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SYMMETRIC AND DUAL PARACONSISTENT LOGICS

Abstract. Two new first-order paraconsistent logics with De Morgan-type negations and co-implication, called symmetric paraconsistent logic (SPL) and dual paraconsistent logic (DPL), are introduced as Gentzen-type sequent calculi. The logic SPL is symmetric in the sense that the rule of contraposition is admissible in cut-free SPL. By using this symmetry property, a simpler cut-free sequent calculus for SPL is obtained. The logic DPL is not symmetric, but it has the duality principle. Simple semantics for SPL and DPL are introduced, and the completeness theorems with respect to these semantics are proved. The cut-elimination theorems for SPL and DPL are proved in two ways: One is a syntactical way which is based on the embedding theorems of SPL and DPL into Gentzen's LK, and the other is a semantical way which is based on the completeness theorems.

Keywords: symmetric paraconsistent logic, dual paraconsistent logic, sequent calculus, cut-elimination, completeness.

1. Introduction

1.1. De Morgan-type negations: Symmetry versus duality

Paraconsistent logics are logics which have the desirable property of paraconsistency (see, e.g., [4], [10], and the references therein). *Paraconsistency* is, roughly speaking, a property of negations (or negation-like operators), and is known to be useful for representing inconsistency-tolerant

reasoning more appropriately. Examples of paraconsistent negations (i.e., negations enjoying paraconsistency) are De Morgan-type negations, such as strong negation [8], negations based on four-valued logic [2] and negations based on bilattice logics [1].

The De Morgan-type negations have the common characteristic axioms of the De Morgan laws: $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$ and $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$. These axioms imply the fact that the rule of contraposition

$$\frac{\Gamma \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Gamma} \text{ (cont)}$$

(where $\sim\Delta$ stands for $\{\sim\alpha \mid \alpha \in \Delta\}$) is admissible in a sequent calculus for the $\rightarrow$-free fragment of the logic in question. These $\rightarrow$-free fragments are symmetric with respect to $\sim$ in this sense. By virtue of this symmetry property, a simpler sequent calculus or axiomatization can be obtained for these logics. The De Morgan laws without the axioms for $\rightarrow$ also imply the *duality principle*. The duality principle for the $\rightarrow$-free fragment of Gentzen's sequent calculus LK for classical logic is well-known: If $\rightarrow$-free LK $\vdash \alpha \Rightarrow \beta$, then $\rightarrow$-free LK $\vdash \tilde{\beta} \Rightarrow \tilde{\alpha}$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are, respectively, obtained from α and β by replacing every occurrence of $\wedge$, $\vee$, $\forall$ and $\exists$ by those of $\vee$, $\wedge$, $\exists$ and $\forall$, respectively. This principle also holds for a sequent calculus for Belnap and Dunn's four-valued logic.

On the other hand, an extended logic (or sequent calculus) with $\rightarrow$ which has the symmetry property or duality principle has not been studied yet. The reason may be that the De Morgan-like laws with respect to $\rightarrow$ and the De Morgan dual counterpart connective of $\rightarrow$ have not been considered. In this paper, such extended logics are proposed by using a co-implication connective $\leftarrow$ as the De Morgan dual counterpart of $\rightarrow$.

1.2. Combination with co-implication

The *co-implication* connective $\leftarrow$ has been studied by many researchers in the context of *Heyting-Brouwer logic* (*H-B logic* for short) or equivalently *bi-intuitionistic logic*, which is, roughly speaking, an extension of (positive) intuitionistic logic with $\leftarrow$ (see, e.g., [3, 5, 7, 11] and the references therein). The connective $\leftarrow$ is known as a subtraction (or difference) operator, since $\leftarrow$ has the informal interpretation $\alpha \leftarrow \beta := \alpha \wedge \neg\beta$ where $\neg$ is classical negation. The notion of subtraction is considered to be very important in the area of computer science, in particular in database

theory and software development. Indeed, subtraction is used as a basic operation of databases, and the difference of two or more program codes (or data) is frequently checked in computer systems. Thus, combining co-implication with paraconsistent negations is also an interesting issue for computer science.

Combining co-implication with paraconsistent negations has been studied in [16] in an intuitionistic setting, although the motivation is different from the present paper's one. In [16], two extended H-B logics with certain packages of axioms concerning $\leftarrow, \rightarrow$ and $\sim$ are investigated: One is

A1: $\sim(\alpha \rightarrow \beta) \leftrightarrow \sim\beta \leftarrow \sim\alpha$
  (negated implication as contraposed co-implication)

A2: $\sim(\alpha \leftarrow \beta) \leftrightarrow \sim\beta \rightarrow \sim\alpha$
  (negated co-implication as contraposed implication)

and another is

A3: $\sim(\alpha \rightarrow \beta) \leftrightarrow \alpha \leftarrow \beta$ (negated implication as co-implication)

A4: $\sim(\alpha \leftarrow \beta) \leftrightarrow \alpha \rightarrow \beta$ (negated co-implication as implication).

It is shown in [16] that some intuitionistic cut-free display calculi for the logics with these axioms are complete with respect to a Kripke semantics. However, standard cut-free Gentzen-type sequent calculi, first-order versions and classical logic based versions for logics with {A1, A2} or {A3, A4} have not been studied yet. For the first set of axioms, constructing a standard cut-free Gentzen-type sequent calculus for H-B logic is known as difficult [3].

1.3. Our approach

The aim of this paper is to obtain cut-free Gentzen-type systems and simple semantics for classical first-order (symmetric and dual) paraconsistent logics with both De Morgan-type negations and co-implication. The result of this paper is thus regarded as a continuation of the work [16] based on classical first-order logic rather than propositional (positive) intuitionistic logic. Based upon classical logic, a very simple framework with the natural properties of symmetry and duality can be obtained.

The results of this paper are summarized as follows. Two new first-order paraconsistent logics with De Morgan-type negations and co-impli-

cation, called *symmetric paraconsistent logic* (SPL) and *dual paraconsistent logic* (DPL), are introduced as Gentzen-type sequent calculi by extending LK. The logic SPL has the inference rules which correspond to the laws A1 and A2. SPL is symmetric, i.e., the rule (cont) is admissible in cut-free SPL. By using (cont), a simpler cut-free system SPL$^-$ can also be obtained for SPL. The logic DPL has the inference rules which correspond to the laws A3 and A4. DPL is not symmetric, but it has the duality principle. These proposed logics are regarded as natural variants of the H-B logic, and as extensions of the existing paraconsistent logics with De Morgan-type negations, i.e., the $\{\wedge, \vee, \sim\}$-fragments of Nelson's paraconsistent logics, Belnap and Dunn's four-valued logic and Arieli and Avron's bilattice logics. Simple semantics for SPL and DPL are introduced, and the completeness theorems with respect to these semantics are proved using Schütte's method [13]. This method simultaneously provides a semantical proof of the cut-elimination theorems for SPL and DPL. The cut-elimination theorems for SPL and DPL are also proved by using the embedding theorems of SPL and DPL into LK.

2. Symmetric paraconsistent logic

Prior to a detailed discussion, the language $\mathcal{L}$ used in this paper is introduced below. *Formulas* are constructed from predicate symbols p, q, ..., (countably many) individual variables x, y, ..., individual constants c, d, ..., function symbols f, g, ..., $\rightarrow$ (implication), $\leftarrow$ (co-implication or subtraction), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (classical negation), $\sim$ (paraconsistent negation), $\forall$ (universal quantifier) and $\exists$ (existential quantifier). Small letters t, s, ... are used to denote terms, Greek small letters α, β, ... are used to denote formulas, and Greek capital letters Γ, Δ, ... are used to represent finite (possibly empty) sets of formulas. An expression $\alpha[t/x]$ means the formula which is obtained from a formula α by replacing all free occurrences of the individual variable x in α by the term t, but avoiding a clash of variables. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. The symbol $\equiv$ is used to denote the equality of sets of symbols. An expression $L \vdash \Gamma \Rightarrow \Delta$ means that the sequent $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus L, and L in this expression will occasionally be omitted.

Firstly, we define the sequent calculus LK for classical logic, and secondly we define SPL by extending LK with $\sim$.

DEFINITION 2.1 (LK). The initial sequents of LK are of the form:

$$\alpha \Rightarrow \alpha\,.$$

The structural inference rules of LK are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (cut)$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (we\text{-}left) \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (we\text{-}right).$$

The logical inference rules of LK are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to left) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\ (\to right)$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge left) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge right)$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee left) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee right)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg \alpha, \Gamma \Rightarrow \Delta}\ (\neg left) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \alpha}\ (\neg right)$$

$$\frac{\alpha[t/x], \Gamma \Rightarrow \Delta}{\forall x \alpha, \Gamma \Rightarrow \Delta}\ (\forall left) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[z/x]}{\Gamma \Rightarrow \Delta, \forall x \alpha}\ (\forall right)$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\exists x \alpha, \Gamma \Rightarrow \Delta}\ (\exists left) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[t/x]}{\Gamma \Rightarrow \Delta, \exists x \alpha}\ (\exists right)$$

where t in ($\forall$left) and ($\exists$right) is a term, and z in ($\forall$right) and ($\exists$left) is an individual variable which has the eigenvariable condition, i.e., z does not occur as a free individual variable in the lower sequent of the rule.

DEFINITION 2.2 (SPL). A sequent calculus SPL for symmetric paraconsistent logic is obtained from LK by adding logical inference rules of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta}\ (\leftarrow left) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \leftarrow \beta}\ (\leftarrow right)$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim \sim \alpha, \Gamma \Rightarrow \Delta}\ (\sim left) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim \sim \alpha}\ (\sim right)$$

$$\frac{\sim\beta, \Gamma \Rightarrow \Delta, \sim\alpha}{\sim(\alpha \to \beta), \Gamma \Rightarrow \Delta} \ (\sim\to left)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\beta \quad \sim\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim(\alpha \to \beta)} \ (\sim\to right)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\beta \quad \sim\alpha, \Sigma \Rightarrow \Pi}{\sim(\alpha \leftarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \ (\sim\leftarrow left)$$

$$\frac{\sim\beta, \Gamma \Rightarrow \Delta, \sim\alpha}{\Gamma \Rightarrow \Delta, \sim(\alpha \leftarrow \beta)} \ (\sim\leftarrow right)$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \ (\sim\wedge left) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta)} \ (\sim\wedge right)$$

$$\frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \ (\sim\vee left) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)} \ (\sim\vee right)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\alpha}{\sim\neg\alpha, \Gamma \Rightarrow \Delta} \ (\sim\neg left) \qquad \frac{\sim\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\neg\alpha} \ (\sim\neg right)$$

$$\frac{\sim\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim\forall x\alpha, \Gamma \Rightarrow \Delta} \ (\sim\forall left) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha[t/x]}{\Gamma \Rightarrow \Delta, \sim\forall x\alpha} \ (\sim\forall right)$$

$$\frac{\sim\alpha[t/x], \Gamma \Rightarrow \Delta}{\sim\exists x\alpha, \Gamma \Rightarrow \Delta} \ (\sim\exists left) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim\exists x\alpha} \ (\sim\exists right)$$

where t and z in the quantifier rules are an arbitrary term and an individual variable with the eigenvariable condition, respectively.

Note that LK + {($\leftarrow$left), ($\leftarrow$right)} is equivalent to Crolard's sequent calculus SLK for (classical) subtractive logic [5]. It is also remarked that the connective $\leftarrow$, which is characterized by the inference rules ($\leftarrow$left) and ($\leftarrow$right), is definable in LK, i.e., SLK and LK are theorem-equivalent. Moreover, the $\{\wedge, \vee, \sim\}$-fragment of SPL is a common fragment of Belnap and Dunn's four-valued logic [2] and Arieli and Avron's bilattice logics [1].

An expression $\alpha \Leftrightarrow \beta$ means the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

PROPOSITION 2.3. *The following sequents are provable in* SPL:

1. $\sim\sim\alpha \Leftrightarrow \alpha$,
2. $\sim\neg\alpha \Leftrightarrow \neg\sim\alpha$,

3. $\sim(\alpha \rightarrow \beta) \Leftrightarrow (\sim\beta \leftarrow \sim\alpha)$
   (*negated implication as contraposed co-implication*),

4. $\sim(\alpha \leftarrow \beta) \Leftrightarrow (\sim\beta \rightarrow \sim\alpha)$
   (*negated co-implication as contraposed implication*),

5. $\sim(\alpha \wedge \beta) \Leftrightarrow \sim\alpha \vee \sim\beta$,

6. $\sim(\alpha \vee \beta) \Leftrightarrow \sim\alpha \wedge \sim\beta$,

7. $\sim(\forall x\alpha) \Leftrightarrow \exists x(\sim\alpha)$,

8. $\sim(\exists x\alpha) \Leftrightarrow \forall x(\sim\alpha)$.

In the laws addressed in Proposition 2.3, $\leftarrow$, $\vee$ and $\exists$ are regarded as the De Morgan duals (w.r.t. $\sim$) of $\rightarrow$, $\wedge$ and $\forall$, respectively. As mentioned in [16], the laws 3 and 4 in Proposition 2.3 were suggested by Restall.

PROPOSITION 2.4. *The rules*

$$\frac{\sim\sim\alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \; (\sim left^{-1}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim\sim\alpha}{\Gamma \Rightarrow \Delta, \alpha} \; (\sim right^{-1})$$

are admissible in cut-free SPL.

PROOF. Straightforward. $\dashv$

We then obtain the characteristic property of SPL as follows.

THEOREM 2.5 (Admissibility of contraposition). *The rule of contraposition*

$$\frac{\Gamma \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Gamma} \; (cont)$$

is admissible in cut-free SPL.

PROOF. By induction on the proof P of $\Gamma \Rightarrow \Delta$ in cut-free SPL. We distinguish the cases according to the last inference of P. We show only the following case.

Case ($\sim \leftarrow$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim\beta \quad \sim\alpha, \Sigma \Rightarrow \Pi}{\sim(\alpha \leftarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \; (\sim \leftarrow left).$$

By the hypothesis of induction, we have $\text{SPL}-(\text{cut}) \vdash \sim\Pi \Rightarrow \sim\Sigma, \sim\sim\alpha$ and $\text{SPL}-(\text{cut}) \vdash \sim\sim\beta, \sim\Delta \Rightarrow \sim\Gamma$. We then obtain the required fact:

$$
\cfrac{
\cfrac{\vdots \quad\quad \atop {\sim\Pi \Rightarrow \sim\Sigma, \sim\sim\alpha}}{\sim\Pi \Rightarrow \sim\Sigma, \alpha} \;(\sim\text{right}^{-1}) \quad
\cfrac{\vdots \quad\quad \atop {\sim\sim\beta, \sim\Delta \Rightarrow \sim\Gamma}}{\beta, \sim\Delta \Rightarrow \sim\Gamma} \;(\sim\text{left}^{-1})
}{
\cfrac{\sim\Pi, \sim\Delta \Rightarrow \sim\Sigma, \sim\Gamma, \alpha \leftarrow \beta}{\sim\Pi, \sim\Delta \Rightarrow \sim\Sigma, \sim\Gamma, \sim\sim(\alpha \leftarrow \beta)} \;(\sim\text{right})
} \;(\leftarrow\text{right})
$$

where $(\sim\text{right}^{-1})$ and $(\sim\text{left}^{-1})$ are admissible in cut-free SPL by Proposition 2.4.   ⊣

Note that the rule of contraposition with respect to Nelson's strong negation is not admissible in the standard cut-free sequent calculi for Nelson's logics. (For a system with contraposable strong negation see [9] and for the standard cut-free sequent calculi for Nelson's logics see [15])

DEFINITION 2.6 (SPL^-). The system SPL^- is defined as LK + $\{(\leftarrow\text{left}),$ $(\leftarrow\text{right}), (\sim\text{left}), (\sim\text{right}), (\text{cont})\}$.

THEOREM 2.7 (Cut-free equivalence between SPL^- and SPL).
The systems $\text{SPL} - (\text{cut})$ *and* $\text{SPL}^- - (\text{cut})$ *are theorem-equivalent, i.e., for any sequent S,* $\text{SPL} - (\text{cut}) \vdash S$ *iff* $\text{SPL}^- - (\text{cut}) \vdash S$.

PROOF. ($\Longrightarrow$): By using the rule (cont). ($\Longleftarrow$): By Theorem 2.5.   ⊣

3. Embedding and cut-elimination

In order to prove the cut-elimination theorem for SPL, we give an embedding f of SPL into LK, which is a modified extension of the embedding of Nelson's logic N3 into (positive) intuitionistic logic. For the embedding of Nelson's logic, see [6, 12, 14].

DEFINITION 3.1. We fix a countable set AT of atomic formulas, and define the set $AT' := \{p' \mid p \in AT\}$ of atomic formulas. The set FO_{SPL} of formulas of SPL is obtained from the language $\mathcal{L}$ by using AT. The set FO_{LK} of formulas of LK is obtained from FO_{SPL} by adding AT' and deleting the formulas with $\sim$ or $\leftarrow$.

A mapping f from FO_{SPL} to FO_{LK} is defined inductively as follows:

1. $f(p) := p$ and $f(\sim p) := p' \in AT'$ for any $p \in AT$,

2. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$, where $\circ \in \{\wedge, \vee, \rightarrow\}$,

3. $f(\alpha \leftarrow \beta) := f(\alpha) \wedge \neg f(\beta)$,

4. $f(\circ \alpha) := \circ f(\alpha)$, where $\circ \in \{\neg, \forall x, \exists x\}$,

5. $f(\sim(\alpha \wedge \beta)) := f(\sim \alpha) \vee f(\sim \beta)$,

6. $f(\sim(\alpha \vee \beta)) := f(\sim \alpha) \wedge f(\sim \beta)$,

7. $f(\sim(\alpha \rightarrow \beta)) := f(\sim \beta) \leftarrow f(\sim \alpha)$ (i.e., $f(\sim \beta) \wedge \neg f(\sim \alpha)$),

8. $f(\sim(\alpha \leftarrow \beta)) := f(\sim \beta) \rightarrow f(\sim \alpha)$,

9. $f(\sim(\forall x \alpha)) := \exists x f(\sim \alpha)$,

10. $f(\sim(\exists x \alpha)) := \forall x f(\sim \alpha)$,

11. $f(\sim(\neg \alpha)) := \neg f(\sim \alpha)$,

12. $f(\sim \sim \alpha) := f(\alpha)$.

Let Γ be a set of formulas in FO_{SPL}. Then, an expression $f(\Gamma)$ means the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

THEOREM 3.2 (Embedding of SPL into LK). *Let Γ and Δ be sets of formulas in FO_{SPL}, and f be the mapping defined in Definition 3.1.*

(1) *SPL $\vdash \Gamma \Rightarrow \Delta$ iff LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$.*

(2) *LK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ iff SPL $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.*

PROOF. We show only (1), since (2) can be obtained by observing the proof of (1). We show only the direction ($\Longrightarrow$) by induction on the proof P of $\Gamma \Rightarrow \Delta$ in SPL. We distinguish the cases according to the last inference of P. We show some cases.

Case ($\leftarrow$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \leftarrow \beta} \ (\leftarrow \mathrm{right}).$$

By the hypothesis of induction, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$ and LK $\vdash f(\beta), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact as follows.

$$
\frac{
\begin{array}{cc}
\vdots \\
f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \\
\vdots \; (\mathrm{we-left/right}) \\
f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), f(\alpha)
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\dfrac{f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\Sigma) \Rightarrow f(\Pi), \neg f(\beta)} \; (\neg\mathrm{right}) \\
\vdots \; (\mathrm{we-left/right}) \\
f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), \neg f(\beta)
\end{array}
}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), f(\alpha) \wedge \neg f(\beta)} \; (\wedge\mathrm{right})
$$

where $f(\alpha) \wedge \neg f(\beta) = f(\alpha \leftarrow \beta)$.

Case ($\sim \leftarrow$left): The last inference of P is of the form:

$$
\frac{\Gamma \Rightarrow \Delta, \sim\beta \quad \sim\alpha, \Sigma \Rightarrow \Pi}{\sim(\alpha \leftarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \; (\sim \leftarrow \mathrm{left}).
$$

By the hypothesis of induction, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)$ and LK $\vdash f(\sim\alpha), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$
\frac{
\begin{array}{c} \vdots \\ f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta) \end{array}
\quad
\begin{array}{c} \vdots \\ f(\sim\alpha), f(\Sigma) \Rightarrow f(\Pi) \end{array}
}{f(\sim\beta) \to f(\sim\alpha), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)} \; (\to \mathrm{left})
$$

where $f(\sim\beta) \to f(\sim\alpha) = f(\sim(\alpha \leftarrow \beta))$.

Case ($\sim \to$right): The last inference of P is of the form:

$$
\frac{\Gamma \Rightarrow \Delta, \sim\beta \quad \sim\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim(\alpha \to \beta)} \; (\sim \to \mathrm{right}).
$$

By the hypothesis of induction, we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta)$ and LK $\vdash f(\sim\alpha), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact as follows.

$$
\frac{
\begin{array}{cc}
\vdots \\
f(\Gamma) \Rightarrow f(\Delta), f(\sim\beta) \\
\vdots \; (\mathrm{we-left/right}) \\
f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), f(\sim\beta)
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\dfrac{f(\sim\alpha), f(\Sigma) \Rightarrow f(\Pi)}{f(\Sigma) \Rightarrow f(\Pi), \neg f(\sim\alpha)} \; (\neg\mathrm{right}) \\
\vdots \; (\mathrm{we-left/right}) \\
f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), \neg f(\sim\alpha)
\end{array}
}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), f(\sim\beta) \wedge \neg f(\sim\alpha)} \; (\wedge\mathrm{right})
$$

where $f(\sim\beta) \wedge \neg f(\sim\alpha) = f(\sim\beta) \leftarrow f(\sim\alpha) = f(\sim(\alpha \to \beta))$. $\dashv$

THEOREM 3.3 (Cut-elimination for SPL). *The rule* (cut) *is admissible in cut-free* SPL.

PROOF. Suppose that SPL $\vdash \Gamma \Rightarrow \Delta$. Then we have LK $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 3.2 (1). We obtain LK$-$(cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the well-known cut-elimination theorem for LK. By Theorem 3.2 (2), we obtain the required fact SPL$-$(cut) $\vdash \Gamma \Rightarrow \Delta$. ⊣

THEOREM 3.4 (Cut-elimination for SPL$^-$). *The rule* (cut) *is admissible in cut-free* SPL$^-$.

PROOF. By theorems 2.7 and 3.3. ⊣

Using Theorems 3.3 and 3.4, the paraconsistency of SPL and SPL$^-$ w.r.t. $\sim$ is shown.

DEFINITION 3.5. Let $\sharp$ be a unary connective. A sequent calculus L is called *explosive* with respect to $\sharp$ if for any formulas α and β, the sequents of the form $\alpha, \sharp\alpha \Rightarrow \beta$ are provable in L. It is called *paraconsistent* with respect to $\sharp$ if it is not explosive with respect to $\sharp$.

THEOREM 3.6 (Paraconsistency for SPL and SPL$^-$). *Let L be* SPL *or* SPL$^-$. *L is paraconsistent with respect to* $\sim$.

PROOF. Let p and q be distinct atomic formulas. Then, the sequent $p, \sim p \Rightarrow q$ is not provable in L. The unprovability of this sequent is guaranteed by Theorems 3.3 and 3.4. ⊣

It is remarked that SPL and SPL$^-$ are explosive with respect to $\neg$.

Since (first-order) LK is known as undecidable, the extensions SPL and SPL$^-$ of LK are also undecidable. On the other hand, the *monadic fragment* of LK, the fragment in which all predicate symbols are one-place and there are no function symbols, is known to be decidable. This fact implies the following theorem.

THEOREM 3.7 (Decidability of the monadic fragments).
The monadic fragments of SPL *and* SPL$^-$ *are decidable.*

PROOF. By (a slightly modified version of) Theorem 3.2, the provability relation of the fragments can be transformed into that of the monadic fragment of LK. Since the monadic fragment of LK is decidable, the monadic fragments of SPL and SPL$^-$ are also decidable. ⊣

Similarly, we can also obtain the following theorem.

THEOREM 3.8 (Decidability of the propositional fragments).
The propositional fragments of SPL *and* SPL$^-$ *are decidable.*

4. Dual paraconsistent logic

DEFINITION 4.1 (DPL). A sequent calculus DPL for dual paraconsistent logic is obtained from SPL by replacing the inference rules ($\sim \to$left), ($\sim \to$right), ($\sim \leftarrow$left) and ($\sim \leftarrow$right) by the inference rules of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\sim(\alpha \to \beta), \Gamma \Rightarrow \Delta} \ (\sim \to \mathit{left}^d)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim(\alpha \to \beta)} \ (\sim \to \mathit{right}^d)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\sim(\alpha \leftarrow \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \ (\sim \leftarrow \mathit{left}^d)$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \leftarrow \beta)} \ (\sim \leftarrow \mathit{right}^d).$$

PROPOSITION 4.2. *The sequents 1, 2, 5–8 in Proposition* 2.3 *and the following sequents are provable in* DPL:

3'. $\sim(\alpha \to \beta) \Leftrightarrow (\alpha \leftarrow \beta)$ *(negated implication as co-implication).*

4'. $\sim(\alpha \leftarrow \beta) \Leftrightarrow (\alpha \to \beta)$ *(negated co-implication as implication).*

Proposition 2.4 holds for DPL, but Theorem 2.5 does not hold for DPL.

DEFINITION 4.3. The set FO_{DPL} of formulas of DPL is the same as FO_{SPL} in Definition 3.1. The set FO_{LK} of formulas of LK is defined in Definition 3.1
   A mapping f from FO_{DPL} to FO_{LK} is obtained from the mapping defined in Definition 3.1 by replacing the conditions 11 and 12 by the following conditions:

11'. $f(\sim(\alpha \to \beta)) := f(\alpha) \leftarrow f(\beta)$ (i.e., $f(\alpha) \wedge \neg f(\beta)$),

12'. $f(\sim(\alpha \leftarrow \beta)) := f(\alpha) \to f(\beta)$.

THEOREM 4.4 (Embedding of DPL into LK). *Let Γ and Δ be sets of formulas in FO_{DPL}, and f be the mapping defined in Definition 4.3.*

(1) $\mathrm{DPL} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

(2) $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ *iff* $\mathrm{DPL} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

Using Theorem 4.4, we can derive the following theorem.

THEOREM 4.5 (Cut-elimination for DPL). *The rule* (cut) *is admissible in cut-free* DPL.

We then present the characteristic property of DPL as follows.

THEOREM 4.6 (Duality for DPL). *Suppose that $\tilde{\alpha}$ and $\tilde{\beta}$ are the formulas obtained from formulas α and β, respectively, by replacing every occurrence of $\wedge, \vee, \forall, \exists, \rightarrow$ and $\leftarrow$ by $\vee, \wedge, \exists, \forall, \leftarrow$ and $\rightarrow$, respectively.*

(1) *if* $\mathrm{DPL} - (\mathrm{cut}) \vdash \alpha \Rightarrow \beta$, *then* $\mathrm{DPL} - (\mathrm{cut}) \vdash \tilde{\beta} \Rightarrow \tilde{\alpha}$.

(2) *if* $\mathrm{DPL} - (\mathrm{cut}) \vdash \alpha \Leftrightarrow \beta$, *then* $\mathrm{DPL} - (\mathrm{cut}) \vdash \tilde{\beta} \Leftrightarrow \tilde{\alpha}$.

PROOF. We show only (1), since (2) is derived from (1). By the hypothesis, we have a cut-free proof P of $\alpha \Rightarrow \beta$ in $\mathrm{DPL} - (\mathrm{cut})$. We replace all the sequents of P by the converse sequents (i.e., the succedent and antecedent are exchanged), and replace all the occurrences of $\wedge, \vee, \rightarrow$ and $\leftarrow$ by those of $\vee, \wedge, \leftarrow$ and $\rightarrow$, respectively. We then obtain a proof of $\tilde{\beta} \Rightarrow \tilde{\alpha}$ in $\mathrm{DPL} - (\mathrm{cut})$.   ⊣

THEOREM 4.7 (Paraconsistency for DPL). DPL *is paraconsistent with respect to* $\sim$.

THEOREM 4.8 (Decidability of the monadic and propositional fragments). *The monadic and propositional fragments of* DPL *are both decidable.*

5. Semantics and completeness

In this section, the semantics and completeness for SPL and DPL are discussed. For the sake of simplicity of the discussion, the language without individual constants and function symbols is adopted in this section. The same names $\mathcal{L}$, SPL and DPL are used for the reduced language and the corresponding subsystems, respectively. Let Γ be a

set $\{\alpha_1, \ldots, \alpha_m\}$ ($m \geq 0$) of formulas, and p be a fixed atomic formula. Then Γ^* is defined as $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, and $\neg(p \rightarrow p)$ if $m = 0$. Also Γ_* is defined as $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, and $p \rightarrow p$ if $m = 0$.

First, the semantics and completeness for SPL are discussed. The semantics and completeness for DPL can also be obtained similarly, and hence the completeness proof for DPL is omitted here.

DEFINITION 5.1. $\mathcal{A} := \langle U, I^+, I^- \rangle$ is called a *model* if the following conditions hold:

1. U is a non-empty set,

2. I^+ and I^- are mappings such that $p^{I^+}, p^{I^-} \subseteq U^n$ (i.e., p^{I^+} and p^{I^-} are n-ary relations on U) for an n-ary predicate symbol p.

We introduce the notation $\underline{u}$ for the name of $u \in U$, and write $\mathcal{L}[\mathcal{A}]$ for the language obtained from $\mathcal{L}$ by adding the names of all the elements of U. A formula α is called a *closed formula* if α has no free individual variable. A formula of the form $\forall x_1 \cdots \forall x_m \alpha$ is called the *universal closure* of α if the free variables of α are x_1, $\ldots$, x_m. We write $cl(\alpha)$ for the universal closure of α.

DEFINITION 5.2. Let $\mathcal{A} := \langle U, I^+, I^- \rangle$ be a model. The satisfaction relations $\mathcal{A} \models^+ \alpha$ and $\mathcal{A} \models^- \alpha$ for any closed formula α of $\mathcal{L}[\mathcal{A}]$ are defined inductively as follows:

1. $[\mathcal{A} \models^+ p(\underline{u}_1, \ldots, \underline{u}_n)$ iff $(u_1, \ldots, u_n) \in p^{I^+}]$ and $[\mathcal{A} \models^- p(\underline{u}_1, \ldots, \underline{u}_n)$ iff $(u_1, \ldots, u_n) \in p^{I^-}]$ for any n-ary atomic formula $p(\underline{u}_1, \ldots, \underline{u}_n)$,

2. $\mathcal{A} \models^+ \alpha \wedge \beta$ iff $\mathcal{A} \models^+ \alpha$ and $\mathcal{A} \models^+ \beta$,

3. $\mathcal{A} \models^+ \alpha \vee \beta$ iff $\mathcal{A} \models^+ \alpha$ or $\mathcal{A} \models^+ \beta$,

4. $\mathcal{A} \models^+ \alpha \rightarrow \beta$ iff not-$(\mathcal{A} \models^+ \alpha)$ or $\mathcal{A} \models^+ \beta$,

5. $\mathcal{A} \models^+ \alpha \leftarrow \beta$ iff $\mathcal{A} \models^+ \alpha$ and not-$(\mathcal{A} \models^+ \beta)$,

6. $\mathcal{A} \models^+ \neg\alpha$ iff not-$(\mathcal{A} \models^+ \alpha)$,

7. $\mathcal{A} \models^+ {\sim}\alpha$ iff $\mathcal{A} \models^- \alpha$,

8. $\mathcal{A} \models^+ \forall x\alpha$ iff $\mathcal{A} \models^+ \alpha[\underline{u}/x]$ for all $u \in U$,

9. $\mathcal{A} \models^+ \exists x\alpha$ iff $\mathcal{A} \models^+ \alpha[\underline{u}/x]$ for some $u \in U$,

10. $\mathcal{A} \models^- \alpha \wedge \beta$ iff $\mathcal{A} \models^- \alpha$ or $\mathcal{A} \models^- \beta$,

11. $\mathcal{A} \models^- \alpha \vee \beta$ iff $\mathcal{A} \models^- \alpha$ and $\mathcal{A} \models^- \beta$,

12. $\mathcal{A} \models^- \alpha \rightarrow \beta$ iff not-$(\mathcal{A} \models^- \alpha)$ and $\mathcal{A} \models^- \beta$,

13. $\mathcal{A} \models^- \alpha \leftarrow \beta$ iff $\mathcal{A} \models^- \alpha$ or not-$(\mathcal{A} \models^- \beta)$,

14. $\mathcal{A} \models^- \neg\alpha$ iff not-$(\mathcal{A} \models^- \alpha)$,

15. $\mathcal{A} \models^- \sim\alpha$ iff $\mathcal{A} \models^+ \alpha$,

16. $\mathcal{A} \models^- \forall x\alpha$ iff $\mathcal{A} \models^- \alpha[\underline{u}/x]$ for some $u \in U$,

17. $\mathcal{A} \models^- \exists x\alpha$ iff $\mathcal{A} \models^- \alpha[\underline{u}/x]$ for all $u \in U$.

The satisfaction relations $\mathcal{A} \models^+ \alpha$ and $\mathcal{A} \models^- \alpha$ for any formula α of $\mathcal{L}$ are defined by $(\mathcal{A} \models^+ \alpha$ iff $\mathcal{A} \models^+ cl(\alpha))$ and $(\mathcal{A} \models^- \alpha$ iff $\mathcal{A} \models^- cl(\alpha))$. A formula α of $\mathcal{L}$ is called *valid* if $\mathcal{A} \models^+ \alpha$ holds for any model $\mathcal{A}$. A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}$ is called valid if so is the formula $\Gamma_* \rightarrow \Delta^*$.

The intended meanings of the satisfaction relations $\models^+$ and $\models^-$ are *verification* (or *provability*, or *support of truth*) and *falsification* (or *refutability*, or *support of falsity*), respectively.

THEOREM 5.3 (Soundness for SPL). *For any sequent S, if* SPL $\vdash S$, *then S is valid.*

PROOF. By induction on the proof P of S. We distinguish the cases according to the last inference of P. We show only the following case.

(Case ($\sim \exists$right)): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim \alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim \exists x\alpha} \ (\sim \exists\text{right}).$$

We show that "$\Gamma \Rightarrow \Delta, \sim \alpha[z/x]$ is valid" implies "$\Gamma \Rightarrow \Delta, \sim \exists x\alpha$ is valid". By the hypothesis, (i): $\forall z_1 \cdots \forall z_n \forall z(\Gamma_* \rightarrow (\Delta^* \vee (\sim \alpha[z/x])))$ (where z_1, …, z_n are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim \exists x\alpha$) is valid. We show that $\mathcal{A} \models^+ \forall z_1 \cdots \forall z_n(\Gamma_* \rightarrow (\Delta^* \vee (\sim \exists x\alpha)))$ for any model $\mathcal{A} := \langle U, I^+, I^- \rangle$, i.e., we show that for any u_1, …, $u_n \in U$, $\mathcal{A} \models^+ \underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee (\sim \exists x\underline{\alpha}))$, where $\underline{\Gamma}_*$, $\underline{\Delta}^*$ and $\underline{\alpha}$ are respectively obtained from Γ_*, Δ^* and α by replacing z_1, …, z_n by $\underline{u}_1$, …, $\underline{u}_n$.[1] By (i), we have $\mathcal{A} \models^+ (\underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee (\sim \underline{\alpha}[z/x])))[\underline{w}/z]$ for any $w \in U$. By

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[1] We note that $(\sim \exists x\alpha)[\underline{w}/z_1, \ldots, \underline{u}_n/z_n]$ (the simultaneous substitution) is equivalent to $\sim \exists x(\alpha[\underline{u}_1/z_1, \ldots, \underline{u}_n/z_n])$, i.e., $\sim \exists x\underline{\alpha}$.

the eigenvariable condition, z is not occurring freely in $\underline{\Gamma}_*, \underline{\Delta}^*$ and $\underline{\alpha}$. Thus, $\underline{\Gamma}_*[\underline{w}/z]$ and $\underline{\Delta}^*[\underline{w}/z]$ are equivalent to $\underline{\Gamma}_*$ and $\underline{\Delta}^*$ respectively, and $\underline{\alpha}[z/x][\underline{w}/z]$ is equivalent to $\underline{\alpha}[\underline{w}/z][\underline{w}/x]$, i.e., $\underline{\alpha}[\underline{w}/x]$. Therefore, for any $w \in U$, we have that (a): $\mathcal{A} \models^+ \underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee \sim \underline{\alpha}[\underline{w}/x])$. Suppose that (b): [$\mathcal{A} \models^+ \underline{\Gamma}_*$ and not ($\mathcal{A} \models^+ \underline{\Delta}^*$)]. Then, by (a), we have that for any $w \in U$, $\mathcal{A} \models^+ \sim \underline{\alpha}[\underline{w}/x]$, i.e., $\mathcal{A} \models^- \underline{\alpha}[\underline{w}/x]$. Therefore, we obtain (c): $\mathcal{A} \models^- \exists x \underline{\alpha}$, and hence $\mathcal{A} \models^+ \sim \exists x \underline{\alpha}$. This means that (b) implies (c), i.e., $\mathcal{A} \models^+ \underline{\Gamma}_*$ implies ($\mathcal{A} \models^+ \underline{\Delta}^*$ or $\mathcal{A} \models^+ \sim \exists x \underline{\alpha}$). Therefore, we have the required fact that $\mathcal{A} \models^+ \underline{\Gamma}_* \rightarrow (\underline{\Delta}^* \vee (\sim \exists x \underline{\alpha}))$ for any $u_1, \ldots, u_n \in U$. $\dashv$

Now, we start to prove the completeness theorem.

DEFINITION 5.4. A sequent $\Gamma \Rightarrow \Delta$ is called *saturated* if for any formulas α and β,

(s1) $\alpha \wedge \beta \in \Gamma$ implies ($\alpha \in \Gamma$ and $\beta \in \Gamma$),

(s2) $\alpha \wedge \beta \in \Delta$ implies ($\alpha \in \Delta$ or $\beta \in \Delta$),

(s3) $\alpha \vee \beta \in \Gamma$ implies ($\alpha \in \Gamma$ or $\beta \in \Gamma$),

(s4) $\alpha \vee \beta \in \Delta$ implies ($\alpha \in \Delta$ and $\beta \in \Delta$),

(s5) $\alpha \rightarrow \beta \in \Gamma$ implies ($\alpha \in \Delta$ or $\beta \in \Gamma$),

(s6) $\alpha \rightarrow \beta \in \Delta$ implies ($\alpha \in \Gamma$ and $\beta \in \Delta$),

(s7) $\alpha \leftarrow \beta \in \Gamma$ implies ($\alpha \in \Gamma$ and $\beta \in \Delta$),

(s8) $\alpha \leftarrow \beta \in \Delta$ implies ($\alpha \in \Delta$ or $\beta \in \Gamma$),

(s9) $\neg \alpha \in \Gamma$ implies $\alpha \in \Delta$,

(s10) $\neg \alpha \in \Delta$ implies $\alpha \in \Gamma$,

(s11) $\forall x \alpha \in \Gamma$ implies ($\alpha[y/x] \in \Gamma$ for any individual variable y),

(s12) $\forall x \alpha \in \Delta$ implies ($\alpha[z/x] \in \Delta$ for some individual variable z),

(s13) $\exists x \alpha \in \Gamma$ implies ($\alpha[z/x] \in \Gamma$ for some individual variable z),

(s14) $\exists x \alpha \in \Delta$ implies ($\alpha[y/x] \in \Delta$ for any individual variable y),

(s15) $\sim \sim \alpha \in \Gamma$ implies $\alpha \in \Gamma$,

(s16) $\sim \sim \alpha \in \Delta$ implies $\alpha \in \Delta$,

(s17) $\sim (\alpha \wedge \beta) \in \Gamma$ implies ($\sim \alpha \in \Gamma$ or $\sim \beta \in \Gamma$),

(s18) $\sim(\alpha \wedge \beta) \in \Delta$ implies ($\sim \alpha \in \Delta$ and $\sim \beta \in \Delta$),

(s19) $\sim(\alpha \vee \beta) \in \Gamma$ implies ($\sim \alpha \in \Gamma$ and $\sim \beta \in \Gamma$),

(s20) $\sim(\alpha \vee \beta) \in \Delta$ implies ($\sim \alpha \in \Delta$ or $\sim \beta \in \Delta$),

(s21) $\sim(\alpha \to \beta) \in \Gamma$ implies ($\sim \alpha \in \Delta$ and $\sim \beta \in \Gamma$),

(s22) $\sim(\alpha \to \beta) \in \Delta$ implies ($\sim \alpha \in \Gamma$ or $\sim \beta \in \Delta$),

(s23) $\sim(\alpha \leftarrow \beta) \in \Gamma$ implies ($\sim \alpha \in \Gamma$ or $\sim \beta \in \Delta$),

(s24) $\sim(\alpha \leftarrow \beta) \in \Delta$ implies ($\sim \alpha \in \Delta$ and $\sim \beta \in \Gamma$),

(s25) $\sim \neg \alpha \in \Gamma$ implies $\sim \alpha \in \Delta$,

(s26) $\sim \neg \alpha \in \Delta$ implies $\sim \alpha \in \Gamma$,

(s27) $\sim \forall x \alpha \in \Gamma$ implies ($\sim \alpha[z/x] \in \Gamma$ for some individual variable z),

(s28) $\sim \forall x \alpha \in \Delta$ implies ($\sim \alpha[y/x] \in \Delta$ for any individual variable y),

(s29) $\sim \exists x \alpha \in \Gamma$ implies ($\sim \alpha[y/x] \in \Gamma$ for any individual variable y),

(s30) $\sim \exists x \alpha \in \Delta$ implies ($\sim \alpha[z/x] \in \Delta$ for some individual variable z).

DEFINITION 5.5. An expression $\Gamma \Rightarrow \Delta$ is called an *infinite sequent* if Γ or Δ are infinite (countable) sets of formulas. An infinite sequent $\Gamma \Rightarrow \Delta$ is called provable if a sequent $\Gamma' \Rightarrow \Delta'$ is provable, where Γ' and Δ' are finite subsets of Γ and Δ respectively.

DEFINITION 5.6. A *decomposition* of a sequent (or infinite sequent) S is defined as being of the form S' or $S'; S''$ by

(1a) $\Gamma \Rightarrow \Delta, \alpha \wedge \beta, \alpha$; $\Gamma \Rightarrow \Delta, \alpha \wedge \beta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \wedge \beta$,

(1b) $\alpha, \beta, \alpha \wedge \beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha \wedge \beta, \Gamma \Rightarrow \Delta$,

(2a) $\Gamma \Rightarrow \Delta, \alpha \vee \beta, \alpha, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \vee \beta$,

(2b) $\alpha, \alpha \vee \beta, \Gamma \Rightarrow \Delta$; $\beta, \alpha \vee \beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha \vee \beta, \Gamma \Rightarrow \Delta$,

(3a) $\alpha, \Gamma \Rightarrow \Delta, \alpha \to \beta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \to \beta$,

(3b) $\alpha \to \beta, \Gamma \Rightarrow \Delta, \alpha$; $\beta, \alpha \to \beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha \to \beta, \Gamma \Rightarrow \Delta$,

(4a) $\Gamma \Rightarrow \Delta, \alpha \leftarrow \beta, \alpha$; $\beta, \Gamma \Rightarrow \Delta, \alpha \leftarrow \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha \leftarrow \beta$,

(4b) $\alpha, \alpha \leftarrow \beta, \Gamma \Rightarrow \Delta, \beta$ is a decomposition of $\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta$,

(5a) $\alpha, \Gamma \Rightarrow \Delta, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\alpha$,

(5b) $\neg\alpha, \Gamma \Rightarrow \Delta, \alpha$ is a decomposition of $\neg\alpha, \Gamma \Rightarrow \Delta$,

(6a) $\Gamma \Rightarrow \Delta, \forall x\alpha, \alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \forall x\alpha$, where z is a *fresh* free individual variable, i.e., z is not occurring in $\Gamma \Rightarrow \Delta, \forall x\alpha$,

(6b) $\alpha[y_1/x], \ldots, \alpha[y_m/x], \forall x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\forall x\alpha, \Gamma \Rightarrow \Delta$, where y_1, $\ldots$, y_m are the free individual variables occurring in $\forall x\alpha, \Gamma \Rightarrow \Delta$,[2]

(7a) $\Gamma \Rightarrow \Delta, \exists x\alpha, \alpha[y_1/x], \ldots, \alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \exists x\alpha$ where y_1, $\ldots$, y_m are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \exists x\alpha$,

(7b) $\alpha[z/x], \exists x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\exists x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,

(8a) $\alpha, \sim\sim\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim\sim\alpha, \Gamma \Rightarrow \Delta$,

(8b) $\Gamma \Rightarrow \Delta, \sim\sim\alpha, \alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim\sim\alpha$,

(9a) $\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta), \sim\alpha, \sim\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta)$,

(9b) $\sim\alpha, \sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$; $\sim\beta, \sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta$,

(10a) $\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta), \sim\alpha$; $\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta), \sim\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)$,

(10b) $\sim\alpha, \sim\beta, \sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta$,

(11a) $\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta), \sim\beta$; $\sim\alpha, \Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \rightarrow \beta)$,

(11b) $\sim\beta, \sim(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta, \sim\alpha$ is a decomposition of $\sim(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta$,

(12a) $\sim\beta, \Gamma \Rightarrow \Delta, \sim(\alpha \leftarrow \beta), \sim\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim(\alpha \leftarrow \beta)$,

(12b) $\sim(\alpha \leftarrow \beta), \Gamma \Rightarrow \Delta, \sim\beta$; $\sim\alpha, \sim(\alpha \leftarrow \beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim(\alpha \leftarrow \beta), \Gamma \Rightarrow \Delta$,

(13a) $\sim\alpha, \Gamma \Rightarrow \Delta, \sim\neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim\neg\alpha$,

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[2]If $\forall x\alpha, \Gamma \Rightarrow \Delta$ has no free individual variable, then we replace x in α by any variable from $\mathcal{L}$. Such a condition is also adopted in (7a), (14a) and (15b).

(13b) $\sim \neg \alpha, \Gamma \Rightarrow \Delta, \sim \alpha$ is a decomposition of $\sim \neg \alpha, \Gamma \Rightarrow \Delta$,

(14a) $\Gamma \Rightarrow \Delta, \sim \forall x \alpha, \sim \alpha[y_1/x], \ldots, \sim \alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim \forall x \alpha$, where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim \forall x \alpha$,

(14b) $\sim \alpha[z/x], \sim \forall x \alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim \forall x \alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,

(15a) $\Gamma \Rightarrow \Delta, \sim \exists x \alpha, \sim \alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim \exists x \alpha$ where z is a fresh free individual variable,

(15b) $\sim \alpha[y_1/x], \ldots, \sim \alpha[y_m/x], \sim \exists x \alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim \exists x \alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\sim \exists x \alpha, \Gamma \Rightarrow \Delta$.

DEFINITION 5.7. A *decomposition tree of S* is a tree which is the result of some repeated decomposition of S.

In other words, a decomposition tree corresponds to a bottom up proof search tree of SPL$-$(cut). In every decomposition of S (i.e., S' or $S'; S''$), if S is unprovable in SPL$-$(cut), then so is S' or S''.

LEMMA 5.8. *Let $\Gamma \Rightarrow \Delta$ be a given unprovable sequent in SPL$-$(cut). There exists an unprovable, saturated (infinite) sequent $\Gamma^\omega \Rightarrow \Delta^\omega$ such that $\Gamma \subseteq \Gamma^\omega$ and $\Delta \subseteq \Delta^\omega$.*

PROOF. Let $\Gamma \Rightarrow \Delta$ be an unprovable sequent in SPL$-$(cut). We construct $\Gamma^\omega \Rightarrow \Delta^\omega$ from $\Gamma \Rightarrow \Delta$ as follows.

(1) We apply the decomposition procedure from Definition 5.6 to $\Gamma \Rightarrow \Delta$, in the following order, skipping the decomposition procedures which are not applicable to the formulas in $\Gamma \Rightarrow \Delta$.

$$(1a) \longrightarrow (1b) \longrightarrow (2a) \longrightarrow \cdots \longrightarrow (15b).$$

In such a decomposition process, one of the decomposed elements S' and S'' of S is an unprovable sequent.

(2) We repeat the same procedure (1), infinitely often. Then, we obtain an infinite decomposition tree with finitely many branches.

(3) By König's Lemma, we have an infinite path of this decomposition tree as follows.

$$\Gamma_0 \Rightarrow \Delta_0 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \infty,$$

where $\Gamma_0 \Rightarrow \Delta_0$ is $\Gamma \Rightarrow \Delta$. In this sequence of sequents on the infinite path, we have that $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \cdots$.

(4) We put $\Gamma^\omega := \bigcup_{i=0}^\infty \Gamma_i$ and $\Delta^\omega := \bigcup_{i=0}^\infty \Delta_i$.[3]

Then, we have that $\Gamma \subseteq \Gamma^\omega$ and $\Delta \subseteq \Delta^\omega$, and can verify that $\Gamma^\omega \Rightarrow \Delta^\omega$ is an unprovable, saturated sequent.   ⊣

LEMMA 5.9. *Let $\Gamma \Rightarrow \Delta$ be an unprovable sequent in SPL−(cut), and $\Gamma^\omega \Rightarrow \Delta^\omega$ be an unprovable, saturated sequent constructed from $\Gamma \Rightarrow \Delta$ by Lemma 5.8. We define a canonical model $\mathcal{A} := \langle U, I^+, I^- \rangle$ as follows:*

$$U := \{z \mid z \text{ is a free individual variable occurring in } \Gamma^\omega \Rightarrow \Delta^\omega\},$$

$$p^{I^+} := \{(z_1, \ldots, z_m) \mid p(z_1, \ldots, z_m) \in \Gamma^\omega\},$$

$$p^{I^-} := \{(z_1, \ldots, z_m) \mid \sim p(z_1, \ldots, z_m) \in \Gamma^\omega\}.$$

Then, for any formula α,

(1) $[(\alpha \in \Gamma^\omega \text{ implies } \mathcal{A} \models^+ \underline{\alpha}) \text{ and } (\alpha \in \Delta^\omega \text{ implies not-}(\mathcal{A} \models^+ \underline{\alpha}))]$,

(2) $[(\sim \alpha \in \Gamma^\omega \text{ implies } \mathcal{A} \models^- \underline{\alpha}) \text{ and } (\sim \alpha \in \Delta^\omega \text{ implies not-}(\mathcal{A} \models^- \underline{\alpha}))]$

where $\underline{\alpha}$ is obtained from α by replacing every individual variable x occurring in α by the name $\underline{x}$.

PROOF. By (simultaneous) induction on the complexity of α.

- Base step: Obvious by the definitions of I^+ and I^-.
- Induction step for (1): We show some cases.

(Case $\alpha \equiv \beta \leftarrow \gamma$): First, we show that $\beta \leftarrow \gamma \in \Gamma^\omega$ implies $\mathcal{A} \models^+ \underline{\beta} \leftarrow \underline{\gamma}$. Suppose $\beta \leftarrow \gamma \in \Gamma^\omega$. Then, we obtain $[\beta \in \Gamma^\omega \text{ and } \gamma \in \Delta^\omega]$ by Definition 5.4 (s7). By the induction hypothesis for (1), we obtain $[\mathcal{A} \models^+ \underline{\beta} \text{ and not-}(\mathcal{A} \models^+ \underline{\gamma})]$. This means $\mathcal{A} \models^+ \underline{\beta} \leftarrow \underline{\gamma}$. Second, we show that $\beta \leftarrow \gamma \in \Delta^\omega$ implies not-$(\mathcal{A} \models^+ \underline{\beta} \leftarrow \underline{\gamma})$. Suppose $\beta \leftarrow \gamma \in \Delta^\omega$. Then, we obtain $[\beta \in \Delta^\omega \text{ or } \gamma \in \Gamma^\omega]$ by Definition 5.4 (s8). By the induction hypothesis for (1), we obtain $[\text{not-}(\mathcal{A} \models^+ \underline{\beta}) \text{ or } \mathcal{A} \models^+ \underline{\gamma}]$. This means not-$(\mathcal{A} \models^+ \underline{\beta} \leftarrow \underline{\gamma})$.

(Case $\alpha \equiv \sim \beta$): First, we show that $\sim \beta \in \Gamma^\omega$ implies $\mathcal{A} \models^+ \sim \underline{\beta}$. Suppose $\sim \beta \in \Gamma^\omega$. Then we obtain $\mathcal{A} \models^- \underline{\beta}$ by the induction hypothesis for (2). Thus, we have $\mathcal{A} \models^+ \sim \underline{\beta}$. Second, we show that $\sim \beta \in \Delta^\omega$

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[3]We note that $\Gamma^\omega \cap \Delta^\omega = \emptyset$.

implies not-($\mathcal{A} \models^+ \sim \underline{\beta}$). Suppose $\sim \beta \in \Delta^\omega$. Then, we obtain not-($\mathcal{A} \models^- \underline{\beta}$) by the induction hypothesis for (2). Thus, we have not-($\mathcal{A} \models^+ \sim \underline{\beta}$).

• Induction step for (2): We show some cases.

(Case $\alpha \equiv \beta \to \gamma$): First, we show that $\sim(\beta \to \gamma) \in \Gamma^\omega$ implies $\mathcal{A} \models^- \underline{\beta \to \gamma}$. Suppose $\sim(\beta \to \gamma) \in \Gamma^\omega$. Then, we obtain [$\sim \beta \in \Delta^\omega$ and $\sim \gamma \in \Gamma^\omega$] by Definition 5.4 (s21). By the induction hypothesis for (2), we obtain [not-($\mathcal{A} \models^- \underline{\beta}$) and $\mathcal{A} \models^- \underline{\gamma}$]. This means $\mathcal{A} \models^- \underline{\beta \to \gamma}$. Second, we show that $\sim(\beta \to \gamma) \in \Delta^\omega$ implies not-($\mathcal{A} \models^- \underline{\beta \to \gamma}$). Suppose $\sim(\beta \to \gamma) \in \Delta^\omega$. Then, we obtain [$\sim \beta \in \Gamma^\omega$ or $\sim \gamma \in \Delta^\omega$] by Definition 5.4 (s22). By the induction hypothesis for (2), we obtain [$\mathcal{A} \models^- \underline{\beta}$ or not-($\mathcal{A} \models^- \underline{\gamma}$)]. This means not-($\mathcal{A} \models^- \underline{\beta \to \gamma}$).

(Case $\alpha \equiv \beta \leftarrow \gamma$): First, we show that $\sim(\beta \leftarrow \gamma) \in \Gamma^\omega$ implies $\mathcal{A} \models^- \underline{\beta \leftarrow \gamma}$. Suppose $\sim(\beta \leftarrow \gamma) \in \Gamma^\omega$. Then, we obtain [$\sim \beta \in \Gamma^\omega$ or $\sim \gamma \in \Delta^\omega$] by Definition 5.4 (s23). By the induction hypothesis for (2), we obtain [$\mathcal{A} \models^- \underline{\beta}$ or not-($\mathcal{A} \models^- \underline{\gamma}$)]. This means $\mathcal{A} \models^- \underline{\beta \leftarrow \gamma}$. Second, we show that $\sim(\beta \leftarrow \gamma) \in \Delta^\omega$ implies not-($\mathcal{A} \models^- \underline{\beta \leftarrow \gamma}$). Suppose $\sim(\beta \leftarrow \gamma) \in \Delta^\omega$. Then, we obtain [$\sim \beta \in \Delta^\omega$ and $\sim \gamma \in \Gamma^\omega$] by Definition 5.4 (s24). By the induction hypothesis for (2), we obtain [not-($\mathcal{A} \models^- \underline{\beta}$) and $\mathcal{A} \models^- \underline{\gamma}$]. This means not-($\mathcal{A} \models^- \underline{\beta \leftarrow \gamma}$).

(Case $\alpha \equiv \sim \beta$): First, we show that $\sim \sim \beta \in \Gamma^\omega$ implies $\mathcal{A} \models^- \sim \underline{\beta}$. Suppose $\sim \sim \beta \in \Gamma^\omega$. Then, we obtain $\beta \in \Gamma^\omega$ by Definition 5.4 (s15). By the induction hypothesis for (1) and $\beta \in \Gamma^\omega$, we obtain $\mathcal{A} \models^+ \underline{\beta}$, and hence $\mathcal{A} \models^- \sim \underline{\beta}$. Second, we show that $\sim \sim \beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^- \sim \underline{\beta}$). Suppose $\sim \sim \beta \in \Delta^\omega$. Then, we obtain $\beta \in \Delta^\omega$ by Definition 5.4 (s16). By the induction hypothesis for (1) and $\beta \in \Delta^\omega$, we obtain not-($\mathcal{A} \models^+ \underline{\beta}$) and hence not-($\mathcal{A} \models^- \sim \underline{\beta}$).

(Case $\alpha \equiv \forall x \beta$): First, we show that $\sim \forall x \beta \in \Gamma^\omega$ implies $\mathcal{A} \models^- \forall x \underline{\beta}$. Suppose $\sim \forall x \beta \in \Gamma^\omega$. Then, we obtain $\sim \beta[z/x] \in \Gamma^\omega$ for some $z \in \overline{U}$, by Definition 5.4 (s27). By the induction hypothesis for (2), we obtain that $\mathcal{A} \models^- \underline{\beta}[\underline{z}/x]$ for some $z \in U$. This means $\mathcal{A} \models^- \forall x \underline{\beta}$. Second, we show that $\sim \forall x \beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^- \forall x \underline{\beta}$). Suppose $\sim \forall x \beta \in \Delta^\omega$. Then, we obtain [$\sim \beta[y_i/x] \in \Delta^\omega$ for all $y_i \in \overline{U}$] by Definition 5.4 (s28). By the induction hypothesis for (2), we obtain not-($\mathcal{A} \models^- \underline{\beta}[\underline{y_i}/x]$) for all $y_i \in U$. This means not-($\mathcal{A} \models^- \forall x \underline{\beta}$).

(Case $\alpha \equiv \exists x \beta$): First, we show that $\sim \exists x \beta \in \Gamma^\omega$ implies $\mathcal{A} \models^- \exists x \underline{\beta}$. Suppose $\sim \exists x \beta \in \Gamma^\omega$. Then we obtain [$\sim \beta[y_i/x] \in \Gamma^\omega$ for all $y_i \in \overline{U}$] by Definition 5.4 (s29). By the induction hypothesis, we obtain that

$\mathcal{A} \models^- \underline{\beta}[\underline{y_i}/x]$ for all $y_i \in U$. This means $\mathcal{A} \models^- \exists x \underline{\beta}$. Second, we show that $\sim \exists x \beta \in \Delta^\omega$ implies not-$(\mathcal{A} \models^- \exists x \underline{\beta})$. Suppose $\sim \exists x \beta \in \Delta^\omega$. Then, we obtain $[\sim \beta[z/x] \in \Delta^\omega$ for some $z \in U]$ by Definition 5.4 (s30). By the induction hypothesis for (2), we obtain not-$(\mathcal{A} \models^- \underline{\beta}[\underline{z}/x])$ for some $z \in U$. This means not-$(\mathcal{A} \models^- \exists x \underline{\beta})$. $\dashv$

THEOREM 5.10 (Strong completeness for SPL). *For any sequent S, if S is valid, then* $\mathrm{SPL} - (\mathrm{cut}) \vdash S$.

PROOF. We prove the following: if $\Gamma \Rightarrow \Delta$ is unprovable in SPL$-$(cut), then there exists a model $\mathcal{A}$ such that $\Gamma \Rightarrow \Delta$ is not valid in $\mathcal{A}$. Suppose that $\Gamma \Rightarrow \Delta$ is not provable in SPL$-$(cut). Then, by Lemma 5.9, we can construct a canonical model $\mathcal{A}$ with the condition (1) in this lemma. Thus, we have $\mathcal{A} \models^+ \underline{\gamma}$ and not $(\mathcal{A} \models^+ \underline{\delta})$ for any $\gamma \in \Gamma \subseteq \Gamma^\omega$ and any $\delta \in \Delta \subseteq \Delta^\omega$. Hence, we obtain "not-$(\mathcal{A} \models^+ \underline{\Gamma}_* \to \underline{\Delta}^*)$", and hence "not-$(\mathcal{A} \models^+ cl(\Gamma_* \to \Delta^*))$". Therefore, $\Gamma \Rightarrow \Delta$ is not valid in $\mathcal{A}$. $\dashv$

Combining Theorem 5.10 and Theorem 5.3, we can obtain an alternative (semantical) proof of the cut-elimination theorem for SPL.

DEFINITION 5.11. The semantics of DPL is obtained from that of SPL by replacing the conditions 12 and 13 in Definition 5.2 by

12′. $\mathcal{A} \models^- \alpha \to \beta$ iff $\mathcal{A} \models^+ \alpha$ and not-$(\mathcal{A} \models^+ \beta)$,

13′. $\mathcal{A} \models^- \alpha \leftarrow \beta$ iff not-$(\mathcal{A} \models^+ \alpha)$ or $\mathcal{A} \models^+ \beta$.

The definition of the validity of formulas and sequents in DPL is analogous to the definition of these notions in SPL. In the case of DLP we use the term "*d-validity*", in order to distinguish validity in DPL from validity in SPL.

THEOREM 5.12 (Soundness for DPL). *For any sequent S, if* $\mathrm{DPL} \vdash S$, *then S is d-valid.*

THEOREM 5.13 (Strong completeness for DPL). *For any sequent S, if S is d-valid, then* $\mathrm{DPL}-(\mathrm{cut}) \vdash S$.

References

[1] O. Arieli, and A. Avron, "Reasoning with logical bilattices", *Journal of Logic, Language and Information* 5 (1996): 25–63.

[2] N. D. Belnap, "A useful four-valued logic", pp. 7–37 in: *Modern Uses of Multiple-Valued Logic*, G. Epstein and J. M. Dunn (eds.), Dordrecht: Reidel, 1977.

[3] L. Buisman and R. Goré, "A cut-free sequent calculus for bi-intuitionistic logic", pp. 90–106 in: *Proceedings of TABLEAUX 2007*, Lecture Notes in Artificial Intelligence 4545, 2007.

[4] N. C. A. da Costa, J.-Y. Béziau, and O. A. S. Bueno, "Aspects of paraconsistent logic", *Bulletin of the IGPL* 3, 4 (1995): 597–614.

[5] T. Crolard, "Subtractive logic", *Theoretical Computer Science* 254 (2001): 151–185.

[6] Y. Gurevich, "Intuitionistic logic with strong negation", *Studia Logica* 36 (1977): 49–59.

[7] P. Łukowski, "A deductive-reductive form of logic: general theory and intuitionistic case", *Logic and Logical Philosophy* 10 (2002): 59–78.

[8] D. Nelson, "Constructible falsity", *Journal of Symbolic Logic* 14 (1949): 16–26.

[9] D. Nelson, "Negation and separation of concepts in constructive systems", pp. 208-225 in: A. Heyting (ed.), *Constructivity in Mathematics*, North-Holland, Amsterdam, 1959.

[10] G. Priest, "Paraconsistent Logic", pp. 287–393 in: *Handbook of Philosophical Logic* (Second Edition), Vol. 6, D. Gabbay and F. Guenthner (eds.), Kluwer Academic Publishers, Dordrecht, 2002.

[11] C. Rauszer, "A formalization of the propositional calculus of H-B logic", *Studia Logica* 33 (1974): 23–34.

[12] W. Rautenberg, *Klassische und nicht-klassische Aussagenlogik*, Vieweg, Braunschweig, 1979.

[13] G. Takeuti, *Proof Theory*, North-Holland Publishing Company, 1975.

[14] N. N. Vorob'ev, "A constructive propositional calculus with strong nega-
tion" (in Russian), *Doklady Akademii Nauk SSR* 85 (1952): 465–468.

[15] H. Wansing, *The logic of information structures*, Lecture Notes in Artificial
Intelligence 681, 163 pages, 1993.

[16] H. Wansing, "Constructive negation, implication, and co-implication",
Journal of Applied Non-Classical Logics 18 (2008): 341–364.

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