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# DYADIC DEONTIC LOGIC AND SEMANTIC TABLEAUX

**Abstract.** The purpose of this paper is to develop a class of semantic tableau systems for some dyadic deontic logics. We will consider 16 different pure dyadic deontic tableau systems and 32 different alethic dyadic deontic tableau systems. Possible world semantics is used to interpret our formal languages. Some relationships between our systems and well known dyadic deontic logics in the literature are pointed out and soundness results are obtained for every tableau system. Completeness results are obtained for all 16 pure dyadic deontic systems and for 16 alethic dyadic deontic systems.

*Keywords*: dyadic deontic logic, modal logic, semantic tableau, analytic tableau, conditional obligation, commitment, Lennart Åqvist, Melvin Fitting, Graham Priest.

## 1. Introduction

Many of our most interesting normative sentences seem to be conditional in nature. Consider, for instance, the following examples:

- *If you have promised to do something, you should keep your promise*,
- *If you have borrowed an item from someone, you should return it*,
- *If you have hurt someone, you should apologize*,
- *If you want to be treated in a certain way, you should treat others similarly*,

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- *It is permitted that you do A only if you accept that everyone does A in similar circumstances,*
- *If you are drunk, it is forbidden that you drive.*

However, it is not at all obvious how such sentences should be formalized logically. According to one tradition, we should introduce certain dyadic deontic operators that can be used to symbolize at least some sentences of this kind adequately. The purpose of this essay is to develop semantic tableau systems for some logical systems that include some dyadic deontic operators.

This is not the place to consider all the philosophical arguments for these systems in detail, but I will mention five nice features that they have.

(1) They can be used to avoid Arthur Prior's paradoxes of derived obligation. E.g. none of our systems include the following sentence

$$Fp \to O[p]q$$

So, you are not commited to doing anything given that you have done something forbidden (see [22]).

(2) They can be used to give a fairly plausible formalization of so called contrary-to-duty obligations (obligations that tell us what ought to be the case given that something forbidden is the case). Hence, they can be used to give a reasonable (although not entirely problem-free) solution to Roderick M. Chisholm's contrary-to-duty imperative paradox (see [2]).

(3) None of our systems include the following sentence

$$O[p]r \to O[p \wedge q]r$$

So, they can be used to avoid the problem of overridable conditional obligations. It may be true that it is obligatory that you meet your friend for lunch given that you have promised to meet her for lunch, while it is false that it is obligatory that you meet your friend for lunch given that you have promised to meet her for lunch and your son needs you to drive him to the hospital (because he suddenly became ill).

(4) By defining unconditional obligation, permission and prohibition in terms of conditional obligation, permission and prohibition we can

derive many monadic deontic systems as special cases of our dyadic systems. So, we can preserve many of the nice features that such systems have while at the same time avoiding several problems.

(5) We can give some interesting definitions of the concepts *better than*, *at least as good as* and *equally good as* in terms of our dyadic deontic notions. Hence, we can see how dyadic deontic logic may be connected to a logic of preference in a systematic, rigorous and interesting way. I think that these features, among others, show that our systems are philosophically interesting enough to warrant this investigation.

All in all 16 different pure dyadic deontic tableau systems and 32 alethic dyadic deontic tableau systems are introduced. All of the 48 logics that we discuss in this essay have been described axiomatically by Lennart Åqvist in e.g. [33, 34, 35]: some explicitly, some only implicitly. However, as far as I know, no one has devised any tableau systems for these logics. So, I think that this investigation is also justified from a logical perspective.[1]

The modal parts of the tableau systems that are described in this essay are similar to systems developed by Melvin Fitting in e.g. [5, 6] and by Graham Priest in [21]. Their propositional part can be traced back to Raymond Smullyan ([24, 25, 26, 27]) and Richard Jeffrey ([13]). I think that this form is particularly elegant and easy to apply. We use possible world semantics similar to the kind introduced by Saul Kripke ([14, 15, 16]) to interpret our systems.

In [23] I described a set of counterfactual tableau systems. Technically some of the systems considered in this essay are very similar to some systems introduced in that paper. So, we can apply many of our results about these counterfactual systems more or less directly to our dyadic systems. However, there are also some differences. I think that these differences together with other facts about the dyadic deontic systems considered in this essay motivate a separate investigation of them. Let us mention some facts that justify this investigation.

(1) It may not be immediately obvious to everyone that there are many similarities between several different classical counterfactual logics

[1] Åqvist uses a slightly different notation than we do and his languages do not contain the operators $F$ and $\diamondminus$. So, in a strict sense none of our systems is deductively equivalent to any of his. But I will ignore such subtle differences in this essay. Similar remarks apply to our comparisons with other systems from the literature (see Section 7).

and several classical dyadic deontic systems (this point is of course not new, but it is worth reiterating).

(2) We introduce some pure dyadic deontic systems that do not have any analogues in [23] and show how these are related to each other.

(3) There are some principles that seem reasonable for the subjunctive conditional but not for conditional obligation. E.g.

$$(p \mathrel{\Box\!\!\rightarrow} q) \rightarrow (p \rightarrow q)$$

seems reasonable, but

$$O[p]q \rightarrow (p \rightarrow q)$$

does not seem plausible. So, even though there are similarities between counterfactual logic and dyadic deontic logic there are also important differences. The informal interpretation of our systems and our semantics is, of course, also different.

(4) We consider an alternative semantics that we didn't examine in [23], one that uses models with a preference relation.

(5) We show how our systems are related to some famous deontic systems in the literature.

(6) I fill in some gaps in the soundness and completeness proofs that I intentionally cut out in [23]. We extend the soundness and completeness results to our set of 16 pure dyadic deontic systems. We also obtain soundness results for several systems with respect to our supplemented models that include a dyadic preference relation.

(7) We mention several new theorems and some new derived rules.

(8) We show how the concepts *better than*, *at least as good as* and *equally good as* can be defined in our systems and mention several intuitively plausible theorems that include these notions.

The essay is divided into seven parts. Part 2 deals with syntax and Part 3 with semantics. In Part 4 I describe the tableau systems that are the main focus of the essay and Part 5 contains soundness and completeness proofs. In Part 6 I say some things about the relationships between these systems. Section 7 ends with some notes on how our dyadic systems are related to some other systems described in the literature. I have not been able to establish that the systems including $T\alpha 0$ are complete. Hopefully someone will be able to prove this in the future or show that they are not complete and find some other tableau rule that exactly corresponds to the semantic condition $T\alpha 0$.[2]

## 2. Syntax

We use two different languages, $L_1$ and $L_2$, in this essay.

**Alphabet.** The alphabet of the language $L_1$ consists of 1–4 and that of $L_2$ of 1–5:

1. A denumerably infinite set Prop of proposition letters: $p$, $q$, $r$, $s$, $p_1$, $q_1$, $r_1$, $s_1$, $p_2$, $q_2$, $r_2$, $s_2$, ....
2. The primitive truth-functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication) and $\leftrightarrow$ (material equivalence).
3. Three deontic operators: $O$ (dyadic obligation), $P$ (dyadic permission), and $F$ (dyadic prohibition).
4. $\top$ (verum), $\bot$ (falsum) and the brackets (, ), [ and ].
5. Three alethic operators: $\Box$ (necessity), $\Diamond$ (possibility) and $\ominus$ (impossibility).

**Sentences.** The language $L_1$ is the set of well-formed formulas (wffs) or sentences generated by 1–4 and 6 below and $L_2$ is the set of wffs or sentences generated by 1–6 below.

1. Every proposition letter, $\top$ and $\bot$ are wffs.
2. If $A$ is a wff, so is $\neg A$.
3. If $A$ and $B$ are wffs, so are $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.
4. If $A$ and $B$ are wffs, so are $O[A]B$, $P[A]B$ and $F[A]B$.
5. If $A$ is a wff, so are $\Box A$, $\Diamond A$ and $\ominus A$.
6. Nothing else is a wff.

[2]The first person to produce a formal deontic system seems to have been Ernst Mally in [20]. However, modern deontic logic is often traced back to Georg Henrik von Wright's [32]. Pioneering contributions to dyadic deontic logic are [4, 10, 29, 30, 18, 19, 31]. For an introduction to deontic logic and more references, see for instance [11, 12, 33, 34, 35].

Formulas $O[A]B$, $P[A]B$ and $F[A]B$ represent a kind of conditional obligation, permission and prohibition, respectively. $O[A]B$ is to be read "$B$ is obligatory given that $A$" or "$A$ commits us to $B$" or "if $A$ then it is obligatory that $B$"; $P[A]B$ – "$B$ is permitted given that $A$" or "if $A$ then it is permitted that $B$"; $F[A]B$ – "$B$ is forbidden given that $A$" or "if $A$ then it is forbidden that $B$".[3]

Capital letters "$A$", "$B$", "$C$", ... are used to represent arbitrary (not necessarily atomic) formulas of the object language. The upper case Greek letter $\Sigma$ represents an arbitrary set of formulas. The empty set is denoted by $\emptyset$. Outer brackets around sentences are usually dropped if the result is not ambiguous. "$L_1$" is used in our pure dyadic deontic tableau systems and "$L_2$" is used in our alethic dyadic deontic tableau systems.

**Definitions.**

1. $OA =_{\mathrm{df}} O[\top]A$
2. $PA =_{\mathrm{df}} P[\top]A$
3. $FA =_{\mathrm{df}} F[\top]A$
4. $O'[B]A =_{\mathrm{df}} P[B]\top \wedge O[B]A$
5. $P'[B]A =_{\mathrm{df}} \neg O'[B]\neg A$ (or $O[B]\bot \vee P[B]A$)
6. $F'[B]A =_{\mathrm{df}} \neg P'[B]A$ (or $O'[B]\neg A$ or $(P[B]\top \wedge F[B]A)$)
7. $A \geq B =_{\mathrm{df}} O[A \vee B]\bot \vee P[A \vee B]A$ (or $P'[A \vee B]A$)
8. $A > B =_{\mathrm{df}} P[A \vee B]\top \wedge O[A \vee B]\neg B$ (or $O'[A \vee B]\neg B$)
9. $A = B =_{\mathrm{df}} O[A \vee B]\bot \vee (P[A \vee B]A \wedge P[A \vee B]B)$
   (or $P'[A \vee B]A \wedge P'[A \vee B]B$)

*Definitions* 1–3 are definitions of the monadic deontic operators $O$ ("it is obligatory that"), $P$ ("it is permitted that") and $F$ ("it is forbidden that") in terms of our dyadic deontic operators and verum. The *definitions* 4–9 will be used to compare our systems with some other systems in the literature (see Section 7). $O'[B]A$, $P'[B]A$ correspond to Åqvist's symbols $O_B^{dfl}A$ and $P_B^{dfl}A$, respectively. These symbols provide us with

[3]Other symbols are sometimes used for the same notions, e.g. instead of $O[A]B$ and $P[A]B$ some write $O_A B$ and $P_A B$, some $O(B/A)$ and $P(B/A)$, and others $(A\,O\,B)$ and $(A\,P\,B)$.

some alternative readings of the concepts of conditional obligation, permission and prohibition. $A \geq B$ is to be read "$A$ is at least as good as (better than or equally good as) $B$", $A > B$ – "$A$ is better than $B$" and $A = B$ – "$A$ is as good as (equally good as) $B$". So, our dyadic operators can be used to give us a possible interpretation of the concepts *better than*, *at least as good as* and *equally good as*.

## 3. Semantics

### 3.1. Basic notions

**Frames.** We will consider two kinds of frames in this essay: (ordinary) frames and supplemented frames. An ordinary frame $F$ is a relational structure $\langle W, \{R_A : A \in L\}\rangle$, where $W$ is a non-empty set of possible worlds and $\{R_A : A \in L\}$ is a set of dyadic relations on $W$, one for each sentence, $A$, in $L$, where $L$ is the language we are interested in ($L_1$ or $L_2$). So, for every $A$ in $L$, $R_A \subseteq W \times W$.

A supplemented frame $F_s$ is a relational structure $\langle W, \{R_A : A \in L\}, \geq\rangle$, where $W$ and $\{R_A : A \in L\}$ are exactly as in an ordinary frame and $\geq$ is a preference relation defined over the elements in $W$, i.e. $\geq \subseteq W \times W$. If it is clear that we are talking about a supplemented frame, we will sometimes drop the subscript. Intuitively, $w \geq w'$ means that $w$ is at least as good as $w'$.

**Models.** We will discuss two kinds of models in this essay: (ordinary) models and supplemented models. An ordinary model $M$ is a pair $\langle F, V\rangle$ where: $F$ is an (ordinary) frame, and $V$ is a valuation or interpretation function, which assigns a truth-value $T$ (true) or $F$ (false) to every proposition letter $p$ in each world $w \in W$.

A supplemented model $M_s$ is a pair $\langle F_s, V\rangle$ where: $F_s$ is a supplemented frame, and $V$ is exactly as in an ordinary model.

We will sometimes drop the subscript if it is clear from the context that we are talking about supplemented models (or if we are talking about any model whatsoever, ordinary or supplemented).

We shall also speak of a model directly as a relational structure $\langle W, \{R_A : A \in L\}, V\rangle$ ($\langle W, \{R_A : A \in L\}, \geq, V\rangle$) to save space.

In a supplemented model the accessibility relations can be defined in terms of the preference relation over our possible worlds in the following way, for every $A \in L$:

$\gamma 0.$ $xR_A y$ iff $\Vdash_{M,y} A \mathbin{\&} \forall z(\Vdash_{M,z} A \Rightarrow y \geq z)$.

**Truth conditions.** Let $M$ be any model (ordinary or supplemented), let $w$ be any member of $W$ and let $A$ be in $L$ ($L_1$ or $L_2$). To mean that $A$ is true at a possible world $w$ in the model $M$ we write $\Vdash_{M,w} A$. The truth conditions for proposition letters, $\top$, $\bot$ and sentences built by truth functional connectives are the usual ones. $\Box A$, $\Diamond A$ and $\diamondminus A$ are interpreted as in [23]. The truth conditions for the remaining sentences in $L$ are given by the following clauses:

1. $\Vdash_{M,w} O[A]B$ iff for all $w' \in W$ such that $wR_A w'$: $\Vdash_{M,w'} B$
2. $\Vdash_{M,w} P[A]B$ iff for at least one $w' \in W$ such that $wR_A w'$: $\Vdash_{M,w'} B$
3. $\Vdash_{M,w} F[A]B$ iff for all $w' \in W$ such that $wR_A w'$: $\Vdash_{M,w'} \neg B$.

**Validity, entailment, countermodel, satisfiability etc.** The concepts of validity, entailment, countermodel, satisfiability etc. can be defined in the usual way.[4] $\Vdash_{\mathcal{M}} B$ says that $B$ is valid in $\mathcal{M}$, where $\mathcal{M}$ is a class of models, and $\Sigma \Vdash_{\mathcal{M}} B$ says that $B$ is a consequence of $\Sigma$ in $\mathcal{M}$.

### 3.2. Conditions on a model

We will consider several different conditions on our models divided into three groups in this essay. The first group of conditions is used to define a set of pure dyadic deontic systems (table 1), the second a set of alethic dyadic deontic systems based on ordinary models (table 2) and the third a set of alethic dyadic deontic systems based on supplemented models (table 3).

The variables $x$, $y$, $z$ are taken to range over $W$. Corresponding to the conditions in group 1 there are four different tableau rules (see Section 4.2.3).

$\|A\|^M = \{w \in W : \Vdash_{M,w} A\}$, i.e. $\|A\|^M$ is the set of all worlds in $M$ where $A$ is true, or, in other words, the set of all $A$-worlds in $M$. Corresponding to the conditions in group 2 there are six different tableau rules (see Sections 4.2.4 and 4.2.5).

[4] For an idea of how to do this, see [21].

| | For all $A$ and for all $B$ |
|---|---|
| $CD\tau$ | $\forall x\forall y\forall z((xR_Ay \mathbin{\&} yR_Bz) \Rightarrow xR_Bz)$ |
| $CD\epsilon$ | $\forall x\forall y\forall z((xR_Ay \mathbin{\&} xR_Bz) \Rightarrow yR_Bz)$ |
| $CD\rho'$ | $\forall x\forall y(xR_Ay \Rightarrow yR_Ay)$ |
| $CD\sigma'$ | $\forall x\forall y\forall z((xR_Ay \mathbin{\&} yR_Az) \Rightarrow zR_Ay)$ |

Table 1. Conditions on a model (group 1)

| | For all $A$ and for all $B$ |
|---|---|
| $Ca6$ | $\forall x\forall y\forall z(xR_Ay \Rightarrow zR_Ay)$ |
| $C\alpha 0$ | $\Vert A\Vert^M = \Vert B\Vert^M \Rightarrow R_A = R_B$ |
| $C\alpha 1$ | $\forall x\forall y(xR_Ay \Rightarrow \Vdash_{M,\,y} A)$ |
| $C\alpha 2$ | $\forall x\forall y((xR_Ay \mathbin{\&} \Vdash_{M,\,y} B) \Rightarrow xR_{A\wedge B}y)$ |
| $C\alpha 3$ | $\forall x((\Vert A\Vert^M \neq \emptyset) \Rightarrow \exists y xR_Ay)$ |
| $C\alpha 4$ | $\forall x\forall y\forall z((xR_Ay \mathbin{\&} \Vdash_{M,\,y} B) \Rightarrow (xR_{A\wedge B}z \Rightarrow (xR_Az \mathbin{\&} \Vdash_{M,\,z} B)))$ |

Table 2. Conditions on a model (group 2)

### 3.3. Classification of some models

The conditions introduced in Section 3.2 can be used to obtain a categorization of the set of all models into various kinds. In general, we shall say that $\mathcal{M}(C_1, \ldots, C_n)$ is the class of (all) models that satisfy the conditions $C_1, \ldots, C_n$. $\mathcal{M}(V)$ will denote the class of all models.

A model that satisfies any combination of the conditions in group 1 but no other conditions will be called a pure dyadic deontic model and a model that satisfies at least condition *Ca6* (that is *Ca6* plus any other combination of conditions) will be called an alethic dyadic deontic model.

$\mathcal{M}(G)$ stands for the class of all models that satisfy all conditions in group 1 and group 2. In Section 5 we will show that the tableau system *TG* is sound with respect to $\mathcal{M}(G)$.

Following Åqvist we will also use the following classification of different models (table 4).

### 3.4. Relations between our conditions

Not all combinations of the conditions in Section 3.2 are independent as the following theorem shows.

| | For all A and for all B |
|---|---|
| $\gamma 0$ | $xR_A y$ iff $\Vdash_{M,y} A \,\&\, \forall z(\Vdash_{M,z} A \Rightarrow y \geq z)$ |
| $\delta 1$ | $\forall x\, x \geq x$ |
| $\delta 2$ | If $\|A\|^M \neq \emptyset$ then $\{x \in \|A\|^M : (\forall y \in \|A\|^M)\, x \geq y\} \neq \emptyset$ |
| $\delta 3$ | $\forall x \forall y \forall z((x \geq y \,\&\, y \geq z) \Rightarrow x \geq z)$ |
| $\delta 4$ | $\forall x \forall y(x \geq y$ or $y \geq x$ or $(x \geq y \,\&\, y \geq x))$ |

Table 3. Conditions on a model (group 3)

| The class of all... | = | the class of models that satisfy... |
|---|---|---|
| minimal H-models | | *Ca6* |
| H-models | | $\gamma 0$ |
| H1-models | | $\gamma 0$ and $\delta 1$ |
| H2-models | | $\gamma 0$, $\delta 1$ and $\delta 2$ |
| H3-models | | $\gamma 0$, $\delta 1$, $\delta 2$ and $\delta 3$ |
| strong H3-models | | $\gamma 0$, $\delta 1$, $\delta 2$, $\delta 3$ and $\delta 4$ |

Table 4. Classification of supplemented models

Theorem 1. (i) *All models that satisfy condition $CD\epsilon$ also satisfy conditions $CD\rho'$ and $CD\sigma'$.*

(ii) *All models that satisfy Ca6 also satisfy $CD\tau$, $CD\epsilon$, $CD\rho'$ and $CD\sigma'$.*

(iii) *If M is any H-model or H1-model, then M satisfies Ca6, $C\alpha 0$, $C\alpha 1$ and $C\alpha 2$.*

(iv) *If M is any H2-model, M satisfies Ca6, $C\alpha 0$, $C\alpha 1$, $C\alpha 2$ and $C\alpha 3$.*

(v) *If M is any H3-model or strong H3-model, then M satisfies Ca6, $C\alpha 0$, $C\alpha 1$, $C\alpha 2$, $C\alpha 3$ and $C\alpha 4$.*

(vi) *Let M be any H-model. Then the truth conditions for dyadic obligation, permission and prohibition, can be equivalently stated in the following way:*

$$\Vdash_{M,x} O[A]B \text{ iff } \forall y((\Vdash_{M,y} A \,\&\, \forall z(\Vdash_{M,z} A \Rightarrow y \geq z)) \Rightarrow \Vdash_{M,y} B),$$
$$\Vdash_{M,x} P[A]B \text{ iff } \exists y((\Vdash_{M,y} A \,\&\, \forall z(\Vdash_{M,z} A \Rightarrow y \geq z)) \,\&\, \Vdash_{M,y} B),$$
$$\Vdash_{M,x} F[A]B \text{ iff } \forall y((\Vdash_{M,y} A \,\&\, \forall z(\Vdash_{M,z} A \Rightarrow y \geq z)) \Rightarrow \Vdash_{M,y} \neg B),$$

*where $x$, $y$ and $z$ range over possible worlds.*

Proof. Left to the reader. ⊣

According to part (vi) in Theorem 1 we can say that it is obligatory that $B$ given $A$ iff $B$ is true in all the best $A$-worlds, it is permitted that $B$ given $A$ iff $B$ is true in at least one of the best $A$-worlds and it is forbidden that $B$ given that $A$ iff $B$ is false in all the best $A$-worlds.

Obviously, $\forall x\forall y\forall z((xR_Ay \;\&\; yR_Bz) \Rightarrow xR_Bz)$ entails $\forall x\forall y\forall z((xR_Ay \;\&\; yR_Az) \Rightarrow xR_Az)$ and $\forall x\forall y\forall z((xR_Ay \;\&\; xR_Bz) \Rightarrow yR_Bz)$ entails $\forall x\forall y\forall z((xR_Ay \;\&\; xR_Az) \Rightarrow yR_Az)$. It follows from Theorem 1 that some classes of models will turn out to be identical even though they are defined in different ways, as our next theorem shows.

THEOREM 2. (i) $\mathcal{M}(CD\epsilon) = \mathcal{M}(CD\rho', CD\epsilon) = \mathcal{M}(CD\sigma', CD\epsilon) = \mathcal{M}(CD\rho', CD\sigma', CD\epsilon)$.

(ii) $\mathcal{M}(CD\tau, CD\epsilon) = \mathcal{M}(CD\rho', CD\tau, CD\epsilon) = \mathcal{M}(CD\sigma', CD\tau, CD\epsilon) = \mathcal{M}(CD\rho', CD\sigma', CD\tau, CD\epsilon)$.

(iii) *Let $\mathcal{C}$ denote any combination of the conditions in group 1 (there are 16 of them). Then $\mathcal{M}(Ca6) = \mathcal{M}(Ca6, \mathcal{C})$.*

PROOF. This follows immediately from Theorem 1. ⊣

$\mathcal{M}(Ca6) = \mathcal{M}(Ca6, CD\tau)$ and $\mathcal{M}(Ca6) = \mathcal{M}(Ca6, CD\tau, CD\epsilon)$ are examples of instances of part (iii) in Theorem 2. There are other identities too as the reader may easily verify, but these suffice to illustrate our point.

### 3.5. Logical systems

The set of all sentences in $L$ that are valid in a class of models $\mathcal{M}$ is called the logical system of (the system of or the logic of) $\mathcal{M}$, in symbols $S(\mathcal{M}) = \{A \in L : \Vdash_{\mathcal{M}} A\}$.

THEOREM 3. (i) $S(\mathcal{M}(CD\epsilon)) = S(\mathcal{M}(CD\rho', CD\epsilon)) = S(\mathcal{M}(CD\sigma', CD\epsilon)) = S(\mathcal{M}(CD\rho', CD\sigma', CD\epsilon))$.

(ii) $S(\mathcal{M}(CD\tau, CD\epsilon)) = S(\mathcal{M}(CD\rho', CD\tau, CD\epsilon)) = S(\mathcal{M}(CD\sigma', CD\tau, CD\epsilon)) = S(\mathcal{M}(CD\rho', CD\sigma', CD\tau, CD\epsilon))$.

(iii) *Let $\mathcal{C}$ be as in Theorem 2. Then $S(\mathcal{M}(Ca6)) = S(\mathcal{M}(Ca6, \mathcal{C}))$.*

PROOF. This follows immediately from Theorem 2. ⊣

Part (iii) in Theorem 3 is a summary of several different identities. Above we saw, for instance, that $\mathcal{M}(Ca6) = \mathcal{M}(Ca6, CD\tau)$ and $\mathcal{M}(Ca6) = \mathcal{M}(Ca6, CD\tau, CD\epsilon)$. So, it is easy to see that $S(\mathcal{M}(Ca6)) = S(\mathcal{M}(Ca6, CD\tau))$ and $S(\mathcal{M}(Ca6)) = S(\mathcal{M}(Ca6, CD\tau, CD\epsilon))$ are examples of instances of part (iii). (In the next section we will develop tableau systems that correspond to these systems.)

## 4. Semantic tableaux and deontic logic

### 4.1. Semantic tableaux

The kind of semantic tableaux systems I use is inspired by Fitting and Priest (see e.g. [5], [6] and [21]). The propositional part is similar to systems introduced by Raymond Smullyan ([24, 25, 26, 27]) and Richard Jeffrey ([13]).[5]

The concepts of semantic tableau, branch, open and closed branch etc. are defined as in Priest's [21].

### 4.2. Tableau rules

#### 4.2.1. Propositional rules and Alethic rules

We use the same propositional rules as in [21] and the same alethic rules as in [23].

#### 4.2.2. Dyadic deontic rules

There are six dyadic deontic rules (see table 5), two for each dyadic deontic operator.

#### 4.2.3. Dyadic deontic accessibility rules

The dyadic deontic accessibility rules correspond to the conditions on our models discussed in Section 3.2, table 1. There are four rules of this kind, displayed in table 6.

[5] See [23], footnote 1 for some references.

| D O-pos ($O$) | D P-pos ($P$) | D F-pos ($F$) |
|---|---|---|
| $O[A]B, i$ | $P[A]B, i$ | $F[A]B, i$ |
| $ir_A j$ | $\downarrow$ | $\downarrow$ |
| $\downarrow$ | $ir_A j$ | $O[A]\neg B, i$ |
| $B, j$ | $B, j$ | |
| | where j is new | |
| **D O-neg ($\neg O$)** | **D P-neg ($\neg P$)** | **D F-neg ($\neg F$)** |
| $\neg O[A]B, i$ | $\neg P[A]B, i$ | $\neg F[A]B, i$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $P[A]\neg B, i$ | $O[A]\neg B, i$ | $P[A]B, i$ |

Table 5. Dyadic deontic rules

| D Trans ($TD\tau$) | D Euclid ($TD\epsilon$) | D Al ref ($TD\rho'$) | D Al sym ($TD\sigma'$) |
|---|---|---|---|
| $ir_A j$ | $ir_A j$ | $ir_A j$ | $ir_A j$ |
| $jr_B k$ | $ir_B k$ | $\downarrow$ | $jr_A k$ |
| $\downarrow$ | $\downarrow$ | $jr_A j$ | $\downarrow$ |
| $ir_B k$ | $jr_B k$ | | $kr_A j$ |

Table 6. Dyadic deontic accessibility rules

#### 4.2.4. Basic alethic dyadic deontic rules

There are two basic alethic dyadic deontic rules: *CUT* and *Ta6* (see table 7). These rules are included in every alethic system. The *CUT* rule is interpreted as in [23].

From *Ta6* all rules in table 6 can be derived. I call this rule "*Ta6*" because it corresponds to Åqvist's axiom $a6$ in e.g. Åqvist [35], p. 237. According to this rule we may add $kr_A j$ for any $k$ on any open branch on which $ir_A j$ occurs. *Ta6* corresponds to condition *Ca6* in Section 3.2, table 2.

| *CUT* | *Ta6* |
|---|---|
| $*$ | $ir_A j$ |
| $\swarrow \quad \searrow$ | $\downarrow$ |
| $\neg A, i \quad A, i$ | $kr_A j$ |
| for every A | |

Table 7. Basic alethic dyadic deontic rules

#### 4.2.5. Alethic dyadic deontic accessibility rules

There are five different alethic dyadic deontic accessibility rules (see table 8) that correspond to the conditions $C\alpha0$–$C\alpha4$ in Section 3.2, table 2. These rules are similar to the accessibility rules found in [23], Section 4.2.5 and they are interpreted similarly.

| $T\alpha0$ | $T\alpha1$ | $T\alpha2$ | $T\alpha3$ | $T\alpha4$ |
|---|---|---|---|---|
| If A is of the form $\Box(A \leftrightarrow B) \rightarrow (O[A]C \leftrightarrow O[B]C)$, A, i can be added to any open branch on which i occurs. | $ir_A j$<br>$\downarrow$<br>$A, j$ | $ir_A j$<br>$B, j$<br>$\downarrow$<br>$ir_{A \wedge B} j$ | $A, i$<br>$\downarrow$<br>$jr_A k$<br>where k is new | $ir_A j$<br>$B, j$<br>$ir_{A \wedge B} k$<br>$\downarrow$<br>$ir_A k$<br>$B, k$ |

Table 8. Alethic dyadic deontic accessibility rules

#### 4.2.6. Derived rules

Let us mention some derived rules that can be used to abbreviate our proofs.

The Global Assumption Rule ($GA$). If $A$ has a tableau proof, then $A, i$ can be added as a line to any open branch of a tableau, for any $i$.

THEOREM 4. *The Global Assumption Rule is admissible in any alethic dyadic deontic system, i.e. GA can be added without expanding the class of provable sentences.*

PROOF. Left to the reader. Use $CUT$. ⊣

$GA$ together with the theorem schema

$$\Box(A \leftrightarrow B) \rightarrow (O[A]C \leftrightarrow O[B]C)$$

can be used to obtain several useful derived rules.

By using these derived rules our tableau proofs can become significantly shorter.

THEOREM 5. *Every rule in table 9 is admissible in any alethic dyadic deontic system that contains $T\alpha0$.*

PROOF. Left to the reader. ⊣

| $DR1$ | $DR3$ | $DR5$ |
|---|---|---|
| $\Box(A \leftrightarrow B), i$ | $\Box(A \leftrightarrow B), i$ | $\Box(A \leftrightarrow B), i$ |
| $O[A]C, i$ | $P[A]C, i$ | $F[A]C, i$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $O[B]C, i$ | $P[B]C, i$ | $F[B]C, i$ |
| $DR2$ | $DR4$ | $DR6$ |
| $\Box(A \leftrightarrow B), i$ | $\Box(A \leftrightarrow B), i$ | $\Box(A \leftrightarrow B), i$ |
| $O[B]C, i$ | $P[B]C, i$ | $F[B]C, i$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $O[A]C, i$ | $P[A]C, i$ | $F[A]C, i$ |

Table 9. Derived rules (group I)

| $DR7$ | $DR8$ | $DR9$ |
|---|---|---|
| $\Box(A \rightarrow B), i$ | $O[A]B, i$ | $\Box(A \rightarrow B), i$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $O[A]B, i$ | $O[\top](A \rightarrow B), i$ | $O[\top](A \rightarrow B), i$ |

Table 10. Derived rules (group II)

THEOREM 6. (i) *DR7 is admissible in any alethic dyadic deontic system that contains $T\alpha 1$.*

(ii) *DR8 is admissible in any system of this kind that includes $T\alpha 0$ and $T\alpha 2$.*

(iii) *DR9 is admissible in any system of this kind that contains $T\alpha 0$, $T\alpha 1$ and $T\alpha 2$ (see table 10).*

PROOF. Left to the reader. ⊣

### 4.3. Conventions for applying rules

We use similar conventions for applying our rules as can be found in [23], Section 4.3.

### 4.4. Tableau systems

By a dyadic deontic tableau system we mean a set of (primitive) tableau rules that includes all dyadic deontic and all propositional rules. By an alethic dyadic deontic tableau system we mean a set of (primitive) tableau rules that includes all propositional rules, all dyadic deontic rules, all alethic rules and the rules *Ta6* and *CUT*. By a pure dyadic deontic tableau system we mean a dyadic deontic tableau system that does not contain anything other than the propositional rules and the rules in table 5 and 6 in Section 4.2. The smallest dyadic deontic system, which we call *DDL*, does not contain any accessibility rules. By adding dyadic deontic accessibility rules to *DDL* we obtain extensions of this system. Since there are four different dyadic deontic accessibility rules (see table 6), we have 16 different pure dyadic deontic tableau systems. Not more than 10 of them are distinct however (see Section 6). The smallest alethic dyadic deontic system is called *ADDL*. By adding alethic dyadic deontic accessibility rules to *ADDL*, we obtain 32 different extensions of this system. For there are five such rules (see table 8).[6]

We will call the tableau system that includes all rules we discuss in this essay *TG*. *TG* is our strongest system (see Section 6).

*Example.* *DDLTD*$\tau$ is the (pure) dyadic deontic tableau system that includes all propositional rules, all dyadic deontic rules and *TD*$\tau$, and *ADDLT*$\alpha$*0T*$\alpha$*1T*$\alpha$*2* is the alethic dyadic deontic tableau system that includes all propositional rules, all dyadic deontic rules, all alethic rules, *CUT*, *Ta6*, *T*$\alpha$*0*, *T*$\alpha$*1* and *T*$\alpha$*2*. If it doesn't lead to any ambiguity, we can drop the letters "*T*", "*D*" and "$\alpha$" in our names. So, *DDL*$\tau$ = *DDLTD*$\tau$ and *ADDL*012 = *ADDLT*$\alpha$*0T*$\alpha$*1T*$\alpha$*2*. And if it is clear from the context that we are speaking of a dyadic deontic system or an alethic dyadic deontic system, respectively, we may drop the letters in the beginning of the name. So, $\tau$ may be used as an abbreviation of *DDLTD*$\tau$ and *012* of *ADDLT*$\alpha$*0T*$\alpha$*1T*$\alpha$*2* in certain contexts.

[6]There are of course many other combinations of our rules. If we suppose that every system includes all propositional rules, all dyadic deontic rules and all alethic rules, there are still over 2000 (2048 to be exact) different combinations of our remaining rules. (If we count the global assumption rule as a separate rule there are 4096 different combinations.) However, many of the systems that result from such combinations are deductively equivalent (i.e. contain exactly the same set of theorems). E.g. everything that can be proved in our strongest pure dyadic deontic system can also be proved in any alethic dyadic deontic system; in the light of *Ta6* it is redundant to add any dyadic deontic accessibility rule to any alethic dyadic deontic system. So, we will focus on the systems we have mentioned.

### 4.5. Proofs, derivations and syntactic validity

The concepts of proof, theorem, derivability etc. can now be defined in the usual way (see e.g. [21] for an idea of how to do it). $\vdash_S B$ says that $B$ is provable in the tableau system $S$ (or that $B$ is a theorem in $S$), and $\Sigma \vdash_S B$ that $B$ is derivable from $\Sigma$ in $S$.

### 4.6. Logical systems

Let $S'$ be a pure dyadic deontic tableau system and $S''$ be an alethic dyadic deontic tableau system. Then $\mathrm{L}(S') = \{A \in L_1 : \vdash_{S'} A\}$ (the logic of $S'$), and $\mathrm{L}(S'') = \{A \in L_2 : \vdash_{S''} A\}$ (the logic of $S''$).

### 4.7. Examples

In this section we will consider some examples of theorems in some systems. Proofs are usually easy and are left to the reader.

| System | Theorem | Condition on $M$ |
|---|---|---|
| $DDL$ | $O[r](p \to q) \to (O[r]p \to O[r]q)$ | – |
| $DDL\tau$ | $O[r]p \to O[s]O[r]p$ | $CD\tau$ |
| $DDL\epsilon$ | $P[r]O[s]p \to O[s]p$ | $CD\epsilon$ |
| $DDL\rho'$ | $O[r](O[r]p \to p)$ | $CD\rho'$ |
| $DDL\sigma'$ | $O[r](P[r]O[r]p \to p)$ | $CD\sigma'$ |

Table 11. Examples of theorems in some dyadic deontic tableau systems

| System | Theorem | Condition on $M$ |
|---|---|---|
| $ADDL$ | $\Box q \to O[p]q$ | – |
| $ADDL$ | $O[p]q \to \Box O[p]q$ | $Ca6$ |
| $ADDL0$ | $\Box(p \leftrightarrow q) \to (O[p]r \leftrightarrow O[q]r)$ | $C\alpha 0$ |
| $ADDL1$ | $O[p]p$ | $C\alpha 1$ |
| $ADDL2$ | $O[p \wedge q]r \to O[p](q \to r)$ | $C\alpha 2$ |
| $ADDL3$ | $\Diamond p \to (O[p]q \to P[p]q)$ | $C\alpha 3$ |
| $ADDL4$ | $P[p]q \to (O[p](q \to r) \to O[p \wedge q]r)$ | $C\alpha 4$ |

Table 12. Examples of theorems in some alethic dyadic deontic tableau systems

Theorem 7. *The sentences in tables 11 and 12 are theorems in the indicated systems.*

Proof. Left to the reader. ⊣

Theorem 8. (i) *Let $A$ be a sentence in table 9 in* [23] *(except the first) and let $t(A)$ be the result of replacing every subsentence in $A$ of the form $C \Box\!\!\rightarrow B$ by $O[C]B$, $C \Diamond\!\!\rightarrow B$ by $P[C]B$, and $C \Box\!\!\rightarrow \neg B$ by $F[C]B$. Then $t(A)$ is a theorem in DDL.*

(ii) *Let $A$ be a sentence in table 13 in* [23] *and let $t(A)$ be defined as in* (i)*. Then $t(A)$ is a theorem in ADDL.*

(iii) *Let $A$ be a sentence in table 10 or in table 14 in* [23] *and let $t(A)$ be defined as in* (i)*. Then $t(A)$ is a theorem in ADDL3.*

(iv) *Let $A$ be a sentence in table 11 in* [23] *and let $t(A)$ be the result of replacing every subsentence in $A$ of the form $C \Box\!\!\rightarrow B$ by $O[C]B$, $C \Diamond\!\!\rightarrow B$ by $P[C]B$, $C \Box\!\!\Rightarrow B$ by $O'[C]B$, and $C \Diamond\!\!\Rightarrow B$ by $P'[C]B$. Then $t(A)$ is a theorem in ADDL1+Def. 4–5.*

(v) *Let $A$ be a sentence in table 15 in [23] and let $t(A)$ be defined as in* (i)*. Then $t(A)$ is a theorem in ADDL13.*

(vi) *Let $A$ be a sentence in table 9 in [23] (except the first) and let $t(A)$ be the result of replacing every subsentence in $A$ of the form $C \Box\!\!\rightarrow B$ by $O'[C]B$, $C \Diamond\!\!\rightarrow B$ by $P'[C]B$, and $C \Box\!\!\rightarrow \neg B$ by $F'[C]B$. Then $t(A)$ is a theorem in DDL+Def. 4–6.*

Proof. Left to the reader. ⊣

Theorem 9. *The smallest normal monadic deontic system* **OK** *conceived of as a set of sentences is included in DDL+Def. 1–3.*[7]

Proof. Left to the reader. ⊣

Many other monadic deontic systems are also included in our dyadic systems in a similar way.

Theorem 10. (i) *The following sentences are theorems in DDL:*

$$(P[s]p \wedge O[s](p \rightarrow (q \vee r))) \rightarrow (P[s]q \vee P[s]r)$$
$$(P[s]p \wedge O[s](p \rightarrow (q \wedge r))) \rightarrow (P[s]q \wedge P[s]r)$$

[7] See e.g. [33, 34, 35] for a discussion of this system.

(ii) *The following sentences are theorems in ADDL+Def. 4–6*:

$$P[p]q \to \Box P[p]q$$
$$F[p]q \to \Box F[p]q$$
$$\Box p \to O[\neg p]p$$
$$P[p]p \to \Diamond p$$
$$\Diamond\!\!\!\!\!-\, p \to O[p]\neg p$$
$$O'[p]q \to \Box O'[p]q$$
$$P'[p]q \to \Box P'[p]q$$
$$F'[p]q \to \Box F'[p]q$$
$$(P[s]p \wedge \Box(p \to (q \vee r))) \to (P[s]q \vee P[s]r)$$
$$(P[s]p \wedge \Box(p \to (q \wedge r))) \to (P[s]q \wedge P[s]r)$$

(iii) *The following sentences are theorems in ADDL0+Def. 4–6*:

$$\Box(p \leftrightarrow q) \to (O'[p]r \leftrightarrow O'[q]r)$$
$$\Box(p \leftrightarrow q) \to (P'[p]r \leftrightarrow P'[q]r)$$
$$\Box(p \leftrightarrow q) \to (F'[p]r \leftrightarrow F'[q]r)$$

(iv) *The following sentences are theorems in ADDL1*:

$$P[p]\top \leftrightarrow P[p]p$$
$$O[\neg p]\bot \leftrightarrow O[\neg p]p$$
$$O[p]\bot \leftrightarrow O[p]\neg p$$

(v) *The following sentences are theorems in ADDL3*:

$$\Diamond p \to P[p]\top$$
$$O[\neg p]\bot \to \Box p$$
$$O[p]\bot \to \Diamond\!\!\!\!\!-\, p$$
$$\Diamond s \to ((O[s]p \wedge O[s](p \to (q \vee r))) \to (P[s]q \vee P[s]r))$$
$$\Diamond s \to ((O[s]p \wedge \Box(p \to (q \vee r))) \to (P[s]q \vee P[s]r))$$

(vi) *The following sentences are theorems in ADDL02+Def. 1–3*:

$$O[p]q \to O(p \to q)$$
$$(O[p](q \vee r) \wedge (Fq \wedge Fr)) \to Fp$$

$$O[p]q \to (Op \to Oq)$$
$$((Op \wedge Oq) \wedge O[p \wedge q]r) \to Or$$
$$(O[p]q \wedge Op) \to Oq$$
$$((Op \vee Oq) \wedge O[p \vee q]r) \to Or$$
$$O[p]q \to (Pp \to Pq)$$
$$(Op \wedge O[p](q \wedge r)) \to (Oq \wedge Or)$$
$$(O[p]q \wedge Pp) \to Pq$$
$$(Fr \wedge O[p \vee q]r) \to (Fp \wedge Fq)$$
$$O[p]q \to (Fq \to Fp)$$
$$((Pp \vee Pq) \wedge O[p \vee q]r) \to Pr$$
$$(O[p]q \wedge Fq) \to Fp$$
$$(O[p](q \wedge r) \wedge (Fq \vee Fr)) \to Fp$$

(vii) *The following sentences are theorems in ADDL13+Def. 4–6*:

$$\Box p \leftrightarrow O[\neg p]\bot$$
$$\ominus p \leftrightarrow O[p]\bot$$
$$\Diamond p \leftrightarrow P[p]\top$$
$$O'[p]q \leftrightarrow (\Diamond p \wedge O[p]q)$$
$$P'[p]q \leftrightarrow (\Diamond p \to P[p]q)$$
$$F'[p]q \leftrightarrow (\Diamond p \wedge F[p]q)$$

(viii) *The following sentence is a theorem in ADDL24*:

$$P[p]q \to (O[p \wedge q]r \leftrightarrow O[p](q \to r))$$

Proof. Left to the reader. ⊣

## 5. Soundness and completeness theorems

The concepts of soundness and completeness are defined as in [23], Section 5.

Let $S = STA_1...TA_n$, where $STA_1...TA_n$ is the tableau system constructed from $DDL$ by adding the tableau rules $TA_1$, ..., $TA_n$, if $S$ is a pure dyadic deontic system, and $STA_1...A_n$ is the tableau system constructed from *DDL* by adding the tableau rules $TA_1, \ldots, TA_n$ and *CUT*, if $S$ is an alethic dyadic deontic system. Then we shall say that the class of models, $\mathcal{M}$, corresponds to $S$ just in case $\mathcal{M} = \mathcal{M}(CA_1...CA_n)$, i.e. the class of models that satisfy $CA_1, \ldots,$ and $CA_n$.

### 5.1. Soundness theorems

Let $M$ be any model, and $b$ be any branch of a tableau. Then $b$ (or, to be more precise, the set of sentences on $b$) is satisfiable in $M$ iff there is a function, $f$, from the set of natural numbers $\{0, 1, 2, 3, \ldots\}$ to $W$ such that: (i) $A$ is true at $f(i)$ in $M$, for every node $A, i$ on $b$, and (ii) if $ir_Aj$ is on $b$, then $f(i)R_Af(j)$ in $M$. If $f$ fulfills these conditions, we say that $f$ shows that $b$ is satisfiable in $M$.

Lemma 11 (Soundness lemma). *Let $b$ be any branch of a tableau, and $M$ be any model. If $b$ is satisfiable in $M$, and a tableau rule is applied to it, then it produces at least one extension, $b'$, of $b$ such that $b'$ is satisfiable in $M$.*

Proof. First the Soundness lemma is proved for *DDL*. Then it is extended to the other systems. This is done as usual.

Propositional rules. The proof is standard (see e.g. [21]).

Dyadic deontic rules, *CUT*, $T\alpha 0$, $T\alpha 1$, $T\alpha 3$. The proof is similar to proofs found in [23], Section 5.1, just replace $A \;\Box\!\!\rightarrow B$ by $O[A]B$ and $A \;\Diamond\!\!\rightarrow B$ by $P[A]B$.

(D Al ref). Assume that $ir_Aj$ is on $b$, and that we apply D Al ref to give an extended branch, $b'$, of $b$ including $jr_Aj$. Since $b$ is satisfiable in $M$, $f(i)R_Af(j)$. Accordingly, $f(j)R_Af(j)$, since $M$ satisfies condition $CD\rho'$. So, D Al ref produces at least one extension, $b'$, of $b$ such that $b'$ is satisfiable in $M$.

(D Al sym). Suppose that $ir_Aj$ and $jr_Ak$ are on $b$, and that we apply D Al sym to give an extended branch, $b'$, of $b$ containing $kr_Aj$. Since $b$ is satisfiable in $M$, $f(i)R_Af(j)$ and $f(j)R_Af(k)$. Hence, $f(k)R_Af(j)$, since $M$ satisfies condition $CD\sigma'$. Consequently, D Al sym produces at least one extension, $b'$, of $b$ such that $b'$ is satisfiable in $M$.

($Ta6$). Suppose that $ir_Aj$ is on $b$, and that we apply $Ta6$ to give an extended branch, $b'$, of $b$ containing $kr_Aj$. Since $b$ is satisfiable in $M$, $f(i)R_Af(j)$. Accordingly, $f(k)R_Af(j)$, since $M$ satisfies condition $Ca6$.

Consequently, *Ta6* produces at least one extension, $b'$, of $b$ such that $b'$ is satisfiable in $M$.

($T\alpha 2$). Suppose that $ir_A j$ and $B, j$ are on $b$, and that we apply $T\alpha 2$ to give an extended branch, $b'$, of $b$ containing $ir_{A\wedge B} j$. Since $b$ is satisfiable in $M$, $f(i)R_A f(j)$ and $B$ is true at $f(j)$. Accordingly, $f(i)R_{A\wedge B} f(j)$, since $M$ satisfies condition $C\alpha 2$. In conclusion, $T\alpha 2$ produces at least one extension, $b'$, of $b$ such that $b'$ is satisfiable in $M$.

($T\alpha 4$). Suppose that $ir_A j$, $B, j$ and $iR_{A\wedge B} k$ are on $b$, and that we apply $T\alpha 4$ to give an extended branch, $b'$, of $b$ containing $ir_A k$ and $B, k$. Since $b$ is satisfiable in $M$, $f(i)R_A f(j)$, $f(i)R_{A\wedge B} f(k)$ and $B$ is true at $f(j)$. Accordingly, $f(i)R_A f(k)$ and $B$ is true at $f(k)$, since $M$ satisfies condition $C\alpha 4$. In conclusion, $T\alpha 4$ produces at least one extension, $b'$, of $b$ such that $b'$ is satisfiable in $M$. Remaining cases are left to the reader. They are proved similarly. ⊣

THEOREM 12 (Soundness theorem I). *Let $S$ be any of the 48 tableau systems we discuss in this essay. Then $S$ is sound with respect to the class of models $\mathcal{M}$ that corresponds to $S$. For finite $\Sigma$, if $\Sigma \vdash_S B$, then $\Sigma \Vdash_{\mathcal{M}} B$.*[8]

PROOF. The proof is essentially the same as the proof that certain normal modal systems are sound. (See e.g. [21], especially chapters 1 and 2.) ⊣

THEOREM 13 (Soundness theorem II).
(i) *ADDL012 is sound with respect to the class of all H-models and also with respect to the class of all H1-models.*

(ii) *ADDL0123 is sound with respect to the class of all H2-models.*

(iii) *TG (ADDL01234) is sound with respect to the class of all H3-models and also with respect to the class of all strong H3-models.*

PROOF. This follows from Soundness theorem I and Theorem 1 in Section 3.4. ⊣

[8]This result can be extended to the infinite case. For an idea of how to do this see [23].

## 5.2. Completeness theorems

Let $b$ be an open branch of a tableau. The model, $M = \langle W, \{R_A : A \in L\}, V\rangle$, induced by $b$ is defined as follows. $W = \{wi : i \text{ occurs on } b\}$, $wiR_Awj$ iff $ir_Aj$ occurs on $b$. If $p, i$ occurs on $b$, then $p$ is true in $wi$; if $\neg p, i$ occurs on $b$, then $p$ is false in $wi$.

Lemma 14 (Completeness lemma). *Let $b$ be any open branch in a complete tableau and let $M = \langle W, \{R_A : A \in L\}, V\rangle$ be the model induced by $b$. Then*

(i) *$A$ is true at $wi$, if $A, i$ is on $b$,*

(ii) *$A$ is false at $wi$, if $\neg A, i$ is on $b$.*

Proof. The proof is easy. See [21] and [23, Section 5.2], just replace $A \mathrel{\Box\!\!\rightarrow} B$ by $O[A]B$ and $A \mathrel{\Diamond\!\!\rightarrow} B$ by $P[A]B$, for an idea of how to do it. ⊣

Theorem 15 (Completeness theorem I). *For finite $\Sigma$, if $\Sigma \Vdash_{\mathcal{M}(V)} B$, then $\Sigma \vdash_{DDL} B$.*[9]

Proof. The proof is similar to the proof that certain normal modal systems are complete. (See e.g. [21], especially chapter 1 and 2.) ⊣

Theorem 16 (Completeness theorem II). *Let $S$ be any of the remaining 31 systems we discuss in this essay, not including $T\alpha 0$, and let $\mathcal{M}$ in each case be the corresponding class of models. For finite $\Sigma$, if $\Sigma \Vdash_{\mathcal{M}} B$, then $\Sigma \vdash_S B$.*[10]

Proof. The proof is standard. We just have to check that the model induced by the open branch, $b$, is of the right kind in every case.

The proofs of $C\alpha 1$, $C\alpha 2$, $C\alpha 3$ are similar to proofs found in [23], Section 5.2.

($CD\rho'$). Suppose that $wiR_Awj$, where $wi$, $wj \in W$. Then $ir_Aj$ occurs on $b$ (by the definition of an induced model). Since the tableau is complete, D Al ref has been applied and $jr_Aj$ occurs on $b$. Hence, $wjR_Awj$, as required (by the definition of an induced model).

[9] For an idea of how to extend this result to the case with an infinite set of premises, see [23].

[10] This result can be extended to the case with an infinite set of premises. See [23] for an idea of how to do this.

($CD\sigma'$). Suppose that $wiR_Awj$ and $wjR_Awk$, where $wi$, $wj$, $wk$ $\in W$. Then $ir_Aj$ and $jr_Ak$ occur on $b$ (by the definition of an induced model). Since the tableau is complete, D Al sym has been applied and $kr_Aj$ occurs on $b$. It follows that $wkR_Awj$, as required (by the definition of an induced model).

($Ca6$) Suppose that $wiR_Awj$ and that $wi$, $wj$, $wk \in W$. Then $ir_Aj$ is on $b$ (by the definition of an induced model). Since the tableau is complete *Ta6* has been applied and $kr_Aj$ is on $b$, for every $k$ on $b$. Hence, $wkR_Awj$ for every $wk$, as required (by the definition of an induced model).

($C\alpha 4$). Let $wi$, $wj$, $wk \in W$. Suppose that $wiR_Awj$, $wiR_{A\wedge B}wk$ and that $B$ is true at $wj$. Then $ir_Aj$ and $ir_{A\wedge B}k$ (by the definition of an induced model). Since the tableau is complete *CUT* has been applied and either $B, j$ or $\neg B, j$ is on $b$. Assume that $\neg B, j$ is on $b$. Then $B$ is false at $wj$ (by the completeness lemma). But this is absurd. So, $B, j$ is on $b$. Since the tableau is complete $T\alpha 4$ has been applied and $ir_Ak$ and $B, k$ occur on $b$. It follows that $wiR_Awk$ and that $B$ is true at $wk$, as required (by the definition of an induced model and the completeness lemma). Remaining cases are left to the reader. ⊣

## 6. Relations between our tableau systems

Figure 1 contains all the 16 pure dyadic deontic tableau systems that we discuss in this essay and it depicts some important relationships between these systems. Inclusions are marked by lines. Systems higher up are extensions of systems lower down, systems lower down are included in systems higher up. $\tau\epsilon$ is the strongest system, it includes every system, while *DDL* is the weakest system. *DDL* is included in every system. Two systems are non-comparable just in case neither is included in the other. So, $\tau$ and $\rho'$ are examples of non-comparable systems. Systems displayed in the boxes are systems that are deductively equivalent to the system with which the box is associated. E.g. all of the following systems are deductively equivalent: $\tau\epsilon$, $\rho'\tau\epsilon$, $\sigma'\tau\epsilon$ and $\rho'\sigma'\tau\epsilon$.

That there are at most these systems and that the inclusion relations that are displayed in figure 1 hold follows from our results in sections 3.4 and 3.5 and the soundness and completeness proofs in Section 5. Proofs of distinctness are left to the reader.

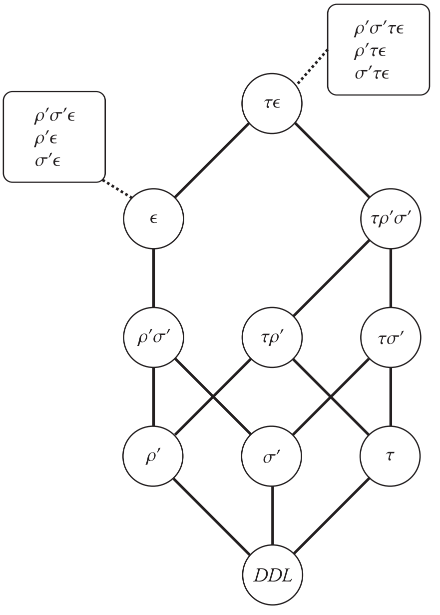


Figure 1. Some relations between our tableau systems

All alethic dyadic deontic systems are stronger than our strongest pure dyadic deontic tableau system. Exactly how these systems are related to each other remains, however, an open question.

## 7. Relations to some other systems

In [33], [34, chapters V and VI], and [35] Åqvist introduces a set of axioms and inference rules that can be used to construct 16 different 'pure' dyadic deontic axiomatic systems and 32 different alethic dyadic deontic axiomatic systems. These systems can be seen to match our systems in a very precise sense.[11]

Let $f$ be a function such that: $f(TD\tau) = a4$ $(O[B]A \to O[C]O[B]A)$, $f(TD\epsilon) = a5$ $(P[C]O[B]A \to O[B]A)$, $f(TD\rho') = a3$ $(O[B](O[B]A \to A))$, $f(TD\sigma') = O[B](P[B]O[B]A \to A)$, $f(Ta6) = a6$ $(O[B]A \to \Box O[B]A)$, $f(T\alpha i) = \alpha i$, for $i = 0$ $(\Box(A \leftrightarrow B) \to (O[A]C \leftrightarrow O[B]C))$,

[11] Throughout this section we will ignore minor differences between our syntax and the syntax used by the thinkers we discuss.

for $i = 1$ ($O[A]A$), for $i = 2$ ($O[A \wedge B]C \to O[A](B \to C)$), for $i = 3$ ($\Diamond A \to (O[A]B \to P[A]B)$), for $i = 4$ ($P[A]B \to (O[A](B \to C) \to O[A \wedge B]C)$). Let $S$ be a pure dyadic deontic tableau system and let $A$ be a pure axiomatic system. Then we shall say that $A$ corresponds to $S$ just in case $A$ includes all and only the axioms (axiom schemas) $f(t)$ such that $t$ is a tableau rule in $S$ plus $a0$ (all truth-functional tautologies), $a1$ ($P[B]A \leftrightarrow \neg O[B]\neg A$), $a2$ ($O[B](A \to C) \to (O[B]A \to O[B]C)$), $F[B]A \leftrightarrow O[B]\neg A$ and $A$ has $R1$ (modus ponens) and $R2$ ($A/O[B]A$) as the only primitive rules. Let $S$ be an alethic dyadic deontic tableau system and $A$ an alethic dyadic deontic axiomatic system. Then we shall say that $A$ corresponds to $S$ precisely when $A$ includes all and only the axioms (axiom schemas) $f(t)$ such that $t$ is a tableau rule in $S$ plus $a0$–$a2$, $a7$ ($\Box A \to O[B]A$) and $a8$ (an appropriate set of S5-schemata), $F[B]A \leftrightarrow O[B]\neg A$ and $A$ has $R1$ and $R2''$ ($A/\Box A$) as the only primitive rules. Then we can state the following theorem.

Theorem 17. (i) *Let $S$ be any of our 32 tableau systems not including Tc0 and let $A$ be the axiomatic system that corresponds to $S$ (as defined above). Then $S$ and $A$ are deductively equivalent.*

(ii) *Let $S$ be any of the remaining 16 systems and $A$ the axiomatic system that corresponds to $S$. Then, if $B$ is a theorem in $S$, $B$ is a theorem in $A$.*

Proof. This follows directly from our soundness and completeness results in this essay and from the soundness and completeness results found in Åqvist's [33], [34, chapters V and VI], or [35]. ⊣

Theorem 18. (i) *Our tableau system TG contains all axioms in the system vK+ introduced by Franz von Kutschera, i.e. all of the following sentences* (see [31] and [34, p. 205])*:*

$$P[B]A \leftrightarrow \neg O[B]\neg A$$
$$O[A]A$$
$$O[\neg A]A \to O[B]A$$
$$(O[\neg(A \to B)](A \to B) \wedge O[C]A) \to O[C]B$$
$$(O[B]A \wedge O[B]C) \to O[B](A \wedge C)$$
$$\neg O[A]\neg B \to (O[A \wedge B]C \leftrightarrow O[A](B \to C))$$
$$O[\neg A]A \to A$$

$$\Box A \leftrightarrow O[\neg A]A$$
$$\Diamond A \leftrightarrow \neg \Box \neg A$$

(ii) *If we add definitions 4 and 5 to our language, TG includes all axioms in the system DFL based on work by Sven Danielsson, Bas van Fraassen and David Lewis, i.e.*

$$O[B]A \leftrightarrow (P'[B]\bot \vee O'[B]A)$$
$$P[B]A \leftrightarrow (O'[B]\top \wedge P'[B]A)$$
$$P'[B]A \leftrightarrow \neg O'[B]\neg A$$
$$O'[B]A \to P'[B]A$$
$$O'[B](A \to C) \to (O'[B]A \to O'[B]C)$$
$$O'[B]A \to \Box O'[B]A$$
$$\Box A \to (P'[B]\bot \vee O'[B]A)$$
$$\Box(A \leftrightarrow B) \to (O'[A]C \leftrightarrow O'[B]C)$$
$$P'[A]\bot \vee O'[A]A$$
$$(P'[A \wedge B]\bot \vee O'[A \wedge B]C) \to (P'[A]\bot \vee O'[A](B \to C))$$
$$\Diamond A \to O'[A]\top$$
$$(O'[A]\top \wedge P'[A]B) \to ((P'[A]\bot \vee O'[A](B \to C)) \to (P'[A \wedge B]\bot \vee O'[A \wedge B]C))$$

*all the axioms in Lewis's system Lw (and hence in the weaker systems CO, CD, CU, CA, CDA and CUA too), i.e.*

$$P'[C]A \leftrightarrow \neg O'[C]\neg A$$
$$O'[C](A \wedge B) \leftrightarrow (O'[C]A \wedge O'[C]B)$$
$$O'[C]A \to P'[C]A$$
$$O'[C]\top \to O'[C]C$$
$$O'[C]\top \to O'[B \vee C]\top$$
$$(O'[B]A \wedge O'[C]A) \to O'[B \vee C]A$$
$$(P'[C]\bot \wedge O'[B \vee C]A) \to O'[B]A$$
$$(P'[B \vee C]B \wedge O'[B \vee C]A) \to O[B]A$$
$$O'[\top]\top$$
$$A \to O'[A]\top$$

$$O'[A]\top \to P'[P'[A]\bot]\bot$$
$$O'[B]A \to P'[\neg O'[B]A]\bot$$
$$P'[B]A \to P'[\neg P'[B]A]\bot$$

*all axioms in van Fraassen's system vF, i.e.*

$$P'[B]A \leftrightarrow \neg O'[B]\neg A$$
$$O'[B](A \to C) \to (O'[B]A \to O'[B]C)$$
$$O'[B]A \to P'[B]A$$
$$O'[A]B \to O'[A](B \wedge A)$$
$$O'[A \vee B]\neg B \to (O'[B \vee C]\neg C \to O'[A \vee C]\neg C)$$
$$P'[A \vee B]A \to (O'[B \vee C]\neg C \to O'[A \vee C]\neg C)$$
$$O'[A \vee B]\neg B \to (P'[B \vee C]B \to O'[A \vee C]\neg C)$$

*and the theorems G1–G7, i.e.* (see [4, 29, 30, 19] and [34, pp. 212, 226–227, 232, 234]):

$$O[A]\bot \to \Box\neg A$$
$$P[A]B \to (P[A \wedge B]C \to P[A](B \wedge C))$$
$$O[A \vee B]\neg B \to O[A \vee B \vee C]\neg B$$
$$(O[A \vee B]\neg B \wedge P[B \vee C]B) \to O[A \vee B \vee C]\neg C$$
$$P[A \vee B]A \to P[A \vee B \vee C]\top$$
$$P[A \vee B]A \to P[A \vee C]\top$$
$$(P[A \vee B]\top \wedge O[A \vee B]\neg B) \to P[A \vee B]A$$

(iii) *If we add definitions 4–9 to our language, TG includes all axioms in Åqvist's system for preference PR and the theorems p0–p23, i.e.* (see [34, pp. 241–243, 253–254]):

$$((A \geq B) \wedge (B \geq C)) \to (A \geq C)$$
$$(A \geq B) \vee (B \geq A)$$
$$(A > B) \leftrightarrow \neg(B \geq A)$$
$$(A = B) \leftrightarrow ((A \geq B) \wedge (B \geq A))$$
$$(((A \to C) \wedge B) > (\neg(A \to C) \wedge B)) \to$$
$$(((A \wedge B) > (\neg A \wedge B)) \to ((C \wedge B) > (\neg C \wedge B)))$$

$$(A \geq B) \rightarrow \Box(A \geq B)$$

$$(A > B) \rightarrow \Box(A > B)$$

$$\Box A \rightarrow ((\bot \geq B) \vee ((A \wedge B) > (\neg A \wedge B)))$$

$$(A \geq B) \leftrightarrow (A \geq (\neg A \wedge B))$$

$$(A > B) \rightarrow \Diamond(A \vee B)$$

$$\Box(A \leftrightarrow B) \rightarrow (((A \geq C) \leftrightarrow (B \geq C)) \wedge ((C \geq A) \leftrightarrow (C \geq B)))$$

$$\Diamond A \rightarrow (A > \bot)$$

$$((\bot \geq (A \wedge B)) \vee ((C \wedge A \wedge B) > (\neg C \wedge A \wedge B))) \rightarrow \\ ((\bot \geq A) \vee (((B \rightarrow C) \wedge A) > (\neg(B \rightarrow C) \wedge A)))$$

$$((A > \bot) \wedge ((B \wedge A) \geq (\neg B \wedge A))) \rightarrow \\ \textit{the converse of the preceding sentence}$$

$$A \geq A$$

$$A = A$$

$$(A > B) \rightarrow (A \geq B)$$

$$(A = B) \rightarrow (A \geq B)$$

$$(A \geq B) \leftrightarrow ((A > B) \vee (A = B))$$

$$(A > B) \rightarrow ((B > C) \rightarrow (A > C))$$

$$((A > B) \wedge (B = C)) \rightarrow (A > C)$$

$$(A \geq B) \rightarrow ((B > C) \rightarrow (A > C))$$

$$(A = B) \rightarrow ((B > C) \rightarrow (A > C))$$

$$(A > B) \rightarrow ((B \geq C) \rightarrow (A > C))$$

$$(A = B) \rightarrow ((B = C) \rightarrow (A = C))$$

$$(A = B) \rightarrow \neg(A > B)$$

$$(A = B) \rightarrow (B = A)$$

$$(A > B) \rightarrow \neg(B > A)$$

$$\neg(A > A)$$

$$((A > B) \vee (B > A)) \vee (A = B)$$

$$(B \geq A) \vee (A > B)$$

$$A \geq \bot$$

$$\neg(\bot > A)$$

$$(A = \bot) \leftrightarrow (\bot \geq A)$$

$$\Diamond A \leftrightarrow (A > \bot)$$
$$\Box A \leftrightarrow (\bot \geq \neg A)$$
$$(A \geq B) \rightarrow (A = (A \vee B))$$
$$\Box(A \rightarrow B) \rightarrow (B \geq A)$$
$$O[B]A \leftrightarrow ((\bot \geq B) \vee ((A \wedge B) > (\neg A \wedge B)))$$
$$P[B]A \leftrightarrow ((B > \bot) \wedge ((A \wedge B) \geq (\neg A \wedge B)))$$
$$O'[B]A \leftrightarrow ((A \wedge B) > (\neg A \wedge B))$$
$$P'[B]A \leftrightarrow ((A \wedge B) \geq (\neg A \wedge B))$$

Proof. Left to the reader. ⊣

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