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THE CRITICS OF PARACONSISTENCY AND OF MANY-VALUEDNESS AND THE GEOMETRY OF OPPOSITIONS

Abstract. In 1995 Slater argued both against Priest’s paraconsistent system LP (1979) and against paraconsistency in general, invoking the fundamental opposition relations ruling the classical logical square. Around 2002 Béziau constructed a double defence of paraconsistency (logical and philosophical), relying, in its philosophical part, on Sesmat’s (1951) and Blanche’s (1953) “logical hexagon”, a geometrical, conservative extension of the logical square, and proposing a new (tri-dimensional) “solid of opposition”, meant to shed new light on the point raised by Slater. By using n -opposition theory (NOT) we analyse Beziau’s anti-Slater move and show both its right intuitions and its technical limits. Moreover, we suggest that Slater’s criticism is much akin to a well-known one by Suszko (1975) against the conceivability of many-valued logics. This last criticism has been addressed by Malinowski (1990) and Shramko and Wansing (2005), who developed a family of tenable logical counter-examples to it: trans-Suszkie systems are *radically* many-valued. This family of new logics has some strange logical features, essentially: each system has more than one consequence operator. We show that a new, deeper part of the aforementioned geometry of logical oppositions (NOT), the “logical poly-simplexes of dimension m ”, generates new logical-geometrical structures, essentially many-valued, which could be a very natural (and intuitive) geometrical counterpart to the “strange”, new, non-Suszkie logics of Malinowski, Shramko and Wansing. By a similar move, the geometry of opposition therefore sheds light both on the foundations of paraconsistent logics and on those of many-valued logics.

Keywords: n -opposition theory, logical poly-simplexes, paraconsistent logic, many-valued logic, trans-Suszkie logic, geometry of opposition, Slater, Priest, Béziau, Suszko, Shramko, Wansing.



1. Slater against the idea of paraconsistency (1995)

In a very short paper, Hartley Slater dismisses paraconsistent logics (the formal systems supporting non-trivial inconsistency) as a whole as oxymoronic, i.e. built mistakenly on an untenable basis, a house of cards. This attack is very problematic for the paraconsistentist because the underlying reasoning may seem, at first glance, unobjectionable and the conclusion is worse than harsh.

Slater makes a quick (6 lines!) and abstract argument, implicitly involving the traditional, “square-theoretical” notions of “contradiction” and of “subcontrariety” and showing the “geometrical” impossibility of paraconsistent logics. Then he exhibits a concrete example of this, Priest’s logic LP. He recalls how this paraconsistent (and “dialethic”) logic functions (via its axioms), then he exhibits a major problem with it, which embodies the abstract starting problem. Further, Slater recalls that Priest is aware *in abstracto* of this problem, for he himself has criticised it harshly with the help of Routley in da Costa’s paraconsistent system C_1 and has conceived LP expressly in order to avoid this problem. This means two things, Slater suggests: that LP is a total failure, for it misses its main explicit target—the one already missed unconsciously by da Costa—and that, consequently, *any* paraconsistent logic is destined to fail in the same way, for no axiomatic or semantic self-reform will be able to truly avoid the problem stated abstractly by Slater at the beginning of his reasoning. In order to prove definitely both points by figuring out and refuting Priest’s possible answers to this criticism, Slater makes an excursus over a similar past debate where Copeland criticised “relevant logic”, a famous sub-family of paraconsistent logic. The comparison of the two debates shows the definite failure of the whole paraconsistent project, because Priest manifested, differently from previous relevantists, an *explicit* intention of defending the point that was problematic in both cases (in the first one it was unclear and related to confusions) and because his deepest arguments on ‘truth’ and ‘contradiction’ are untenable (Slater relies on the authority of Tarski, Montague and Goodstein). Slater concludes that truly speaking, that is without fallacious conceptual face-lifts, “there are no paraconsistent logics”.

We will consider two ways of re-stating Slater’s reasoning. One is philosophical, employing a transcendental argument (we mean by “transcendental” a “necessary condition of possibility”): something, a tran-

scendental structure which cannot be demonstrated directly, is demonstrated *indirectly* by showing that anyone trying to think beyond it is destined to contradict himself. Slater implicitly recalls the official definition of paraconsistency, one involving the notion of contradiction: he then recalls that paraconsistency is defined by its partisans as the possibility of having a non-trivial true contradiction (“A” and “not A”, both true). Slater claims that, by virtue of the logical principles and operators embodied by the “logical square”, the transcendental, unchangeable heart of standard logic, paraconsistency as such, i.e. a “true contradiction”, is just impossible. Paraconsistency, truly speaking, in its real axiomatic instances such as C_1 or LP, deals with “subcontrariety”, not with “contradiction” (nor “contrariety”, the third square-theoretical relation). In some sense Slater’s argument can be reformulated geometrically by saying that “we cannot do whatever we want with a square structure” (such as Aristotle-Apuleius’ “logical square”).

A second possible way of restating Slater’s criticism is purely logical. We source it from Paoli (cf. [19]), for it seems both very clear and true to the facts: Paoli discussed it with Slater, and scholars debating this issue seem to find a consensus on this reconstruction. It says that Slater’s reasoning consists of four premisses (explicit or implicit):

- (1) contradictories cannot be true together;
- (2) a sentence and its negation are contradictories;
- (3) if L is a paraconsistent logic, then, in the semantics for L , there are “inconsistent” valuations which assign both A and $\neg A$ a “designated value”, for some formula A ;¹
- (4) if A and B both receive a designated value, under some valuation v , in the semantics for L , then A and B can be true together according to L .

From these follow (in Slater’s argument), with no particular deduction rule other than the simple and usual ones, two consequences:

- (5) in paraconsistent logics, A and $\neg A$ may not be contradictories (from (1), (3), (4));
- (6) thus, paraconsistent “negations” are not negations (from (2), (5)).

¹We will return later to the notion of “designated values” (cf. §6.1).



These two consequences lead (if accepted) to rejecting paraconsistent logics. So, because the deduction is a very simple one, anyone wanting to deny Slater's consequences will have to reject at least one of the four premisses.

Again, Slater's criticism is astonishing: in a few lines it denies (in some sense) the work of quite a lot of bright people (mostly professional mathematicians) over several decades now, published in official scientific journals. The only possible understanding of this could be that these works have (maybe) some value, but surely not the one they are claiming to have! They are not researches in "paraconsistency" (for there is no such thing as "paraconsistent logics" ...).

2. Some answers to Slater's attack

Besides Béziau's two answers, which we are going to discuss to some extent in the next section (cf. §3), there have been at least five direct answers to Slater. We recall them briefly in this section: the first four are of a technical-logical kind, the fifth (by Dutilh Novaes) is a philosophical one.

2.1. Restall's answer (1997)

Greg Restall seems to be the first to have given an answer to Slater (written in 1995, published in 1997). His first strategy consists in distinguishing several kinds of paraconsistency, blaming Slater for having missed such a distinction. Then he will show that Slater is wrong (for different reasons) in 2 ways.

In this respect, he first defines a "paraconsistent logic" as being one which rejects the validity of the *ex falso quodlibet*, "EFQ" (i.e. $A, \neg A \vdash B$), or, equivalently, as a logic which allows inconsistent but not trivial theories; he stresses that paraconsistent logics in general are about logical consequence, not about what is (ontologically, metaphysically) necessary or possible or impossible ... In order to make Slater's reasoning clear (with respect to its major ambiguity), he then separates (i) "regular dialetheism" (the position of Priest), which assumes the existence of true contradictions, from (ii) "light dialetheism", the position that assumes that there are *possibly true* contradictions (a light dialetheist is not committed to believe ontologically in contradictions). These two constitute together a global position which Restall calls "dialetheic para-

consistency”, from which one must separate (iii) “non-dialetheic paraconsistency” (the position of many relevant logicians and of Restall himself), according to which there are no true contradictions at all, but which nevertheless rejects the EFQ (in this tri-partition, Restall admits to ignoring where to put the well-known school of the Brazilian paraconsistentists; he consequently won’t plead their cause with respect to Slater’s attack). This distinction changes a lot, for, Restall claims, Slater’s argument fails then essentially in one case and is circular in the other. (a) First, it has no bite at all over the non-dialetheic position (Restall and relevance logic): the latter rejects, by virtue of its ontological non-commitment with contradiction, Slater’s fourth premiss: the existence of valuations assigning truth to a given contradiction does not imply the *reality* of that contradiction. (b) In the second case, dialetheic paraconsistency, because contradiction is recognised as being possible, the previous strategy (rejection of the fourth premiss) won’t succeed. According to Restall, the best strategy here is to accept Slater’s conclusion (instead of rejecting it), while showing that it is not a deadly position. In the case of light dialetheism, such logicians will have to avoid committing some expressions (like saying that A and $\neg A$ are contradictory—so they will have to reject the second premiss of Slater) otherwise they will collapse into the regular dialetheist position; they will be committed to the existence of true contradictions. In the case of regular dialetheist (i.e. Priest), Restall recognises that Slater’s argument does indeed show that, for them, some other contradictions (desired or not) must be true. For instance, in some sense their negation “ $\neg A$ ” is (also) not the negation of “ A ” (Restall concedes this point to Slater); or also: one can prove that for the regular, dialetheist paraconsistentist their very logic is (also) false. But Restall shows that this need not deter the dialetheic paraconsistentist from their position: Priest’s formalism for LP in 1979 already predicted the possibility of being itself (partly) contradictory (even at the meta-level). Inside LP an atom A and its negation $\neg A$ can be together both a contradiction (against Slater) and not a contradiction (in agreement with Slater). The only thing Priest wants (and needs) to reject is deriving strong (trivialising) contradictions like “ $0 = 1$ ”. And Slater’s argument cannot force Priest to derive that.

Summing up, Slater is wrong with respect to non-dialetheic paraconsistency (it is immune to his attack) but is right with respect to regular dialetheism, but the latter is prepared to endorse the “bad” consequences Slater brandishes.



2.2. Priest's own answer (1999)

As we saw and will see again, several answers to Slater consisted in separating their own approach to paraconsistency from Priest's one and in claiming that Slater's attack did not concern (did not hit successfully) those other approaches. Graham Priest's position has been defended *per se* by Restall. As we saw, the latter said that the adequate answer for a holder of the strong dialetheist position consists in assuming Slater's reasoning just as it is: regular (i.e. strong) dialetheism is able to endorse the self-contradictions Slater brandishes in order to frighten it (cf. *supra*). The essence of this is that Priest himself has answered Slater by refusing the first premiss of his reasoning and therefore Slater's argument, with respect to Priest, must be seen as question-begging.

2.3. Brown's answer (1999)

Another answer to Slater came from the Canadian school of paraconsistency. Bryson Brown deliberately does not face the main point in Slater's argument. Instead, he claims that Slater's own argument (partly) misses its target, as there is an important class of paraconsistent logics, the preservationist logics (of Jennings and Schotch), which are not submitted to Slater's objection (this argument is similar to Restall's first one). And there are some dialethic logics which can be reinterpreted as preservationist (non-dialethic) logics. Therefore, one can separate the discussion of paraconsistent logic from the discussion of the tenability of dialetheism. And the existence and the interest of systems of paraconsistent logic does not depend on a defense of dialetheism (we will not detail Brown's argument).

2.4. Paoli's answer (2003)

Francesco Paoli offered another interesting technical answer. First of all, he proposed the convincing reconstruction of Slater's argument we rely on (cf. *supra*). Starting from there, he proposes to examine the ways of rejecting the third of Slater's premisses (whereas Priest rejected the first and Restall the fourth), committing this way himself with the already mentioned usual strategy consisting in exhibiting, in front of Slater, his/her own independence with respect to Priest's dialetheist approach to paraconsistency. So, Paoli proposes a tripartition of paraconsistent

logics into (1) “dialethic” (as in Priest), (2) “non-dialethic” (as in Restall) and (3) “proof-theoretic paraconsistency” (Paoli’s own approach). Paoli shows that Slater’s attack is, in some sense, just an instance of a more general position of hostility towards “deviant logics”, namely the position of Quine (in *Philosophy of Logic*, 1970, ch. 6). So he decides to face Quine before facing (as a particular case) Slater. With this respect, Paoli builds a “criterium of genuine rivalry between logics”: CGR. First, he distinguishes between two aspects of the meaning of a logical constant in a given logic: its operational meaning and its global meaning. Then he can exhibit the CGR: there can be genuine rivalry between two logics (despite Quine) whenever each constant in the first has the same operational meaning as its counterpart in the second, although differences in global meaning arise in at least one case. Relying on this CGR, he can finally show that there exists at least one paraconsistent logic which, according to CGR, genuinely rivals classical logic: that is “subexponential linear logic without additive constants”. So, this paraconsistent logic, which has rejected Slater’s third premiss, is immune to Slater’s (and Quine’s) criticism and is a genuine alternative rival to classical logic.

2.5. Dutilh Novaes’ answer (2007)

In contrast to the four preceding answers (which were logical), Catarina Dutilh Novaes faces Slater’s attack to paraconsistency from a more philosophical point of view. She draws attention to the fact that paraconsistent logicians (and philosophers) tend to admit, without questioning it, that the major task (or problem) of constructing and/or thinking of a paraconsistent system concerns its “negation operator”. She claims that, however, a closer examination of the fundamentals here shows that negation is not the real problem of paraconsistency. There is in fact a rather systematic confusion of the concepts of negation and contradiction, whereas the two ought to be kept distinguished. Therefore, she proposes to assume that the real challenge for paraconsistent thinking is that of elaborating not a new concept of negation (this is relatively easy and has already been done in several acceptable ways), but that of elaborating and clarifying a new concept of contradiction. And this problem seems much harder. But this is the real price to be paid, the condition for facing with confidence Slater’s rough attack on the very possibility of true paraconsistency.

We saw both a philosophical comment and some logical answers to Slater. But, among the logical answers, and with respect to the axiomatic reconstruction of Slater's argument we relied on, we saw no answer representing the Brazilian school of paraconsistency. This has been done by Jean-Yves Béziau.

3. Béziau's two answers to Slater (2006, 2003)

Béziau wrote two papers against Slater around 2002. These two form a unique argument, articulated in two parts, one more logical, the other more philosophical. First (2006), he analyses in a logically technical and precise way, using his, Alves' and da Costa's "bivaluation theory", what is going on with Slater, but also with da Costa and Priest: by using very precise and powerful logical definitions of "contradiction" and formal translation-rules between logical systems, and by a powerful general logical theorem over paraconsistent systems in general, he shows how the problem must be restated against Slater and his confusions. Once he has given such a general precise analysis, he tackles the problem anew (2003), this time from a more philosophical and linguistical point of view, the problem being to defend and to explain the interest of subcontrariety: by summoning a "geometry of the logical oppositions" and by making some claims about the relations between the opposition-forming operators and the kinds of logical negations of contemporary logic, among which lies the paraconsistent one. Béziau's "geometrical answer" (the one developed in his second paper) may seem to some logically strange—the arguments there are of a new kind—but it has the merit of going right to the heart of Slater's criticism, the notion of subcontrariety. Moreover, as we will see in the next section, he has partly involuntarily opened a new fundamental branch of pure logic, one which seems to show that logic is in fact an autonomous (so to say Bourbakian) new family of abstract structures, parallel to topology, algebra, etc (and not just a branch of algebra). And this, as we will see, has great relevance with respect to the "Slater debate".

3.1. Béziau's first answer, logical (published later in 2006)

Béziau's first answer complies in advance with Dutilh Novaes' posterior remark: it makes the effort of giving a new, more refined definition of

“contradiction”. He starts by showing that Slater, in his paper, used bad (i.e. incomplete, truncated) definitions. For instance, Slater defines contradiction as “the impossibility of being true together”, whereas the true, traditional Aristotelian definition is “the impossibility of being true together *and the impossibility of being false together*”. The main thesis of Béziau’s paper is that if one uses good definitions instead—which implies one co-define contrariety and subcontrariety to stress their mathematical symmetry—then Slater’s claims are either false or, at best, tautological. But preliminarily, by means of considerations over the translatability conditions between logical systems, Béziau recalls that paraconsistent logic is not “just switching names” (as Slater rudely states his charge against paraconsistency). It is instead the emergence of an entirely new phenomenon, historically comparable to the emergence of non-Euclidean geometry, where the mathematical meaning of “straight line” changed drastically: the meaning of several classical fundamental notions of logic do change indeed. Once this provable and proven essential epistemological point is recalled and restated, and after having recalled the classical opposition theory (the logical square) and its explicit and complete definitions of contradiction, contrariety and subcontrariety, Béziau starts analysing the opposition definitions inside the framework of da Costa’s system C_1 . He shows that its “negation operator” is a subcontrary-forming operator (as Priest and Slater reproachfully claimed) but *relative to the semantics (given by “bivaluation theory”) of this system*: in this respect, Slater is totally mistaken, for he totally missed this technical essential point. Then Béziau analyses the opposition relations inside Priest’s system LP and shows that Slater’s claim thereupon (“Priest’s paraconsistency deals with subcontrariety”) is partly false and partly true. It is generally false because LP’s negation operator, as C_1 ’s negation operator, is relative to that logic (LP) and cannot be translated simply so into classical logic as Slater implicitly and very mistakenly does: so LP’s negation operator is not the *classical* subcontrariety-forming operator (as Slater claims), it is a LP subcontrariety-forming operator.

Nevertheless, Slater is right when he points out the presence of an *illicit* “trick” in Priest’s logic: an ambiguity, due to a play with the “designated values” (cf. §6.1), in the use of the concept of truth, for by Priest in some sense truth is “1” and in some sense truth is “1 or $\frac{1}{2}$ ”, and this cannot be. This means that either truth is “1 or $\frac{1}{2}$ ”—as per LP’s set of designated values—in which case the system LP is paraconsistent

but its negation is (only) a subcontrary-forming operator (Slater is right) *from the point of view of LP* (Slater is nevertheless partially wrong); or truth is only “1”—as per LP’s explicit definition of its truth predicate—in which case the negation operator of LP is a contradiction-forming operator (as Priest claims) *from the point of view of LP*, but then LP is not paraconsistent (so, Slater is, in some sense—that he only confusedly perceived—right).

Having examined with precise definitions the opposition relations inside (1) classical logic, (2) da Costa’s system C_1 and (3) Priest’s system LP, Béziau can now express what he thinks to be the real question at stake: that is, knowing whether a paraconsistent negation *in general* can be a contradiction-forming operator *from the point of view of its own semantics*. The previous analysis has already shown that neither da Costa’s C_1 nor Priest’s LP can. But Béziau shows more generally, by a powerful theorem, that it is not possible for a paraconsistent negation operator in general to be a contradiction-forming operator (as Slater asks provokingly) *from the point of view of its own semantics*.²

As a first corollary, only classical negation is a contradiction-forming operator, hence the tautological, uninteresting value of Slater’s thesis according to Béziau. But then, if we additionally followed Slater’s drastic anti-paraconsistent philosophical criteria (i.e. equating negation to contradiction), we should say—which is commonsensically absurd—that not even the intuitionist negation—a fully recognised one!—is a “negation”, because intuitionist negation is, from the point of view of opposition theory, a *contrariety*-forming operator.

²Remark—this point will turn out to be important later—that Béziau reaches his theorem (i) by using, against Priest, Suszko’s remarks on many-valued logics (his ideas on the binarity of the designated-undesignated subsets of the set of all truth-values of a given system), (ii) by using a very general definition of “logic”, one grounded on Béziau’s notion of “universal logic” and (iii) by using a general Béziau-Dacostian definition of contradiction, i.e. one in terms of “bivaluation theory”: he recognises that, following Malinowski’s anti-Suszko strategy (i.e. q -logics, cf. §6.2 *infra*) we could escape Suszko’s restriction and thus Béziau’s theorem would be less general, so it could be seemingly possible to look for paraconsistent systems with a contradiction-forming negation. This would be another anti-Slater strategy, one accepting Slater’s second premiss. But he leaves aside this case as being very remote and very non-standard: Béziau maintains, in his anti-Slater strategy, the refusal of the second premiss. Which means that, in some sense, Béziau recognises Slater’s idea that “paraconsistent” negations can be only subcontrariety-forming operators.

As a second corollary—this concerns the “Brazilian-Australian cold war”—the negation operator of Priest’s LP system has no superiority over that of da Costa’s C_1 system.

So Slater is, *stricto sensu*, wrong, but a bit right nevertheless(!), for there is indeed a strong link between paraconsistency and subcontrariety. So, by refusing Slater’s second premiss, the one which equates negation and contradiction, the remaining problem for Béziau is philosophical: taking subcontrariety, and hence paraconsistency, seriously in general.

3.2. Béziau’s second answer, philosophico-geometrical (2003)

Béziau’s second strategy, deepening his rejection of Slater’s second premiss *via* the geometry implicit in opposition theory, will internalise linguistic and philosophic deep arguments over the naturalness (against Slater) of paraconsistency, taken (in partial agreement with Slater) as a subcontrariety-forming operator.

Firstly, coming back explicitly to the geometry of opposition theory, Béziau draws the attention over a famous linguistic problem, the non-lexicalisation of the “O” position of the Aristotelian-Apuleian traditional AEIO square, that is: this place, and this one only, has no name—whereas, for instance, “A” is “all” or “necessary”, “I” is “some” or “possible”, etc. This phenomenon, relative to linguistics, psychology and even law (and popularised in 1989 by Larry Horn), turns out to be related to an unresolved antinomy between the square of opposition and a triangular model for contrariety (a problem already present in Aristotle, cf. [10], pp. 15–21). Now, this confusion, which gave rise, by reaction, to the philosophical and logical ideas of Vasil’ev, one of the forerunners of paraconsistent and of many-valued logics (cf. [14]), is dissipated by the adoption of a geometrical, conservative extension of the logical square, that is Sesmat’s (1951) and Blanché’s (1953) “logical hexagon” ([25] and [5], cf. Fig. 1): for this hexagon integrates the two conflicting models, the logical hexagon contains both the square and the triangle of contrariety.

Secondly, Béziau remarks that, modally speaking, the non-lexicalised “O” position can be read (among others in S5) as a paraconsistent negation (for the modality “ $\neg\Box$ ”, taken as a negation “ \sim ”, has all the properties such a paraconsistent negation “ \sim ” must have (cf. [2]). Now, one could be tempted to take the fact that the paraconsistent position in the logical square is not lexicalised as an argument against paraconsis-

tency. Béziau takes it the other way round. In fact, the logical square and paraconsistency can offer each other mutual help (against Slater): (1) the paraconsistent negation seems more natural in the context of the square, for there is a *natural place* for paraconsistency, the “O” corner; (2) the square seems more natural if we observe that the mysterious “O” corner, viewed as a problematic empty place, is in fact the paraconsistent negation (structuralism—a position endorsed by Blanché, [6], and therefore seemingly by Béziau—commands us to respect and fulfil the “empty places” of any regular structure: natural linguistic evolution is largely contingent, rational structuralism is formally constrained and clarifying). Béziau argues then that the best synthesis of all this consists in choosing to rely on the logical hexagon, in contrast to Slater (who ignores it): it is a better mathematical solution for it, has more symmetry axes than the square, it conserves all the good points of the square (mainly the full expression of all the four Aristotelian relations) and additionally it integrates the triangular model of contrariety, present in many natural languages and in many conceptual systems. But Béziau goes further. In fact, in discussing modal logic geometrically—by invoking and comparing logical squares and hexagons—he discovers, again, by an implicitly structuralist move, i.e. by taking into account the *null* modalities “ α ” and “ $\neg\alpha$ ”, two further logical hexagons (i.e. two new decorations of the hexagonal structure), of which he shows that one expresses paracomplete (i.e. intuitionist) negation, by a (blue) contrariety segment $\alpha \text{---} \Box\neg\alpha$, whereas the other expresses paraconsistent negation, by a (green) subcontrariety segment $\alpha \text{---} \neg\Box\alpha$ (cf. Fig. 1).

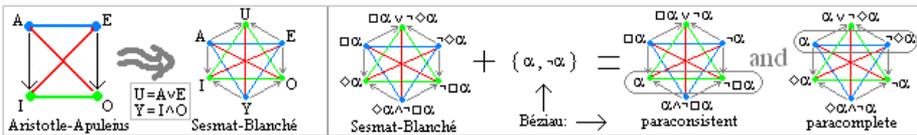


Figure 1. The square, Sesmat-Blanché’s hexagon and Béziau’s two new hexagons

The next step consists, invoking logical duality and mathematical symmetry (structuralism again, this time Bourbakian), in arguing that one must take subcontrariety as an *opposition* relation in every respect, against Aristotle’s solution, followed by Slater, which considered “contradiction” and “contrariety” but not “subcontrariety” (Aristotle had no name for this relation). This last solution, i.e. neglecting subcontrari-

ety, is “asymmetrical”: that is, mathematically bizarre. On this basis, Béziau then summons against Slater a strong analogy between the three kinds of opposition in Aristotle’s so reconstructed theory (contradiction, contrariety, subcontrariety) and the three kinds of negation theorised by Miró Quesada (classical, paracomplete, paraconsistent). But for these reasons, stressing—against Slater—the importance of subcontrariety inside a triple of globally “symmetric” oppositions, which is analogous to a triple of negations, Béziau does not want to admit subalternation (i.e. implication) as being a fourth opposition relation, claiming that this would be intuitively absurd (cf. Fig. 2).

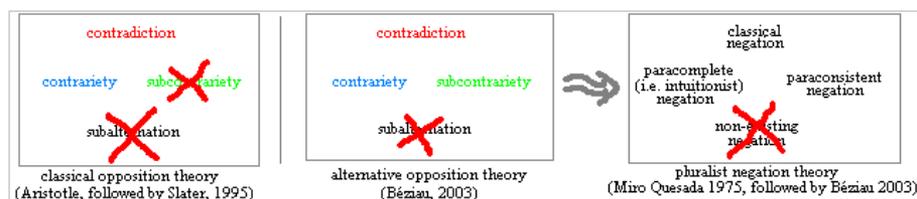


Figure 2. Béziau’s modification (against Slater) of Aristotle’s opposition theory

Therefore, instead of logical hexagons (which would include the 6 arrows constituting the perimeter of each of them), he speaks of “logical stars” (or “Blanché’s stars”) taking thus only 3 of the 4 possible Aristotelian relations (only the three conventional colors: red, blue and green). Then, because these 3 logical stars have pairwise some vertices in common, he has the idea of looking for a whole structure, which seemingly cannot be 2-dimensional, containing them all. As there are 12 different vertices (6 among the 18 are repeated), Béziau says that the corresponding 3D solid must be a “stellar dodecahedron”, also known as “Escher’s solid”, a solid obtained by constructing a pentagonal pyramid or spike over each of the 12 pentagonal faces of a dodecahedron (cf. Fig. 3).

(Remark that Béziau never drew such a solid.) With this geometrical move, supposed to be analogous—*mutatis mutandis*—to Aristotle’s and Apuleius’ one, philosophically speaking, Béziau thinks he has found a “transcendental structure” alternative both to the logical square (of Aristotle and Slater) and to its conservative extension, the logical hexagon (of Sesmat and Blanché). And therefore he thinks to have opened philosophically, seated on a robust theory of paraconsistency (in terms of bi-

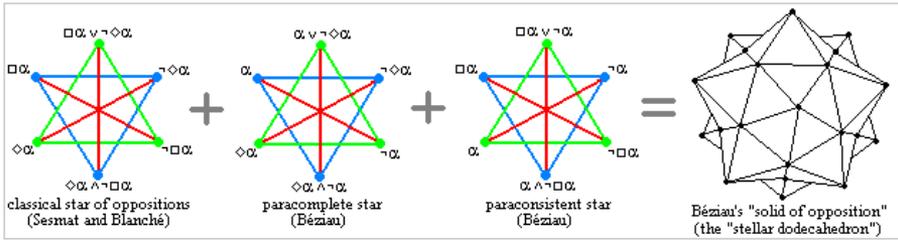


Figure 3. Beziau’s “solid of oppositions” made of three “logical stars”

valuation theory), the possibility of escaping Slater’s transcendental way of eliminating the thinkability of paraconsistency (the argument over the irrelevance of subcontrariety).

These new elements seem to confirm that Slater’s reflections (i.e. his transcendental anti-paraconsistent argument relying on the logical square and presupposing that subcontrariety—the true face of paraconsistent “negation”—is not an opposition and therefore not a negation) are a limited (and badly conceived) case in the whole geometry of opposition. So Slater’s argument is defeated, according to Béziau, both logically and philosophically.

4. The science of the geometry of logical oppositions: “*n*-opposition theory” (N.O.T.) (2004)

As it happens, Béziau’s intuition turned out to be more true than he had imagined, for far from having just superseded the logical square, it opened to a new branch of mathematics and of logic.

4.1. Correcting Béziau

First of all, the solid of opposition is not the one Béziau had imagined: the three hexagons known by him, plus a fourth one, compose an elegant logical cuboctahedron (cf. [13]). But the real closure of the hexagons (in fact the “geometrical closure” of S5) contains two more hexagons, discovered meanwhile by Hans Smessaert ([28]), and turns out to be Régis Pellissier’s logical tetraicosahedron (cf. [20] and Fig. 4).

Another point where Béziau had to be corrected is that the geometrical expression of 4-opposition (“quadritomy” as he called it in [4]) is

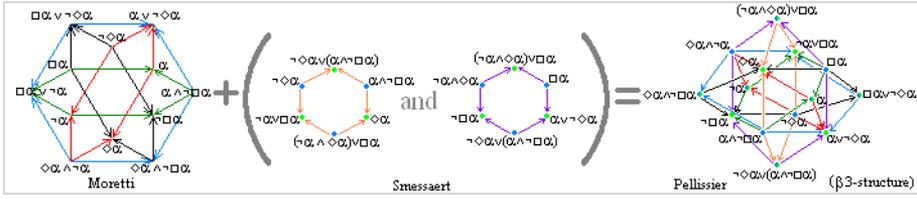


Figure 4. The real solid of opposition is the “logical tetraicosahedron”

not made of two squares but of two tetrahedra (cf. [13]). This “logical bi-tetrahedron” of 4-opposition gives, once the subalternation arrows are duly added, a “logical cube” (cf. Fig. 5).

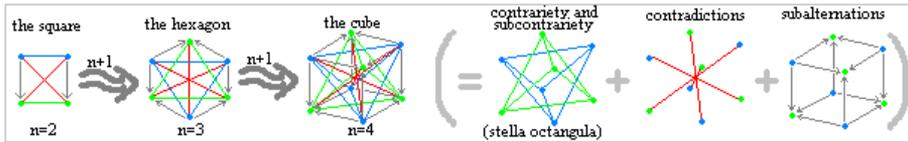


Figure 5. The geometrical expression of 4-opposition is the “logical cube”

4.2. αn -, βn - and γ -structures and $(\gamma \rightarrow \beta n \rightarrow \alpha k)$ -translation

Now, this behaviour of putting square, hexagon and cube in a row is stable: it is based on the expression of n -contrariety by means of geometrical (blue) “simplexes” of dimension m (with $m = n - 1$). Adding to this a “central symmetry” for expressing contradiction (red diagonals), one gets, point by point, (green) simplexes expressing subcontrariety. If we add the subalternation arrows (neglected by Béziau)—whose construction rule is very simple: each blue point implies all the green ones except its contradictory one—this gives, as a final result (in fact an algorithm), all the possible n -oppositions, that is the series of the “logical bi-simplexes of dimension m ” (or series of the “ αn -structures”, cf. [13] and Fig. 6).

This algorithm is geometrically very powerful: each term of the series is a natural generalisation of the previous one, and in each all the essential strong properties of (Aristotle’s) opposition theory (“oppositional closure”, “negational closure” and “Aristotelian closure”, cf. [17]), are conserved and expressed.

Now, as the logical hexagon “gathers” three logical squares and as the tetraicosahedron gathers a cube, six hexagons and 18 squares, it has been

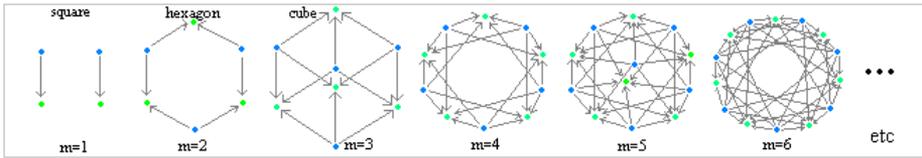


Figure 6. Square, hexagon, cube, ... the “logical bi-simplexes of dim. m ”

proven that there is a series of logical gatherings (or “ βn -structures”), of which the logical hexagon and the logical tetraicosahedron are instances for $n = 2$ and $n = 3$ respectively. These “gatherings” are the *complete* geometrical counterpart of any modal system (but also of non-modal conceptual systems) provided this respects a very common property (cf. [20], [15]).

A third very important family of opposition structures is that of the so-called “modal graphs” (a notion theorised by NOT but grounded on ideas of A. Prior). By contrast with the two previous families (the α - and the β -structures), which can be ordered each in a series and whose geometrical forms are totally constrained (finitely fractal and increasingly symmetric), the modal graphs (or γ -structures) cannot be put in a linear series, for they can take almost any form, provided they respect central symmetry of the contradictories. And they have no colours: it is not at their level that the opposition kinds (blue contrariety, green subcontrariety, etc.) must be studied (this is a common, harmful mistake which NOT helps avoid). Modal graphs encode the “identity” of each modal system (in other words, each system of modal logic has one and only one modal graph, finite or not): effectively, the modal graph expresses the fundamental (network) properties of the “basic modalities” constituting them (cf. [13], [20])

The NOT establishes, *via* a general $(\gamma \rightarrow \beta n \rightarrow \alpha k)$ -translation rule (for $2 \leq k \leq n + 1$), that each *finite* modal graph can be translated (in principle) into one and only one logical βn -structure (or gathering), and hence *all* its inner αn -structures can be obtained. Notice that this implies that 2 different modal graphs, different in shape but equivalent in “oppositional complexity”, may happen to map to the same β -structure: in this case they are “the same” from the point of view of a new kind of logical class of equivalence.

For the general meaning of NOT, in [17] seven main points are given. For a direct, important result of NOT for paraconsistency and intuitionism, cf. [21].³

4.3. The specific meaning of NOT for the Slater debate

As we saw, NOT originated very recently inside the “Slater debate” because of Béziau’s singular (geometrical-philosophical) reaction to it (this reaction constituting, in Paoli’s terms, the refusal of Slater’s second premiss, cf. §3). Relatively to that, NOT shows only that Béziau’s second answer to Slater is partly right, partly wrong. Partly right, because he perceived the importance of geometry for studying the fundamental oppositional laws of logic (whereas Slater simply relied on the implicitly alleged transcendental uniqueness of the logical square). Partly wrong, because he did not seize the real shapes (going into infinite) of the new theory (and fell instead into some kind of misleading rivalry to Aristotle-Slater and Sesmat-Blanché in the quest for *the* transcendental geometric structure of logic). To which extent can we then come back *newly* to the question of Slater’s attack? The first point to be judged is seemingly Béziau’s claim that subcontrariety is an opposition, and that subalternation is not one. Relatively to that, again, NOT seems to show that Béziau was partly right and partly wrong. Partly right because it seems true, in the light of the many discoveries and structures of NOT, that *subcontrariety is indeed a kind of opposition* (dual of contrariety, Pellissier’s result seems to confirm this topologically). Partly wrong, because the fact that, precisely because he voluntarily neglected subalternation (he used stars instead of hexagons, cf. *supra*) Béziau missed the NOT (beginning by missing the logical cuboctahedron), seems a symptom that something conceptually important is going on between implication (i.e.

³In [21] Pellissier, working on Heyting and co-Heyting algebras in order to better understand inside NOT a particular property of the logical square (which is, in some sense, a degenerated case of logical bi-simplex) from the point of view of category theory, showed that paraconsistency (as well as its dual, paracompleteness—i.e. intuitionism) can be found deeply inscribed in a topological fundamental instance of the logical hexagon. This shows in NOT style a *syntactic* (topological) existence of paraconsistency inside the geometry of S5, whereas Béziau’s second argument concerned a *semantic* (modal logical) presence of paraconsistency, as identified to $\neg\Box\alpha$, in S5. So, Béziau’s pioneering intuition seems to be good (paraconsistency must be looked for geometrically), but not exactly as simple as he thought.



subalternation) and opposition. Something still unclear. In other words, from the point of view of NOT, *subalternation is undoubtedly one of the four main ingredients of opposition: it is, in fact, an opposition kind* (cf. Fig. 7).

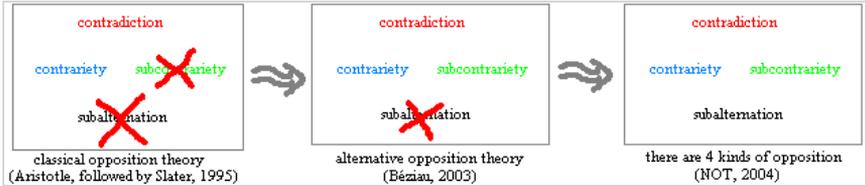


Figure 7. Three models of opposition: 2, 3 or 4 opposition relations

Remark that, philosophically, this is not so shocking or strange: subalternation (an order relation) *opposes* a “first” to a “second” (“first me, then you”).

As for the transcendental, if one looks for one such thing, it is (at least so far) the whole notion of logical bi-simplex, and each logical bi-simplex, no matter how big (i.e. no matter how m -dimensional), is made of four and only four elements (four “colours”). But this, the fact of having this strong quaternary invariant, despite the formal unexpected (infinite) richness of the geometry of opposition, suggests that there still could be more to be discovered. And this conjecture (or suspicion) of ours is important for the “Slater debate”. In fact, before starting discussing logical invariants as do Slater (by opposition to the “mad idea” of paraconsistency, he who claims that negation is subjected to one such invariant) and Béziau (who exhibits a more liberal one), we must be sure we really have one in our hands. So, in order to deepen Béziau’s fertile geometric starting intuition, and according to his Universal Logic research line (“try to break any logical invariant you can find, so to find more fundamental ones”), we must try to break the invariants of NOT. At that price only we will be allowed to bring some new light over this debate at the heart of paraconsistency.

5. Radicalising NOT: the logical p -simplexes

One could say that, by the logical bi-simplexes, the whole NOT remains “Slaterian”. As a matter of fact, NOT so far, despite the appearances,

remains very classical in two points, bivaluedness and oppositional quaternarity (i.e. Aristotelian closure). The answer, in [15], to this challenge has produced an unexpected part of NOT, a radicalisation of the general theory of geometric opposition.

5.1. The Aristotelian 2^2 -semantics and its 2^2 -lattice

It seems clear that this quaternarity comes from the “bi-” element in the concept of “logical bi-simplex”. With this respect, could we get, somehow, “logical tri-simplexes” (and so on) instead? In order to think this question, we need to make a step back. A useful trick, in order to understand where we are, is to reconstruct Aristotle’s opposition theory as being ruled by an implicit “ask-answer” semantic game. Aristotle’s 4 opposition relations are generated combinatorially simply by asking 2 questions, each admitting one of two possible answers (cf. Fig. 8).

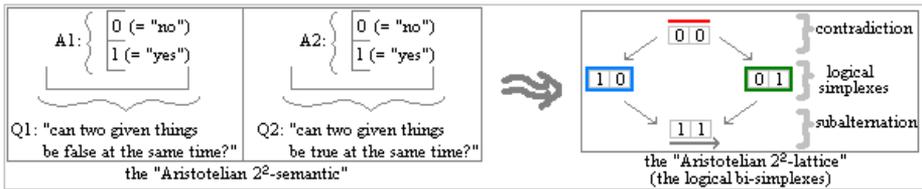


Figure 8. The Aristotelian 2^2 -semantics and its 2^2 -lattice give bi-simplexes

This game generalises the *traditional* definition of the three opposition relations, that is “contradiction” (for two things it is “the impossibility for them to be both false and the impossibility for them to be both true”), “contrariety” (for two things: “the possibility of being both false and the impossibility of being both true”) and “subcontrariety” (for two things: the impossibility of being both false and the possibility of being both true”), for it defines the same way also the fourth term, “subalternation” (for two things: “the possibility of being both false and the possibility of being both true”), provided a small *ad hoc* axiom for distinguishing it from logical equivalence and for endorsing its asymmetry. This is a considerable gain, it gives a unified semantic treatment of opposition! Let us call it “Aristotelian 2^2 -semantics” and let us call the lattice ordering its possible $[x|y]$ outcomes the “Aristotelian 2^2 -lattice”. Now, here is the second gain, this game can itself be generalised, by changing

can see that the first two tri-simplexes are the logical tri-segment (the extension of the square) and the logical tri-triangle (the extension of the hexagon, cf. Fig. 10).

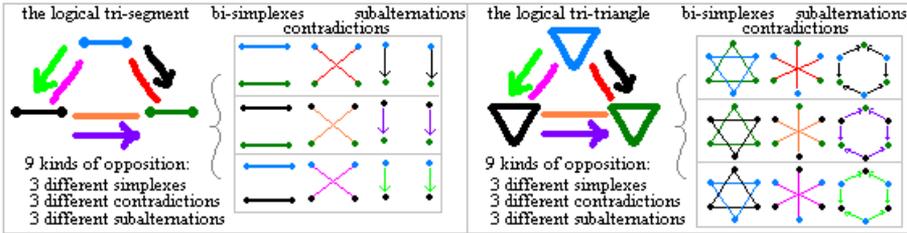


Figure 10. The logical tri-simplexes of dim 1 and 2: tri-segment and tri-triangle

The tri-segment and tri-triangle can be represented as a whole (cf. Fig. 11).

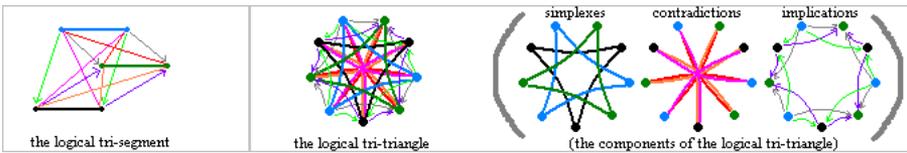


Figure 11. The whole representation of the logical tri-segment and tri-triangle

In the same way, augmenting the parameter n (the number of opposed terms) we have the logical tri-tetrahedron (i.e. the logical tri-simplex of dimension 3), the logical tri-simplex of dimension 4, etc. The series of the logical tri-simplexes (of dimension m , with $m = n - 1$) is an infinite one. We will not enter here into the technical details of the tri-simplicial semantics, which give the functioning rules of each of the 9 kinds of opposition: like the $[\frac{1}{2}|0]$, $[1|\frac{1}{2}]$, etc. The important idea to be grasped here is that the 5 new opposition relations (of the 9) are: a black new logical simplex $[\frac{1}{2}|\frac{1}{2}]$, two new contradictory negations (a rose $[\frac{1}{2}|0]$ and an orange $[0|\frac{1}{2}]$ diagonal, “sub-contradictions”) and two subalternations (a light green $[1|\frac{1}{2}]$ and a violet $[\frac{1}{2}|1]$ arrow, “sub-implications”). And that, because of the very semantic ask-answer game at their origin ($p = 3$), the logical tri-simplexes of dimension m are 3-valued logical structures. Remark (with respect to Slater and to Dutilh Novaes) that we have here not one, but 3 kinds of “contradiction” (3 different symmetry centres): we are building a new model of contradiction.



5.3. The logical poly-simplexes (of dimension m)

Now, it turns out that the Aristotelian p^2 -semantics (with their respective lattices) generate a whole series of “logical p -simplexes (or logical poly-simplexes) of dimension m ”, of which the bi-simplexes and the tri-simplexes are only the terms for $p = 2$ and $p = 3$ (cf. Fig. 12, where we fixed arbitrarily $m = 3$).⁴

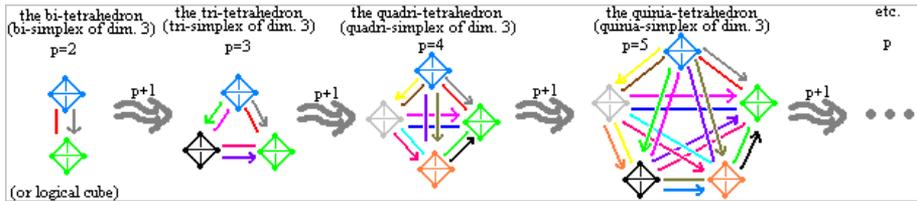


Figure 12. The series of the logical p -simplexes (here: of dimension 3)

Again, a proper treatment of these structures requires the establishment of many detailed and effective semantic specifications (we do it partially in [15]). Here we limit ourselves to remark that each logical p -simplex, whatever its dimension m , is a p -valued logical structure and generates a conceptual universe where p^2 kinds of oppositions (i.e. p^2 colours) are available, instead of the “transcendental” 4 of Aristotle, Sesmat-Blanché, Béziau and of the logical bi-simplexes in general). Remark also that any logical p -simplex has not just one kind of contradiction and one kind of subalternation relation, but C_p^2 different kinds of contradictions and C_p^2 different kinds of subalternations.

So, we succeeded in breaking the geometrical-logical invariant! The transcendental limit of opposition is made more distant. And with respect to Slater, because of the big contextual shift, the concept of sub-contrariety (to which paraconsistency in some sense is tied, from the point of view of Béziau) is both clarified (it is a “logical simplex”) and complexified (it admits interpolation).

⁴To these structures I proposed in [15], Pellissier has given a full-fledged “decoration technique” (draft unpublished), analogous to the one he gave to my logical bi-simplexes (cf. [20]), based this time not on sets but on a new kind of “sheaves”.

5.4. Towards the Aristotelian p^q -semantics and p^q -lattice

In what preceded we only changed the p parameter (the number of eligible answers) of the general Aristotelian p^q -semantics. What happens if we change q (the number of questions to be answered)? The result of the simplest change of q gives the following Aristotelian 2^3 -semantics, with its 2^3 -lattice, which orders all the eight possible [A1|A2|A3] answers, and eight kinds of opposition (cf. Fig. 13).

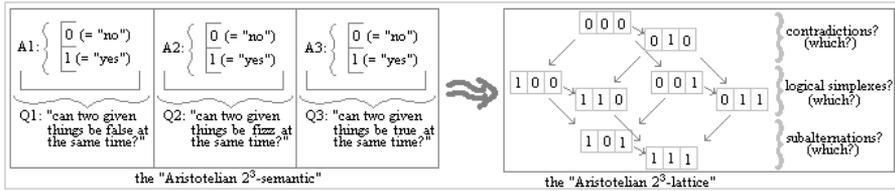


Figure 13. The Aristotelian 2^3 -semantics and its (cubic) 2^3 -lattice

A further increase of q (i.e. a further meta-theoretical semantic question) gives the Aristotelian 2^4 -semantics with its Aristotelian 2^4 -lattice, which turns out to be a hyper-cube. In this case, the number of possible oppositions is $2^4=16$. What kind of oppositional structures are required, is still unclear at the present day. As a sign of this, we still are unable to put “opposition colors” to the outcomes of the oppositional combinations. Formally speaking, the Aristotelian 2^2 -lattice is isomorphic to a square, the Aristotelian 2^3 -lattice is isomorphic to a cube, the Aristotelian 2^4 -lattice is isomorphic to a hyper-cube and it can be easily shown that this gives rise to an infinite series of Aristotelian 2^q -lattices shaped like the series of the m -dimensional hyper-cubes. These series and that of the p^2 -lattices of the logical poly-simplexes can and must be combined: this gives the general theory of the Aristotelian p^q -semantics and p^q -lattices. We know in advance that the shape of the corresponding hyper-cubic lattice will be determined by the q parameter, whereas the p parameter will determine the length of each of the squares composing the hyper-cubic simplex. But actually we do not know much more (these issues are still being investigated). So, opposition as a whole (which is interesting for Béziau against Slater) *must* be understood in terms of general Aristotelian p^q -semantics and p^q -lattices. And we are still far from a serious understanding of this new field, except for knowing that it seems to be the key to the concept of opposition (and hence of contradiction).



5.5. The Aristotelian p^q -semantics and Béziau’s strategy

Now we can come back to our starting question and try to see if we can understand better the “geometrical way” developed by Béziau against Slater’s attack to paraconsistency (the refusal of Slater’s second premiss).

A new, relevant point which has emerged from the theory of the logical poly-simplexes of dimension m is that, by means of its Aristotelian p^2 -semantics, each logical p -simplex (whatever its dim. m) generates not 4 but p^2 kinds of opposition, each time distributed in a p^2 -lattice. Remark that all p^2 -lattices are square (i.e. lozenge) shaped. This distribution suggests that there are 3 invariant “meta-kinds” of oppositions: logical contradictions (the $[x|y]$ terms in the upper half of the lattice, with $x + y < 1$), logical simplexes (the $[x|y]$ terms in the diagonal of the lattice, with $x + y = 1$) and logical implications (the $[x|y]$ terms in the lower half of the lattice, with $x + y > 1$). This knowledge relates to Béziau’s anti-Slater strategy for thinking paraconsistent negation (cf. §3.2), by suggesting a deep link between opposition kinds and negation kinds (cf. Fig. 14).

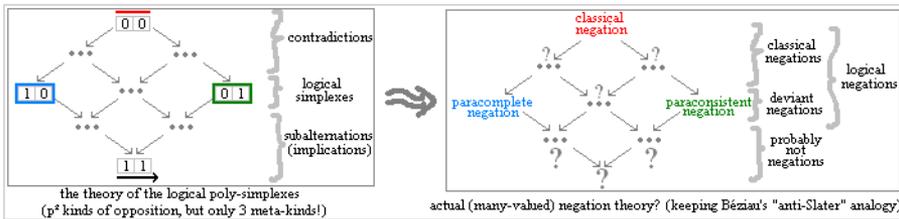


Figure 14. The oppositions according to the p -simplexes, and the negations

In some sense it seems to confirm Béziau’s intuition (cf. §3.2), while making it more complex. Moreover, we saw that within the logical p -simplexes (for $p > 2$), the notion of “contradiction” (like those of “contrariety”, “subcontrariety” and “subalternation”) gets *diffracted*. But despite these nice advances, we must be aware that this knowledge will be probably changed (i.e. specified) in the future, when we will understand the opposition systems generated by changing q in the Aristotelian p^q -semantics. There could be further changes in the very notion of Aristotelian p^q -semantics (we could become aware of new parameters, beside the n , p and q). We must stop here our “big dive”, and be contented with the present, urgent task of ruminating on all these results of NOT.

In fact, first of all our possible problems, the logical poly-simplexes, of which the logical bi-simplexes are just a particular case, are p -valued logics (i.e. many-valued). Béziau's discussion of Slater's argument, pushing us into the geometry of oppositions, therefore takes us at least potentially to many-valued logics. This was partly unexpected. But are such logics themselves OK? What are we thus committed to? In fact, as we are going to see, here as well there has been a critical earthquake, akin to the Slaterian one.

6. The poly-simplexes and Malinowski, Shramko and Wansing's "trans-Suszkian" strategies

There are important links between paraconsistency and many-valuedness, because of the "architectural" parallelism of the logically independent PNC (principle of non-contradiction) and PEM (principle of excluded middle). A sign of this is the fact that Priest's paraconsistent system LP (the one debated by Slater) is 3-valued and a similar thing is also the case with Vasil'ev's "imaginary logic".⁵ And, as we already mentioned, part of Slater's (and Béziau's) criticism against Priest's LP concerns its being a 3-valued logic with, at the meta-level, a binary (not a ternary!) distinction (designated, non-designated) of the possible truth-values. Now, there has been a violent (and impressive) attack, by Roman Suszko (1975, 1977), against the very idea of many-valued logic (remember that Béziau's theorem relies on Suszko—cf. §3.1 *supra*). Very interesting answers have been given to this attack by Malinowski, Frankowski, Shramko, and Wansing, which seem to have, possibly, rather deep links with our logical poly-simplexes and therefore with Béziau's geometrical anti-Slater strategy.

6.1. Suszko's Thesis and Reduction (1975): with Slater!

In many-valued logics, instead of a set of truth-values $V = \{0, 1\}$, one adopts a set like $V = \{0, \frac{1}{2}, 1\}$, $V = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $V = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ or $V = \{0, \dots, 1\}$. The logical connectives take this into account, by

⁵On Vasil'ev cf. [14]. Another similar example is given by the well-known system FDE (First Degree Entailment), which can be viewed both as a 2-valued paraconsistent logic with a ternary accessibility relation or as a 4-valued paraconsistent logic (cf. [23]).

modifying the definitional truth-tables (by incorporating the new truth-values, cf. [23], ch. 7).

However, Suszko remarked that the so-called *many-valued logics*, despite the apparent non-binarity of the set V (it can contain *more* than 2 elements), keep secretly binarity, at a meta-level of the theory of the logical matrices they rely on, in the distinction between the subset D^+ of “designated” and the subset $\overline{D^+}$ (or $V \setminus \{D^+\}$) of the so-called “non-designated” truth-values of V . In fact, in standard many-valued logics, the consequence relation “ \models ” (an operator which transfers, under any valuation “ v ”, the “designated truth-value” of the antecedent over the consequent, and which by contraposition transfers the “non-designated truth-value” of the consequent over the antecedent), is

$$A \models B \quad \text{iff} \quad \forall v : v(A) \in D^+ \Rightarrow v(B) \in D^+ .$$

Claiming that a truth-value can only be called “logical” when it is necessary in order to define a consequence relation, Suszko affirms that, truly speaking, there are 2 *and only* 2 logical truth values, the “true” and the “false”, whatever their names: all other *interpolated* truth values, i.e. those typical of many-valued logics, bear no effect on the consequence relation (which only needs to know which values are designated and which are not); they only have a non-logical use as “references”, and therefore they must be called “algebraic values” and not “logical (truth-)values”. Therefore, there is no many-valued logic at all, “many-valued logic” being only “a huge scientific deceit perpetrated by Lukasiewicz”. This “bomb” (for it is one), at least as loud as Slater’s one, is called “Suszko’s thesis”. Moreover, Suszko demonstrated a powerful theorem showing that the so-called many-valued logics can in principle be translated (i.e. reduced), sometimes (but not always) even rather easily, into an equivalent 2-valued system *via* a suited semantics. This is called “Suszko’s reduction”. Of course, Suszko’s reduction makes Suszko’s thesis more impressive and more convincing. Remark that Suszko’s results have been deepened and made even more general, more severe, by some logicians—usually paraconsistent scholars, disturbed by Suszko—who were trying to face it, such as da Costa, Bueno, Béziau, Tsuji, Caleiro, Carnielli, Coniglio and Marcos. Recall also that Béziau’s aforementioned theorem ([3]), directed against Slater, assumes Suszko’s results (against Priest).

Now—and that’s our main concern here—there seems to be a clear similarity, and resonance, between Suszko’s point and Slater’s one. In-

deed, both do claim, so to say, that: “we cannot play seriously (i.e. at a fundamental level) with the basic principles (i.e. the PNC and the PEM) of logic”. They both hold a similar “transcendentalist” position about the logical principles, Slater with the PNC, Suszko with the PEM. Together they say that the relativisations of the PEM (by many-valued logics) and of the PNC (by paraconsistent logics), despite the appearances, are fake and superficial. So, the question of knowing if Suszko’s attack obtains becomes rather important for the question we are dealing with about Slater’s attack; even more so, if we keep in mind that the geometry of opposition, NOT, summoned against Slater by Béziau, revealed itself committed, as we saw by the discovery at its core of the logical poly-simplexes, to many-valuedness. Is then many-valued logic, and with it our new-born logical poly-simplexes of dim. m , really “only a mad idea”?

6.2. Malinowski (1990) against Suszko: q -logics

Most of the many answers to Suszko have first of all simply tried to understand his quite complex reasoning, and sometimes put forward brave but sketchy ideas of possible future ways of hopeful counter-attack. The first substantial change seems to have occurred when G. Malinowski (1990), himself a pupil of Suszko, in addition, built a *concrete*, logical, explicit counter-example to it. The main idea has been to face Suszko’s criticism according to which, at a meta-level, bivalence is kept in the matricial distinction between designated and non-designated truth-values. Accordingly, Malinowski has built a logic where the set V of the truth-values is divided not into two but into *three disjunct subsets*: the D^+ one of designated values, a D^- one of so-called “anti-designated values”, and a third one, $V \setminus \{D^+ \cup D^-\}$, containing the remaining truth-values ([12]).

With the help of this device (which *per se* was not new), Malinowski was able to exhibit a *second consequence relation* (this was new!) which also needs, in order to exist, the third truth value. So, as a theorem of Malinowski demonstrates, his logical system is *radically* three-valued, in a way that *cannot be reduced* “à la Suszko”, to a 2-valued one. Malinowski has thus countered Suszko’s attack, he could only be questioned over the (slight) strangeness of the logic so generated.

Notice that Malinowski has subsequently been objected to by Tsuji ([30]) who, invoking Béziau’s notion of any abstract logical structure

$\langle L, \vdash \rangle$ (in his words, just a set with a consequence relation), has demonstrated by a theorem that, within that context, Malinowski, without being totally false, partly misses his target when answering Suszko: in fact, Tsuji’s theorem demonstrates that “an abstract logical structure $\langle L, \vdash \rangle$ is characterized by a class of two-valued models iff \vdash satisfies reflexivity”. No reflexivity outside two-valuedness!

6.3. Shramko and Wansing (2005–2008) against Suszko

Shramko and Wansing ([26]) radicalise Malinowski, against Suszko, by taking a further step: they admit an increase not only in the number of *logical* values (which reach three, thanks to Malinowski’s use of the set of the anti-designated values V^- in order to define a second consequence relation, \models_q), but also in the number of the entailment relations of a same logical system, which can even be more than two (if there are enough “*-designated” subsets of V), whereas Malinowski took in consideration only one extra consequence relation. This move, in return, allows having more than three *logical* (inferential) truth-values. Incidentally, as for Tsuji’s argument, Shramko and Wansing discard it after showing that it is hiddenly but strongly *circular*: in fact, by invoking (in his theorem) Béziau’s definition of an “abstract logical structure” $\langle L, \vdash \rangle$ Tsuji presupposes (as Béziau does in his Universal Logic approach) that each logical system has *one* (and only one) consequence relation “ \vdash ”, which is exactly the point to be debated, when radicalising Malinowski! As a first step of such a possible renewal of many-valued logic by radicalising Malinowski’s strategy, Shramko and Wansing exhibit a logic which has four consequence relations: a *t*-consequence, an *f*-consequence, Malinowski’s *q*-consequence and Frankowski’s *p*-consequence. And, this is a major point against Tsuji’s criticism to Malinowski (cf. §6.2), each is *reflexive* (in fact, Tarskian). They also exhibit a so-called “bi-consequence logic” (constructed over a tri-lattice). Then they generalise this by constructing a logically very general *n*-valued logics. Many-valued logic, transfigured, is therefore regained by a change of the notion of logical system.

6.4. The verdict on Suszko: multiplicity regained

On this question of the thinkability of many-valued logics, the current result seems to be at least tripartite: (1) Suszko is right with respect

to many existing systems of contemporary many-valued logics (including the classical ones he fought against: the ones of Łukasiewicz, etc.): they are, fundamentally speaking, 2-valued; and this is a major discovery in logic; (2) but Suszko is nevertheless wrong in general, for “*truly* many-valued (i.e. trans-Suszonian) logical systems” are by now possible: Malinowski, Frankowski, Shramko and Wansing explore them currently, thanks to Suszko’s appealing, if not anymore devastating, rude challenge, a challenge brightly faced; (3) these non-Suszonian, radically many-valued logics allow, however, a strange multiplicity of logical entities which traditionally were unique (*in primis* the “ \models ”): this still needs to be better understood and (possibly) nourished with more familiar intuitions.

6.5. The logical poly-simplexes with Malinowski, Shramko and Wansing: do they express trans-Suszonian logics?

What was previously seen about the changes in the theory of the logical matrices can be easily represented, using the bi-simplexes of NOT (a square for Suszko, an hexagon for Malinowski, a cube for Shramko and Wansing, . . .). This seems to suggest, more radically, ways of generalising Shramko and Wansing’s research line by the use of the geometry of opposition (the NOT can give them ideas). Moreover, our multiplicity (i.e. the p^q -semantics and the p^q -lattices) resemble curiously their multiplicity, the new plurality of consequence relations and logical values, of which it could be therefore the geometrical counterpart. In other words, we are tempted to see the logical poly-simplexes, with their geometrical concrete features, as embodying the geometry of the “trans-Suszonian many-valued logics”. And the future developments of NOT through its “ q ” parameter could inspire ideas to the trans-Suszonian logicians. But this seminal conjecture, maybe seminal, will have to be explored elsewhere.

7. Conclusion: NOT, the PNC and the PEM

In this paper, after recalling the general context of the debate, we concentrated on the two articulated answers, logical and philosophical, given to Slater by Béziau, and in particular we concentrated on the “geometric turn” the latter gave to the discussion over the foundations of paraconsistent logics. In order to do that, we recalled and discussed how Béziau’s



own strategy gave rise, beyond Béziau (but thanks to him), to the discovery of a new branch of mathematics, n -opposition theory. So, the task of judging the quality of Béziau's *geometric* (and philosophical) answer to Slater (the logical answer seems already clear and powerful) depends on the final shape, the mathematical closure of this last theory (NOT), which indeed is still young but already flourishing. This closure is not yet at hand, due to its amazing richness in results and surprises. What can be already said, however, is that the logical square (Slater's implicit fetish) has been relativised by NOT. The general p^q semantics and lattice are the actual transcendental structures of the geometry of logical opposition. Remark that the existence of the logical poly-simplexes seems to show that, contrary to what was believed until a very recent date by seemingly almost all logicians, the same logical system, provided it is many-valued, can have several contradiction-forming operators (this point is against Slater): the logical tri-simplexes have three, the logical quadri-simplexes have six, and so on (and the same remark can be done as for the subalternation-forming operators). We recalled how Malinowski, Frankowski, Shramko and Wansing have recently succeeded in showing that *there exist* logical systems *radically many-valued*, i.e. such that they defeat Suszko's arguments, and preventing the possible objection according to which commitment to many-valuedness could be "deadly" for NOT's logical seriousness in exploring the transcendental foundations of logic. Remarking that such systems have several consequence operators, which is a "strangeness" in some sense shared by the logical poly-simplexes, we finished by suggesting the possibility, of examining whether the rich logical geometry we (very partially) exhibited here could be put into a close relation with the strange new logical panorama offered to us by Shramko and Wansing. So, the final (provisory) lesson seems to be that—as Restall, Priest, Brown and Paoli already showed us with their logical anti-Slater arguments—we should feel "cool" and not frightened with respect to "transcendental constraints" (such as Suszko's or Slater's). For, as it seems, there are ways to radicalise logical "non-standardness" beyond the limits such transcendentially-minded logicians and philosophers (Suszko, Slater) set to the logically possible. Moreover, we should feel encouraged to try to have *concrete*, even visual intuitions of the non-standard (like the geometries of NOT).

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References

- [1] Béziau, J.-Y., “New light on the square of oppositions and its nameless corner”, *Logical Investigations* 10 (2003): 218–233.
- [2] Béziau, J.-Y., “Paraconsistent logic from a modal viewpoint”, *Journal of Applied Logic* 3 (2005): 7–14.
- [3] Béziau, J.-Y., “Paraconsistent logic! (A reply to Slater)”, *Sorites* 17 (2006).
- [4] Béziau, J.-Y., “Adventures in the paraconsistent jungle”, in: J.-Y. Béziau, W. A. Carnielli and D. Gabbay (eds.), *Handbook of Paraconsistency*, King’s College, London, 2007.
- [5] Blanché, R., “Sur l’opposition des concepts”, *Theoria* 19 (1953).
- [6] Blanché, R., *Structures intellectuelles. Essai sur l’organisation systématique des concepts*, Vrin, Paris, 1966.
- [7] Brown, B., “Yes, Virginia, there really are paraconsistent logics”, *Journal of Philosophical Logic* 28 (1999): 489–500.
- [8] Dutilh Novaes, C., “Contradiction: the real challenge for paraconsistent logic”, in: J.-Y. Béziau, W. A. Carnielli and D. Gabbay (eds.), *Handbook of Paraconsistency*, King’s College, London, 2007.
- [9] Frankowski, S., “ p -consequence versus q -consequence”, *Bulletin of the Section of Logic* 33 (2004): 197–208.
- [10] Gardies, J.-L., *Essai sur la logique des modalités*, PUF, Paris, 1979.
- [11] Horn, L., *A natural history of negation*, UCP, Chicago, 1989.
- [12] Malinowski, G., “Towards the concept of logical many-valuedness”, *Folia Philosophica* 7 (1990): 97–103.
- [13] Moretti, A., “Geometry for modalities? Yes. Through n -opposition theory”, in: J.-Y. Béziau, A. Costa-Leite and A. Facchini (eds.), *Aspects of Universal Logic*, University of Neuchâtel, Neuchâtel, December 2004.
- [14] Moretti, A., “A graphical decision procedure and some paraconsistent theorems for the Vasil’evian logic IL2”, in: J.-Y. Béziau, W. A. Carnielli and D. Gabbay (eds.), *Handbook of Paraconsistency*, King’s College, London, 2007.

- [15] Moretti, A., *The Geometry of Logical Opposition*, PhD Thesis, University of Neuchâtel, Switzerland (March 2009).
- [16] Moretti, A., “The geometry of standard deontic logic”, *Logica Universalis* 3, 1 (2009).
- [17] Moretti, A., “A geometrical method for classifying and studying quantifiers and modalities” (draft).
- [18] Moretti, A., “Correcting McNamara’s ‘deontic dodecagon for DWE’: through NOT”, submitted to *Logique et Analyse*.
- [19] Paoli, F., “Quine and Slater on paraconsistency and deviance”, *Journal of Philosophical Logic* 32 (2003): 531–548.
- [20] Pellissier, R., “‘Setting’ n -opposition”, *Logica Universalis* 2 (2008).
- [21] Pellissier, R., “2-opposition and the topological hexagon”, in: J.-Y. Béziau and G. Payette (eds.), *New perspectives on the square of opposition*, Peter Lang, Bern, 2010.
- [22] Priest, G., “The logic of paradox”, *Journal of Philosophical Logic*, 8 (1979): 219–241.
- [23] Priest, G., *An Introduction to Non-Classical Logic*, CUP, Cambridge, 2001.
- [24] Restall, G., “Paraconsistent logics!”, *Bulletin of the Section of Logic* 26 (1997): 156–173.
- [25] Sesmat, A., *Logique II*, Hermann, Paris, 1951.
- [26] Shramko, Y., and H. Wansing, “Suszko’s thesis, inferential many-valuedness, and the notion of a logical system”, *Studia Logica* 88 (2008).
- [27] Slater, B.H., “Paraconsistent logics?”, *Journal of Philosophical Logic*, 24 (1995): 451–454.
- [28] Smessaert, H., “On the 3D visualisation of logical relations”, *Logica Universalis* 3, 2 (2009).
- [29] Suszko, R., “Remarks on Łukasiewicz’s three-valued logic”, *Bulletin of the Section of Logic* 4 (1975): 87–90.
- [30] Tsuji, M., “Many-valued logics and Suszko’s thesis revisited”, *Studia Logica* 60 (1998): 299–309.