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SIMPLIFIED KRIPKE STYLE SEMANTICS FOR SOME VERY WEAK MODAL LOGICS

Abstract. In the present paper¹ we examine very weak modal logics **C1**, **D1**, **E1**, **S0.5^o**, **S0.5^o+**(D)****, **S0.5** and some of their versions which are closed under replacement of tautological equivalents (rte-versions). We give semantics for these logics, formulated by means of Kripke style models of the form $\langle w, A, V \rangle$, where w is a «distinguished» world, A is a set of *worlds* which are alternatives to w , and V is a valuation which for formulae and all worlds, V preserves classical conditions for truth-value operators; (ii) for the world w and any formula φ , $V(\Box\varphi, w) = 1$ iff $\forall x \in A V(\varphi, x) = 1$; (iii) for other worlds formula $\lceil \Box\varphi \rceil$ has an arbitrary value. Moreover, for rte-versions of considered logics we must add the following condition: (iv) $V(\Box\chi, w) = V(\Box\chi[\varphi/\psi], w)$, if φ and ψ are tautological equivalent. Finally, for **C1**, **D1** and **E1** we must add *queer* models of the form $\langle w, V \rangle$ in which: (i) holds and (ii') $V(\Box\varphi, w) = 0$, for any formula φ . We prove that considered logics are determined by some classes of above models.

Keywords: Simplified Kripke style semantics, very weak modal logics.

1. Preliminaries. Some historical notes

Modal formulae are formed in the standard way from the set At of propositional letters: ' p ', ' q ', ' p_0 ', ' p_1 ', ' p_2 ', ...; truth-value operators: ' \neg ',

¹This article is the final version of a draft paper [14], mentioned in the references of the papers [13] and [15].

‘ \vee ’, ‘ \wedge ’, ‘ \supset ’, and ‘ \equiv ’ (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively); the modal operator ‘ \Box ’ (necessity; the possibility sign ‘ \Diamond ’ is the abbreviation of ‘ $\neg\Box\neg$ ’); and brackets. Let For be the set of all modal formulae. For any set Γ of formulae we put $\Box\Gamma := \{\ulcorner\Box\varphi\urcorner : \varphi \in \Gamma\}$.

Let Taut be the set of all classical tautologies (without the modal operator) and—as in [3, 4]—let PL be the set of modal formulae which are instances of classical tautologies.

Let Σ be a set of modal formulae. Also as in [3], Σ is a *modal system* iff $\text{PL} \subseteq \Sigma$ and Σ is closed under the following rule of detachment for ‘ \supset ’ (*modus ponens*), i.e., for any formulae φ and ψ :

$$\text{if } \varphi \text{ and } \ulcorner\varphi \supset \psi\urcorner \text{ are members of } \Sigma, \text{ so is } \psi. \quad (\text{MP})$$

We say that a modal system is *congruential* iff it is closed under the following rule of congruence:

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \Sigma, \text{ then } \ulcorner\Box\varphi \equiv \Box\psi\urcorner \in \Sigma. \quad (\text{RE})$$

Notice that a modal system Σ is congruential iff it is closed under replacement

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi^{[\varphi/\psi]} \in \Sigma, \quad (\text{RRE})$$

or equivalently

$$\text{if } \ulcorner\varphi \equiv \psi\urcorner \in \Sigma, \text{ then } \ulcorner\chi^{[\varphi/\psi]} \equiv \chi\urcorner \in \Sigma, \quad (\text{RRE}')$$

where $\chi^{[\varphi/\psi]}$ is any formula that results from χ by replacing one or more occurrences of φ , in χ , by ψ .

A modal system Σ is called *regular* iff it is closed under the following regularity rule:

$$\text{if } \ulcorner(\varphi \wedge \psi) \supset \chi\urcorner \in \Sigma, \text{ then } \ulcorner(\Box\varphi \wedge \Box\psi) \supset \Box\chi\urcorner \in \Sigma. \quad (\text{RR})$$

A modal system Σ is regular iff it contains all instances of

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \quad (\text{K})$$

and is closed under the following *monotonic* rule

$$\text{if } \ulcorner\varphi \supset \psi\urcorner \in \Sigma \text{ then } \ulcorner\Box\varphi \supset \Box\psi\urcorner \in \Sigma, \quad (\text{RM})$$

iff it is closed under (RM) and contains all instances of

$$(\Box p \wedge \Box q) \supset \Box(p \wedge q) \quad (\text{C})$$

iff it is closed under (RE) and contains all instances of

$$\Box(p \wedge q) \equiv (\Box p \wedge \Box q) \quad (\text{R})$$

We say that a modal system Σ is *normal* iff it contains all instances of (K) and is closed under the following rule:

$$\text{if } \varphi \in \Sigma, \text{ then } \ulcorner \Box \varphi \urcorner \in \Sigma. \quad (\text{RN})$$

A modal system Σ is *normal* iff it is regular and contains the following formula

$$\Box(p \supset p) \quad (\text{N})$$

iff it contains (N) and all instances of (K), and is closed under (RE).

A set Σ of modal formulae is a *logic* iff Σ is a modal system and is closed under the following rule of uniform substitution:

$$\text{if } \varphi \in \Sigma \text{ then } s\varphi \in \Sigma, \quad (\text{US})$$

where $s\varphi$ is the result of uniform substitution of formulae for propositional letters in φ . Of course, the set PL is the smallest modal system and it is a logic.

In [9] Lemmon set out the logic **S0.5** and two groups of non-normal modal logics called the “D” and “E” systems.

Firstly, the logic **S0.5** is the smallest modal logic which includes $\Box\text{Taut}$, and contains (K) and the following formula:

$$\Box p \supset p \quad (\text{T})$$

The logic **S0.5**^o is associated with Lemmon’s **S0.5** (for these logics see e.g. [4, 9, 16]). **S0.5**^o is the smallest logic which includes $\Box\text{Taut}$ and contains (K). Thus, **S0.5** is **S0.5**^o plus (T). Of course, by (US), **S0.5** and **S0.5**^o include the set $\Box\text{PL}$, and **S0.5**^o \subsetneq **S0.5** (see Fact 4.1).

Secondly, Lemmon “consider a series of Lewis modal systems E1, E2, E3, E4, and E5, which are intended as possible epistemic counterparts to the five systems S0.5, S2, S3, S4, and S5. A distinguishing mark of E-systems is that in none of them is there *any* thesis of the form $L\alpha$ ” [9, p. 181–182] (in our text $L\alpha := \Box\varphi$). All E-systems—just like all S-systems—are logics that contain (K) and (T), include the set Taut, and are closed under the rules: (MP) and (US) (so they include the set PL). Moreover, the logics **E2–E5** are regular. For example, **E2** is the smallest regular modal logic which contains (T). **E3** is the smallest modal logic

which is closed under the rule **RM** and contains **(T)** and the following formula:

$$\Box(p \supset q) \supset \Box(\Box p \supset \Box q) \quad (\text{sK})$$

Thus, by **PL**, **(sK)** and **(T)**, the logic **E3** contains **(K)**. So it is regular.

The logic **E1** is closed neither under **(RM)** nor under **(RR)**. It is the smallest logic which contains **(K)** and **(T)**, and includes the following set of formulae:

$$M_{\text{Taut}} := \{\ulcorner \Box \varphi \supset \Box \psi \urcorner : \ulcorner \varphi \supset \psi \urcorner \in \text{Taut}\}.$$

Thus, **E1** also includes the following sets of formulae.

$$\begin{aligned} M_{\text{PL}} &:= \{\ulcorner \Box \varphi \supset \Box \psi \urcorner : \ulcorner \varphi \supset \psi \urcorner \in \text{PL}\}, \\ R_{\text{PL}} &:= \{\ulcorner (\Box \varphi \wedge \Box \psi) \supset \Box \chi \urcorner : \ulcorner (\varphi \wedge \psi) \supset \chi \urcorner \in \text{PL}\}, \\ E_{\text{PL}} &:= \{\ulcorner \Box \varphi \equiv \Box \psi \urcorner : \ulcorner \varphi \equiv \psi \urcorner \in \text{PL}\}. \end{aligned}$$

We have **E1** \subsetneq **S0.5** (see Fact 4.1).

Thirdly, the five D-logics, **D1**, **D2**, **D3**, **D4** and **D5**, were associated with the five E-logics. “The distinguishing feature of D-systems is that axiom **(T)** of the corresponding E-systems is weakened to **(D)**” [9, p. 184]

$$\Box p \supset \neg \Box \neg p \quad (\text{D})$$

Precisely, **D1** is the smallest logic which contains **(K)** and **(D)**, and includes the set M_{Taut} . Thus, the logic **D1** also includes the sets M_{PL} , R_{PL} and E_{PL} . We have **D1** \subsetneq **E1** (see Fact 4.1). The logics **D2–D5** are regular, e.g. **D2** is the smallest regular modal logic which contains **(D)**. We have **D2** \subsetneq **E2**.

In [10] the logic **C2** is examined. It is **E2** without **(T)** and **(D)**. Precisely, **C2** is the smallest regular logic. We have **C2** \subsetneq **D2**.

By analogy to **C2**, in [16] by ‘**C1**’ Routley denoted the system **E1** without **(T)** and **(D)**, i.e., **C1** is the smallest modal logic which contains **(K)** and includes the set M_{Taut} . So **C1** includes M_{PL} , R_{PL} and E_{PL} . We have **C1** \subsetneq **D1** and **C1** \subsetneq **S0.5**^o (see Fact 4.1).

As in [2, 4], we say that a modal system Σ is *closed under replacement of tautological equivalents* iff for all $\varphi, \psi, \chi \in \text{For}$:

$$\text{if } \ulcorner \varphi \equiv \psi \urcorner \in \text{PL} \text{ and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma. \quad (\text{rte})$$

or equivalently

$$\text{if } \ulcorner \varphi \equiv \psi \urcorner \in \text{PL}, \text{ then } \chi \in \Sigma \text{ iff } \chi[\varphi/\psi] \in \Sigma. \quad (\text{rte}')$$

Thus, by PL, a modal system is closed under (rte) iff it includes the following set of formulae:

$$\text{RE}_{\text{PL}} := \{\ulcorner \chi[\varphi/\psi] \equiv \chi^\top : \ulcorner \varphi \equiv \psi^\top \in \text{PL} \}.$$

In [2] a modal logic is called *classical modal* iff it contains (K) and (N), and is closed under (rte).² Notice that

LEMMA 1.1. *If Σ is closed under (rte) and (N) $\in \Sigma$, then $\Box\text{PL} \subseteq \Sigma$.*

PROOF. For any $\tau \in \text{PL}$ we have that $\ulcorner (p \supset p) \equiv \tau^\top \in \text{PL}$. Hence $\ulcorner \Box \tau^\top \in \Sigma$, by (rte) for $\chi := (\text{N})$, $\varphi := 'p \supset p'$ and $\psi := \tau'$. \dashv

The non-congruential logics **S0.9**^o, **S0.9**, **S1**^o, **S1**, **S2**^o, **S2**, **S3** and **S3.5** are examples of “classical modal logics” in the sense of [2]. For details concerning these logics see [4, 9] and Appendix A.

2. Some very weak systems

2.1. Very weak t-regular systems

Any modal system which includes the set R_{PL} we will call *t-regular*. Thus, the set R_{PL} replaces the rule (RR) in the formulation of regular systems. Of course, if Σ is a t-regular system and Σ' is a modal system such that $\Sigma \subseteq \Sigma'$, then Σ' is also a t-regular.

LEMMA 2.1. *All t-regular systems include the sets M_{PL} and E_{PL} .*

PROOF. If $\ulcorner \varphi \supset \psi^\top \in \text{PL}$, then also $\ulcorner (\varphi \wedge \varphi) \supset \psi^\top \in \text{PL}$. So we use R_{PL} and PL. Moreover, If $\ulcorner \varphi \equiv \psi^\top \in \text{PL}$, then also $\ulcorner \varphi \supset \psi^\top \in \text{PL}$ and $\ulcorner \psi \supset \varphi^\top \in \text{PL}$. So we use M_{PL} and PL. \dashv

LEMMA 2.2. *All instances of (K), (C), (R) and*

$$(\Box(p \supset q) \wedge \Box(q \supset r)) \supset \Box(p \supset r) \tag{X}$$

are members of all t-regular systems.

PROOF. Since $\ulcorner ((\varphi \supset \psi) \wedge \varphi) \supset \psi^\top$, $\ulcorner (\varphi \wedge \psi) \supset \varphi^\top$, $\ulcorner (\varphi \wedge \psi) \supset \psi^\top$, $\ulcorner (\varphi \wedge \psi) \supset (\varphi \wedge \psi)^\top$ and $\ulcorner ((\varphi \supset \psi) \wedge (\psi \supset \chi)) \supset (\varphi \supset \chi)^\top$ belong to PL and all t-regular systems include R_{PL} and M_{PL} . \dashv

²In [3, 4] the expression ‘classical modal’ was referred to ‘congruential’.

LEMMA 2.3. *For any system Σ the following conditions are equivalent:*

- (a) Σ is *t*-regular,
- (b) Σ contains all instances of **(K)** and includes the set M_{PL} ,
- (c) Σ contains all instances of **(C)** and includes the set M_{PL} ,
- (d) Σ contains all instances of **(X)** and includes the set M_{PL} .

PROOF. “(a) \Rightarrow (b)”, “(a) \Rightarrow (c)”, “(a) \Rightarrow (d)” By lemmas 2.1 and 2.2.

“(c) \Rightarrow (a)” If $\lceil \varphi \wedge \psi \rceil \supset \chi^\top \in PL$, then $\lceil \Box(\varphi \wedge \psi) \rceil \supset \Box\chi^\top \in \Sigma$, since $M_{PL} \subseteq \Sigma$. Hence $\lceil (\Box\varphi \wedge \Box\psi) \rceil \supset \Box\chi^\top \in \Sigma$, by **(C)** and PL.

“(b) \Rightarrow (a)” If $\lceil \varphi \wedge \psi \rceil \supset \chi^\top \in PL$, then $\lceil \varphi \rceil \supset (\psi \supset \chi)^\top \in PL$, by PL. Hence $\lceil \Box\varphi \rceil \supset \Box(\psi \supset \chi)^\top \in \Sigma$, by $M_{PL} \subseteq \Sigma$. So $\lceil \Box\varphi \rceil \supset (\Box\psi \supset \Box\chi)^\top \in \Sigma$, by **(K)** and PL.

“(d) \Rightarrow (b)” By **(X)**, $\lceil (\Box(\tau \supset \varphi) \wedge \Box(\varphi \supset \psi)) \rceil \supset \Box(\tau \supset \psi)^\top \in \Sigma$, for any $\tau \in \text{Taut}$. Since $\lceil \varphi \equiv (\tau \supset \varphi)^\top \rceil \in PL$ and $E_{PL} \subseteq \Sigma$, so $\lceil \Box\varphi \equiv \Box(\tau \supset \varphi)^\top \rceil \in \Sigma$. Similarly for ψ . Hence $\lceil \Box(\varphi \supset \psi) \rceil \supset (\Box\varphi \supset \Box\psi)^\top \in \Sigma$, by PL. \dashv

All *t*-regular systems contain all instances of the following formulae:

$$\Diamond p \equiv \neg \Box \neg p \quad (\text{df } \Diamond)$$

$$\Box p \equiv \neg \Diamond \neg p \quad (\text{df } \Box)$$

$$\Diamond(p \vee q) \equiv (\Diamond p \vee \Diamond q) \quad (\mathbf{R}^\Diamond)$$

$$\Diamond(p \supset q) \equiv (\Box p \supset \Diamond q) \quad (\mathbf{R}^{\Diamond\Box})$$

The logics **C1**, **D1** and **E1** are *t*-regular (for these logics see p. 274). The logic **C1** is the smallest *t*-regular system.

Notice that **E1** contains the following formula:

$$p \supset \Diamond p \quad (\mathbf{T}^\Diamond)$$

and **(D)**. Moreover, by $(\mathbf{R}^{\Diamond\Box})$, **D1** contains the following formula:

$$\Diamond(p \supset p) \quad (\mathbf{P})$$

In this paper by **C1**+(**T_q**) we denote the smallest *t*-regular logic which contains the following formula

$$\Box p \supset (p \vee \Box q) \quad (\mathbf{T}_q)$$

For t-regular logics the formula (\mathbf{T}_q) may be replaced by

$$\neg \Box(q \wedge \neg q) \supset (\Box p \supset p) \quad (\mathbf{T}'_q)$$

$$\Diamond(q \supset q) \supset (\Box p \supset p) \quad (\mathbf{T}''_q)$$

The name ' \mathbf{T}_q ' is an abbreviation for 'quasi- \mathbf{T} ', because (\mathbf{T}) and (\mathbf{T}_q) are valid in all reflexive and quasi-reflexive standard models, respectively.³ We have that $\mathbf{C1} \subsetneq \mathbf{D1} \subsetneq \mathbf{E1}$ and $\mathbf{C1} \subsetneq \mathbf{C1}+(\mathbf{T}_q) \subsetneq \mathbf{E1}$ (see Fact 4.1).

Notice that the logic $\mathbf{C1}$ plus two axioms (\mathbf{D}) and (\mathbf{T}_q) equals $\mathbf{E1}$ (i.e. $\mathbf{E1} = \mathbf{C1}+(\mathbf{D})+(\mathbf{T}_q) = \mathbf{D1}+(\mathbf{T}_q)$). Indeed, by $\mathbf{C1}$ and (\mathbf{D}) we obtain (\mathbf{P}) . Hence we have (\mathbf{T}) , by (\mathbf{T}''_q) , (MP) and (US).

In this paper we prove that the logics $\mathbf{C1}$, $\mathbf{D1}$, $\mathbf{C1}+(\mathbf{T}_q)$ and $\mathbf{E1}$ are not closed under (rte). For example, the formula ' $\Box \Box p \equiv \Box \Box \neg \neg p$ ' is not a member of these logics (see Remark 3.2 and Fact 4.1).

2.2. Very weak t-normal systems

Any modal system which contains all instances of (\mathbf{K}) and includes the set $\Box\text{PL}$ will be called *t-normal*. Thus, the set $\Box\text{PL}$ replaces the rule (\mathbf{RN}) in the formulation of normal systems. Of course, if Σ is a t-normal system and Σ' is a modal system such that $\Sigma \subseteq \Sigma'$, then Σ' is also a t-normal.

LEMMA 2.4. *For any system Σ the following conditions are equivalent:*

- (a) Σ is t-normal,
- (b) Σ is t-regular and contains (\mathbf{N}) .

PROOF. “(a) \Rightarrow (b)” $(\mathbf{N}) \in \Box\text{PL}$. Moreover, if $\ulcorner (\varphi \wedge \psi) \supset \chi \urcorner \in \text{PL}$, then $\ulcorner \varphi \supset (\psi \supset \chi) \urcorner \in \text{PL}$, by PL and (MP). Hence $\ulcorner \Box(\varphi \supset (\psi \supset \chi)) \urcorner \in \Sigma$, since $\Box\text{PL} \subseteq \Sigma$. So $\ulcorner \Box\varphi \supset (\Box\psi \supset \Box\chi) \urcorner \in \Sigma$ and $\ulcorner (\Box\varphi \wedge \Box\psi) \supset \Box\chi \urcorner \in \Sigma$, by (\mathbf{K}) , PL and (MP).

“(b) \Rightarrow (a)” By Lemma 2.3, Σ contains all instances of (\mathbf{K}) and includes the set M_{PL} . Let $\tau \in \text{PL}$. Then $\ulcorner (p \supset p) \supset \tau \urcorner \in \text{PL}$. So $\ulcorner (\mathbf{N}) \supset \Box\tau \urcorner \in \Sigma$, since $M_{\text{PL}} \subseteq \Sigma$. Thus, $\Box\text{PL} \subseteq \Sigma$. \dashv

³In any quasi-reflexive standard frame an *accessibility* relation R on a set W of worlds is such that $\forall x, y \in W (xRy \Rightarrow xRx)$. See [3, p. 92, Exercise 3.51], where instead of ‘quasi-reflexive’ the term ‘reverse secondary reflexive’ is used.

The logic $\mathbf{S0.5}^\circ$ is the smallest t-normal system; $\mathbf{S0.5}$ is the smallest t-normal logic which contains (\mathbf{T}) (for these logics see p. 273). Of course, $\mathbf{S0.5}$ contains (\mathbf{T}°) and (\mathbf{D}) .

In the present paper by $\mathbf{S0.5}^\circ+(\mathbf{D})$ we denote the smallest t-normal logic which contains (\mathbf{D}) , i.e. $\mathbf{S0.5}^\circ$ plus (\mathbf{D}) . Of course, $\mathbf{S0.5}^\circ+(\mathbf{D})$ contains (\mathbf{P}) . Moreover, by $\mathbf{S0.5}^\circ+(\mathbf{T}_q)$ we denote the smallest t-normal logic which contains (\mathbf{T}_q) , i.e. $\mathbf{S0.5}^\circ$ plus the axiom (\mathbf{T}_q) .

We have that $\mathbf{S0.5}^\circ \subsetneq \mathbf{S0.5}^\circ+(\mathbf{D}) \subsetneq \mathbf{S0.5}$, besides $\mathbf{S0.5}^\circ \subsetneq \mathbf{S0.5}^\circ+(\mathbf{T}_q) \subsetneq \mathbf{S0.5}$ and $\mathbf{C1}+(\mathbf{T}_q) \subsetneq \mathbf{S0.5}^\circ+(\mathbf{T}_q)$ (see Fact 4.1).

Notice that the logic $\mathbf{S0.5}^\circ$ plus two axioms (\mathbf{D}) and (\mathbf{T}_q) is equals $\mathbf{S0.5}$ (i.e. $\mathbf{S0.5} = \mathbf{S0.5}^\circ+(\mathbf{D})+(\mathbf{T}_q)$). Indeed, from $\mathbf{S0.5}^\circ$ and (\mathbf{D}) we obtain (\mathbf{P}) , and hence (\mathbf{T}) , by (\mathbf{T}'_q) , (\mathbf{MP}) and (\mathbf{US}) .

In this paper we prove that $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ+(\mathbf{T}_q)$, $\mathbf{S0.5}^\circ+(\mathbf{D})$ and $\mathbf{S0.5}$ are not closed under (\mathbf{rte}) . For example, the formula ' $\Box\Box p \equiv \Box\Box\neg\neg p$ ' is not a member of these logics (see Remark 3.2 and Fact 4.1).

2.3. Very weak t-normal rte-systems

By *rte-systems* we mean modal systems which are closed under (\mathbf{rte}) . By Lemma 1.1 we have

LEMMA 2.5. *If a rte-system contains (\mathbf{N}) and all instances of (\mathbf{K}) , then it is t-normal.*

Let $\mathbf{S0.5}^\circ_{\mathbf{rte}}$, $\mathbf{S0.5}_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{D})$ and $\mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{T}_q)$ be, respectively, such versions of the logics $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}$, $\mathbf{S0.5}^\circ+(\mathbf{D})$ and $\mathbf{S0.5}^\circ+(\mathbf{T}_q)$ that are closed under (\mathbf{rte}) . Thus, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$ is the smallest t-normal rte-system, and $\mathbf{S0.5}_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{D})$ and $\mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{T}_q)$ are the smallest t-normal rte-logics which contain (\mathbf{T}) , (\mathbf{D}) and (\mathbf{T}_q) , respectively.⁴ We have that $\mathbf{S0.5}^\circ_{\mathbf{rte}} \subsetneq \mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{D}) \subsetneq \mathbf{S0.5}_{\mathbf{rte}}$ and $\mathbf{S0.5}^\circ_{\mathbf{rte}} \subsetneq \mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{T}_q) \subsetneq \mathbf{S0.5}_{\mathbf{rte}}$ (see Fact 4.1).

2.4. Very weak t-regular rte-systems

Let $\mathbf{C1}_{\mathbf{rte}}$, $\mathbf{D1}_{\mathbf{rte}}$, $\mathbf{E1}_{\mathbf{rte}}$ and $\mathbf{E1}_{\mathbf{rte}}+(\mathbf{T}_q)$ be, respectively, such versions of the logics $\mathbf{C1}$, $\mathbf{D1}$, $\mathbf{E1}$ and $\mathbf{C1}+(\mathbf{T}_q)$ that are closed under (\mathbf{rte}) . The

⁴Thus, $\mathbf{S0.5}^\circ_{\mathbf{rte}}$ is the smallest classical modal logic in the sense of [2], and $\mathbf{S0.5}_{\mathbf{rte}}$, $\mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{D})$ and $\mathbf{S0.5}^\circ_{\mathbf{rte}}+(\mathbf{T}_q)$ are the smallest classical modal logics (in the sense of [2]) which contain (\mathbf{T}) , (\mathbf{D}) and (\mathbf{T}_q) , respectively.

logic $\mathbf{C1}_{\text{rte}}$ is the smallest t-regular rte-system. The logics $\mathbf{D1}_{\text{rte}}$, $\mathbf{E1}_{\text{rte}}$ and $\mathbf{E1}_{\text{rte}}+(\mathbf{T}_q)$ are smallest t-regular rte-logics which contain (\mathbf{T}) , (\mathbf{D}) and (\mathbf{T}_q) , respectively. We have that $\mathbf{C1}_{\text{rte}} \subsetneq \mathbf{D1}_{\text{rte}} \subsetneq \mathbf{E1}_{\text{rte}}$ and $\mathbf{C1}_{\text{rte}} \subsetneq \mathbf{E1}_{\text{rte}}+(\mathbf{T}_q) \subsetneq \mathbf{E1}_{\text{rte}}$ (see Fact 4.1).

3. Semantics for very weak systems

3.1. Models for very weak t-normal and t-regular systems

For very weak t-normal modal systems we are using the following semantics, which consists of “t-normal models”.

A *model for very weak t-normal systems* (or *t-normal model*) is any triple $\langle w, A, V \rangle$ in which

1. w is a «distinguished» (normal) world,
2. A is a set of *worlds* which are alternatives to the world w ,
3. V is a valuation from $\text{For} \times (\{w\} \cup A)$ to $\{0, 1\}$:
 - (i) for all formulae and all worlds, V preserves classical conditions for truth-value operators,
 - (ii) for the world w and any $\varphi \in \text{For}$

$$(V_{\square}) \quad V(\square\varphi, w) = 1 \text{ iff } \forall_{x \in A} V(\varphi, x) = 1,$$
 - (iii) for every world from $A \setminus \{w\}$, formulae $\lceil \square\varphi \rceil$ have arbitrary values.

A formula φ is *true* in a t-normal model $\langle w, A, V \rangle$ iff $V(\varphi, w) = 1$. We say that a formula is *t-normal valid* iff it is true in all t-normal models.

We say that a t-normal model $\langle w, A, V \rangle$ is *self-associate* (resp. *empty*, *non-empty*) iff $w \in A$ (resp. $A = \emptyset$, $A \neq \emptyset$). Let \mathbf{nM} be the class of all t-normal models. Moreover, let \mathbf{nM}^{sa} (resp. \mathbf{nM}^{\emptyset} , \mathbf{nM}^+) be the class of t-normal models which are self-associate (resp. empty, non-empty). Of course, $\mathbf{nM}^{\text{sa}} \subsetneq \mathbf{nM}^+$ and $\mathbf{nM}^{\emptyset} \cap \mathbf{nM}^+ = \emptyset$.

Remark 3.1. We may also use the class of models of the form $\langle W, w, A, V \rangle$, where W is a non-empty set of *worlds*, $w \in W$, $A \subseteq W$, and w , A and V are as mentioned above. Of course, the triple $\langle w, A, V \rangle$ may be identified with the quadruple $\langle W, w, A, V \rangle$ such that $W = \{w\} \cup A$. \dashv

In the case of very weak t -regular systems we broaden the class of t -normal models by the class of *queer* models of the form $\langle w, V \rangle$ with only one (queer) world w and a valuation $V: \text{For} \times \{w\} \rightarrow \{0, 1\}$ which satisfies classical conditions for truth-value operators and such that

(ii') for the world w and any $\varphi \in \text{For}$

$$V(\Box \varphi, w) = 0.$$

Of course, a queer model $\langle w, V \rangle$ may be identified with the valuation $V: \text{For} \rightarrow \{0, 1\}$ such that $V(\varphi) = V(\varphi, w)$, for any φ from For .

Let \mathbf{qM} be the class of all queer models and we put $\mathbf{rM} := \mathbf{nM} \cup \mathbf{qM}$, i.e. \mathbf{rM} is the class of models for very weak t -regular systems.

A formula φ is *true* in a queer model $\langle w, V \rangle$ iff $V(\varphi, w) = 1$. We say that a formula is *t -regular valid* iff it is true in all models from \mathbf{rM} . We have the following lemmas.

LEMMA 3.1. 1. If $\varphi \in \text{PL}$, then $V(\varphi, x) = 1$, for any world x in any model from \mathbf{rM} . So all formulae from PL are t -regular valid.

2. All formulae from $\Box\text{PL}$ are t -normal valid.

3. All formulae from the sets M_{PL} R_{PL} and E_{PL} are t -regular valid.

LEMMA 3.2. 1. All instances of formulae (\mathbf{K}) and (\mathbf{R}) are t -regular valid.

2. All instances of the formulae (\mathbf{T}) and (\mathbf{T}_q) are true in any model from $\mathbf{nM}^{\text{sa}} \cup \mathbf{qM}$.

3. All instances of the formula (\mathbf{D}) are true in all models from $\mathbf{nM}^+ \cup \mathbf{qM}$.

4. All instances of the formula (\mathbf{T}_q) are true in all models from \mathbf{nM}^\emptyset .

FACT 3.3. Let $\ulcorner \varphi \equiv \psi \urcorner \in \text{PL}$. Then for any classical formula χ (without the modal operator) following holds: $V(\chi, x) = V(\chi[\varphi/\psi], x)$, for any world x in any model from \mathbf{rM} .

Remark 3.2. Let $w \neq a$, $A := \{w, a\}$ and V be an arbitrary valuation such that $V(\Box p, a) = 1$ and $V(\Box \neg p, a) = 0$. Then $\langle w, A, V \rangle$ belongs to \mathbf{nM}^{sa} and the formula ' $\Box \Box p \equiv \Box \Box \neg p$ ' is not true in this model. \dashv

3.2. Models for very weak t-normal and t-regular rte-systems

For very weak t-normal rte-systems we are using t&rte-normal models, where by a *t&rte-normal model* we mean a t-normal model $\langle w, A, V \rangle$ which satisfies the following condition:

- (iv) for all formulae φ, ψ and χ : if $\ulcorner \varphi \equiv \psi \urcorner \in \text{PL}$ and $\forall_{x \in A} V(\chi, x) = 1$, then $\forall_{x \in A} V(\chi[\varphi/\psi], x) = 1$.

Of course, the condition (iv) is equivalent to the following:

- (iv') for all formulae φ, ψ and χ : if $\ulcorner \varphi \equiv \psi \urcorner \in \text{PL}$, then $\forall_{x \in A} V(\chi, x) = 1$ iff $\forall_{x \in A} V(\chi[\varphi/\psi], x) = 1$.

Moreover, by (V_{\Box}) , the condition (iv) is equivalent to the following one:

- (iv'') for all formulae φ, ψ and χ : if $\ulcorner \varphi \equiv \psi \urcorner \in \text{PL}$, then $V(\Box \chi, w) = V(\Box \chi[\varphi/\psi], w)$.

Let \mathbf{nM}_{rte} be the class of all t&rte-normal models. Moreover, let $\mathbf{nM}_{\text{rte}}^{\text{sa}}$ (resp. $\mathbf{nM}_{\text{rte}}^{\emptyset}$, $\mathbf{nM}_{\text{rte}}^+$) be the class of t&rte-normal models which are self-associate (resp. empty, non-empty).

In the case of very weak t-regular rte-systems we broaden the class of t&rte-normal models by queer models. We put $\mathbf{rM}_{\text{rte}} := \mathbf{nM}_{\text{rte}} \cup \mathbf{qM}$, i.e. \mathbf{rM}_{rte} is the class of models for very weak t&rte-regular systems.

We say that a formula is *t&rte-normal valid* (resp. *t&rte-regular valid*) iff it is true in all models from \mathbf{nM}_{rte} (resp. \mathbf{rM}_{rte}).

We have the following lemma.

LEMMA 3.4. *If $\ulcorner \varphi \equiv \psi \urcorner \in \text{PL}$, then $V(\chi, w) = V(\chi[\varphi/\psi], w)$ in all t&rte-normal models and all queer models. So all formulae from RE_{PL} are t&rte-regular valid.*

4. Determination theorems

Let \mathbf{C} be any class of considered models. We say that a formula φ is \mathbf{C} -*valid* (written $\models_{\mathbf{C}} \varphi$) iff φ is true in all models from \mathbf{C} .

Let Σ be an arbitrary modal system. We say that Σ is *sound* with respect to \mathbf{C} iff $\Sigma \subseteq \{\varphi \in \text{For} : \models_{\mathbf{C}} \varphi\}$. We say that Σ is *complete* with respect to \mathbf{C} iff $\Sigma \supseteq \{\varphi \in \text{For} : \models_{\mathbf{C}} \varphi\}$. We say that Σ is *determined* by \mathbf{C} iff $\Sigma = \{\varphi \in \text{For} : \models_{\mathbf{C}} \varphi\}$, i.e., Σ is sound and complete with respect to \mathbf{C} .

4.1. Soundness

By lemmas 3.1, 3.2 and 3.4 we obtain the following facts.

- FACT 4.1. 1. **C1** is sound with respect to the class \mathbf{rM} .
2. **D1** is sound with respect to the class $\mathbf{nM}^+ \cup \mathbf{qM}$.
3. **E1** is sound with respect to the class $\mathbf{nM}^{\text{sa}} \cup \mathbf{qM}$.
4. **C1**+ (\mathbf{T}_q) is sound with respect to the class $\mathbf{nM}^{\text{sa}} \cup \mathbf{nM}^\emptyset \cup \mathbf{qM}$.
5. **S0.5** $^\circ$ is sound with respect to the class \mathbf{nM} .
6. **S0.5** $^\circ$ + (\mathbf{D}) is sound with respect to the class \mathbf{nM}^+ .
7. **S0.5** is sound with respect to the class \mathbf{nM}^{sa} .
8. **S0.5** $^\circ$ + (\mathbf{T}_q) is sound with respect to the class $\mathbf{nM}^{\text{sa}} \cup \mathbf{nM}^\emptyset$.
9. **C1** $_{\text{rte}}$ is sound with respect to the class \mathbf{rM}_{rte} .
10. **D1** $_{\text{rte}}$ is sound with respect to the class $\mathbf{nM}_{\text{rte}}^+ \cup \mathbf{qM}$.
11. **E1** $_{\text{rte}}$ is sound with respect to the class $\mathbf{nM}_{\text{rte}}^+ \cup \mathbf{qM}$.
12. **E1** $_{\text{rte}}$ + (\mathbf{T}_q) is sound with respect to the class $\mathbf{nM}_{\text{rte}}^+ \cup \mathbf{qM}$.
13. **S0.5** $^\circ_{\text{rte}}$ is sound with respect to the class \mathbf{nM}_{rte} .
14. **S0.5** $^\circ_{\text{rte}}$ + (\mathbf{D}) is sound with respect to the class $\mathbf{nM}_{\text{rte}}^+$.
15. **S0.5** $_{\text{rte}}$ is sound with respect to the class $\mathbf{nM}_{\text{rte}}^{\text{sa}}$.
16. **S0.5** $^\circ_{\text{rte}}$ + (\mathbf{T}_q) is sound with respect to the class $\mathbf{nM}_{\text{rte}}^{\text{sa}} \cup \mathbf{nM}_{\text{rte}}^\emptyset$.

For completeness of considered very weak logics we use canonical models method.

4.2. Notions and facts concerning maximal consistent sets

For the following definitions see, for example, [3, 2.4 and 2.6]. Let Σ and Σ' be any modal systems, and $\Gamma \subseteq \text{For}$.

Σ is *consistent* iff $\Sigma \neq \text{For}$; equivalently in the light of PL, iff ' $p \wedge \neg p$ ' does not belong to Σ . For example, all modal logics from Section 2 are consistent.

A formula φ is *deducible* from Γ in Σ (written $\Gamma \vdash_\Sigma \varphi$) iff for some $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ ($n \geq 0$) we have $\ulcorner (\psi_1 \wedge \dots \wedge \psi_n) \supset \varphi \urcorner \in \Sigma$. We have $\vdash_{\text{PL}} \subseteq \vdash_\Sigma$. Moreover, $\Sigma \vdash_\Sigma \varphi$ iff $\varphi \in \Sigma$ iff $\emptyset \vdash_\Sigma \varphi$.

A set Γ is Σ -*consistent* iff for some $\varphi \in \text{For}$, $\Gamma \not\vdash_\Sigma \varphi$; equivalently in the light of PL, iff $\Gamma \not\vdash_\Sigma p \wedge \neg p$. We have (see e.g. [3]):

- If Γ is Σ -consistent, then Σ is consistent.
- Σ is consistent iff Σ is Σ -consistent.
- If Γ is Σ -consistent and $\Sigma' \subseteq \Sigma$, then Γ is Σ' -consistent; so, Γ is PL-consistent.

We say that Γ is Σ -maximal iff Γ is Σ -consistent and Γ has only Σ -inconsistent proper extensions. Let Max_Σ be the set of all Σ -maximal sets.

LEMMA 4.2 ([3]). *Let $\Gamma \in \text{Max}_\Sigma$. Then*

1. $\Sigma \subseteq \Gamma$ and Γ is a modal system.
2. $\Gamma \vdash_\Sigma \varphi$ iff $\varphi \in \Gamma$.
3. $\lceil \neg\varphi \rceil \in \Gamma$ iff $\varphi \notin \Gamma$.
4. $\lceil \varphi \wedge \psi \rceil \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.
5. $\lceil \varphi \vee \psi \rceil \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.
6. $\lceil \varphi \supset \psi \rceil \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.
7. $\lceil \varphi \equiv \psi \rceil \in \Gamma$ iff either $\varphi, \psi \in \Gamma$ or $\varphi, \psi \notin \Gamma$.

LEMMA 4.3. *If $\Gamma \in \text{Max}_\Sigma$ and $\Sigma' \subseteq \Sigma$, then $\Gamma \in \text{Max}_{\Sigma'}$. So $\Gamma \in \text{Max}_{\text{PL}}$.*

PROOF. Let $\Gamma \in \text{Max}_\Sigma$ and $\Sigma' \subseteq \Sigma$. Then Γ is Σ' -consistent and PL-consistent. Moreover, suppose that $\Gamma \cup \{\varphi\}$ is Σ' -consistent. Then $\Gamma \cup \{\varphi\}$ is also PL-consistent. So $\lceil \neg\varphi \rceil \notin \Gamma$. Therefore $\varphi \in \Gamma$, by Lemma 4.2.3. Hence $\Gamma \cup \{\varphi\} = \Gamma$. Thus Γ be Σ' -maximal. \dashv

LEMMA 4.4 ([3]). 1. $\Gamma \vdash_\Sigma \varphi$ iff $\varphi \in \Delta$, for any Δ such that $\Delta \in \text{Max}_\Sigma$ and $\Gamma \subseteq \Delta$.

2. $\varphi \in \Sigma$ iff $\varphi \in \Delta$, for any $\Delta \in \text{Max}_\Sigma$.

4.3. Canonical models

For completeness of very weak logics we need two following auxiliary lemmas.

LEMMA 4.5. *Let Σ be a t -regular consistent system and let Γ be a Σ -maximal set such that $\Gamma \cap \Box \text{For} \neq \emptyset$, i.e. $\{\psi \in \text{For} : \ulcorner \Box \psi^\urcorner \in \Gamma\} \neq \emptyset$.⁵ Then for every $\varphi \in \text{For}$ the following conditions are equivalent:*

- (a) $\ulcorner \Box \varphi^\urcorner \in \Gamma$.
- (b) $\Gamma \vdash_\Sigma \Box \varphi$.
- (c) $\{\psi : \ulcorner \Box \psi^\urcorner \in \Gamma\} \vdash_{\text{PL}} \varphi$.
- (d) $\varphi \in \Delta$, for any PL-maximal set Δ such that $\{\psi : \ulcorner \Box \psi^\urcorner \in \Gamma\} \subseteq \Delta$.

PROOF. “(a) \Leftrightarrow (b)” Lemma 4.2.2.

“(a) \Rightarrow (d)” It is trivial, since for any $\Gamma, \Delta \subseteq \text{For}$, if $\ulcorner \Box \varphi^\urcorner \in \Gamma$ and $\{\psi \in \text{For} : \ulcorner \Box \psi^\urcorner \in \Gamma\} \subseteq \Delta$, then $\varphi \in \Delta$.

“(d) \Leftrightarrow (c)” By Lemma 4.4.1.

“(c) \Rightarrow (b)” Either $\varphi \in \text{PL}$ or for some $\psi_1, \dots, \psi_n \in \{\psi : \ulcorner \Box \psi^\urcorner \in \Gamma\}$, $n > 0$, we have $\ulcorner (\psi_1 \wedge \dots \wedge \psi_n) \urcorner \supset \varphi^\urcorner \in \text{PL}$. But the first case entails the second case. Hence $\ulcorner (\Box \psi_1 \wedge \dots \wedge \Box \psi_n) \urcorner \supset \Box \varphi^\urcorner \in \Sigma$, since $\text{R}_{\text{PL}} \subseteq \Sigma$. But Γ contains each of $\ulcorner \Box \psi_1^\urcorner, \dots, \ulcorner \Box \psi_n^\urcorner$, so $\Gamma \vdash_\Sigma \Box \varphi$. \dashv

Let Σ be a t -regular system, $\Gamma \in \text{Max}_\Sigma$ and $\{\psi : \ulcorner \Box \psi^\urcorner \in \Gamma\} \neq \emptyset$. We say that $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is a *canonical model for Σ and Γ* iff it satisfies these conditions:

- $w_\Gamma := \Gamma$,
- $A_\Gamma := \{\Delta \in \text{Max}_{\text{PL}} : \forall \psi \in \text{For} (\ulcorner \Box \psi^\urcorner \in \Gamma \Rightarrow \psi \in \Delta)\}$,
- $V_\Gamma : \text{For} \times (\{w_\Gamma\} \cup A_\Gamma) \rightarrow \{0, 1\}$ is the valuation such that for all $\varphi \in \text{For}$ and $\Delta \in \{w_\Gamma\} \cup A_\Gamma$

$$V_\Gamma(\varphi, \Delta) := \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 4.6. *For any t -regular system Σ and any $\Gamma \in \text{Max}_\Sigma$ such that $\{\psi : \ulcorner \Box \psi^\urcorner \in \Gamma\} \neq \emptyset$ it holds that:*

- (a) $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is a t -normal model.
- (b) If Σ contains all instances of **(T)**, then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is self-associate.

⁵Notice that all t -normal systems satisfy these assumptions. Firstly, all t -normal systems are t -regular. Secondly, for any t -normal system Σ , if Γ is Σ -maximal, then $\{\psi : \ulcorner \Box \psi^\urcorner \in \Gamma\} \neq \emptyset$, since $\Box \text{PL} \subseteq \Sigma \subseteq \Gamma$, by Lemma 4.2.1.

- (c) If Σ contains all instances of **(D)**, then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is non-empty.
- (d) If Σ contains all instances of **(T_q)**, then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is either empty or self-associate.
- (e) If Σ is a rte-system, then $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle$ is t&rte-normal model.

PROOF. (a) Thanks to properties of maximal sets (see Lemma 4.2), for every $\Delta \in \{w_\Gamma\} \cup A_\Gamma$ the assignment $V_\Gamma(\cdot, \Delta)$ preserves classical conditions for truth-value operators. We prove that for w_Γ the assignment $V_\Gamma(\cdot, w_\Gamma)$ satisfies the condition (V_\square) .

For any $\varphi \in \text{For}$: $V_\Gamma(\square\varphi, w_\Gamma) = 1$ iff $\lceil \square\varphi^\top \in \Gamma$ (by definition of V_Γ) iff for every $\Delta \in \text{Max}_{\text{PL}}$ for which $\{\psi \in \text{For} : \lceil \square\psi^\top \in \Gamma\} \subseteq \Delta$ we have $\varphi \in \Delta$ (by Lemma 4.5) iff for every $\Delta \in A_\Gamma$, $\varphi \in \Delta$ (by definition of A_Γ) iff for every $\Delta \in A_\Gamma$, $V_\Gamma(\varphi, \Delta) = 1$ (by definition of V_Γ).

(b) We show that $w_\Gamma \in A_\Gamma$. Firstly, by Lemma 4.3, $\Gamma \in \text{Max}_{\text{PL}}$ -maximal. Secondly, for any $\psi \in \text{For}$, $\lceil \square\psi \supset \psi^\top \in \Gamma$, by Lemma 4.2.1. So, if $\lceil \square\psi^\top \in \Gamma$, then $\psi \in \Gamma$, by Lemma 4.2.6.

(c) For some φ_0 we have $\lceil \square\varphi_0^\top \in \Gamma$. By Lemma 4.2.1, $\lceil \square\varphi_0 \supset \neg\square\neg\varphi_0^\top \in \Gamma$. Hence, by lemmas 4.2.6 and 4.2.1, $\lceil \neg\square\neg\varphi_0^\top \in \Gamma$ and $\lceil \square\neg\varphi_0^\top \notin \Gamma$. Therefore, by Lemma 4.5, $\lceil \neg\varphi_0^\top \notin \Delta_0$, for some Δ_0 such that Δ_0 is PL-maximal and $\{\psi : \lceil \square\psi^\top \in \Gamma\} \subseteq \Delta_0$. Hence $\Delta_0 \in A_\Gamma$. Thus, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^+$.

(d) We show that $w_\Gamma \in A_\Gamma$ or $A_\Gamma = \emptyset$. Notice that, by lemmas 4.2.1 and 4.2.6, $\lceil \neg\square(q \wedge \neg q) \supset (\square\psi \supset \psi)^\top \in \Gamma$, for any formula ψ . Suppose that $A_\Gamma \neq \emptyset$. Then $\lceil \square(q \wedge \neg q) \rceil \notin \Gamma$, by Lemma 4.5, since $\lceil q \wedge \neg q \rceil \notin \Delta$, for any Δ which is PL-consistent. So, $\lceil \neg\square(q \wedge \neg q) \rceil \in \Gamma$. Therefore $\lceil \square\psi \supset \psi^\top \in \Gamma$. Hence $w_\Gamma \in A_\Gamma$, as in (b).

(e) Suppose that $\lceil \varphi \equiv \psi^\top \in \text{PL}$. Then $\lceil \square\chi[\varphi/\psi] \equiv \square\chi^\top \in \Sigma$, since $\text{RE}_{\text{PL}} \subseteq \Sigma$. So also $\lceil \square\chi[\varphi/\psi] \equiv \square\chi^\top \in \Gamma$, by Lemma 4.2.1. Thus, $V(\square\chi, w) = V(\square\chi[\varphi/\psi], w)$, by definition of V_Γ . \dashv

Let Σ be a t-regular system, $\Gamma \in \text{Max}_\Sigma$ and $\{\psi : \lceil \square\psi^\top \in \Gamma\} = \emptyset$. We say that $\langle w_\Gamma, V_\Gamma \rangle$ is a *canonical model for Σ and Γ* iff it satisfies these conditions:

- $w_\Gamma := \Gamma$,
- $V_\Gamma: \text{For} \times \{w_\Gamma\} \rightarrow \{0, 1\}$ is the valuation such that

$$V_\Gamma(\varphi, w_\Gamma) := \begin{cases} 1 & \text{if } \varphi \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 4.7. *For any t -regular system Σ and any $\Gamma \in \text{Max}_\Sigma$ such that $\{\psi : \lceil \Box \psi \rceil \in \Gamma\} = \emptyset$: $\langle w_\Gamma, V_\Gamma \rangle$ is a queer model.*

PROOF. Thanks to properties of maximal sets in modal systems (see Lemma 4.2), the assignment V_Γ preserves classical conditions for truth-value operators. Moreover, for any $\varphi \in \text{For}$ we have: $\lceil \Box \varphi \rceil \notin \Gamma$. So, $V_\Gamma(\Box \varphi, w_\Gamma) = 0$. \dashv

4.4. Completeness

By lemmas 4.4.2 and 4.6 for very weak t -normal and t -normal rte -systems we obtain

THEOREM 4.8. 1. $\mathbf{S0.5}^\circ$ is complete with respect to the class \mathbf{nM} .

2. $\mathbf{S0.5}^\circ + (\mathbf{D})$ is complete with respect to the class \mathbf{nM}^+ .

3. $\mathbf{S0.5}^\circ + (\mathbf{T}_q)$ is complete with respect to the class $\mathbf{nM}^{\text{sa}} \cup \mathbf{nM}^\emptyset$.

4. $\mathbf{S0.5}$ is complete with respect to the class \mathbf{nM}^{sa} .

5. $\mathbf{S0.5}_{\text{rte}}^\circ$ is complete with respect to the class \mathbf{nM}_{rte} .

6. $\mathbf{S0.5}_{\text{rte}}^\circ + (\mathbf{D})$ is complete with respect to the class $\mathbf{nM}_{\text{rte}}^+$.

7. $\mathbf{S0.5}_{\text{rte}}^\circ + (\mathbf{T}_q)$ is complete with respect to the class $\mathbf{nM}_{\text{rte}}^{\text{sa}} \cup \mathbf{nM}_{\text{rte}}^\emptyset$.

8. $\mathbf{S0.5}_{\text{rte}}$ is complete with respect to the class $\mathbf{nM}_{\text{rte}}^{\text{sa}}$.

PROOF. The logics $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ + (\mathbf{D})$, $\mathbf{S0.5}^\circ + (\mathbf{T}_q)$ and $\mathbf{S0.5}$ are consistent and t -regular. Moreover, for any t -normal logic Λ , if $\Gamma \in \text{Max}_\Lambda$, then $\{\psi : \lceil \Box \psi \rceil \in \Gamma\} \neq \emptyset$, since $\Box \text{PL} \subseteq \Lambda \subseteq \Gamma$.

1. Let φ be an arbitrary formula such that $\models_{\mathbf{nM}} \varphi$. Let Γ be an arbitrary $\mathbf{S0.5}^\circ$ -maximal set. By Lemma 4.6a, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}$. Thus, $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$, by definitions of w_Γ and V_Γ . So, we have shown that φ belongs to all $\mathbf{S0.5}^\circ$ -maximal sets. Hence $\varphi \in \mathbf{S0.5}^\circ$, by Lemma 4.4.2.

2. By Lemma 4.6c, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^+$. The rest as in 1.

3. By Lemma 4.6d, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^+ \cup \mathbf{nM}^\emptyset$. The rest as in 1.

4. By Lemma 4.6b, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^{\text{sa}}$. The rest as in 1.

5. By Lemma 4.6e, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}$. The rest as in 1.

6. By Lemma 4.6ce, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}^+$. The rest as in 1.

7. By Lemma 4.6de, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}^{\text{sa}} \cup \mathbf{nM}_{\text{rte}}^\emptyset$. The rest as in 1.

8. By Lemma 4.6be, $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}^{\text{sa}}$. The rest as in 1. \dashv

By lemmas 4.4.2, 4.6 and 4.7 for very weak t-regular and t-regular rte-systems we obtain

- THEOREM 4.9.** 1. **C1** is complete with respect to the class **rM**.
2. **D1** is complete with respect to the class $\mathbf{nM}^+ \cup \mathbf{qM}$.
 3. **C1**+**(T_q)** is complete with respect to the class $\mathbf{nM}^{\text{sa}} \cup \mathbf{nM}^\emptyset \cup \mathbf{qM}$.
 4. **E1** is complete with respect to the class $\mathbf{nM}^{\text{sa}} \cup \mathbf{qM}$.
 5. **C1**_{rte} is complete with respect to the class **rM**_{rte}.
 6. **D1**_{rte} is complete with respect to the class $\mathbf{nM}_{\text{rte}}^+ \cup \mathbf{qM}$.
 7. **E1**_{rte}+**(T_q)** is complete with respect to $\mathbf{nM}_{\text{rte}}^{\text{sa}} \cup \mathbf{nM}_{\text{rte}}^\emptyset \cup \mathbf{qM}$.
 8. **E1**_{rte} is complete with respect to $\mathbf{nM}_{\text{rte}}^{\text{sa}} \cup \mathbf{qM}$.

PROOF. 1. Let φ be an arbitrary formula such that $\models_{\mathbf{rM}} \varphi$. Let Γ be an arbitrary **C1**-maximal set. In both alternative cases from lemmas 4.6 and 4.7, either $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. Thus, in both cases we have $V_\Gamma(\varphi, w_\Gamma) = 1$. Hence $\varphi \in \Gamma$, by definitions of w_Γ and V_Γ . So, we have shown that φ belongs to all **C1**-maximal sets. Hence $\varphi \in \mathbf{C1}$, by Lemma 4.4.2.

2. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^+$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1.
3. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^{\text{sa}} \cup \mathbf{nM}^\emptyset$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1.
4. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}^{\text{sa}}$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1.
5. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1.
6. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}^+$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1.
7. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}^{\text{sa}} \cup \mathbf{nM}_{\text{rte}}^\emptyset$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1.
8. $\langle w_\Gamma, A_\Gamma, V_\Gamma \rangle \in \mathbf{nM}_{\text{rte}}^{\text{sa}}$ or $\langle w_\Gamma, V_\Gamma \rangle \in \mathbf{qM}$. The rest as in 1. \dashv

A. Location of very weak modal logics

A.1. Strict implication and strict equivalence

In original Lewis' works (see e.g. [12]) the primitive modal operator is the possibility sign ' \diamond '. The necessity sign ' \square ' is the abbreviation of ' $\neg \diamond \neg$ '. Moreover, for the connective of strict implication ' \rightarrow ' was used ' $\ulcorner \varphi \rightarrow \psi \urcorner$ ' as an abbreviation of a formula ' $\ulcorner \neg \diamond (\varphi \wedge \neg \psi) \urcorner$ '.

In this paper—as in [9]—the primitive modal operator is ' \square ' and ' $\ulcorner \varphi \rightarrow \psi \urcorner$ ' is an abbreviation of ' $\ulcorner \square (\varphi \supset \psi) \urcorner$ '. Moreover, in this paper—as in [12] and [9]—a strict equivalence ' $\ulcorner \varphi \leftrightarrow \psi \urcorner$ ' is an abbreviation of ' $\ulcorner (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \urcorner$ '.

LEMMA A.1. *For any modal system Σ and any $\varphi, \psi \in \text{For}$:*

$$\text{if } \ulcorner \varphi \varepsilon\text{-}\exists \psi \urcorner \in \Sigma, \text{ then } \ulcorner \varphi \rightarrow \psi \urcorner, \ulcorner \psi \rightarrow \varphi \urcorner \in \Sigma.$$

PROOF. Let $\ulcorner \varphi \varepsilon\text{-}\exists \psi \urcorner \in \Sigma$, i.e., $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \Sigma$. Hence $\ulcorner \Box(\varphi \supset \psi) \urcorner, \ulcorner \Box(\psi \supset \varphi) \urcorner \in \Sigma$, by PL, i.e., $\ulcorner \varphi \rightarrow \psi \urcorner, \ulcorner \psi \rightarrow \varphi \urcorner \in \Sigma$. \dashv

LEMMA A.2. *For any t -regular system Σ and any $\varphi, \psi \in \text{For}$:*

$$\ulcorner \varphi \varepsilon\text{-}\exists \psi \urcorner \in \Sigma \text{ iff } \ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma.$$

PROOF. If $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \Sigma$, then $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma$, by (MP) and since $\text{R}_{\text{PL}} \subseteq \Sigma$. If $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma$, then $\ulcorner \Box(\varphi \supset \psi) \urcorner, \ulcorner \Box(\psi \supset \varphi) \urcorner \in \Sigma$, since PL, $\text{M}_{\text{PL}} \subseteq \Sigma$. So, $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \Sigma$, by PL. \dashv

LEMMA A.3 ([4, 9]). *If Σ is closed under the following rule*

$$\text{if } \ulcorner \Box \varphi \urcorner \in \Sigma, \text{ then } \varphi \in \Sigma, \quad (\text{RN}_*)$$

then Σ is closed under the strict version of modus ponens

$$\text{if } \ulcorner \varphi \rightarrow \psi \urcorner \in \Sigma \text{ and } \varphi \in \Sigma, \text{ then } \psi \in \Sigma. \quad (\text{SMP})$$

Hence, any modal system which contains all instances of (T) is also closed under (RN_{}) and (SMP).*

LEMMA A.4 ([4]). *Let Σ be a rte-system which is closed under (SMP). Then Σ is closed under (RN_{*}).*

PROOF. Let $\ulcorner \Box \varphi \urcorner \in \Sigma$ and $\tau \in \text{PL} \subseteq \Sigma$. Then $\ulcorner \varphi \equiv (\tau \supset \varphi) \urcorner \in \text{PL}$, so $\ulcorner \Box(\tau \supset \varphi) \urcorner \in \Sigma$, by (rte). So $\varphi \in \Sigma$, by (SMP). \dashv

LEMMA A.5. *Let Σ be any system which is closed under (SMP) and includes M_{PL} . Then Σ is closed under (RN_{*}).*

PROOF. Let $\ulcorner \Box \varphi \urcorner \in \Sigma$ and $\tau \in \text{PL} \subseteq \Sigma$. Then $\ulcorner \varphi \supset (\tau \supset \varphi) \urcorner \in \text{PL}$, so $\ulcorner \Box \varphi \supset \Box(\tau \supset \varphi) \urcorner \in \Sigma$, since $\text{M}_{\text{PL}} \subseteq \Sigma$. Thus, $\ulcorner \Box(\tau \supset \varphi) \urcorner \in \Sigma$, by (MP), and $\varphi \in \Sigma$, by (SMP). \dashv

A.2. Strict classical modal systems

Imitating [4], we say that a modal system Σ is *strict_T classical* (“*traditionally strict classical*”) iff $\Box\text{PL} \subseteq \Sigma$ and Σ is closed under “traditional replacement rule for strict equivalents”:

$$\text{if } \ulcorner \varphi \varepsilon\exists \psi \urcorner \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma. \quad (\text{RRSE}_T)$$

Moreover, a modal system Σ is called *strict classical* iff $\Box\text{PL} \subseteq \Sigma$ and Σ is closed under the following replacement rule:

$$\text{if } \ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma \text{ and } \chi \in \Sigma, \text{ then } \chi[\varphi/\psi] \in \Sigma. \quad (\text{RRSE})$$

We obtain that for modal logics which contain **(K)** and/or **(X)**, the above notions are equivalent (see Lemma A.9).

LEMMA A.6 ([4]). *Let Σ be strict_T or strict classical. Then Σ is also a rte-system.*

PROOF. Suppose that $\ulcorner \varphi \equiv \psi \urcorner \in \text{PL}$ and $\chi \in \Sigma$. Since $\Box\text{PL} \subseteq \Sigma$, so we have that $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma$ and $\ulcorner \Box(\varphi \supset \psi) \wedge \Box(\psi \supset \varphi) \urcorner \in \Sigma$, by PL. Hence $\chi[\varphi/\psi] \in \Sigma$ follows by **(RRSE)** or by **(RRSE_T)**, respectively. \dashv

By definitions we have the following lemma.

LEMMA A.7. *Let Σ be strict_T or strict classical and let Σ contain all instances of **(K)**. Then Σ is t-normal.*

Now notice that

LEMMA A.8 ([4, 9]). *Let Σ be strict_T or strict classical and let Σ contain all instances of **(X)** (resp. $\Box\text{(X)}$). Then Σ contains all instances of **(K)** (resp. $\Box\text{(K)}$).*

PROOF. Let $\varphi, \psi \in \text{For}$. Since $\Box\text{PL} \subseteq \Sigma$ and $\ulcorner \varphi \equiv (\tau \supset \varphi) \urcorner \in \text{PL}$, for any $\tau \in \text{Taut}$, so we have $\ulcorner \varphi \varepsilon\exists (\tau \supset \varphi) \urcorner, \ulcorner \Box(\varphi \equiv (\tau \supset \varphi)) \urcorner \in \Sigma$, by PL. Similarly for ψ . Let Σ contain all instances of **(X)**. Then $\ulcorner (\Box(\tau \supset \varphi) \wedge \Box(\varphi \supset \psi)) \supset \Box(\tau \supset \psi) \urcorner \in \Sigma$. Hence $\ulcorner \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi) \urcorner \in \Sigma$, by PL and either **(RRSE_T)** or **(RRSE)**.

Let Σ contain all instances of $\Box\text{(X)}$. Then $\ulcorner \Box((\Box(\tau \supset \varphi) \wedge \Box(\varphi \supset \psi)) \supset \Box(\tau \supset \psi)) \urcorner \in \Sigma$. Hence $\ulcorner \Box(\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)) \urcorner \in \Sigma$, by PL and either **(RRSE_T)** or **(RRSE)**. \dashv

By lemmas 2.4, A.2, A.7 and A.8 we have the following lemma.

LEMMA A.9 ([4]). *For any modal system Σ which contains all instances of (K) or (X): Σ is strict_T classical iff Σ is strict classical.*

Moreover, we obtain

LEMMA A.10 ([4]). 1. *If Σ is strict_T classical, then it is also closed under the following “traditional” rule of congruence for strict equivalence*

$$\text{if } \ulcorner \varphi \varepsilon\exists \psi \urcorner \in \Sigma, \text{ then } \ulcorner \Box \varphi \varepsilon\exists \Box \psi \urcorner \in \Sigma. \quad (\text{RSE}_T)$$

2. *If Σ is strict classical, then it is also closed under the following rule of congruence for strict equivalence*

$$\text{if } \ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma, \text{ then } \ulcorner \Box(\Box \varphi \equiv \Box \psi) \urcorner \in \Sigma. \quad (\text{RSE})$$

PROOF. 1. Since $\Box\text{PL} \subseteq \Sigma$, we have that $\ulcorner \Box \varphi \varepsilon\exists \Box \varphi \urcorner \in \Sigma$, by PL. Hence if $\ulcorner \varphi \varepsilon\exists \psi \urcorner \in \Sigma$, then $\ulcorner \Box \varphi \varepsilon\exists \Box \psi \urcorner \in \Sigma$, by (RRSE_T).

2. Since $\Box\text{PL} \subseteq \Sigma$, we have that $\ulcorner \Box(\Box \varphi \equiv \Box \varphi) \urcorner \in \Sigma$. Hence if $\ulcorner \Box(\varphi \equiv \psi) \urcorner \in \Sigma$, then $\ulcorner \Box(\Box \varphi \equiv \Box \psi) \urcorner \in \Sigma$, by (RRSE). \dashv

LEMMA A.11 ([4, 9]). *Let Σ be a t-normal system which closed under (RSE_T). Then*

1. *Σ is also closed under the following rule of replacement*

$$\text{if } \ulcorner \varphi \varepsilon\exists \psi \urcorner \in \Sigma, \text{ then } \ulcorner \chi[\varphi/\psi] \varepsilon\exists \chi \urcorner \in \Sigma, \quad (\text{RRSE}'_T)$$

2. *If Σ is also closed under (SMP), then Σ is closed under (RRSE_T).*

PROOF. 1. By induction.

2. Let $\ulcorner \varphi \varepsilon\exists \psi \urcorner \in \Sigma$ and $\chi \in \Sigma$. Then $\ulcorner \chi[\varphi/\psi] \varepsilon\exists \chi \urcorner \in \Sigma$, by 1. Hence $\ulcorner \chi \rightarrow \chi[\varphi/\psi] \urcorner \in \Sigma$, by Lemma A.1. So $\chi[\varphi/\psi] \in \Sigma$, by (SMP). \dashv

A.3. The logics S0.9, S0.9^o, S1 and S1^o

In [9] Lemmon provided a simple axiomatization of the Lewis' logic **S1**, where it is the smallest strict_T classical modal logic which contains formulae $\Box(\mathbf{X})$, **(T)** and $\Box(\mathbf{T})$. Of course, the logic **S1** contains also **(X)** and, by Lemma A.8, the formulae **(K)** and $\Box(\mathbf{K})$. So **S1** is strict classical and it is a t-normal rte-logic (see lemmas A.6, A.7 and A.9).

In [9] Lemmon also introduced the logic $\mathbf{S0.9}$, where it was meant as the smallest modal logic which included $\Box\text{Taut}$, contained formulae $\Box(\mathbf{K})$, (\mathbf{T}) and $\Box(\mathbf{T})$, and is closed under (\mathbf{RSE}_T) . So $\mathbf{S0.9}$ contains (\mathbf{K}) and is t-normal. Hence, contains (\mathbf{X}) , since $\mathbf{S0.9}$ is also t-regular. Moreover, by lemmas A.8 and A.10.1, we obtain that $\mathbf{S0.9} \subseteq \mathbf{S1}$. In [7] it was proved that $\mathbf{S0.9} \neq \mathbf{S1}$, since $\Box(\mathbf{X}) \notin \mathbf{S0.9}$ (see also [4]).

“The other two systems, $\mathbf{S1}^\circ$ and $\mathbf{S0.9}^\circ$, are often loosely described as $\mathbf{S1}$ and $\mathbf{S0.9}$ minus the schema T” [4, p. 12]. In [4] the Feys’ logic $\mathbf{S1}^\circ$ from [5] is described as the smallest strict_T classical modal logic which contains the formulae (\mathbf{X}) and $\Box(\mathbf{X})$, and is closed under (\mathbf{SMP}) . Thus, $\mathbf{S1}^\circ$ contains (\mathbf{K}) and $\Box(\mathbf{K})$, by Lemma A.8. So, it is also a strict classical rte-logic.

Moreover, in [4] the logic $\mathbf{S0.9}^\circ$ is described as the smallest strict_T classical modal logic which contains the formulae (\mathbf{K}) and $\Box(\mathbf{K})$, and is closed under (\mathbf{SMP}) .

Thus we have the following axiomatizations (of course, in each case PL, (\mathbf{MP}) and (\mathbf{US}) are added as default items):

- $\mathbf{S0.9}$: $\Box\text{Taut}$, $\Box(\mathbf{K})$, (\mathbf{T}) , $\Box(\mathbf{T})$ and (\mathbf{RSE}_T) ,
- $\mathbf{S0.9}^\circ$: $\Box\text{Taut}$, (\mathbf{K}) , $\Box(\mathbf{K})$, (\mathbf{RRSE}_T) and (\mathbf{SMP}) ,
- $\mathbf{S1}$: $\Box\text{Taut}$, $\Box(\mathbf{X})$, (\mathbf{T}) , $\Box(\mathbf{T})$ and (\mathbf{RRSE}_T) ,
- $\mathbf{S1}^\circ$: $\Box\text{Taut}$, (\mathbf{X}) , $\Box(\mathbf{X})$, (\mathbf{RRSE}_T) and (\mathbf{SMP}) .

By Lemma A.10 the logic $\mathbf{S0.9}^\circ$ is also closed under the rules (\mathbf{RSE}_T) and (\mathbf{RSE}) . So $\mathbf{S0.9}^\circ \subsetneq \mathbf{S0.9}$, since $\mathbf{S0.9}$ is also closed under (\mathbf{SMP}) and (\mathbf{T}) , $\Box(\mathbf{T}) \notin \mathbf{S0.9}^\circ$. Hence $\Box(\mathbf{X}) \notin \mathbf{S0.9}^\circ$, since $\Box(\mathbf{X}) \notin \mathbf{S0.9}$. Therefore $\mathbf{S0.9}^\circ \subsetneq \mathbf{S1}^\circ$. Moreover, because $\mathbf{S1}$ is also closed under (\mathbf{SMP}) , but (\mathbf{T}) , $\Box(\mathbf{T}) \notin \mathbf{S1}^\circ$, so we have $\mathbf{S1}^\circ \subsetneq \mathbf{S1}$.

By Lemma A.6, the logics $\mathbf{S0.9}^\circ$, $\mathbf{S1}$ and $\mathbf{S1}^\circ$ are a t-normal rte-logic. Moreover, by lemmas A.3, A.11, A.9 and A.6, we have:

COROLLARY A.12 ([4]). $\mathbf{S0.9}$ is strict_T and strict classical, and it is a t-normal rte-logic.

Notice that using lemmas given in sections A.1 and A.2 as well as Lemma 1.1 we obtain the following facts.

FACT A.13 ([4]). 1. $\mathbf{S0.9}$ is the smallest rte-logic which is closed under (\mathbf{RN}_*) and (\mathbf{RRSE}) (resp. (\mathbf{RRSE}_T)), and contains the formulae (\mathbf{N}) , $\Box(\mathbf{T})$ and $\Box(\mathbf{K})$.

2. $\mathbf{S0.9}^\circ$ is the smallest rte-logic which is closed under (\mathbf{RN}_*) and (\mathbf{RRSE}) (resp. (\mathbf{RRSE}_T)), and contains the formulae (\mathbf{N}) and $\Box(\mathbf{K})$.
3. $\mathbf{S1}$ is the smallest rte-logic which is closed under (\mathbf{RN}_*) and (\mathbf{RRSE}) (resp. (\mathbf{RRSE}_T)), and contains the formulae (\mathbf{N}) , $\Box(\mathbf{T})$ and $\Box(\mathbf{X})$.
4. $\mathbf{S1}^\circ$ is the smallest rte-logic which is closed under (\mathbf{RN}_*) and (\mathbf{RRSE}) (resp. (\mathbf{RRSE}_T)), and contains the formulae (\mathbf{N}) and $\Box(\mathbf{X})$.

FACT A.14. 1. $\mathbf{S0.9}$ is the smallest strict (resp. strict_T) classical logic which is closed under (\mathbf{RN}_*) , and contains the formulae $\Box(\mathbf{T})$ and $\Box(\mathbf{K})$.

2. $\mathbf{S0.9}^\circ$ is the smallest strict (resp. strict_T) classical logic which is closed under (\mathbf{RN}_*) , and contains the formula $\Box(\mathbf{K})$.
3. $\mathbf{S1}$ is the smallest strict (resp. strict_T) classical logic which is closed under (\mathbf{RN}_*) , and contains the formulae $\Box(\mathbf{T})$ and $\Box(\mathbf{X})$.
4. $\mathbf{S1}^\circ$ is the smallest strict (resp. strict_T) classical logic which is closed under (\mathbf{RN}_*) , and contains the formula $\Box(\mathbf{X})$.

A.4. The logics $\mathbf{S2}$, $\mathbf{S2}^\circ$, $\mathbf{S3}$, $\mathbf{S3.5}$, $\mathbf{S4}$ and $\mathbf{S5}$

We say the a modal logic Λ is closed under *Becker's rule* iff

$$\text{if } \ulcorner \varphi \rightarrow \psi \urcorner \in \Lambda, \text{ then } \ulcorner \Box \varphi \rightarrow \Box \psi \urcorner \in \Lambda. \quad (\mathbf{RB})$$

In [9] (see also [1]) the logic $\mathbf{S2}$ is described as the smallest modal logic which includes $\Box\text{Taut}$, contains the formulae (\mathbf{T}) , $\Box(\mathbf{T})$, and $\Box(\mathbf{K})$, and is closed under (\mathbf{RB}) . Of course, $\mathbf{S2}$ includes $\Box\text{PL}$, contains (\mathbf{K}) and, by Lemma A.3, it is closed under (\mathbf{RN}_*) and (\mathbf{SMP}) .

Moreover, in [1] the logic $\mathbf{S2}^\circ$ is described as the smallest modal logic which includes $\Box\text{Taut}$, contains $\Box(\mathbf{K})$, and is closed under (\mathbf{RB}) and (\mathbf{RN}_*) . Of course, $\mathbf{S2}^\circ$ includes $\Box\text{PL}$, contains (\mathbf{K}) and, by Lemma A.3, it is closed under (\mathbf{SMP}) . So $\mathbf{S2}^\circ \subsetneq \mathbf{S2}$. For example $(\mathbf{T}), \Box(\mathbf{T}) \notin \mathbf{S2}^\circ$.

Moreover, by (\mathbf{RB}) and PL , the logics $\mathbf{S2}$ and $\mathbf{S2}^\circ$ are closed under (\mathbf{RSE}_T) . Thus, by lemmas A.3, A.11 and A.9, the logics $\mathbf{S2}$ and $\mathbf{S2}^\circ$ are strict_T and strict classical, but they are not congruential.

In [4] the *Lewis version* $\mathbf{Lew}(\Lambda)$ of a logic Λ understood as the smallest modal logic which includes Λ and contains the formula (\mathbf{N}) . We have: $\mathbf{S2}^\circ = \mathbf{Lew}(\mathbf{C2})$ and $\mathbf{S2} = \mathbf{Lew}(\mathbf{E2})$. Moreover, for every $\varphi \in \text{For}$: $\varphi \in \mathbf{C2}$ iff $\ulcorner \Box \varphi \urcorner \in \mathbf{S2}^\circ$; $\varphi \in \mathbf{E2}$ iff $\ulcorner \Box \varphi \urcorner \in \mathbf{S2}$ (see e.g. [4, 8]).

In [9] Lemmon proved that $\Box(\mathbf{X}) \in \mathbf{S2}$. His proof shows that also $\Box(\mathbf{X}) \in \mathbf{S2}^\circ$. We have that $\mathbf{S1}^\circ \subsetneq \mathbf{S2}^\circ$ and $\mathbf{S1} \subsetneq \mathbf{S2}$. For example, the formulae ' $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$ ', ' $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ ' and ' $\Diamond(p \wedge q) \rightarrow \Diamond p$ ' belong to $\mathbf{S2}^\circ$, but they are not members of $\mathbf{S1}$.

In [9] the logic $\mathbf{S3}$ is described as the smallest modal logic which includes \Box Taut and contains the formulae (\mathbf{T}) , $\Box(\mathbf{T})$ and $\Box(\mathbf{sK})$. Of course, $\mathbf{S3}$ contains (\mathbf{sK}) and (\mathbf{K}) . Moreover, it contains also $\Box(\mathbf{K})$.⁶ So $\mathbf{S3}$ is also closed under (\mathbf{RB}) , (\mathbf{RSE}_T) , (\mathbf{RSE}) , and it is strict $_T$ and strict classical. We have $\mathbf{S2} \subsetneq \mathbf{S3}$. For example $(\mathbf{sK}), \Box(\mathbf{sK}) \notin \mathbf{S2}$. We have: $\mathbf{S3} = \mathbf{Lew}(\mathbf{E3})$. Moreover, for every $\varphi \in \text{For}$: $\varphi \in \mathbf{E3}$ iff $\ulcorner \Box\varphi \urcorner \in \mathbf{S3}$ (see e.g. [8]).

Åqvist's logic $\mathbf{S3.5}$ is obtained by adding

$$\Diamond p \supset \Box \Diamond p \tag{5}$$

or equivalently

$$p \supset \Box \Diamond p \tag{B}$$

to Lewis' logic $\mathbf{S3}$ (see e.g. [6, p. 208]). We have that $\mathbf{S3} \subsetneq \mathbf{S3.5}$. For example (5), (B) $\notin \mathbf{S3}$.

In [9] the logic $\mathbf{S4}$ is described as the smallest modal logic which contains the formulae (\mathbf{T}) and (\mathbf{sK}) , and is closed under (\mathbf{RN}) . Of course, $\mathbf{S4}$ contains (\mathbf{K}) , $\Box(\mathbf{K})$, $\Box(\mathbf{sK})$, it is closed under (\mathbf{RB}) and (\mathbf{RSE}_T) , and it is strict $_T$ and strict classical. It is known (see e.g. [9]) that $\mathbf{S4}$ is the smallest normal logic which contains the formulae (\mathbf{T}) and

$$\Box p \supset \Box \Box p \tag{4}$$

We have that $\mathbf{S3} \subsetneq \mathbf{S4}$. For example (4) $\notin \mathbf{S3}$.

Finally, $\mathbf{S5}$ is the smallest normal logic which contains (\mathbf{T}) and (5). Moreover, $\mathbf{S5}$ is the smallest normal logic which contains (\mathbf{T}) , (B) and (4); resp. (D), (B) and (4); resp. (D), (B) and (5); resp. (D) (5) and (\mathbf{T}_q) . It is known that $\mathbf{S3.5} \subsetneq \mathbf{S5}$ and $\mathbf{S4} \subsetneq \mathbf{S5}$. For example $\Box(5) \notin \mathbf{S3.5}$ and (5) $\notin \mathbf{S4}$. Note that $\Box(5)$ strengthens $\mathbf{S3}$ to $\mathbf{S5}$ (see e.g. [6, p. 208]).

⁶Notice the formula ' $\Box(p \supset \Box q) \supset \Box(p \supset q)$ ' belongs to $\mathbf{S2}$ and $\mathbf{S3}$. By the substitution $p/\Box(p \supset q)$ and $q/\Box p \supset \Box q$ we have $\ulcorner \Box(\mathbf{sK}) \supset \Box(\mathbf{K}) \urcorner$.

A.5. Location

Using semantics result we will to situate the logics $\mathbf{C1}$, $\mathbf{D1}$, $\mathbf{C1+(T}_q)$, $\mathbf{E1}$, $\mathbf{S0.5}^\circ$, $\mathbf{S0.5}^\circ+(\mathbf{D})$, $\mathbf{S0.5}^\circ+(\mathbf{T}_q)$, $\mathbf{S0.5}$, $\mathbf{C1}_{rte}$, $\mathbf{D1}_{rte}$, $\mathbf{E1}_{rte}+(\mathbf{T}_q)$, $\mathbf{E1}_{rte}$, $\mathbf{S0.5}^\circ_{rte}$, $\mathbf{S0.5}^\circ_{rte}+(\mathbf{D})$, $\mathbf{S0.5}^\circ_{rte}+(\mathbf{T}_q)$ and $\mathbf{S0.5}_{rte}$ among other logics (see Fig. 1; see also diagrams in [1, p. 3], [3, p. 132], [4, p. 21], [9, p. 186], [10, p. 48] and [11, p. 58]).

Using names of formulae, to simplify notation of normal logics we write the *Lemmon code* $\mathbf{KA}_1 \dots \mathbf{A}_n$ to denote the smallest normal logic containing the formulae $(A_1), \dots, (A_n)$ (see [2, 3]). Thus, for example, $\mathbf{KT4}$ is the smallest normal modal logic which contains (\mathbf{T}) and (4) . We standardly put $\mathbf{T} := \mathbf{KT}$ and $\mathbf{D} := \mathbf{KD}$. We have $\mathbf{S4} = \mathbf{KT4}$, $\mathbf{KT} = \mathbf{KDT}_q$, $\mathbf{KB4} = \mathbf{KB5} = \mathbf{K5T}_q$ and $\mathbf{S5} = \mathbf{KT5} = \mathbf{KTB4} = \mathbf{KDB4} = \mathbf{KDB5} = \mathbf{KD5T}_q$ (see e.g. [9, 10, 4]).

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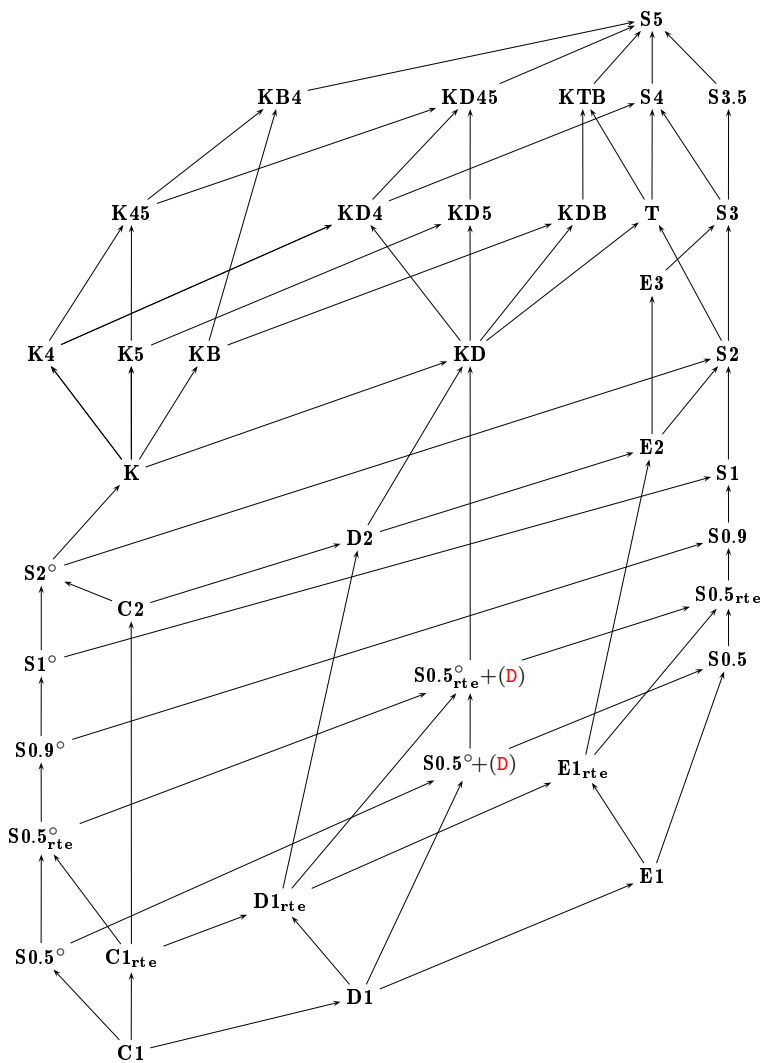


Figure 1. Some t-regular modal logics

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