type, including constructions) on which compound constructions operate. The constructions themselves may occur not only as constituents to be executed in order to arrive at the object, if any, they construct, but also as objects that still other constructions operate on. Thus when a construction C is Trivialized, it is not a constituent to be executed; rather, C itself is an object of predication. *Compound* constructions, which consist of other constituents than just themselves, are *Composition* and *Closure*. Composition is the procedure of applying a function f to an argument A to obtain the value (if any) of f at A. It is *improper* (i.e., does not construct anything) if f is not defined at A. Closure is the procedure of constructing a function by abstracting over variables in the ordinary manner of the λ-calculi. For the sake of simplicity, instead of an exact definition of constructions we now provide their informal description.[24]

Variables are conceived as a special kind of procedures rather than the letters like x, y, …, p, q, …, k, l, m, …, which are *names* of variables. The specific feature of this kind of constructions is that they construct an object of a given type dependently on the total function

[24] The exact definition can be found in Tichý [2008, pp. 63–65], see also Duží *et al* [2010, Definition 1.2, p. 45].

known as *valuation*, so they (and any constructions that contain their free occurrences) v-construct, where v is the parameter of valuations.

Trivialization ${}^0X$, where X is any object (including a construction), is a construction that *mentions* the object X, so that its product is just X. Hence Trivialization constructs X without the mediation of any other construction.[25]

Higher-order constructions can be executed twice over. To this end we define a construction called *Double Execution*, ${}^2X$. If X is a construction that v-constructs a construction Y, and Y v-constructs an object Z, then ${}^2X$ v-constructs Z. Otherwise ${}^2X$ is v-*improper* by failing to v-construct anything.

The other two kinds of molecular constructions correspond to dual operations of lambda calculi, *viz.* application of a function to an argument and creating a function by abstracting over the values of variables. Thus we have:

If X is a construction of an m-ary function f and $X_1, \ldots, X_m$ are constructions of the arguments of f then $[XX_1 \ldots X_m]$ is a construction called *Composition*. It is the procedure of applying the function f v-constructed by X at a tuple-argument A v-constructed by $X_1, \ldots, X_m$. Hence Composition $[XX_1 \ldots X_m]$ v-constructs the value (if any) of f at the argument A, if f is defined at A, otherwise the Composition is v-improper.[26]

Let X be a construction and $x_1, \ldots, x_m$ pairwise distinct variables. Then $[\lambda x_1 \ldots x_m X]$ is a construction called *Closure*.[27] It v-constructs a function f by abstracting over the values of variables in the ordinary manner of λ-calculi.

Constructions, as well as the entities they construct, all receive a type. The ontology of TIL is organized in an infinite, bi-dimensional hierarchy of types. One dimension is made up of non-constructions, i.e., entities unstructured from the algorithmic point of view. The other

[25] If one considers such a construction dispensable, then be sure that it is very useful. Trivialization makes it possible to operate on constructions as on objects of predication. Due to this feature TIL is a robust *hyperintensional* system.

[26] Note that as a construction it contains constituents $X, X_1, \ldots, X_m$. Whereas the term '$[XX_1 \ldots X_m]$' in which the Composition is encoded is interpreted as the very procedure of application, the analogous λ-term is interpreted as the product of the procedure.

[27] The analogous λ-term is called *abstraction*.

dimension of the type hierarchy is made up of constructions, which are structured, higher-order entities constructing lower-order entities. The definitions are inductive, and they proceed in three stages. First, we define the simple types of order 1 comprising non-constructions; then we define constructions; finally, the ramified hierarchy of types is defined.

4.2. Simple hierarchy of types

Types of order 1 are defined over a *base*, i.e., a finite collection of non-empty sets whose members are *atomic* object. The choice of the base depends on the area and language we are going to analyze. In case of arithmetic of naturals, the base can consist, for instance, of two atomic types, $\{o, \nu\}$, where o is the type of truth values $\{\mathbf{T}, \mathbf{F}\}$ and ν is the set of natural numbers. When analyzing ordinary natural language or its vernaculars, we use the so-called *epistemic base* consisting of these atomic types:

o $(= \{\mathbf{T}, \mathbf{F}\})$: the set of *truth-values*,

ι: the set of *individuals* (universe of discourse),

τ: serving as the set of *time moments* and, at the same time, as the set of *real numbers*,

ω: the set of *possible worlds* (logical space).

Compound types defined over the base are *sets of partial functions*.[28]

Definition 1 (*simple types of order* 1). Let B be a base. Then

(i) Members of B are *types of order* 1 *over* B.

(ii) If α, β_1, …, β_m are *types of order* 1 *over* B, then the set of partial functions whose arguments are elements of the types β_1, …, β_m, respectively, and whose values are elements of the type α, denoted by $(\alpha\beta_1 \ldots \beta_m)$, is a *type of order* 1 *over* B.

(iii) A *type of order* 1 *over* B is just what satisfies (i), (ii).

Examples of types:

- Type of *classes* of individuals is $(o\iota)$; (Classes and relations are represented by their *characteristic functions*.)

[28] The following definition reproduces the first part of Tichý's definition; see Tichý [1988, p. 66, Def. 16.1].

- In general, types of classes of objects of a type α are $(o\alpha)$; types of relations between objects of types α, β are $(o\alpha\beta)$. For example, $(o(o\iota))$ is the type of classes of classes of individuals, $(o\tau(o\tau))$ is the type of relations between a number and a class of numbers.
- *Propositions* are functions from possible worlds to the chronologies of truth-values, so their type is $((o\tau)\omega)$. We will write $o_{\tau\omega}$ for short.
- In general, α-*intensions* are functions of type $(\alpha\omega)$ with the domain of possible worlds ω. Most frequently they are functions of type $((\beta\tau)\omega)$ from possible worlds to the chronology of a type β. We write $\beta_{\tau\omega}$ for short. Thus, for instance, the type of *properties of individuals* is $(o\iota)_{\tau\omega}$, the type of *individual offices* or *roles* is $\iota_{\tau\omega}$, the type of relations-in-intensions between individuals is $(o\iota\iota)_{\tau\omega}$. One of the possible types of *propositional attitudes* is $(o\iota o_{\tau\omega})_{\tau\omega}$. $\dashv$

Comments:
1. *A construction must be type-theoretically definite.* Thus X/α means that the object X belongs to a type α, it is an α-object, while $X \rightarrow_v \alpha$ means that the construction X v-constructs an object of the type α. Constructions X also belongs to a type; but we still cannot write X/β, because β cannot be a first-order type. Higher-order types will be specified in the next section.

Examples of first-order types: If $0/\tau$, $>/(o\tau\tau)$, $x \rightarrow_v \tau$, then the Composition $[{}^0\!> x\, {}^0 0]$ v-constructs $\mathbf{T}$ or $\mathbf{F}$ dependently on the valuation of x: if x v-constructs a positive number, the product is $\mathbf{T}$, otherwise $\mathbf{F}$. The Closure $\lambda x[{}^0\!> x\, {}^0 0]$ constructs the function from real numbers to $\{\mathbf{T}, \mathbf{F}\}$ according as a given number x is greater than zero or not. Thus it constructs the (characteristic function of the) class of positive (real) numbers.

It is useful to draw a type-theoretical tree of a construction in order to check whether the construction is well-typed. The type-theoretical tree of the Closure $\lambda x\, [{}^0\!> x\, {}^0 0]$ is as on Fig. 1.

Schematically, the *rules* for typing molecular constructions are:

A) $$\begin{array}{cccccc} [X & X_1 & \ldots & X_m] & \rightarrow & \alpha \\ (\alpha\beta_1 \ldots \beta_m) & \beta_1 & \ldots & \beta_m & & \end{array}$$

B) $$\begin{array}{ccc} \lambda x_1 \ldots x_m X & \rightarrow & (\alpha\beta_1 \ldots \beta_m) \\ \beta_1 \ldots \beta_m \alpha & & \end{array}$$

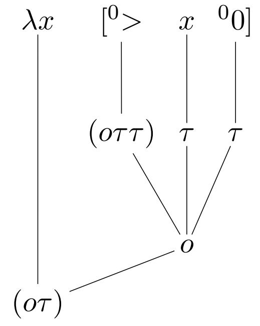


Figure 1.

2. As mentioned above, *constructions are not syntactic objects*. Rather, they are abstract, extra-linguistic procedures. Being abstract, they must be encoded in a language in order to be logically tractable. Yet the terms of the language of constructions are not constructions. The former are linguistic objects while the latter are objectual procedures.

As an example consider again the Closure

$$\lambda x[^0\!> x\, ^0 0].$$

As an extra-linguistic object this construction does not contain any symbols, so it does not contain brackets or 'λ'. The symbols of brackets are the linguistic way to encode the procedure of applying a function to its arguments. Yet we might use any other symbols instead of '[', ']'. Symbols do not matter. What matters are the objects encoded by these symbols. Similarly the symbol 'λx' only indicates that an abstraction over the values of 'x' is a constituent step of a Closure. Thus whereas the term '$\lambda x[^0\!> x\, ^0 0]$' contains two occurrences of the *letter* x the Closure itself contains just one occurrence of the *variable* x. (Recall that variables are what is named by letters such as x, i.e., a kind of construction.)

3. One of the principles of logical analysis of natural language based on TIL is that *empirical expressions denote non-trivial intensions*, i.e. intensions whose values differ in at least two possible worlds. The type of the objects denoted by empirical expressions is thus the type of an intension $\alpha_{\tau\omega}$ for some type α.

For instance, the predicate 'being older than Pope' denotes a non-trivial property of individuals, i.e., an object of type $(o\iota)_{\tau\omega}$. To assign a construction to this expression as its meaning, we first *assign types* to the objects mentioned by the predicate:

Older (than)/$(o\iota\iota)_{\tau\omega}$: the relation-in-intension between two individuals; *Pope*/$\iota_{\tau\omega}$: the individual office; $x/*_1 \to \iota$ (see below 4.3.).

Note that 'Pope' denotes an individual office rather than an individual. Which individual plays the role of Pope is contingent (modal parameter ω) and time-dependent (temporal parameter τ). Yet an office cannot be in the relation of being older with an individual. Thus we must extensionalize the office in order to v-construct an individual holding the office. To this end we make use of TIL explicit intensionalization and temporalization. Let $w/*_n \to \omega$ be a variable ranging over possible worlds and $t/*_n \to \tau$ variable ranging over times. Then the Composition $[[{}^0Pope\, w]t]$, abbreviated as ${}^0Pope_{wt}$ v-constructs the individual (if any) that occupies the office in a given $\langle w, t\rangle$-pair. Similarly, the relation-in-intension must be extensionalized in order to obtain pairs of individuals who are in the relation in a given world w and time t of evaluation: $[[{}^0Older\, w]t]$, abbreviated as ${}^0Older_{wt}$.

Notational conventions. We commonly use variables 'w' (possibly with subscripts 'w_1', 'w_2', ...) and 't' (possibly 't_1', 't_2', ...) as v-constructing elements of type ω and τ, respectively. In general, if C is a construction of an α-intension, type $\alpha_{\tau\omega}$, the intensional descent of the intension is constructed by the Composition $[[C\ w]t]$. Since this kind of Composition is frequently applied, we use the abbreviated notation C_{wt}. Outer brackets are often omitted, in particular the brackets around Closures, if no confusion arises. Thus instead of, e.g., '$[\lambda x[{}^0{+}x\,{}^01]]$' we write simply '$\lambda x[{}^0{+}x\,{}^01]$'.

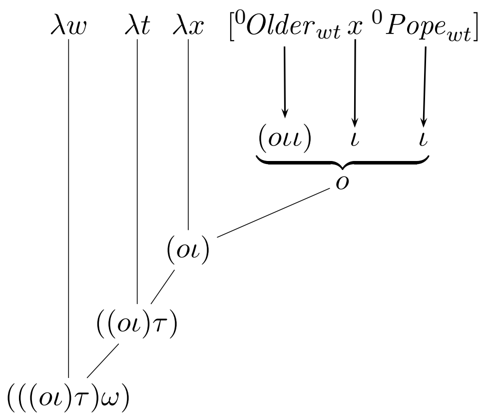
To complete the analysis of 'being older than Pope' we must:

(a) apply the extensionalized relation *Older* to a variable and a holder of the Pope office, $[{}^0Older_{wt}x\,{}^0Pope_{wt}]$,

(b) abstract over the values of 'x' in order to obtain the class of individuals who are in the relation of being older with the current Pope in a given $\langle w, t\rangle$-pair, $\lambda x[{}^0Older_{wt}\, x\,{}^0Pope_{wt}]$,

(c) abstract over the values of variables 'w', 't' in order to intensionalize the class of individuals so that to construct the property: $\lambda w\lambda t\lambda x$ $[{}^0Older_{wt}\, x\,{}^0Pope_{wt}]$;

(d) as always, it is useful to draw a type-theoretical tree in order to check whether the resulting construction is composed in compliance with the typing rules. In order to simplify the tree, we often directly indicate the type of the result of extensionalization. Thus if $C \rightarrow \alpha_{\tau\omega}$, then $[C\ w] \rightarrow_v (\alpha\tau)$ and $[[Cw]t] \rightarrow_v \alpha$ and we directly depict that $C_{wt} \rightarrow_v \alpha$. In our case we have:



or $(o\iota)_{\tau\omega}$ for short: the type of a property of individuals (of being older than the Pope).

4.3. Higher-order types

Types of order 1 (see Definition 1) cannot be used for associating constructions themselves with types. Elements of types of order 1 are non-constructions. Constructions that construct these entities must be of a higher-order type. Similarly, constructions themselves are entities which can be arguments or values of functions. Yet functions as set-theoretic mappings are not constructions. Thus in order to play the role of an object of predication, a construction C must be constructed by another construction D that is of a type of a higher-order than C. For these reasons we need to extend the hierarchy of simple types into a ramified hierarchy of types.

Consider, for instance, the sentence 'Charles calculates 2 + 3'. As always, we start with assigning types to the objects mentioned by the sentence: *Charles*/ι; $2, 3/\tau$; $+/(\tau\tau\tau)$. But what type should be assigned to the relation *Calculate*? It is a relation-in-intension; hence its type should be of a shape $(o\iota?)_{\tau\omega}$. But what is the type of entity Charles is related to when calculating? Let us consider three options. Either he is related to the expression '2 + 3', or to the number 5, or to the very procedure of adding 2 and 3.

The first option is untenable. The sentence does not contain a reference to the notation by means of which Charles realizes his activity of calculating. He may calculate $2 + 3$ when playing with the balls of an abacus. Thus relating him to this particular notation would be a misinterpretation of the character of his activity.

The second option, if not unreasonable, is certainly not plausible as well. To calculate does not mean to be related to a number. Which number, by the way? If it were the number 5 then the activity of calculating is accomplished with the correct result before it even started. Besides, it would entail that if Charles calculates, say, $48 - 43$ then he does the same as when calculating $2+3$. And if Charles is not good in math then he might arrive at the number 6 or any other number. Hence calculating does not relate an individual to the result of calculation.

Thus we vote for the third option. Charles is related to the very procedure of applying the function + to the arguments 2 and 3. He aims at finding the product of this procedure. We have defined such procedures as TIL constructions, in this case the Composition $[{}^0{+}\ {}^0 2\ {}^0 3]$, but we have not defined the type(s) of constructions as yet. Up to now we know that this Composition constructs a number of type τ, $[{}^0{+}\ {}^0 2\ {}^0 3] \to \tau$, but we do not know a higher-order type α to which the Composition itself belongs, $[{}^0{+}\ {}^0 2\ {}^0 3]/\alpha$. In other words, we need the *ramified hierarchy of types*. The definition is inductive and decomposes into three parts:[29]

(i) *Types of order* 1. See Definition 1.

(ii) *Constructions of order* n. The idea is this: a construction of order n constructs an object of a type of order n.

(iii) *Types of order* $n+1$. Let $*_n$ be the set of constructions of order n. Then $*_n$ and the types of order n are types of order $n+1$.
(Further compound types, where some types among the types of arguments or values of functions are of a type of order $n+1$, are of a type of order $n+1$.)

[29]The exact form can be found in Tichý [1988, p. 66] as Def. 16.1 and as Def. 1.7 in Duží *et al* [2010, p. 52].

Examples. 1. A numerical variable x v-constructs numbers, hence the type of an entity v-constructed by x is τ. The variable itself is a construction of order 1, its type being the type of order 2, *viz.* $*_1$. We write $x/*_1 \rightarrow_v \tau$, or just $x \rightarrow \tau$ for short. The Trivialization of x, ${}^0x$, constructs x, i.e., an object of a type of order 2. Thus ${}^0x/*_2 \rightarrow *_1$, so that ${}^0x$ is of a type of order 3.

2. In Section 4.1 we adduced the example of an invalid argument based seemingly on Leibniz's rule *substitutivity of identicals*:

$$3+5=6+2, \text{ Charles calculates } 3+5 \implies \text{ Charles calculates } 6+2.$$

Now we have the means to explain why this argument is invalid. To this end let us analyze the premises:

$$[{}^0\!= [{}^0\!+ {}^0 3\, {}^0 5][{}^0\!+ {}^0 6\, {}^0 2]], \lambda w\lambda t[{}^0\mathit{Calculate}_{wt}\, {}^0\mathit{Charles}\, {}^0[{}^0\!+ {}^0 3\, {}^0 5]]$$

$$\text{Types: } = /(o\tau\tau);\ \ +/(\tau\tau\tau);\ \ 2,3,5,6/\tau;\ \ \mathit{Charles}/\iota;\ \ \mathit{Calculate}/(o\iota *_1)_{\tau\omega}$$

Obviously, Leibniz's rule is not applicable here; the identity concerns the products of the Compositions (here the number 8) while Charles' calculation concerns the construction itself (here the Composition $[{}^0\!+ {}^0 3\, {}^0 5]$). This very Composition is supplied by Trivialization as the second argument of the relation Calculate.[30] In other words, in the first premise the occurrence of both Compositions is extensional whereas the occurrence of the Composition $[{}^0\!+ {}^0 3\, {}^0 5]$ in the second premise is hyperintensional.

5. Concepts as closed constructions

In the outset of this paper we explained why Frege's notion of concept is untenable. In Section 3.2 we generalized the criticism of Frege's conception as follows: Any adjustment of Frege's notion that would not in principle semantically distinguish between two distinct conceptualisations of one and the same set is unsatisfactory.

Now we have got logical machinery for a more adequate explication of concepts. This is due to the fact that between the intensional or extensional level of denotations we have inserted the hyperintensional level of procedural meanings defined as TIL constructions. Moreover, due to the ramified hierarchy of types and the special construction known as

[30] This also demonstrates the principal importance of Trivialization.

Trivialization we can not only use constructions in order to construct an object, but also logically operate on constructions themselves (to mention them as objects of predication). Thus TIL is an extensional logic of hyperintensions and we can introduce the *procedural theory of concepts*.[31]

As mentioned above, Church adjusted Frege's notion of concepts and shifted concepts up to the level of sense. Moreover, he conceived the semantic role of concepts as a universal one. In his [1985, p. 41] Church says:

> [...] anything which is capable of being the sense of some name in some language, actual or possible, is a concept.

Since the sentences were also names for Frege and Church (*viz.* of truth-values) it is reasonable to suppose that for Church the meaning of every meaningful expression E of some language is a concept of the denotation of E. Thus he properly speaking proposed to identify meanings (senses) with concepts. Yet not absolutely: concepts are just "*capable* of being the sense of some name", which means that for Church (and for every realist) concepts are independent of language; they may *become* meanings of some expressions. TIL also adheres to this conception and defines concepts as language-independent abstract complexes, i.e. *algorithmically structured abstract procedures.*

Thus procedurally defined concepts can play the role that the set-theoretical objects cannot play. Unlike sets, procedures can be *executed* in order to produce an object (if any). In this way we explicate Frege's sense characterized as the *mode of presentation* of the object.

We have seen that TIL constructions are such procedures. Yet not every construction is a concept. This is because some constructions are open, i.e. they contain free variables. Whereas the execution of a (non-empty) concept produces an object, open constructions cannot be executed in order to produce an object unless the valuation of free variables is provided.

Consider the following examples:

'Mayor of Dunedin'

'Mayor of it'

[31] See also Materna [1998] and [2004] for earlier versions of the theory of concepts.

The first expression specifies a definite empirical condition that is satisfied by an individual who plays the role of Dunedin's mayor. We say that it denotes an individual role (or office), an object of type $\iota_{\tau\omega}$. The meaning of this expression is a concept of this role. On the other hand, the second expression has a pragmatically incomplete meaning. By itself it does not denote anything. Only when a situation of utterance or a linguistic context makes it possible to determine the city referred to by 'it' can one arrive at an individual who plays the role of its mayor. The analyses reveal these facts:

$$\lambda w \lambda t\, [{}^0 Mayor\_of_{wt}\, {}^0 Dunedin] \rightarrow \iota_{\tau\omega}$$

$$\lambda w \lambda t\, [{}^0 Mayor\_of_{wt}\, it] \rightarrow_v \iota_{\tau\omega}$$

Types: $Dunedin/\iota$;[32] $Mayor\_of/(\iota\iota)_{\tau\omega}$; $it/*_1 \rightarrow_v \iota$. The free variable *it* is a pragmatic variable; its valuation is supplied by an external context, i.e. by a situation of utterance or by a linguistic discourse. Since concepts are definite procedures that can be executed to produce the denoted entity (if any), the second expression does not express a concept.

Another example: while $3+5$ can be safely said to express a concept of the number 8, expressions like $3+x$ or $x+y$ cannot be said to express any concept. On the other hand, the Closure $\lambda x[{}^0{+}\ {}^0 3\, x]$ is a concept of the function that maps a given number to its third successor.

Generalizing: *Concepts are only closed constructions*, i.e. constructions without free occurrences of variables.

Yet *defining* concepts as closed constructions is unsatisfactory as well. The reason is this. Constructions are a bit too fine-grained from the procedural point of view. We need to specify slightly coarser granularity of procedural meanings. In principle, the problem concerns λ-bound variables for which there is no equivalent in natural language. Technically, the quest for the right hyperintensional calibration of senses in terms of procedures is the quest for the right measure(s) of extensionality in the typed lambda-calculus. The lambda-calculus has three rules of conversion: α-, β-, and η-conversion. α-conversion expresses the idea of 'renaming', i.e. replacing one λ-bound variable x by another λ-bound variable y, typically in order to avoid collision of variables (i.e. a free

[32] For the sake of simplicity we conceive proper names of cities as rigidly denoting individuals. This is rather a coarse-grained semantics, for sure. Yet this simplification is irrelevant to the problems we address here.

occurrence of a variable becoming bound). β-conversion expresses the idea of functional application and η-conversion, in loose terms, lifts an application of a function to a variable out by abstracting over the values of this variable.

The η-rewrite rule is similar to the α-rule but radically different from β-rule. The β-rule is the one to watch carefully due to its central role in the evaluation of functional values. It is a well-known fact that β-rule is not generally valid as soon as partial functions are involved. For instance, the following constructions C_1 and C_2 are *not* equivalent as they construct different functions:

$$C_1 \qquad [[\lambda x[\lambda y[{}^0 Div\, y\, x]]]\, [{}^0 Cot\, {}^0\pi]]$$

$$C_2 \qquad [\lambda y[{}^0 Div\, y [{}^0 Cot\, {}^0\pi]]]$$

Types: $x, y \to \tau$; $Div/(\tau\tau\tau)$: the division function; $Cot/(\tau\tau)$: the cotangent function; π/τ.

The construction C_1 is the Composition of the Closure $[\lambda x[\lambda y[{}^0 Div\, y\, x]]]$ with $[{}^0 Cot\, {}^0\pi]$. Since the latter is improper by failing to produce any result (for the function cotangent is undefined at π), the whole C_1 is improper. Now the Closure C_2 is the result of the application of β-rule at C_1 by substituting $[{}^0 Cot\, {}^0\pi]$ for x. Yet a Closure is never improper; it always constructs a function. In this case it is the function of type $(\tau\tau)$ that is undefined for any argument. Though such a function is a degenerated one, it is no less an object.

On the other hand, α- and η-rules are always valid and oftentimes the equivalences between α- and η-equivalent transformations cannot be expressed in natural language. For instance, consider the following α-equivalent constructions ($x_i \to \tau$):

$$\lambda x_1[{}^0\!>x_1\, {}^0 0],\ \lambda x_2[{}^0\!>x_2\, {}^0 0],\ \ldots,\ \lambda x_{56}\, [{}^0\!>x_{56}\, {}^0 0], \ldots.$$

All these infinitely many constructions construct the class of positive real numbers and each of them can serve as the meaning of the expression (*real*) *numbers greater than 0*. Yet they are pairwise distinct, so if concepts were defined as closed constructions we would end up with infinitely many concepts of positive numbers here. This is absurd, because all these Closures can be assigned to one and the same unambiguous expression as its meaning. When executing the respective procedure we follow one and the same instruction: for any number, check whether this number is greater than zero; the fact which variable is used here to supply the number as an input parameter of the procedure is irrelevant. In this way one can potentially reach the infinite characteristic function of the class positive numbers.

To illustrate procedurally irrelevant η-transformations, consider the following constructions (Tom/ι; $Happy/(o\iota)_{\tau\omega}$; $x \rightarrow_v \iota$):

$$\begin{gathered}
\lambda w \lambda t\, [{}^0 Happy_{wt}\, {}^0 Tom], \\
\lambda w \lambda t\, [[\lambda w'\, [{}^0 Happy\, w']]_{wt}\, {}^0 Tom], \\
\lambda w \lambda t\, [[\lambda w'\, [\lambda t'\, [[{}^0 Happy\, w'] t']]]_{wt}\, {}^0 Tom], \\
\lambda w \lambda t\, [[\lambda w'\, [\lambda t'\, [\lambda x\, [[{}^0 Happy\, w'] t'] x]]]_{wt}\, {}^0 Tom].
\end{gathered}$$

Though the number of constituents is increasing, replacing the Trivialization ${}^0 Happy$ by its η-equivalents $\lambda w'\, [{}^0 Happy\, w']$, $[\lambda w'\, [\lambda t'\, [[{}^0 Happy\, w'] t']$, $[\lambda w'\, [\lambda t'\, [\lambda x\, [[{}^0 Happy\, w'] t'] x]]]$ is irrelevant from the procedural point of view. All the above Closures could be assigned to the sentence 'Tom is happy' as its analyses. Yet the sentence is unambiguous. It expresses just one meaning that is one concept.

Our proposal, then, is to introduce a notion of procedural isomorphism designed to obliterate semantically irrelevant procedural differences. Thus we aim at the same goal as Church; to define the degree to which concepts should be fine-grained. Summarising Church's Alternatives mentioned above (see Church [1993]), we have:

- Alternative (0). α-conversion,
- Alternative (1). α- and β-conversion,
- Alternative (1′). α-, β-, and η-conversion,
- Alternative (2). Logical equivalence.

Logical equivalence is no doubt too coarse-grained criterion, as is demonstrated by the well-known problem of belief sentences. Above we showed that while β-conversion is too permissive a rule, at least in its unrestricted form, α- and η-conversions should be included. Thus our (minimalistic) proposal of procedural isomorphism is Alternative (1/2) including α- and η-conversion defined as follows:

Definition 2 (procedurally isomorphic constructions). Let C, D be constructions. Then C, D are *α-equivalent* iff they differ at most by deploying different λ-bound variables. C, D are *η-equivalent* iff one arises from the other by η-reduction or η-expansion. C, D are *procedurally isomorphic* iff there are closed constructions $C_1, \ldots, C_m$, $m \geqslant 1$, such that ${}^0C = {}^0C_1$, ${}^0D = {}^0C_m$, and if $m > 1$, then all C_i, $C_{i+1}(1 \leqslant i < m)$ are either α- or η-equivalent.

For instance, the constructions ${}^0\mathit{Prime}$, $\lambda x\,[{}^0\mathit{Prime}\,x]$, $\lambda y\,[{}^0\mathit{Prime}\,y]$, $\lambda z\,[\lambda x\,[{}^0\mathit{Prime}\,x]\,z]$, are procedurally isomorphic, while $\lambda x\,[[{}^0\mathit{Card}\,\lambda y\,[{}^0\mathit{Divide}\,y\,x]] = {}^02]$ is only equivalent to them; it does construct the set of primes, but does so in a non-isomorphic manner.[33]

Types: $x, y, z \to \nu$, the type of natural numbers; $\mathit{Card}/(\nu(o\nu))$: the number of elements of a set of natural numbers; $\mathit{Divide}/(o\nu\nu)$: the relation of dividing x by y.

Slightly different definitions of procedural isomorphism are thinkable. We are also considering whether it might be philosophically wise to adopt several notions of procedural isomorphism. It is not at all improbable that several degrees of hyperintensional individuation are called for, depending on exactly which sort of hyperintensional context happens to be analyzed.[34]

Above we excluded unrestricted β-conversion for technical reasons. There is another reason why not to adopt this rule. Oftentimes even the β-equivalent constructions have its different counterparts in natural language. This is in particular the case of attitudinal reports in their *de dicto* and *de re* variants. For instance, the difference between

‘Charles believes that Tom is happy’

and

‘Tom is believed by Charles to be happy’

is just the difference between β-equivalent meanings. The former (*de dicto*) receives the analysis

$$\lambda w\lambda t\,[{}^0\mathit{Believe}_{wt}\,{}^0\mathit{Charles}\,\lambda w\lambda t\,[{}^0\mathit{Happy}_{wt}\,{}^0\mathit{Tom}]]$$

[33] In the interest of better readability we often omit Trivialization, the type subscript, and use infix notion when using constructions of identities of α-entities, $=_\alpha/(o\alpha\alpha)$, if no confusion arises.

[34] See also Jespersen [2010] and [2010a].

while the latter (*de re*) expresses

$$\lambda w \lambda t\, [\lambda x\, [^0 Believe_{wt}\, ^0 Charles\, \lambda w \lambda t\, [^0 Happy_{wt}\, x]]^0 Tom]$$

Types: *Charles*, Tom/ι; $Happy/(o\iota)_{\tau\omega}$; $x \rightarrow_v \iota$; $Believe/(o\iota\, o_{\tau\omega})_{\tau\omega}$: an implicit belief (closed under logical equivalence).[35]

Note that the attitudes *de dicto* and *de re* are in general *not* equivalent. For instance, the truth-conditions of slightly altered beliefs are different:

'Charles believes that the Pope is happy'

'The Pope is believed by Charles to be happy'

The analyses are:

$$\lambda w \lambda t\, [^0 Believe_{wt}\, ^0 Charles\, \lambda w \lambda t\, [^0 Happy_{wt}\, ^0 Pope_{wt}]]$$

$$\lambda w \lambda t\, [\lambda x\, [^0 Believe_{wt}\, ^0 Charles\, \lambda w \lambda t\, [^0 Happy_{wt}\, x]]\, ^0 Pope_{wt}]$$

Additional type: $Pope/\iota_{\tau\omega}$: an individual office.

While the former Closure constructs a proposition that can well be true even when the Pope does not exist (the office is vacant), the proposition constructed by the second Closure has a truth-value gap in such a world-time pair. This is due to the fact that in a world w and time t at which the office is vacant the Composition $^0 Pope_{wt}$ is v-improper. Due to compositionality, the whole Composition $[\lambda x\, [^0 Believe_{wt}\, ^0 Charles\, \lambda w \lambda t\, [^0 Happy_{wt}\, x]]\, ^0 Pope_{wt}]$ is then v-improper; it does not v-construct any truth-value.

Yet there is a restricted version of the equivalent β-conversion that is similar to η-conversion and which is usually not expressed in natural language, nor would there be much point in doing so. This is a reduction that consists in substituting a free variable for a λ-bound variable of the same type.[36] For instance, why differentiate between 'Tom is believed by Charles to be happy' and 'Tom has the property of being believed by Charles to be happy'? The latter sentence expresses

$$\lambda w \lambda t\, [\lambda w' \lambda t' \lambda x\, [^0 Believe_{w't'}\, ^0 Charles\, \lambda w \lambda t\, [^0 Happy_{wt}\, x]]_{wt}\, ^0 Tom]$$

[35] Another and perhaps more adequate alternative is an explicit belief that is a relation-in-intension of an individual to a hyperproposition: $(o\iota *_n)_{\tau\omega}$.

[36] In Duží [2004] this kind of β-reduction is called β_{i}-reduction, where by 'i' is meant "innocent".

This is a β-expanded form of

$$\lambda w \lambda t\, [\lambda x\, [{}^0 Believe_{wt}\, {}^0 Charles\, \lambda w \lambda t\, [{}^0 Happy_{wt}\, x]]\, {}^0 Tom].$$

Thus a slightly coarser notion of procedural isomorphism would include this innocent β_{i}-transformation.

Procedural isomorphism is an equivalence relation and thus it induces the factor set of equivalent classes of constructions. In [1998] Materna defined concepts as such equivalence classes.[37] The drawback of this solution is, however, obvious: a concept was construed as a *set*, an outcome that is in a direct opposition to the conception of concepts being structured procedures and not mere set-theoretic entities. This defect has been corrected by Horák in [2002] by another proposal. Duží, Jespersen and Materna explain Horák's solution as follows:

> The solution that Horák puts forward in [2002] is based on exploiting the Quid relation to define a *normalization* procedure resulting in the unique *normal form* of a *construction* C: $NF(C)$. If this procedure is applied to a closed construction C, the result, $NF(C)$, is the simplest member of the *Quid* equivalence class generated by C. The simplest member is defined as the alphabetically first, non-η-reducible construction. For every closed construction C it holds that $NF(C)$ is the *concept induced by* C, the other members of the same equivalence class *pointing to* this concept. In this manner Horák's solution makes it possible to define *concepts as normalized closed constructions*. Their type is always $*_n$, $n \geqslant 1$.
>
> [2010, p. 155]

For instance, the following constructions are procedurally isomorphic and thus belong to the same *Quid* class (a Materna-style concept of the successor function):

$$\lambda x\, [{}^0{+}\, x\, {}^0 1];\ \lambda y\, [{}^0{+}\, y\, {}^0 1];\ \lambda z\, [{}^0{+}\, z\, {}^0 1];\ \lambda x\, [\lambda x\, [{}^0{+}\, x\, {}^0 1]\, x];\ \lambda y\, [\lambda x\, [{}^0{+}\, x\, {}^0 1]\, y].$$

The normal form of these constructions is $\lambda x\, [{}^0{+}\, x\, {}^0 1]$. Thus, $\lambda x\, [{}^0{+}\, x\, {}^0 1]$ is a Horák-style concept of the successor function. Since Horák's solution is more plausible than the previous solution offered by Materna, we adopt this definition:

[37] Materna here called the procedural isomorphism "*Quasi-identity*", abbrev. "*Quid*".

Definition 3 (concept). Concept is a closed construction in its normal form.

In this section we explicated concepts as abstract procedures defined as closed constructions in their normal form. We also briefly described the way of assigning concepts to expressions as their context-invariant meanings. It is readily seen that our conception is immune to the objections raised against Frege's notion of concept in Section 2. Moreover, our definition is more general. It includes not only general concepts but also Church's individual concepts, mathematical concepts of numbers, concepts of propositions, of individual offices, etc.

Thus the question raised in the title of this paper can be answered as follows. Concepts are not sets. Rather, they are abstract ways of producing sets and other objects as their products. This, of course, does not exclude the attempts to define concepts in terms of sets. Yet we are convinced that any plausible definition must be hyper-intensional, inserting concepts at the level of sense rather than denotation. Moreover, to our best knowledge, this hyperintensional level is best explicated procedurally.

Our procedural theory of concepts is not only immune to the objections raised against Frege's theory. In addition, it operates smoothly even in those cases that are traditionally hard-nuts for denotational semantic theories like the sense of non-denoting meaningful terms, definition of synonymous expressions and logic of attitudes. As for attitudes, we distinguish between implicit and explicit attitudes. Implicit attitudes are relations-in-intension of an individual to an α-intension (i.e. $(o\iota\alpha_{\tau\omega})_{\tau\omega}$-objects) and thus they are closed under logical entailment. In particular, some version of the well-known paradox of omniscience is inevitable in case of implicit propositional attitudes. On the other hand, explicit attitudes are hyperintensional, relations-in-intension of an individual to a construction; that is they are objects of type $(o\iota *_n)_{\tau\omega}$. This is the type of attitudes to mathematical objects (recall the example of calculating) and of those attitudes where paradox of omniscience must not arise.[38] The problems of synonymy and non-denoting terms will be solved in the next section.

[38] For details on TIL attitude logic see, in particular, Duží *et al* [2010, Ch. 5].

6. Some applications

Now we can easily explain how it is possible that some meaningful expressions do not denote anything.[39] They express *empty concepts*, that is *improper constructions* that do not construct anything. For instance, 'the greatest prime number' is not a meaningless term. Prior to proving that there is no greatest prime, mathematicians had to understand what is to be proved. Thus the meaning of this term is the Composition $[{}^0Greatest\,{}^0Prime]$; $Greatest/(\tau(o\tau))$: the function that returns as its value the greatest element of a set of numbers; $Prime/(o\tau)$: the set of prime numbers. The Composition $[{}^0Greatest\,{}^0Prime]$ is improper, because the function *Greatest* is undefined on *Prime*. Yet the Composition is no less a construction due to its improperness, and can be assigned to the term 'the greatest prime number' as its meaning.

Empirical expressions do not express empty concepts as some mathematical expressions do. The reason is that *empirical* expressions are defined as those that denote non-constant *intensions*. Thus the concept assigned to an empirical expression as its meaning is never empty; it is a concept of the denoted intension. For instance, 'golden mountain' and 'man taller than the Eiffel tower' denote individual properties, objects of type $(o\iota)_{\tau\omega}$. The fact that these properties are not instantiated in the actual world now (golden mountains and men taller than the Eiffel tower do not exist) does not make them being less-objects than any instantiated property. Similarly 'the first man to run 100 m under 5 s' denotes an individual role/office of type $\iota_{\tau\omega}$. It is a condition to be satisfied by the holder of this role. True, this condition is so hard that it is beyond the capacities of a human being to satisfy; hence the role is currently and actually vacant. Yet from the logical point of view there is nothing to prevent the role of being occupied: such a man *might* exist and the concept of the first man to run 100 m under 5 seconds is not empty absolutely, it is only *empirically empty* in the actual world now.

Even concepts of impossible entities like the property of being older than oneself are not strictly empty. The meaning of the predicate 'being older than oneself' is the Closure $(Older/(o\iota\iota)_{\tau\omega},\ x \rightarrow_v \iota)$

$$\lambda w \lambda t\, [\lambda x\, [{}^0Older_{wt}\, x\, x]]$$

[39] Portions of this section draw on material of Materna [1998, Chapter 7] and Materna [2004, Chapter 1.4].

that constructs a constant property of individuals returning the value **F** for all individuals in all world-time pairs.

Our procedural theory of concepts is thus a powerful theory that complies with our intuitive assumptions and explains many hard-nut phenomena which are often a stumbling block of weaker theories. Yet no explication can presumably satisfy all intuitive criteria and philosophical desiderata connected with the explicated notion. There are some disputable features of our approach as well.

First, since *any* meaningful expression not containing indexicals expresses a concept, we have concepts of any entity of language ontology. Thus even sentences express concepts, *viz.* of truth-values in mathematics and of propositions in case of empirical sentences. This may seem unnatural. People usually connect concepts with non-sentential terms. This is perhaps just a terminological problem. Our over-arching semantic theory is universal and homogeneous. Any meaningful expression regardless whether it is a sentence or a non-sentential term expresses as its context-invariant meaning a construction. In case the meaning is not pragmatically incomplete it expresses a closed construction, i.e., a concept.

Second, *empirical* expressions express concepts of intensions denoted by a given expression. This is correct and defendable. Yet one might raise this objection: Isn't, for instance, the concept of the *highest mountain* the concept of the concrete individual Mt Everest rather than of some abstract individual office? Our answer is no, it is not. The reason is this. The term 'highest mountain' expresses a conceptualization of a *condition* to be satisfied by concrete individuals. Which individual, if any, satisfies the condition is a matter of contingent facts and it cannot be decided by purely logical analysis; empirical investigation is needed to determine the satisfier. Hence *logical* semantics does not investigate empirical facts, as already Carnap in [1947] knew. To illustrate this fact, consider the sentence 'The highest mountain is in Asia'. Does this sentence entail 'Mt Everest is in Asia'? Certainly no; additional assumption is needed, namely that: 'The highest mountain is Mt Everest'.

To distinguish between conditions and their satisfiers, we also use two different notions, *denotation* and *reference in* the actual world now. Whereas reference is out of scope of logical semantics, denotation is the back-end of our semantic schema:

$$\text{expression} \xrightarrow[\textit{expresses}]{} \text{construction/concept} \xrightarrow[\textit{denotes}]{} \text{denotation}$$

Another positive feature of our procedural theory of concepts is the capacity to define *synonymy* and distinguish it of pure *equivalence*. Expressions are *synonymous* iff their meanings are procedurally isomorphic. Expressions are *equivalent* iff they denote one and the same entity. Finally, expressions are *coreferential* iff they refer to one and the same entity in the actual world now. Obviously, synonymy implies equivalence and equivalence implies co-reference, but not *vice versa*.

Thus contra Frege, 'morning star' is not equivalent to 'evening star'. These two terms only happen to be coreferential in the actual world now. Example of equivalent expressions would be, e.g., 'being older than' and 'not being younger or of the same age as'. Note that equivalent expressions do not have to be *logically* equivalent. Synonymous expressions often differ only by grammatical variants that do not reflect different meanings, like 'a man who is coming' and 'a coming man'. As an example of syntactically simple synonymous expressions one can presumably take the predicates 'sky-blue' and 'azure'. Note that provided these expressions are synonymous then they literally express one and the same meaning regardless of the name used to denote the colour. Thus ^{0}*Sky-blue* and ^{0}*Azure* is one and the same construction, hence one and the same concept.

7. Appendix: Gödel's uneasiness with concepts

In his [1944] Gödel says:

> I shall use the term "concept" in the sequel exclusively in this objective sense. One formal difference between the two conceptions of notions would be that any two different definitions of the form $\alpha(x) \equiv \phi(x)$ can be assumed to define two different notions α in the constructivistic sense. [...] For concepts, on the contrary, this is by no means the case, since the same thing may be described in different ways. [...] The difference may be illustrated by the following definition of the number two: "Two is the notion under which fall all pairs and nothing else." There is certainly more than one notion in the constructivistic sense satisfying this condition, but there might be one common "form" or "nature" of all pairs.
>
> [1990, p. 128]

The way Gödel used the notion concept here is not at all clear.[40] This was noticed also by Parsons in his comment:

[40] Portions of this section draw on material presented in Materna [2007].

> By 'concepts' Gödel evidently means objects signified in some way (italics the authors) by predicates. [...] Gödel's remarks about realistic theories of concepts [...] have an inconclusive character, no available theory satisfies him. [1990, p. 110]

Yet Gödel tried to be more definite:

> [t]he meaning of the term 'concept' seems to imply that every propositional function defines a concept. [1990, p. 139]

However, this is hardly helpful, because there are other questions arising here: Is every concept defined by a propositional function? Do equivalent propositional functions define one concept, or two distinct concepts? The answers again depend on the decision whether to adopt constructivistic or set-theoretical view.

Gödel himself is not content. On page 140 one can read that there is a need

> [...] to make the meaning of the terms "class" and "concept" clearer and to set up a consistent theory of classes and concepts as objectively existing entities. [1990, p. 139]

Sure, as we have argued in this paper, a concept cannot be identified with a property or class or with an expression denoting a property or class. Thus we proposed a procedural theory of concepts which can be executed in order to produce a property, class, and generally any entity of our ontology as their products. Concepts defined as closed constructions share some noteworthy common ground with constructions of constructivist logic. Both consist of some atoms combined into a whole by operations (combination rules in intuitionism). But there is also a substantial difference. Whereas TIL constructions are modes of presentation of pre-existing entities, for constructivists/intuitionists constructions are proofs mostly understood as mental acts of proving.

One can ask why Gödel did not accept the constructivistic view. The reason is most likely this: Gödel was undoubtedly a realist. He assumed (and expressed this assumption explicitly, see [1990, p. 129]) that the constructivistic view is a nominalistic one, i.e., essentially a manifestation of a subjectivist philosophy. Indeed, Brouwer's and even Dunmmett's constructions are in principle mental entities and thus they are not acceptable for Gödel (and, for that matter, for any realist) as candidates of being concepts.

Our procedural theory is a realist theory and our suggestion is that the procedural theory of concepts can meet the desiderata expressed by Gödel. Thus this theory can be a solution of Gödel's problem of setting up "a consistent theory of classes and concepts as objectively existing entities". Classes are set-theoretical products of concepts conceived as algorithmically structured procedures known as TIL constructions. These constructions are rigorously defined and available for logical manipulation.

Acknowledgments. This study has been supported by the Grant Agency of Czech Republic; project P401/10/0792. We are grateful to an anonymous reviewer for valuable comments which improved the quality of this paper.

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