However, the standard notion of prime implicate would not be able to handle Example 4.2 since Definition 4.1 uses a classical notion of consequence and thus inconsistent formulae would have the same (complete equivalence class) of prime implicates—namely the empty clause $\emptyset$.[4] But this is exactly what we are trying to avoid in the first place. However, there is a straightforward way to amend the situation—we use a *relevant* notion of prime implicate:

DEFINITION 4.2. A clause D is a relevant implicate of A iff $A \models_{FDE} D$. A relevant implicate D of A is prime iff for all relevant implicates D' of A if $D' \models_{FDE} D$ then $D \models_{FDE} D'$. A set of relevant prime implicates $\{D_1, \ldots, D_n\}$ of A is complete iff $D_1 \wedge \ldots \wedge D_n \models_{FDE} A$. A set of relevant prime implicates $\{D_1, \ldots, D_n\}$ is independent iff for no D_i do we have $D_1 \wedge \ldots \wedge D_{i-1} \wedge D_{i+1} \wedge \ldots \wedge D_n \models_{FDE} D_i$. We say that two sets of formulae Γ and Δ are *FDE* equivalent, written as $\Gamma \equiv_{FDE} \Delta$, iff $\bigwedge \Gamma \models_{FDE} \bigwedge \Delta$ and $\bigwedge \Delta \models_{FDE} \bigwedge \Gamma$.

The dual notion of *relevant prime implicant* of a given formula A can easily be defined: a relevant implicant of a formula A is a *cube* C (conjunction of literals) which *FDE*-entails A. In addition, C is *prime* if it is a minimal cube that *FDE*-entails A.

Since *FDE* is a paraconsistent logic, the empty clause is not a *FDE*-consequence of any (non-empty) inconsistent formula, i.e. for no A do we have $A \models_{FDE} \emptyset$. Given that *resolution* is not a valid form of inference in relevant logic in general, it is easy to see that the set of classical prime implicates (PI) and the set of relevant prime implicates (RPI) may be distinct for a given formula:

Example 4.4. $A = (q \vee r) \wedge ((p \vee q) \wedge (\neg p \vee q))$

[4] There have been other attempts in defining the *content* of propositional data. In [13, 14], Gemes gives several definitions of the *empirical content* of propositional data. However, none of his definition addressed the problem of inconsistent data and thus provided no useful means to individuate the content of inconsistent data. In [39], Tennant applies the Fregean Premise Principle for Content Identity to a relevant logic and shows that even with a relevant logic all contradictions are, in a sense, content equivalent. The author is inclined to take Tennant's result to be a *modus tollens* for rejecting either the Fregean Principle or the use of Tennant's relevant logic for specifying contents. In a separate paper, we'll investigate the possibility of saving the Fregean Principle using the proposal stated in this paper while maintaining that contradictions are NOT content equivalent.

	RPI	PI
$q \vee r$	$\surd$	$\times$
q	$\times$	$\surd$

Figure 7. *RPI* and *PI* of A

Although not every RPI of a given A is a PI of A, it is easy to see that every RPI of a given A is a classical implicate of A (since $\models_{FDE} \subset \models$).

Since any two complete independent sets of relevant prime implicates of a given formula must be FDE equivalent, we can treat them as unique up to equivalence. We'll use the notation $RPI(A)$ to denote any such complete independent set of relevant prime implicates of A. Similarly we use $PI(A)$ for the complete independent set of classical prime implicates of A. We note that $RPI(A)$ is a minimal set (ordered under $\subseteq$) that is both complete and independent. In classical logic, two formulae are equivalent iff their prime implicates are equivalent. This is also true with respect to FDE formulae:

PROPOSITION 4.1. *For any A and B, $A \models_{FDE} B$ and $B \models_{FDE} A$ iff $RPI(A) \equiv_{FDE} RPI(B)$.*

PROOF. ($\Rightarrow$): Suppose A and B are FDE equivalent. Let $RPI(A) = \{D_1, \ldots, D_m\}$ and $RPI(B) = \{E_1, \ldots, E_n\}$. By the transitivity of $\models_{FDE}$ we have, $\bigwedge_{i \leqslant m} D_i \models_{FDE} E_j$ for each $j \leqslant n$. Hence $\bigwedge_{i \leqslant m} D_i \models_{FDE} \bigwedge_{j \leqslant n} E_j$. Similarly we can show that $\bigwedge_{j \leqslant n} E_j \models_{FDE} \bigwedge_{i \leqslant m} D_i$.
($\Leftarrow$): Suppose $RPI(A) \equiv_{FDE} RPI(B)$. Then we have

$$A \models_{FDE} \bigwedge_{i \leqslant m} D_i \models_{FDE} \bigwedge_{j \leqslant n} E_j \models_{FDE} B$$

Similarly we have

$$B \models_{FDE} \bigwedge_{j \leqslant n} E_j \models_{FDE} \bigwedge_{i \leqslant m} D_i \models_{FDE} A$$

By the transitivity of these entailments, it follows that A and B are FDE equivalent. ⊣

An immediate corollary is that standard reduction rules for CNF (DNF) conversion are RPI preserving:

COROLLARY 4.1. *The following equivalences hold:*

1. $RPI(\neg\neg A) \equiv_{FDE} RPI(A)$
2. $RPI(\neg(A \vee B)) \equiv_{FDE} RPI(\neg A \wedge \neg B)$
3. $RPI(\neg(A \wedge B)) \equiv_{FDE} RPI(\neg A \vee \neg B)$
4. $RPI(A \vee (B \wedge C)) \equiv_{FDE} RPI((A \vee B) \wedge (A \vee C))$

The minimality of an RPI ensures that a certain *transitivity* property of RPI holds:

PROPOSITION 4.2. *For any formulae A, B and C, if $C \in RPI(B)$ and $B \in RPI(A)$, then $C \in RPI(A)$.*

PROOF. Given that $B \in RPI(A)$, B must be a clause and thus $B \in RPI(B)$ holds trivially. So if $C \in RPI(B)$, $B \equiv_{FDE} C$ follows immediately from Definition 4.2. Hence $C \in RPI(A)$. ⊣

Just as Belnap's replacement rules can be used as a basis for defining the closure operators C_B and C_B^+, RPIs too can be used as a basis for defining certain Tarskian closure operators:

DEFINITION 4.3. For any A and Γ, define

$$\begin{aligned}
C_{RPI}(A) &= RPI(A) \cup \{A\} \\
C_{RPI}(\Gamma) &= \{B \in C_{RPI}(A) |\ A \in \Gamma\} \\
C^+_{RPI}(\Gamma) &= \bigcup_{\Delta \subseteq_{\text{fin}} C_{RPI}(\Gamma)} \{B \mid B \equiv_{FDE} \bigwedge \Delta\}
\end{aligned}$$

PROPOSITION 4.3. *C_{RPI} and C^+_{RPI} are both Tarskian closure operators. Moreover, C^+_{RPI} is an E-equivalent (U-equivalent) extension of C_{RPI}.*

PROOF. Reflexivity: trivial since $A \in C_{RPI}(A)$ for every $A \in \Gamma$.

Monotonicity: Assume $\Gamma \subseteq \Delta$, then if $B \in C_{RPI}(\Gamma)$, there must exist some $A \in \Gamma$ such that $B \in C_{RPI}(A)$. But $A \in \Delta$ holds, so $B \in C_{RPI}(\Delta)$ as required.

Idempotence: $C_{RPI}(\Gamma) \subseteq C_{RPI}(C_{RPI}(\Gamma))$ is implied by the monotonicity of C_{RPI} above. For $C_{RPI}(C_{RPI}(\Gamma)) \subseteq C_{RPI}(\Gamma)$, we note that Proposition 4.2 gives us the transitivity property of RPI:

$$D \in C_{RPI}(C_{RPI}(\Gamma)) \implies \exists A \in C_{RPI}(\Gamma) : \ D \in C_{RPI}(A)$$

$$
\begin{aligned}
&\Longrightarrow \quad \exists B \in \Gamma : A \in C_{RPI}(B) \\
&\Longrightarrow \quad D \in C_{RPI}(B) \\
&\Longrightarrow \quad D \in C_{RPI}(\Gamma)
\end{aligned}
$$

Reflexivity and monotonicity for C^{+}_{RPI} are straightforward. For idempotence, we verify that $C^{+}_{RPI}(C^{+}_{RPI}(\Gamma)) \subseteq C^{+}_{RPI}(\Gamma)$:

$$
\begin{aligned}
A \in C^{+}_{RPI}(C^{+}_{RPI}(\Gamma)) \quad &\Longrightarrow \quad \exists C_1, \ldots, C_i \in C_{RPI}(C^{+}_{RPI}(\Gamma)) : \\
&\qquad A \equiv_{FDE} C_1 \wedge \ldots \wedge C_i \\
&\Longrightarrow \quad \exists D_1, \ldots, D_i \in C^{+}_{RPI}(\Gamma) : \\
&\qquad \forall j \leqslant i,\ C_j \in C_{RPI}(D_j) \\
&\Longrightarrow \quad \forall j \leqslant i,\ \exists E^1_j, \ldots, E^m_j \in C_{RPI}(\Gamma) : \\
&\qquad D_j \equiv_{FDE} E^1_j \wedge \ldots \wedge E^m_j \\
&\Longrightarrow \quad \forall j \leqslant i,\ \exists F^1_j, \ldots, F^m_j \in \Gamma : \\
&\qquad E^1_j \in C_{RPI}(F^1_j), \ldots, E^m_j \in C_{RPI}(F^m_j) \\
&\Longrightarrow \quad \forall j \leqslant i,\ C_j \in C_{RPI}(E^1_j \wedge \ldots \wedge E^m_j) \\
&\Longrightarrow \quad \forall j \leqslant i,\ \exists k :\ C_j \equiv_{FDE} E^k_j \\
&\Longrightarrow \quad \forall j \leqslant i,\ \exists F^k_j \in \Gamma :\ C_j \in C_{RPI}(F^k_j) \\
&\Longrightarrow \quad C_1, \ldots, C_i \in C_{RPI}(\Gamma) \\
&\Longrightarrow \quad A \in C^{+}_{RPI}(\Gamma)
\end{aligned}
$$

To show that C^{+}_{RPI} is an E-equivalent extension of C_{RPI}, we need to show that

1. C_{RPI} and C^{+}_{RPI} have the same E-consequences.
2. For any Γ, $C_{RPI}(\Gamma) \subseteq C^{+}_{RPI}(\Gamma)$.
3. For any Γ, $C^{+}_{RPI}(C_{RPI}(\Gamma)) = C^{+}_{RPI}(\Gamma)$.

(2) is trivial. For (1) we note that any FDE equivalent formula are also classically equivalent, so an argument similar to Lemma 3.2 suffices to show that C^{+}_{RPI} is an E-equivalent (U-equivalent) extension of C_{RPI}. Finally we verify that $C^{+}_{RPI}(C_{RPI}(\Gamma)) = C^{+}_{RPI}(\Gamma)$:

($\supseteq$): Trivial since C_{RPI} and C^{+}_{RPI} are both Tarskian closure operators.

($\subseteq$): We note that C_{RPI} is idempotent.

$$
A \in C^{+}_{RPI}(C_{RPI}(\Gamma)) \quad \Longrightarrow \quad \exists B_1, \ldots, B_i \in C_{RPI}(C_{RPI}(\Gamma)) :
$$

$$\begin{aligned} & A \equiv_{FDE} (B_1 \wedge \ldots \wedge B_i) \\ \Longrightarrow \quad & \exists B_1, \ldots, B_i \in C_{RPI}(\Gamma) : \\ & A \equiv_{FDE} (B_1 \wedge \ldots \wedge B_i) \\ \Longrightarrow \quad & A \in C^{+}_{RPI}(\Gamma) \quad \dashv \end{aligned}$$

We note that Definition 4.3 makes use of $RPI(A)$ for each A in a given set Γ, but $\bigcup_{A\in\Gamma} RPI(A)$ need not be an independent set of RPIs. In particular redundant (i.e. disjunctively implied) information can be spread across members of Γ. This motivates the following alternative definition:

Definition 4.4. For any Γ and any clause C, we define $C \in RPI^*(\Gamma)$ iff

1. for some $A \in \Gamma$, $C \in RPI(A)$ and
2. for any $B \in \Gamma$ and clause D, if $D \in RPI(B)$ and $D \models_{FDE} C$, then $C \models_{FDE} D$

For any Γ,

$$C^*_{RPI}(\Gamma) = RPI^*(\Gamma) \cup \Gamma$$

Membership for C^*_{RPI} is clearly more stringent than C_{RPI}—a clause C is in $RPI^*(\Gamma)$ only if C is a relevant prime implicate of some member of Γ and no other member of Γ entails a relevant prime implicate stronger than C. This definition is similar to Definition 4.2 for the RPI's of an individual formula. However, C^*_{RPI} is not a closure operator in Tarski's sense. Although both reflexivity and idempotence remain intact, C^*_{RPI} does not have the usual monotonicity property.

Example 4.5. $\Gamma = \{p \wedge (q \vee r)\}$, $p \in C^*_{RPI}(\Gamma)$ and $q \vee r \in C^*_{RPI}(\Gamma)$. But $q \vee r \notin C^*_{RPI}(\Gamma')$ where $\Gamma' = \{p \wedge (q \vee r), q\}$.

The failure of monotonicity should not be regarded as a defect of C^*_{RPI}. Arguably, *implicit information* need not always increase monotonically with respect to supersets; C^*_{RPI} is a possible candidate for specifying the content of a given set of logical expressions. To illustrate the difference between C^*_{RPI} and C_{RPI} consider the following example:

Example 4.6. $\Gamma = \{p, (r \wedge \neg r) \wedge (p \vee q), \neg p\}$

Since $p \in C^*_{RPI}(\Gamma)$ we have $p \vee q \notin C^*_{RPI}(\Gamma)$. However $p \vee q \in C_{RPI}(\Gamma)$ given that $p \vee q \in RPI((r \wedge \neg r) \wedge (p \vee q))$. Note that in Example 4.6 q is an E-consequence of $C_{RPI}(\Gamma)$ but not an E-consequence of $C^*_{RPI}(\Gamma)$. In general, C^*_{RPI} does not yield the same E-consequence (U-consequence) as C_{RPI}.

PROPOSITION 4.4. *For any* Γ,

1. $C^*_{RPI}(\Gamma) \subseteq C_{RPI}(\Gamma)$
2. $C_{PRI}(C^*_{RPI}(\Gamma)) = C_{RPI}(\Gamma)$
3. $C^*_{RPI}(C_{RPI}(\Gamma)) = C_{RPI}(\Gamma)$

PROOF. For (1) it suffices to observe that $RPI^*(\Gamma) \subseteq \bigcup_{A\in\Gamma} RPI(A)$.

(2$\supseteq$): Since $\Gamma \subseteq C^*_{RPI}(\Gamma)$, we have $C_{RPI}(\Gamma) \subseteq C_{RPI}(C^*_{RPI}(\Gamma))$ by the monotonicity of C_{RPI}.

(2$\subseteq$): From (1) we have $C^*_{RPI}(\Gamma) \subseteq C_{RPI}(\Gamma)$ so by the monotonicity of C_{RPI} it follows that $C_{RPI}(C^*_{RPI}(\Gamma)) \subseteq C_{RPI}(C_{RPI}(\Gamma))$. By the idempotence of C_{RPI} we have $C_{RPI}(C^*_{RPI}(\Gamma)) \subseteq C_{RPI}(\Gamma)$.

(3): One direction follows from Definition 4.4 immediately. For the other direction consider an arbitrary but fixed Γ_0. From (1) above we have $C^*_{RPI}(C_{RPI}(\Gamma_0)) \subseteq C_{RPI}(C_{RPI}(\Gamma_0))$. But from Proposition 4.3 C_{RPI} is a Tarskian closure operator and thus $C_{RPI}(C_{RPI}(\Gamma_0)) = C_{RPI}(\Gamma_0)$. Hence $C^*_{RPI}(C_{RPI}(\Gamma_0)) \subseteq C_{RPI}(\Gamma_0)$. Since Γ_0 was arbitrary we conclude that for any Γ, $C^*_{RPI}(C_{RPI}(\Gamma)) \subseteq C_{RPI}(\Gamma)$. ⊣

Returning to examples 4.1 and 4.2, Belnap's replacement rules are complete with respect to the given A and B in these examples, i.e. $C_{RPI}(A) \subset C_B(A)$ and $C_{RPI}(B) \subset C_B(B)$, but the generated implicates are not all prime. So Belnap's replacement rules are unsound with respect to relevant prime implicates. In the general case, Belnap's replacement rules are not complete since they are insufficient to transform formulae into clausal form. Clearly for clause reduction we need the additional rule, $\vdash \neg(B \wedge C) \leftrightarrow (\neg B \vee \neg C)$, to distribute negation over conjunction. However C^+_B is complete with respect to RPI's, i.e. for any Γ, we have $C_{RPI}(\Gamma) \subseteq C^+_B(\Gamma)$. We summarise the relationships of these closure operators in Figure 8.

To illustrate consider $\Gamma = \{(p \wedge q) \vee (p \wedge r), \neg(p \wedge q) \wedge s\}$. Clearly, $p \vee r \in C_B(\Gamma)$ but $p \vee r$ is not an RPI, so $p \vee r \notin C^+_{RPI}(\Gamma)$. Region (1) is non-empty. Moreover $\neg(p \wedge q) \in C_B(\Gamma)$ but $\neg p \vee \neg q \in C^+_{RPI}(\Gamma)$, so region (2) is non-empty. Example 4.6 shows that region (3) is non-empty and

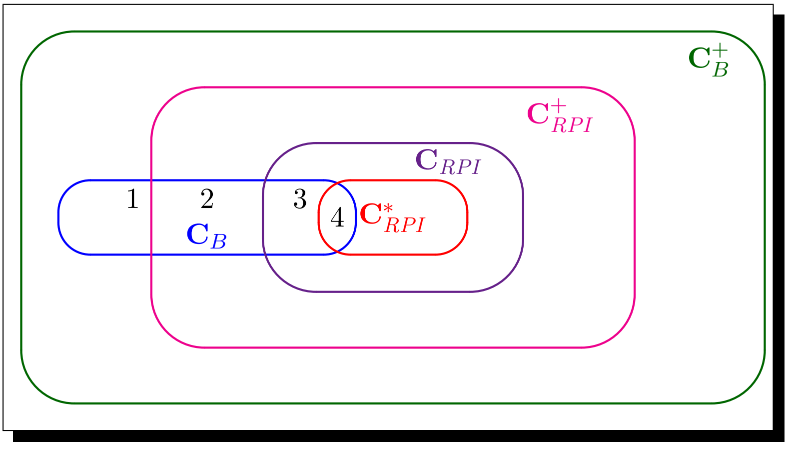

Figure 8. Relationships between Closure Operators

with minor modification it can show that region (4) is also non-empty. To see that $C_{RPI}^+ \subseteq C_B^+$, it suffices to note that

Proposition 4.5. *For any clause D and formula A, if $D \in RPI(A)$, then $E \equiv_{FDE} D$ for some $E \in C_B(A)$.*

Proof. We note that using arguments similar to proofs of Lemma 3.1 and Proposition 3.1, we can show that any A is FDE-equivalent to the conjunction of the leaves of the Belnap's tree $\mathcal{T}^A$, i.e. $le(\mathcal{T}^A) \equiv_{FDE} RPI(A)$. Moreover, each leaf of a Belnap's tree is conjunction free in the sense that each leaf is FDE-equivalent to a clause. Hence if $D \in RPI(A)$, there must be a leaf E of $\mathcal{T}^A$ such that $D \equiv_{FDE} E$. ⊣

We should point out that in adopting the use of either C_{RPI} or C_{RPI}^+ for capturing the informational content of a formula, there is no guarantee that conjunction elimination is a sound strategy for generating RPIs. In general $RPI(A) \cup RPI(B) \neq RPI(A \wedge B)$.

Example 4.7. $A = p \wedge (p \vee q)$

In Example 4.7, it is clear that $RPI(p \wedge (p \vee q)) \subset RPI(p) \cup RPI(p \vee q)$. The containment here is proper. However, we do have the containment $RPI(A \wedge B) \subseteq RPI(A) \cup RPI(B)$ in the general case.

Lemma 4.1. *For any clause C and any formula A and B, if $A \models_{FDE} C$, then $A \wedge B \models_{FDE} C$.*

PROOF. We use the ambi-valuation of Dunn [12] to prove our claim. Assume that $A \models_{FDE} C$. Then we have the implication $1 \in v(A) \Rightarrow 1 \in v(C)$ for any standard 4-valued valuation v of *FDE*. Consider an arbitrary v' such that $1 \in v'(A \wedge B)$. Then it follows that $1 \in v'(A)$ and $1 \in v'(B)$. So on v' in particular, $1 \in v'(C)$. Since v' was arbitrary, we have $A \wedge B \models_{FDE} D$ as required. ⊣

PROPOSITION 4.6. *For any A and B, $RPI(A \wedge B) \subseteq RPI(A) \cup RPI(B)$.*

PROOF. Assume that for an arbitrary clause D we have $D \in RPI(A \wedge B)$ but $D \notin RPI(A) \cup RPI(B)$. Then we have $A \wedge B \models_{FDE} D$ but $D \notin RPI(A)$ and $D \notin RPI(B)$. Then there are 4 cases to consider:

(Case 1) $A \not\models_{FDE} D$ and $B \models_{FDE} D$ but D is not prime for B: it follows that there exists a $D_0 \in RPI(B)$ such that $B \models_{FDE} D_0$ and $D_0 \models_{FDE} D$ but $D \not\models_{FDE} D_0$. By Lemma 4.1, we have $A \wedge B \models_{FDE} D_0$. But given that $D \in RPI(A \wedge B)$ and $D_0 \models_{FDE} D$, $D \models_{FDE} D_0$ holds. This is a contradiction.

(Case 2) $B \not\models_{FDE} D$ and $A \models_{FDE} D$ but D is not prime for A: the proof is similar to case 1 with B replaced with A throughout.

(Case 3) Both $A \models_{FDE} D$ and $B \models_{FDE} D$, but D is prime for neither A nor B: the argument in case (1) suffices to show that case 3 is impossible.

(Case 4) $A \not\models_{FDE} D$ and $B \not\models_{FDE} D$: we make use of the equivalence between *FDE* and *tautological entailment* as described in Anderson and Belnap ([1]). Since $A \not\models_{FDE} D$ and $B \not\models_{FDE} D$, for any arbitrary but fixed $DNF = C_1^A \vee \ldots \vee C_m^A$ of A and $DNF = C_1^B \vee \ldots \vee C_n^B$ of B, there exist some $i \leqslant m$, and some $j \leqslant n$ such that $C_i^A \not\models_{FDE} D$ and $C_j^B \not\models_{FDE} D$. Denote the set of literals occurring in C_i^A as $lit(C_i^A)$. We have $lit(C_i^A) \cap lit(D) = \emptyset$ and $lit(C_j^B) \cap lit(D) = \emptyset$. Hence $(lit(C_i^A) \cup lit(C_j^B)) \cap lit(D) = \emptyset$. Now consider the formula

$$E = \bigvee_{1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n} (C_k^A \wedge C_l^B)$$

Clearly, E is a *DNF* of $A \wedge B$. Since $lit(C_i^A \wedge C_j^B) = (lit(C_i^A) \cup lit(C_j^B))$, we note that $lit(C_i^A \wedge C_j^B) \cap lit(D) = \emptyset$. We define a 4-valued assignment

v on the set of propositional atoms as follows:

$$\begin{cases} 0 \in v(p) \text{ and } 1 \notin v(p) & \text{if } \neg p \in \mathit{lit}(C_i^A \wedge C_j^B) \text{ and} \\ & p \notin \mathit{lit}(C_i^A \wedge C_j^B) \\ 1 \in v(p) \text{ and } 0 \notin v(p) & \text{if } p \in \mathit{lit}(C_i^A \wedge C_j^B) \text{ and} \\ & \neg p \notin \mathit{lit}(C_i^A \wedge C_j^B) \\ 1 \in v(p) \text{ and } 0 \in v(p) & \text{if } p \in \mathit{lit}(C_i^A \wedge C_j^B) \text{ and} \\ & \neg p \in \mathit{lit}(C_i^A \wedge C_j^B) \\ 1 \notin v(p) \text{ and } 0 \notin v(p) & \text{otherwise} \end{cases}$$

Clearly $1 \in v(C_i^A \wedge C_j^B)$ and hence $1 \in v(E)$ but by the disjointness of $\mathit{lit}(C_i^A \wedge C_j^B)$ and $\mathit{lit}(D)$, $1 \notin v(D)$. Hence $A \wedge B \not\models_{FDE} D$. But this contradicts the initial assumption that $D \in RPI(A \wedge B)$. ⊣

4.2. Algorithmic Considerations

Proposition 4.6 shows that in terms of using replacement rules in the style of [∗] or [#] for eliminating conjunctions, the *RPI*s of a child node need not be the *RPI*s of the root node. So although Corollary 4.1 shows that the standard reduction method for *CNF* conversion is indeed complete for generating *RPI*s, there is no guarantee that the clauses obtained are indeed independent. Checking for clause subsumption seems unavoidable and indeed critical when redundant information is presented. However when combined with a clause subsumption check, the standard *CNF* conversion algorithm can provide a sound and complete algorithm for generating *RPI*s.

Algorithm 4.1 RPI Generation

Require: input $A \in \Phi$
Ensure: output $S = RPI(A)$
convert A into $CNF(A)$ using the standard reduction method
for each $C \in CNF(A)$, $S := S \cup \{C\}$ if C is relevant prime, else $S := S$.
return S

Algorithm 4.1 is a naive method for generating *RPI*s. It first generates a set of relevant implicates of A and then prunes the set by removing all non-prime implicates. Clearly we have $CNF(A) \equiv_{FDE} RPI(A)$ given Corollary 4.1. So completeness is ensured in step (1) provided that step (2) does not remove implicates that are also prime (and clearly it doesn't). Although the clause subsumption check may be deployed earlier while $CNF(A)$ is generated, in the worst case the size of $CNF(A)$ can be exponentially related to the size of A, e.g. if $A = (p_1 \wedge p_2) \vee \ldots \vee (p_{2n-1} \wedge p_{2n})$, there are 2^n clauses in the corresponding *CNF*. Our problem is inherently difficult computationally.

4.2.1. *PRI* via Classical *PI* Generation

In what follows, we'll present an alternative algorithm for generating $RPI(A)$ based on ideas from Ramesh *et al* [32, 30, 31] and Arieli and Denecker [3, 4]. The main idea here is to avoid the expensive *CNF* conversion by using *negated normal form* (*NNF*) instead. Once a formula A is converted into $NNF(A)$, we'll make use of Arieli and Denecker's *splitting transform* to convert $NNF(A)$ into a positive (i.e. negation free) formula $\widehat{NNF(A)}$.[5] The conversion will preserve our problem in the sense that for any clause D, $D \in RPI(A)$ iff $\widehat{D} \in PI(\widehat{NNF(A)})$. So in effect our problem is transformed into the classical problem of prime implicate generation for a positive *NNF* formula. The algorithm of Ramesh *et al* [30, 31] can thus be invoked to generate the required *PI*'s via the use of the corresponding *semantic graph*.[6] We recall some of the main definitions from Arieli and Denecker [3, 4].

DEFINITION 4.5. 1. A formula A is in negated normal form (*NNF*) iff no complex subformula of A is in the scope of a negation, i.e. only atomic formulae are within the scope of a negation operator.

2. Let $NNF(A)$ denotes the negated normal form of A. Then the splitting transform of $NNF(A)$, denoted by $\widehat{NNF(A)}$, is the formula obtained by uniformly substituting every unnegated atom p_i occurring

[5] We note that Besnard and Schaub [8] employed the same transform for defining signed systems of paraconsistent reasoning.

[6] We do not have space here to explicate the algorithm or the underlying concept of a *semantic graph* here. The reader can refer to Ramesh *et al* [30, 31] for full details.

in $NNF(A)$ with a new (signed) atom p_i^+ and every negated atom $\neg p_i$ in $NNF(A)$ with a new (signed) atom p_i^- [3, 4]. If $B = \widehat{A}$ for some A, then we define the inverse of splitting transform $\overline{B}$ as the formula obtained by uniformly substituting every signed atom p_i^+ with literal p_i and every signed atom p_i^- with literal $\neg p_i$, i.e. $\overline{\widehat{A}} = A$

3. Let v be an arbitrary 4-valued assignment and $NNF(A)$ an arbitrary NNF formula.[7] Then $\widehat{v}$ is the 2-valued (classical) assignment defined as follows:

    - For all p_i^+ and p_i^- occurring in $\widehat{NNF(A)}$, $\widehat{v}(p_i^+) = 1$ iff $1 \in v(p_i)$ and $\widehat{v}(p_i^-) = 1$ iff $0 \in v(p_i)$.

We note that both the splitting transform and $\widehat{v}$ are well defined and do not depend on A. The following are consequences of Definition 4.5:

PROPOSITION 4.7.

1. *Let v be an arbitrary 4-valued assignment and $NNF(A)$ be an arbitrary NNF formula. Let $\widehat{v}$ be a 2-valued assignment as defined in (3) of Definition 4.5. Then $1 \in v(NNF(A))$ iff $\widehat{v}(\widehat{NNF(A)}) = 1$ (cf. Lemma 3.1 of [3]).*

2. *For any A and B, $A \models_{FDE} B$ iff $\widehat{NNF(A)} \models \widehat{NNF(B)}$ (cf. Theorem 3.1 of [3]).*

3. *For any clause D, $D \in RPI(A)$ iff $\widehat{D} \in PI(\widehat{NNF(A)})$.*

4. *The problem of relevant prime implicate generation is polynomially reducible to classical prime implicate generation.*

PROOF. We note that (2) is a simple corollary of (1). For (1), we use an induction on the structure of $NNF(A)$. There are two base cases with either $NNF(A) = p_i$ or $NNF(A) = \neg p_i$. In the former case we have $1 \in v(p_i) \Leftrightarrow \widehat{v}(p_i^+) = 1$ given by the definition of $\widehat{v}$. In the later case we have $1 \in v(\neg p_i) \Leftrightarrow 0 \in v(p_i) \Leftrightarrow \widehat{v}(p_i^-) = 1$.

For the induction case we have either $NNF(A) = B \wedge C$ or $NNF(A) = B \vee C$. We note that both B and C must be in NNF form and hence the induction hypothesis applies. Thus we have $1 \in v(B) \Leftrightarrow \widehat{v}(\widehat{B}) = 1$ and

[7] The underlying 4-value assignment is based on the same ambi-valuation of Dunn [12] as deployed in the proof of Lemma 4.1. We note in passing that this is not the only semantics available for *FDE*.

$1 \in v(C) \Leftrightarrow \widehat{v}(\widehat{C}) = 1$. So $1 \in v(B \wedge C) \Leftrightarrow [1 \in v(B)$ and$1 \in v(C)] \Leftrightarrow [\widehat{v}(\widehat{B}) = 1$ and $\widehat{v}(\widehat{C}) = 1] \Leftrightarrow \widehat{v}(\widehat{B \wedge C}) = 1$. The case for $B \vee C$ is similar.

(3 $\Rightarrow$): Since $NNF(A) \equiv_{FDE} A$ it suffices for us to consider an arbitrary $D \in RPI(NNF(A))$. Then by (2) above we have $\widehat{NNF(A)} \models \widehat{D}$. This shows that $\widehat{D}$ is an implicate of $\widehat{NNF(A)}$. Toward a contradiction, suppose $\widehat{D}$ is not prime. Then there exists a clause C such that $\widehat{NNF(A)} \models C$ and $C \models \widehat{D}$ but $\widehat{D} \not\models C$. But $\widehat{NNF(A)}$ is negation free and thus neither C nor $\widehat{D}$ are the empty clause, nor are they tautologies. Hence there must be a C' such that $NNF(A) \models_{FDE} C'$ where $\widehat{C'} = C$. But then we have $C' \models_{FDE} D$ but $D \not\models_{FDE} C'$. This contradicts the primeness of D. Hence $\widehat{D} \in PI(\widehat{NNF(A)})$ as required.

(3 $\Leftarrow$): Suppose that $D \notin RPI(NNF(A))$. Then either $NNF(A) \not\models_{FDE} D$ or D is not prime. In the former case, $\widehat{NNF(A)} \not\models \widehat{D}$ follows immediately from (2). So suppose D is a relevant implicate of $NNF(A)$ but is not prime. Then there exists a C such that $NNF(A) \models_{FDE} C$ and $C \models_{FDE} D$ but $D \not\models_{FDE} C$. By (3$\Rightarrow$) and (2) above it follows that $\widehat{C}$ is a prime implicate of $\widehat{NNF(A)}$ but $\widehat{D} \not\models \widehat{C}$. Hence $\widehat{D} \notin PI(\widehat{NNF(A)})$ as required.

(4): We note that both *NNF* conversion and the splitting transform are linearly related to the input formula. Hence by (3) above, the claim follows. ⊣

We note that (4) of Proposition 4.7 in fact answers a question raised by J. Marcos in WCP4—the complexity of deciding whether a given clause is a relevant prime implicate of a given formula is no harder than deciding whether a given clause is just a prime implicate of a given formula. Both decision problems are BH_2-complete (see Marquis [24], Proposition 100).[8]

5. Conclusion

In this paper we have seen that reasoning with inconsistent information can be divided into two distinct stages. In the first stage inconsistent information encoded in a full language can be rewritten in such a way as to

[8] This result is not to be confused with the result of Umans [40], Theorem 5, which is the problem of finding the *shortest implicant*. Umans shows that the shortest implicant problem is Σ_2^p-complete.

facilitate the isolation of the inconsistent part of the information. In the second stage various deduction strategies based on either classical or non-classical logics can then be applied to the rewrite. We note that Belnap's strategy of dividing reasoning into a *preprocessing* stage and a *deduction* stage is akin to a recent approach to *knowledge compilation*. The key to preprocessing data is to ensure that any operation on data must be *content preserving*. However, we find Belnap's suggestion of using *conjunctive containment* wanting. In particular, inconsistent information tends to interact badly with disjunctive and redundant information. Although conjunctive containment generally reduces disjunctive consequences, it is however insufficient. Our remedy is to use a relevant notion of prime implicates as the basis to both preserve information and minimise the potentially harmful disjunctive content of inconsistent information.

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