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COUNTERFACTUALS AND SEMANTIC TABLEAUX

Abstract. The purpose of this paper is to develop a class of semantic tableau systems for some counterfactual logics. All in all I will discuss 1024 systems. Possible world semantics is used to interpret our formal languages. Soundness results are obtained for every tableau system and completeness results for a large subclass of these.

Keywords: Counterfactuals, subjunctive conditionals, conditional logic, modal logic, semantic tableau, analytic tableau, Robert Stalnaker, David Lewis, Melvin Fitting, Graham Priest.

1. Introduction

Conditionals and counterfactuals turn up all the time in philosophy and in every day life. Consider, for instance, the following sentences: 'If I were a brain in a vat, most of my beliefs about my environment would be wrong', 'If I were a cartesian soul, my mind might exist without my body' and 'If hedonism were true, virtue would not have intrinsic value'. If this is true, we ought to be able to analyze and reason with such sentences. Conditional logic or counterfactual logic is a part of logic that deals with conditional sentences such as these. Pioneering contributions to this branch of logic can be found in Robert Stalnaker's [26] and David Lewis's [19]. (For a philosophical introduction to conditionals and more references, see [2].) Philosophers and logician often introduce certain counterfactual operators to help symbolize, at least certain, conditional sentences. The purpose of this essay is

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to develop a set of semantic tableau systems that include some counterfactual operators of this kind. All in all we will consider 1024 systems. If we think of a logic as a set of sentences (provable in an axiomatic system or in a tableau system), many of the most famous counterfactual logics discussed in the literature since Stalnaker and Lewis "invented" this field are included in our set of 1024 logics. All of our systems are more or less intuitively plausible. So, I think it is philosophically warranted to consider them all. For most of our logics there are presently no known tableau systems of the kind used in this essay, at least as far as I know. Hence, I also think that this investigation is justified from a logical perspective.

The modal part of the tableau systems that are described in this essay are similar to systems developed by Melvin Fitting in e.g. [6] and [7] and by Graham Priest in [21] among others. Their propositional part can be traced back to Raymond Smullyan [22, 23, 24, 25] and Richard Jeffrey [15]. I think that this kind of tableau system is particularly elegant and easy to apply and I will assume that the reader is familiar with it.

Possible world semantics similar to the kind introduced by Saul Kripke [16, 17, 18] is used to interpret our systems and I will assume that the reader is familiar with this kind of semantics.

The essay is divided into five parts. Section 2 deals with syntax and Section 3 with semantics. In Section 4 I describe the tableau systems that are the main focus of the essay and Section 5 contains soundness and completeness proofs. Soundness results are obtained for every tableau system and completeness results for every system that does not include the tableau rules Tc0 or Tc0′. I have not been able to prove that the systems including Tc0 or Tc0′ are complete with respect to their corresponding semantics. My conjecture is that they are complete. Hopefully someone will be able to prove this in the future (or show that this conjecture is false and find some other tableau rules that exactly correspond to the semantic conditions Cc0 and Cc0′).

2. Syntax

2.1. Alphabet:

1. A denumerably infinite set Prop of proposition letters: p, q, r, s, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, ...
2. Truth-functional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication) and $\leftrightarrow$ (material equivalence).

3. two counterfactual conditional operators: $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$.
4. $\top$ (verum), $\bot$ (falsum) and the brackets (and).
5. three alethic operators: $\Box$ (necessity), $\Diamond$ (possibility) and $\ominus$ (impossibility).

2.2. Sentences. The language L is the set of well-formed formulas (wffs) or sentences generated by the usual clauses for proposition letters, $\top$, $\bot$ and propositionally compound sentences, plus the following clauses:

1. if A and B are wffs, so are $(A \Box\!\!\rightarrow B)$ and $(A \Diamond\!\!\rightarrow B)$,
2. if A is a wff, so are $\Box A$, $\Diamond A$ and $\ominus A$,
3. nothing else is a wff.

$(A \Box\!\!\rightarrow B)$ is to be read "If it were the case that A, then it would be the case that B", and $(A \Diamond\!\!\rightarrow B)$ "If it were the case that A, then it might be the case that B".

Capital letters 'A', 'B', 'C', '... are used to represent arbitrary (not necessarily atomic) formulas of the object language. The upper case Greek letter 'Σ' represents an arbitrary set of formulas. The empty set is denoted by '$\emptyset$'. Outer brackets around sentences are usually dropped if the result is not ambiguous.

2.3. Definitions.

1. $A \Box\!\!\Rightarrow B := (A \Diamond\!\!\rightarrow \top) \wedge (A \Box\!\!\rightarrow B)$
2. $A \Diamond\!\!\Rightarrow B := \neg(A \Box\!\!\Rightarrow \neg B)$ (or $(A \Box\!\!\rightarrow \bot) \vee (A \Diamond\!\!\rightarrow B)$)

$A \Box\!\!\Rightarrow B$ is an alternative explication of the expression "If it were the case that A, then it would be the case that B", and $A \Diamond\!\!\Rightarrow B$ of "If it were the case that A, then it might be the case that B".

3. Semantics

3.1. Frames. A frame F is a relational structure $\langle W, \{R_A : A \in L\}\rangle$, where W is a non-empty set of possible worlds and $\{R_A : A \in L\}$ is a set of dyadic relations on W, one for each sentence, A, in L. So, for every A in L, $R_A \subseteq W \times W$. When $w\, R_A\, w'$ we say that w is A-related to w', or that w' is A-accessible from w.

3.2. Models. A model M is a pair $\langle F, V\rangle$ where: F is a frame and V is a valuation or interpretation function, which assigns a truth-value T (true) or

F (false) to every proposition letter from Prop in each world from W, i.e., $V\colon \mathrm{Prop} \times W \longrightarrow \{T, F\}$.

We shall also speak of a model M as a relational structure, $\langle W, \{R_A : A \in L\}, V\rangle$, instead of saying that $M = \langle F, V\rangle$ where $F = \langle W, \{R_A : A \in L\}\rangle$, to save space.

3.3. Truth conditions. Let $M = \langle W, \{R_A : A \in L\}, V\rangle$, be any model, let w be any member of W and let A be in L. To mean that A is true at a possible world w in the model M we write $\Vdash_{M,w} A$. The truth conditions for proposition letters, $\top$, $\bot$ and sentences built by truth functional connectives are the usual ones. The truth conditions for the other sentences in L are given by the following clauses:

1. $\Vdash_{M,w} A \,\Box\!\!\rightarrow B$ iff for all $w' \in W$ such that $w \, R_A \, w'$: $\Vdash_{M,w'} B$,
2. $\Vdash_{M,w} A \,\Diamond\!\!\rightarrow B$ iff for at least one $w' \in W$ such that $w \, R_A \, w'$: $\Vdash_{M,w'} B$,
3. $\Vdash_{M,w} \Box A$ iff for all $w' \in W$: $\Vdash_{M,w'} A$,
4. $\Vdash_{M,w} \Diamond A$ iff for at least one $w' \in W$: $\Vdash_{M,w'} A$,
5. $\Vdash_{M,w} \ominus A$ iff for all $w' \in W$: $\Vdash_{M,w'} \neg A$.

3.4. Validity, entailment, countermodel and satisfiability. The concepts of validity, entailment, countermodel, satisfiability etc. can be defined in the usual way. (For an idea of how to do this, see [21].) $\Sigma \Vdash_{\boldsymbol{M}} B$ says that B is a consequence of Σ in $\boldsymbol{M}$ (or that Σ entails B in $\boldsymbol{M}$), where $\boldsymbol{M}$ is a class of models. $\Vdash_{\boldsymbol{M}} B$ says that B is valid in $\boldsymbol{M}$.

3.4. Conditions on a model. We will consider 10 different conditions on our models in this essay (Table 1). Corresponding to the conditions in Table 1 there are 10 different tableau rules (see Section 4.2.5). The variables x, y, z are taken to range over W, and the symbols $\wedge$, $\rightarrow$, $\forall$ and $\exists$ are used in the standard way. $\|A\|^M = \{w \in W\colon \Vdash_{M,w} A\}$, i.e. $\|A\|^M$ is the set of all worlds in M where A is true.

3.5. Classification of some models. The conditions introduced in Section 3.4 can be used to obtain a subcategorization of the set of all models into various kinds. In general, we shall say that $\boldsymbol{M}(C_1, \ldots, C_n)$ is the class of (all) models that satisfy the conditions $C_1, \ldots, C_n$. $\boldsymbol{M}(V)$ will denote the class of all models.

Example 1. $\boldsymbol{M}(\mathrm{Cc0}, \mathrm{Cc1}, \mathrm{Cc2})$ = the class of (all) models that satisfy the conditions Cc0, Cc1 and Cc2.

	For all A and B
Cc0	$\|A\|^M = \|B\|^M \to R_A = R_B$
Cc1	$\forall x \forall y (x\, R_A\, y \to \Vdash_{M,y} A)$
Cc2	$\forall x \forall y ((x\, R_A\, y \wedge \Vdash_{M,y} B) \to x\, R_{A \wedge B}\, y)$
Cc3	$\forall x (\|A\|^M \neq \emptyset \to \exists y\, x\, R_A\, y)$
Cc4	$\forall x \forall y \forall z (x\, R_A\, y \wedge \Vdash_{M,y} B) \to (x\, R_{A \wedge B}\, z \to (x\, R_A\, z \wedge \Vdash_{M,z} B)))$
Cc0′	$\forall x ((\forall y (x\, R_A\, y \to \Vdash_{M,y} B) \wedge \forall y (x\, R_B\, y \to \Vdash_{M,y} A)) \to R_A = R_B)$
Cc5	$\forall x (\Vdash_{M,x} A \to x\, R_A\, x)$
Cc6	$\forall x \forall y ((x\, R_A\, y \wedge \Vdash_{M,x} A) \to x = y)$
Cc7	$\forall x \forall y \forall z ((x\, R_A\, y \wedge x\, R_A\, z) \to y = z)$
Cc8	$\forall x \forall y \forall z (x\, R_A\, y \to z\, R_A\, y)$

Table 1. Conditions on a model

3.6. Logical systems. By imposing different formal conditions on our models we can obtain different logical systems. The set of all sentences in L that are valid in a class of models $\boldsymbol{M}$ is called the logical system of $\boldsymbol{M}$, or the system of $\boldsymbol{M}$ or the logic of $\boldsymbol{M}$, in symbols $\mathrm{S}(\boldsymbol{M}) = \{A \in L : \Vdash_{\boldsymbol{M}} A\}$.

Example 2. $\mathrm{S}(\boldsymbol{M}(\mathrm{Cc0}, \mathrm{Cc1}, \mathrm{Cc2})) = \{A \in L : \Vdash_{\boldsymbol{M}(\mathrm{Cc0},\mathrm{Cc1},\mathrm{Cc2})} A\}$. I.e. the logical system of the class of all models that satisfy the conditions Cc0, Cc1 and Cc2 is (identical with) the set of all well-formed sentences in L that are valid in the class of all models that satisfy Cc0, Cc1 and Cc2.

4. Semantic tableaux and conditional logic

4.1. Semantic tableaux

There are several different kinds of tableaux systems for classical and modal logic in the literature. The one I use is inspired by Fitting and Priest (see e.g. [6], [7] and [21]). The propositional part is similar to systems introduced by Raymond Smullyan [22, 23, 24, 25] and Richard Jeffrey [15].[1]

[1] Evert Beth seems to be the first to introduce the concept of "semantic tableau" and to develop teableau techniques for classical logic (see [3]). See also [4], pp. 186–201, 267–293, and 444–463. According to Smullyan ([25], p. 15), the whole idea ultimately derives from Gerhard Gentzen (see Gentzen's [9] and [10]). This also seems to be Melvin Fitting's opinion (see [8], p. 7). Other early important contributions can be found in Hintikka [12], Lis [20], Smullyan [22, 23, 24, 25] and Jeffrey [15]. Tableau techniques were applied to modal logic already in 1959 by Kripke (see [16]). See also [17, 18], Hughes and Cresswell [14], chapter five, chapter six and chapter fifteen, Fitting [6] and Zeman [28]. See also Hanson's [11] and Åqvist's [29]. For an introduction to tableau methods, see for instance

The concepts of semantic tableau, branch, open and closed branch etc. are defined as in Priest's [21].

4.2. Tableau rules

4.2.1. Propositional rules. All our tableau systems should include some set of propositional rules sufficient to prove all propositionally valid sentences. We use the ones that can be found in [21].

4.2.2. Counterfactual rules. There are four counterfactual rules (see Table 2), two for both counterfactual operators.

$\Box\!\!\rightarrow$-pos ($\Box\!\!\rightarrow$)	$\Diamond\!\!\rightarrow$-pos ($\Diamond\!\!\rightarrow$)	$\Box\!\!\rightarrow$-neg ($\neg\,\Box\!\!\rightarrow$)	$\Diamond\!\!\rightarrow$-neg ($\neg\,\Diamond\!\!\rightarrow$)
$(A \Box\!\!\rightarrow B), i$	$(A \Diamond\!\!\rightarrow B), i$	$\neg(A \Box\!\!\rightarrow B), i$	$\neg(A \Diamond\!\!\rightarrow B), i$
$i\, r_A\, j$	$\downarrow$	$\downarrow$	$\downarrow$
$\downarrow$	$i\, r_A\, j$	$(A \Diamond\!\!\rightarrow \neg B), i$	$(A \Box\!\!\rightarrow \neg B), i$
B, j	B, j		
	where j is new		

Table 2. Counterfactual rules

According to $\Box\!\!\rightarrow$-pos we may write B, j on any (open) branch on which both $A \Box\!\!\rightarrow B, i$ and $i\, r_A\, j$ occur (not necessarily in that order and possibly with other nodes in between), while according to $\Diamond\!\!\rightarrow$-pos we may write $i r_A j$ and B, j on any (open) branch on which $A \Diamond\!\!\rightarrow B, i$ occurs. Note that j must be new to the branch when we apply $\Diamond\!\!\rightarrow$-pos, i.e. j must not occur anywhere on the branch. The other rules are interpreted similarly.

4.2.3. Alethic rules. We use the alethic rules for the standard system S5. So, we don't have to take into account a separate (set of) alethic accessibility relation(s). There are two rules for every alethic operator (Table 3) and they are interpreted as usual.

4.2.4. Basic rules. There are three basic rules that are included in all our systems (Table 4): CUT and two identity rules. (However, in some systems we may use a restricted version of CUT, see 4.3). The CUT rule is special since it can be applied to any A (for any i on an open branch). Hence, it is not a so called "analytic" rule. An application of a rule to a branch, b, is analytic iff all new sentences that are added to b are subsentences (or

Handbook of Tableau Methods [5], which also contains many references to important work in the field.

$\Box$-pos ($\Box$)	$\Diamond$-pos ($\Diamond$)	$\diamondminus$-pos ($\diamondminus$)
$\Box A, i$	$\Diamond A, i$	$\diamondminus A, i$
$\downarrow$	$\downarrow$	$\downarrow$
A, j	A, j	$\Box \neg A, i$
	where j is new	
$\Box$-neg ($\neg \Box$)	**$\Diamond$-neg ($\neg \Diamond$)**	**$\diamondminus$-neg ($\neg \diamondminus$)**
$\neg \Box A, i$	$\neg \Diamond A, i$	$\neg \diamondminus A, i$
$\downarrow$	$\downarrow$	$\downarrow$
$\Diamond \neg A, i$	$\Box \neg A, i$	$\Diamond A, i$

Table 3. Alethic rules

negations of subsentences) of sentences in b and a rule is analytic iff every application of it is analytic. (The concept of a subsentence is used in a standard way.)

We have two identity rules: IdI and IdII (both abbreviated Id). $\alpha(i)$ is a line in a tableau that includes an "i", and $\alpha(j)$ is like $\alpha(i)$ except that "i" is replaced by 'j'. E.g. if $\alpha(i)$ is A, i, $\alpha(j)$ is A, j; if $\alpha(i)$ is $k\, r_A\, i$, $\alpha(j)$ is $k\, r_A\, j$; if $\alpha(i)$ is $i = k$, $\alpha(j)$ is $j = k$. The rule is sound, in the first case, regardless of whether A is atomic or complex. We will, however, only apply the rule to atomic sentences and negations of atomic sentences, in this case.

CUT	Id (IdI)	Id (IdII)
$*$	$\alpha(i)$	$\alpha(i)$
$\swarrow \quad \searrow$	$i = j$	$j = i$
$\neg A, i \quad A, i$	$\downarrow$	$\downarrow$
for every A	$\alpha(j)$	$\alpha(j)$

Table 4. Basic rules

4.2.5. Accessibility rules. There are 10 different accessibility rules (see Table 5) that correspond to the semantic conditions in Section 3.4. According to Tc0 if A is of the form $\Box(A \leftrightarrow B) \to ((A \Box\!\!\rightarrow C) \leftrightarrow (B \Box\!\!\rightarrow C))$ A, i may be added to any open branch on which i occurs. E.g. all of the following expressions are instances of this rule: $\Box(p \leftrightarrow q) \to ((p \Box\!\!\rightarrow r) \leftrightarrow (q \Box\!\!\rightarrow r)), i$; $\Box((\neg p \to q) \leftrightarrow (p \vee q)) \to (((\neg p \to q) \Box\!\!\rightarrow r) \leftrightarrow ((p \vee q) \Box\!\!\rightarrow r)), i$ and $\Box(p \leftrightarrow \neg q) \to ((p \Box\!\!\rightarrow (r \wedge s)) \leftrightarrow (\neg q \Box\!\!\rightarrow (r \wedge s))), i$.

According to Tc1 we may add A, j to any open branch on which $i\, r_A\, j$ occurs, while according to Tc3 we may add $j\, r_A\, k$ to any open branch on which A, i occurs. Note that Tc3 may be applied to any j on the branch but that k must be new, i.e. it must not occur anywhere else on the branch. The other accessibility rules are interpreted similarly.

Tc0	Tc1	Tc2	Tc3	Tc4
If A is of the form $\Box(B \leftrightarrow C) \rightarrow$ $((B \Box\!\!\rightarrow D) \leftrightarrow (C \Box\!\!\rightarrow D))$, A, i can be added to any open branch on which i occurs.	$i\, r_A\, j$ $\downarrow$ A, j	$i\, r_A\, j$ B, j $\downarrow$ $i\, r_{A \wedge B}\, j$	A, i $\downarrow$ $j\, r_A\, k$ where k is new	$i\, r_A\, j$ B, j $i\, r_{A \wedge B}\, k$ $\downarrow$ $i\, r_A\, k$ B, k
Tc0′	**Tc5**	**Tc6**	**Tc7**	**Tc8**
If A is of the form $((B \Box\!\!\rightarrow C) \wedge (C \Box\!\!\rightarrow B)) \rightarrow$ $((B \Box\!\!\rightarrow D) \leftrightarrow (C \Box\!\!\rightarrow D))$, A, i can be added to any open branch on which i occurs.	A, i $\downarrow$ $i\, r_A\, i$	A, i $i\, r_A\, j$ $\downarrow$ $i = j$	$i\, r_A\, j$ $i\, r_A\, k$ $\downarrow$ $j = k$	$i\, r_A\, j$ $\downarrow$ $k\, r_A\, j$

Table 5. Accessibility rules

4.2.6. Derived rules. Finally, let us look at some derived rules that can be used to abbreviate our proofs.

The Global Assumption Rule (GA): *If A has a tableau proof then for any i: A, i can be added as a line to any open branch of a tableau.*

THEOREM 3. *The Global Assumption Rule is admissible in any of our counterfactual tableau systems that includes* CUT *(as all our systems do), i.e.* GA *can be added without expanding the class of provable sentences.*

PROOF. Left to the reader. Use CUT. ⊣

GA together with the theorem schema $\Box(A \leftrightarrow B) \rightarrow ((A \Box\!\!\rightarrow C) \leftrightarrow (B \Box\!\!\rightarrow C))$ (and $\Box(A \leftrightarrow B) \rightarrow ((A \Diamond\!\!\rightarrow C) \leftrightarrow (B \Diamond\!\!\rightarrow C)))$ can be used to obtain several useful derived rules.

If there is a tableau proof of $\Box(A \leftrightarrow B)$, then if $A \Box\!\!\rightarrow C, i$ (or $A \Diamond\!\!\rightarrow C, i$) occurs on a branch, then you may add $B \Box\!\!\rightarrow C, i$ (or $B \Diamond\!\!\rightarrow C, i$) to this branch; and if $B \Box\!\!\rightarrow C, i$ (or $B \Diamond\!\!\rightarrow C, i$) occurs on a branch, then you may add $A \Box\!\!\rightarrow C, i$ (or $A \Diamond\!\!\rightarrow C, i$) to this branch.

Similar derived rules can also be obtained from the theorem schema $((A \Box\!\!\rightarrow B) \wedge (B \Box\!\!\rightarrow A)) \rightarrow ((A \Box\!\!\rightarrow C) \leftrightarrow (B \Box\!\!\rightarrow C))$ (and $((A \Box\!\!\rightarrow B) \wedge (B \Box\!\!\rightarrow A)) \rightarrow ((A \Diamond\!\!\rightarrow C) \leftrightarrow (B \Diamond\!\!\rightarrow C)))$, but I leave that to the reader.

By using these derived rules our tableau proofs can become significantly shorter.

DR1	DR2	DR3	DR4
$\Box(A \leftrightarrow B), i$	$\Box(A \leftrightarrow B), i$	$\Box(A \leftrightarrow B), i$	$\Box(A \leftrightarrow B), i$
$A \mathrel{\Box\!\!\rightarrow} C, i$	$B \mathrel{\Box\!\!\rightarrow} C, i$	$A \mathrel{\Diamond\!\!\rightarrow} C, i$	$B \mathrel{\Diamond\!\!\rightarrow} C, i$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$B \mathrel{\Box\!\!\rightarrow} C, i$	$A \mathrel{\Box\!\!\rightarrow} C, i$	$B \mathrel{\Diamond\!\!\rightarrow} C, i$	$A \mathrel{\Diamond\!\!\rightarrow} C, i$

Table 6. Derived rules (group I)

THEOREM 4. *Every rule in Table 6 is admissible in any counterfactual tableau system that contains* Tc0.

PROOF. Left to the reader. ⊣

DR7	DR8
$\Box(A \rightarrow B), i$	$A \mathrel{\Box\!\!\rightarrow} B, i$
$\downarrow$	$\downarrow$
$A \mathrel{\Box\!\!\rightarrow} B, i$	$(A \rightarrow B), i$

Table 7. Derived rules (group II)

THEOREM 5. DR7 *is admissible in any counterfactual tableau system that contains* Tc1 *and* DR8 *in any that contains* Tc5 (see Table 7).

PROOF. Left to the reader. Use CUT and the indicated rules. ⊣

4.3. Conventions for applying rules

Intuitively we can think of a complete tableau as a tableau where every rule that can be applied has been applied. In our completeness proofs, however, we shall think of a complete tableau as a tableau constructed as follows.

1. For every open branch on the tree, one at a time, start from its root and move towards its tip. Apply any rule that produces something new to the branch. Some rules may have several possible applications, e.g. $\mathrel{\Box\!\!\rightarrow}$. Then make all applications at once.

2. When we have done this for all open branches on the tree, we repeat the procedure. Tc3 is applied if no other rule can be applied, since it introduces

a new "world". The following rules are special: CUT, Tc0 and Tc0′. CUT, Tc0 and Tc0′ are special since they are not analytic and have infinitely many instances. If the system includes CUT, we begin by arranging all sentences in a denumerable sequence $A_1, A_2, \ldots, A_n$. Then we construct a tableau as above but for every cycle n before we conclude n, we split the end of every open branch in the tree and add $\neg A_n, i$ to the left node and A_n, i to the right, for every i on the branch. If the system contains Tc0 we arrange all sentences of the form $\Box(A \leftrightarrow B) \rightarrow ((A \mathbin{\Box\!\!\rightarrow} C) \leftrightarrow (B \mathbin{\Box\!\!\rightarrow} C))$ in a denumerable sequence $A_1, A_2, \ldots, A_n$. We proceed as above, but for every cycle n before we conclude n, we add A_n, i to the end of every open branch in the tree, for every i on the branch. Tc0′ is treated similarly. If there is still something to do according to this "algorithm", the tableau is incomplete; if not, it is complete. In some tableau systems it is possible to restrict the CUT rule in such a way that this rule is only applied to A if A is the antecedent of a counterfactual sentence at a node. This is a good thing if we want to use a tableau system to come up with countermodels to particular sentences or inferences, since we then in many cases get a terminating system (i.e. one that does not generate infinite tableau branches). However, this is not always possible. So, in our soundness and completeness theorems we will use the unrestricted version of CUT.

4.4. Tableau systems

By a counterfactual (tableau) system we mean a set of (primitive) tableau rules that includes all basic, counterfactual, propositional and alethic rules. We call the smallest counterfactual system T and the strongest system TV.

By adding accessibility rules to T, we obtain extensions of this system. Since there are ten different accessibility rules, we have 1024 different counterfactual tableau systems. (However, not all of these systems are distinct.)

Example 6. T012 is the tableau system that includes all propositional rules, all counterfactual rules, all alethic rules, all basic rules and Tc0, Tc1 and Tc2.

4.5. Proofs, derivations, theoremhood etc.

The syntactic concepts of proof, theorem, derivability etc. can essentially be defined as in [21]. (For an extension to the infinite case, see Section 5.3). $\vdash_S A$ says that A is provable in the tableau system S (or that A is a theorem in S), and $\Sigma \vdash_S B$ that B is derivable from Σ in S.

4.6. Logical systems

Let S be a tableau system. Then $L(S)$, the logic of S, is the set of sentences provable in this system, i.e. $L(S) = \{A \in L : \vdash_S A\}$.

Example 7. $L(\mathrm{T012})$, the logic of T012, is the set of all sentences provable in T012, i.e. $L(\mathrm{T012}) = \{A \in L : \vdash_{\mathrm{T012}} A\}$.

4.7. Examples

In this section we will consider some examples of theorems in some systems. Proofs are usually easy and are left to the reader.

T0	$\Box(p \leftrightarrow q) \rightarrow ((p \,\Box\!\!\rightarrow r) \leftrightarrow (q \,\Box\!\!\rightarrow r))$	Cc0
T1	$p \,\Box\!\!\rightarrow p$	Cc1
T2	$((p \wedge q) \,\Box\!\!\rightarrow r) \rightarrow (p \,\Box\!\!\rightarrow (q \rightarrow r))$	Cc2
T3	$\Diamond p \rightarrow ((p \,\Box\!\!\rightarrow q) \rightarrow (p \,\Diamond\!\!\rightarrow q))$	Cc3
T4	$(p \,\Diamond\!\!\rightarrow q) \rightarrow ((p \,\Box\!\!\rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \,\Box\!\!\rightarrow r))$	Cc4
T5	$(p \,\Box\!\!\rightarrow q) \rightarrow (p \rightarrow q)$	Cc5
T6	$(p \wedge q) \rightarrow (p \,\Box\!\!\rightarrow q)$	Cc6
T7	$(p \,\Box\!\!\rightarrow q) \vee (p \,\Box\!\!\rightarrow \neg q)$	Cc7
T8	$(p \,\Box\!\!\rightarrow q) \rightarrow \Box(p \,\Box\!\!\rightarrow q)$	Cc8
T0′	$((p \,\Box\!\!\rightarrow q) \wedge (q \,\Box\!\!\rightarrow p)) \rightarrow ((p \,\Box\!\!\rightarrow r) \leftrightarrow (q \,\Box\!\!\rightarrow r))$	Cc0′

Table 8. Examples of theorems in some counterfactual tableau systems

Theorem 8. *The sentences in Table 8 to 15 are theorems in the indicated systems.*

Proof. Left to the reader. ⊣

5. Soundness and completeness theorems

Let S be any of the tableau systems we discuss in this essay and let $\boldsymbol{M}$ be a class of models. Then S is said to be sound (with respect to $\boldsymbol{M}$) iff every proof-theoretically valid inference in S is semantically valid in $\boldsymbol{M}$, i.e. iff (for every set Σ of sentences in L and every sentence B in L) $\Sigma \vdash_S B$ entails $\Sigma \Vdash_{\boldsymbol{M}} B$. S is said to be complete (with respect to $\boldsymbol{M}$) iff every semantically valid inference in S is proof-theoretically valid in $\boldsymbol{M}$, i.e. iff (for every set Σ of sentences in L and every sentence B in L) $\Sigma \Vdash_{\boldsymbol{M}} B$ entails $\Sigma \vdash_S B$.

$$\begin{array}{l}
\Box q \rightarrow (p \mathrel{\Box\!\!\rightarrow} q) \\
(r \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \rightarrow ((r \mathrel{\Box\!\!\rightarrow} p) \rightarrow (r \mathrel{\Box\!\!\rightarrow} q)) \\
(r \mathrel{\Box\!\!\rightarrow} (p \wedge q)) \leftrightarrow ((r \mathrel{\Box\!\!\rightarrow} p) \wedge (r \mathrel{\Box\!\!\rightarrow} q)) \\
(r \mathrel{\Box\!\!\rightarrow} (p \leftrightarrow q)) \rightarrow ((r \mathrel{\Box\!\!\rightarrow} p) \leftrightarrow (r \mathrel{\Box\!\!\rightarrow} q)) \\
((r \mathrel{\Box\!\!\rightarrow} p) \vee (r \mathrel{\Box\!\!\rightarrow} q)) \rightarrow (r \mathrel{\Box\!\!\rightarrow} (p \vee q)) \\
(r \mathrel{\Box\!\!\rightarrow} (p \leftrightarrow q)) \rightarrow ((r \mathrel{\Diamond\!\!\rightarrow} p) \leftrightarrow (r \mathrel{\Diamond\!\!\rightarrow} q)) \\
(r \mathrel{\Diamond\!\!\rightarrow} (p \wedge q)) \rightarrow ((r \mathrel{\Diamond\!\!\rightarrow} p) \wedge (r \mathrel{\Diamond\!\!\rightarrow} q)) \\
(r \mathrel{\Box\!\!\rightarrow} (p \leftrightarrow q)) \rightarrow ((r \mathrel{\Box\!\!\rightarrow} \neg p) \leftrightarrow (r \mathrel{\Box\!\!\rightarrow} \neg q)) \\
(r \mathrel{\Diamond\!\!\rightarrow} (p \vee q)) \leftrightarrow ((r \mathrel{\Diamond\!\!\rightarrow} p) \vee (r \mathrel{\Diamond\!\!\rightarrow} q)) \\
((r \mathrel{\Box\!\!\rightarrow} (p \vee q)) \wedge (r \mathrel{\Box\!\!\rightarrow} \neg q)) \rightarrow (r \mathrel{\Box\!\!\rightarrow} p) \\
(s \mathrel{\Box\!\!\rightarrow} \neg(p \vee q)) \leftrightarrow ((s \mathrel{\Box\!\!\rightarrow} \neg p) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg q)) \\
(((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} q)) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \wedge q) \rightarrow r))) \rightarrow (s \mathrel{\Box\!\!\rightarrow} r) \\
((s \mathrel{\Box\!\!\rightarrow} \neg p) \vee (s \mathrel{\Box\!\!\rightarrow} \neg q)) \rightarrow (s \mathrel{\Box\!\!\rightarrow} \neg(p \wedge q)) \\
((s \mathrel{\Box\!\!\rightarrow} (p \rightarrow (q \vee r))) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg q) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg r))) \rightarrow (s \mathrel{\Box\!\!\rightarrow} \neg p) \\
((s \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \wedge (s \mathrel{\Box\!\!\rightarrow} p)) \rightarrow (s \mathrel{\Box\!\!\rightarrow} q) \\
(((s \mathrel{\Box\!\!\rightarrow} p) \vee (s \mathrel{\Box\!\!\rightarrow} q)) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \vee q) \rightarrow r))) \rightarrow (s \mathrel{\Box\!\!\rightarrow} r) \\
(s \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \rightarrow ((s \mathrel{\Diamond\!\!\rightarrow} p) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} q)) \\
((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} (p \rightarrow (q \wedge r)))) \rightarrow ((s \mathrel{\Box\!\!\rightarrow} q) \wedge (s \mathrel{\Box\!\!\rightarrow} r)) \\
((s \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \wedge (s \mathrel{\Diamond\!\!\rightarrow} p)) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} q) \\
((s \mathrel{\Box\!\!\rightarrow} \neg r) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \vee q) \rightarrow r))) \rightarrow ((s \mathrel{\Box\!\!\rightarrow} \neg p) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg q)) \\
(s \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \rightarrow ((s \mathrel{\Box\!\!\rightarrow} \neg q) \rightarrow (s \mathrel{\Box\!\!\rightarrow} \neg p)) \\
(((s \mathrel{\Diamond\!\!\rightarrow} p) \vee (s \mathrel{\Diamond\!\!\rightarrow} q)) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \vee q) \rightarrow r))) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} r) \\
((s \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg q)) \rightarrow (s \mathrel{\Box\!\!\rightarrow} \neg p) \\
((s \mathrel{\Box\!\!\rightarrow} (p \rightarrow (q \wedge r))) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg q) \vee (s \mathrel{\Box\!\!\rightarrow} \neg r))) \rightarrow (s \mathrel{\Box\!\!\rightarrow} \neg p) \\
((s \mathrel{\Box\!\!\rightarrow} p) \vee ((s \mathrel{\Diamond\!\!\rightarrow} p) \wedge (s \mathrel{\Diamond\!\!\rightarrow} \neg p))) \vee (s \mathrel{\Box\!\!\rightarrow} \neg p) \\
((s \mathrel{\Box\!\!\rightarrow} (p \rightarrow q)) \wedge (s \mathrel{\Box\!\!\rightarrow} (q \rightarrow r))) \rightarrow (s \mathrel{\Box\!\!\rightarrow} (p \rightarrow r))
\end{array}$$

Table 9. Examples of theorems in T

Let $S = STA_1 \ldots TA_n$, where $STA_1 \ldots TA_n$ is the tableau system constructed from T be adding the accessibility rules TA_1, …, TA_n. Then we shall say that the class of models, $\boldsymbol{M}$, corresponds to S just in case $\boldsymbol{M} = \boldsymbol{M}(CA_1, \ldots, CA_n)$, i.e. the class of models that satisfies CA_1, …, CA_n.

5.1. Soundness theorems

Let M be any model, and b be any branch of a tableau. Then b (or, to be more precise, the set of sentences on b) is satisfiable in M iff there is a

$\Diamond\, p \to ((p \mathrel{\Box\!\!\rightarrow} q) \to \Diamond\, q)$
$\Diamond\, p \to (\Box\, q \to (p \mathrel{\Diamond\!\!\rightarrow} q))$
$\Diamond\, s \to \neg((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg\, p))$
$\Diamond\, s \to ((s \mathrel{\Diamond\!\!\rightarrow} p) \vee (s \mathrel{\Diamond\!\!\rightarrow} \neg\, p))$
$\Diamond\, s \to ((s \mathrel{\Box\!\!\rightarrow} (p \to q)) \to ((s \mathrel{\Box\!\!\rightarrow} p) \to (s \mathrel{\Diamond\!\!\rightarrow} q)))$
$\Diamond\, s \to \neg((s \mathrel{\Box\!\!\rightarrow} (p \vee q)) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg\, p) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg\, q)))$
$\Diamond\, s \to (((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} (p \to q))) \to (s \mathrel{\Diamond\!\!\rightarrow} q))$
$\Diamond\, s \to (((s \mathrel{\Box\!\!\rightarrow} (p \to q)) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg\, q)) \to \neg(s \mathrel{\Box\!\!\rightarrow} p))$
$\Diamond\, s \to ((((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} q)) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \wedge q) \to r))) \to (s \mathrel{\Diamond\!\!\rightarrow} r))$
$\Diamond\, s \to (((s \mathrel{\Box\!\!\rightarrow} (p \to (q \vee r))) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg\, q) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg\, r))) \to \neg(s \mathrel{\Box\!\!\rightarrow} p))$
$\Diamond\, s \to ((((s \mathrel{\Box\!\!\rightarrow} p) \vee (s \mathrel{\Box\!\!\rightarrow} q)) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \vee q) \to r))) \to (s \mathrel{\Diamond\!\!\rightarrow} r))$
$\Diamond\, s \to (((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} (p \to (q \wedge r)))) \to ((s \mathrel{\Diamond\!\!\rightarrow} q) \wedge (s \mathrel{\Diamond\!\!\rightarrow} r)))$
$\Diamond\, s \to (((s \mathrel{\Box\!\!\rightarrow} ((p \vee q) \to r)) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg\, r)) \to (\neg(s \mathrel{\Box\!\!\rightarrow} p) \wedge \neg(s \mathrel{\Box\!\!\rightarrow} q)))$
$\Diamond\, s \to ((((s \mathrel{\Box\!\!\rightarrow} p) \vee (s \mathrel{\Box\!\!\rightarrow} q)) \wedge (s \mathrel{\Box\!\!\rightarrow} ((p \vee q) \to r))) \to (s \mathrel{\Diamond\!\!\rightarrow} r))$
$\Diamond\, s \to (((s \mathrel{\Box\!\!\rightarrow} (p \to (q \wedge r))) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg\, q) \vee (s \mathrel{\Box\!\!\rightarrow} \neg\, r))) \to \neg(s \mathrel{\Box\!\!\rightarrow} p))$

Table 10. Examples of theorems in T3

$\Box(p \to q) \to (p \mathrel{\Box\!\!\rightarrow} q)$	$⟠\, p \to (p \mathrel{\Box\!\!\rightarrow} \bot)$	$\Box\, p \to (\neg\, p \mathrel{\Box\!\!\rightarrow} \bot)$
$(p \mathrel{\Diamond\!\!\rightarrow} \top) \to \Diamond\, p$	$⟠\, p \to (p \mathrel{\Box\!\!\rightarrow} q)$	$⟠\, p \to \neg(p \mathrel{\Diamond\!\!\rightarrow} q)$
$⟠\, p \to \neg(p \mathrel{\Box\!\!\Rightarrow} q)$	$⟠\, p \to (p \mathrel{\Diamond\!\!\Rightarrow} q)$	$⟠(p \wedge \neg\, q) \to (p \mathrel{\Box\!\!\rightarrow} q)$

Table 11. Examples of theorems in T1

function, f, from the natural numbers ($\{0, 1, 2, 3, \dots\}$) to W such that: (i) A is true at $f(i)$ in M, for every node A, i on b, (ii) if $i\, r_A\, j$ is on b, then $f(i)\, R_A\, f(j)$ in M, and (iii) if $i = j$ is on b, then $f(i) = f(j)$. If f fulfills these conditions, we say that f shows that b is satisfiable in M.

LEMMA 9 (Soundness Lemma). *Let b be any branch of a tableau, and M be any model. If b is satisfiable in M, and a tableau rule is applied to it, then it produces at least one extension, b', of b such that b' is satisfiable in M.*

PROOF. First the Soundness Lemma is proved for T. Then it is extended to the other systems. To extend it, all we have to do is check that it still works given the addition of relevant tableau rules. First we check that it works for every single rule, given that M fulfills the corresponding semantic conditions, and then we combine each of the single arguments.

Let f be a function that shows that b is satisfiable in M. The proof proceeds by going through all the tableau rules.

$(p \Box\!\!\rightarrow q) \rightarrow (\Box p \rightarrow q)$	$((p \Box\!\!\rightarrow (q \vee r)) \wedge (\neg q \wedge \neg r)) \rightarrow \neg p$
$((p \Box\!\!\rightarrow q) \wedge p) \rightarrow q$	$((p \wedge q) \wedge (p \Box\!\!\rightarrow r)) \rightarrow r$
$(p \Box\!\!\rightarrow q) \rightarrow (p \rightarrow \Diamond q)$	$(p \wedge (p \Box\!\!\rightarrow (q \wedge r))) \rightarrow q$
$((p \Box\!\!\rightarrow q) \wedge \Box p) \rightarrow \Diamond q$	$(\neg r \wedge ((p \vee q) \Box\!\!\rightarrow r)) \rightarrow (\neg p \wedge \neg q)$
$(p \Box\!\!\rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$	$(p \wedge ((p \vee q) \Box\!\!\rightarrow r)) \rightarrow r$
$((p \Box\!\!\rightarrow q) \wedge \neg q) \rightarrow \neg p$	$((p \Box\!\!\rightarrow (q \wedge r)) \wedge (\neg q \vee \neg r)) \rightarrow \neg p$

Table 12. Examples of theorems in T5

Propositional rules. The proof is standard (see e.g. [21] for an idea of how to do it).

Counterfactual rules. As an illustration we consider the following rules: $\Box\!\!\rightarrow$-pos and $\Diamond\!\!\rightarrow$-pos. The other cases are proved similarly.

($\Box\!\!\rightarrow$-pos): Suppose that $A \Box\!\!\rightarrow B, i$ and $i\, r_A\, j$ are on b, and that we apply the rule $\Box\!\!\rightarrow$ $-$pos to b. Then we get an extension, b', of b that includes the node B, j. Since b is satisfiable in M, $A \Box\!\!\rightarrow B$ is true at $f(i)$ and $f(i)\, R_A\, f(j)$. Hence, by the truth conditions for $A \Box\!\!\rightarrow B$, B is true at $f(j)$, and so $\Box\!\!\rightarrow$ $-$pos produces at least one extension of b, b', such that b' is satisfiable in M.

($\Diamond\!\!\rightarrow$-pos): Suppose that $A \Diamond\!\!\rightarrow B, i$ is on b and we apply $\Diamond\!\!\rightarrow$ $-$pos to get an extension, b', of b that includes nodes of the form $i r_A j$ and B, j. Since b is satisfiable in M, $A \Diamond\!\!\rightarrow B$ is true at $f(i)$. Hence, for some $w \in W$, $f(i)\, R_A\, w$ and B is true at w. Let f' be the same as f except that $f'(j) = w$. Since f and f' differ only at j, f' shows that b is satisfiable in M. Moreover, by definition, $f'(i)\, R_A\, f'(j)$, and B is true at $f'(j)$. Hence, $\Diamond\!\!\rightarrow$ $-$pos produces at least one extension of b, b', such that b' is satisfiable in M.

Basic rules. (CUT): Suppose b is satisfiable in M and we apply CUT. Then we get two branches, one extending b with $\neg A, i$ (the left branch, *bl*) and the other extending b with A, i (the right branch *br*). A is either true or false at $f(i)$. If it is true *br* is satisfiable in M and if it is false *bl* is satisfiable in M. So, CUT produces at least one extension of b that is satisfiable in M.

(IdI): Assume $\alpha(i)$ and $i = j$ are on b and that we obtain $\alpha(j)$ by IdI. f shows that b is satisfiable in M. Hence, $f(i) = f(j)$. If $\alpha(i)$ is A, i, A is true at $f(i)$. So, A is true at $f(j)$, which is what we wanted to show. If $\alpha(i)$ is $k\, r_A\, i$, $f(k)\, R_A\, f(i)$. So, $f(k)\, R_A\, f(j)$, as required. If $\alpha(i)$ is $i = k$, $f(i) = f(k)$. Hence, $f(j) = f(k)$, which is the result we wanted.

(IdII): Left to the reader (see IdI).

Accessibility rules. (Tc0): Any instance of the sentence schema $\Box(A \leftrightarrow B) \rightarrow ((A \Box\!\!\rightarrow C) \leftrightarrow (B \Box\!\!\rightarrow C))$ is true at every world in every model M

$$(p \Diamondright q) \rightarrow \Diamond q$$
$$\diamondminus q \rightarrow (p \boxright \neg q)$$
$$\Box(p \wedge q) \rightarrow ((r \boxright p) \wedge (r \boxright q))$$
$$\Box(p \leftrightarrow q) \rightarrow ((r \boxright p) \leftrightarrow (r \boxright q))$$
$$(\Box p \vee \Box q) \rightarrow (r \boxright (p \vee q))$$
$$\Box(p \leftrightarrow q) \rightarrow ((r \Diamondright p) \leftrightarrow (r \Diamondright q))$$
$$(r \Diamondright (p \wedge q)) \rightarrow (\Diamond p \wedge \Diamond q)$$
$$\Box(p \leftrightarrow q) \rightarrow ((r \boxright \neg p) \leftrightarrow (r \boxright \neg q))$$
$$(r \Diamondright (p \vee q)) \rightarrow (\Diamond p \vee \Diamond q)$$
$$((r \boxright (p \vee q)) \wedge \diamondminus q) \rightarrow (r \boxright p)$$
$$\diamondminus(p \vee q) \rightarrow ((s \boxright \neg p) \wedge (s \boxright \neg q))$$
$$(((s \boxright p) \wedge (s \boxright q)) \wedge \Box((p \wedge q) \rightarrow r)) \rightarrow (s \boxright r)$$
$$(\diamondminus p \vee \diamondminus q) \rightarrow (s \boxright \neg(p \wedge q))$$
$$(\Box(p \rightarrow (q \vee r)) \wedge ((s \boxright \neg q) \wedge (s \boxright \neg r))) \rightarrow (s \boxright \neg p)$$
$$(\Box(p \rightarrow q) \wedge (s \boxright p)) \rightarrow (s \boxright q)$$
$$(((s \boxright p) \vee (s \boxright q)) \wedge \Box((p \vee q) \rightarrow r)) \rightarrow (s \boxright r)$$
$$\Box(p \rightarrow q) \rightarrow ((s \Diamondright p) \rightarrow (s \Diamondright q))$$
$$((s \boxright p) \wedge \Box(p \rightarrow (q \wedge r))) \rightarrow ((s \boxright q) \wedge (s \boxright r))$$
$$(\Box(p \rightarrow q) \wedge (s \Diamondright p)) \rightarrow (s \Diamondright q)$$
$$((s \boxright \neg r) \wedge \Box((p \vee q) \rightarrow r)) \rightarrow ((s \boxright \neg p) \wedge (s \boxright \neg q))$$
$$\Box(p \rightarrow q) \rightarrow ((s \boxright \neg q) \rightarrow (s \boxright \neg p))$$
$$(((s \Diamondright p) \vee (s \Diamondright q)) \wedge \Box((p \vee q) \rightarrow r)) \rightarrow (s \Diamondright r)$$
$$(\Box(p \rightarrow q) \wedge (s \boxright \neg q)) \rightarrow (s \boxright \neg p)$$
$$(\Box(p \rightarrow (q \wedge r)) \wedge ((s \boxright \neg q) \vee (s \boxright \neg r))) \rightarrow (s \boxright \neg p)$$

Table 13. Examples of theorems in T

that satisfies the condition Cc0. For suppose that this sentence is false at some world w in a model M of the appropriate kind. Then $\|A\|^M = \|B\|^M$ and either (i) $A \boxright C$ is true but $B \boxright C$ is false at w in M, or (ii) $B \boxright C$ is true but $A \boxright C$ is false at w in M. Assume (i). Then there is a world w' in M such that $w\, R_B\, w'$ and C is false at w'. Since M satisfies Cc0 and $\|A\|^M = \|B\|^M$, $R_A = R_B$. So, there is a world w' in M such that $w\, R_A\, w'$. But C is true at every world that is A-accessible from w, since $A \boxright C$ is true at w. Accordingly, C is true at w'. But this is absurd. In a similar way it can be shown that (ii) leads to a contradiction. Hence, our assumption is impossible and our original claim verified. So, if we add any instance of $\Box(A \leftrightarrow B) \rightarrow ((A \boxright C) \leftrightarrow (B \boxright C)), i$ to b and b is satisfiable in M, then the branch, b', that results is also satisfiable in M.

$$
\begin{array}{l}
\Diamond s \rightarrow (\Box(p \rightarrow q) \rightarrow ((s \mathrel{\Box\!\!\rightarrow} p) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} q))) \\
\Diamond s \rightarrow \neg((s \mathrel{\Box\!\!\rightarrow} (p \vee q)) \wedge (\diamondminus p \wedge \diamondminus q)) \\
\Diamond s \rightarrow (((s \mathrel{\Box\!\!\rightarrow} p) \wedge \Box(p \rightarrow q)) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} q)) \\
\Diamond s \rightarrow ((\Box(p \rightarrow q) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg q)) \rightarrow \neg(s \mathrel{\Box\!\!\rightarrow} p)) \\
\Diamond s \rightarrow ((((s \mathrel{\Box\!\!\rightarrow} p) \wedge (s \mathrel{\Box\!\!\rightarrow} q)) \wedge \Box((p \wedge q) \rightarrow r)) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} r)) \\
\Diamond s \rightarrow ((\Box(p \rightarrow (q \vee r)) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg q) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg r))) \rightarrow \neg(s \mathrel{\Box\!\!\rightarrow} p)) \\
\Diamond s \rightarrow ((((s \mathrel{\Box\!\!\rightarrow} p) \vee (s \mathrel{\Box\!\!\rightarrow} q)) \wedge \Box((p \vee q) \rightarrow r)) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} r)) \\
\Diamond s \rightarrow (((s \mathrel{\Box\!\!\rightarrow} p) \wedge \Box(p \rightarrow (q \wedge r))) \rightarrow ((s \mathrel{\Diamond\!\!\rightarrow} q) \wedge (s \mathrel{\Diamond\!\!\rightarrow} r))) \\
\Diamond s \rightarrow ((\Box((p \vee q) \rightarrow r) \wedge (s \mathrel{\Box\!\!\rightarrow} \neg r)) \rightarrow (\neg(s \mathrel{\Box\!\!\rightarrow} p) \wedge \neg(s \mathrel{\Box\!\!\rightarrow} q))) \\
\Diamond s \rightarrow ((((s \mathrel{\Box\!\!\rightarrow} p) \vee (s \mathrel{\Box\!\!\rightarrow} q)) \wedge \Box((p \vee q) \rightarrow r)) \rightarrow (s \mathrel{\Diamond\!\!\rightarrow} r)) \\
\Diamond s \rightarrow ((\Box(p \rightarrow (q \wedge r)) \wedge ((s \mathrel{\Box\!\!\rightarrow} \neg q) \vee (s \mathrel{\Box\!\!\rightarrow} \neg r))) \rightarrow \neg(s \mathrel{\Box\!\!\rightarrow} p))
\end{array}
$$

Table 14. Examples of theorems in T3

$\Box p \leftrightarrow (\neg p \mathrel{\Box\!\!\rightarrow} p)$	$\diamondminus p \leftrightarrow (p \mathrel{\Box\!\!\rightarrow} \neg p)$	$\Diamond p \leftrightarrow (p \mathrel{\Diamond\!\!\rightarrow} p)$

Table 15. Examples of theorems in T13

(Tc1): Suppose that ir_Aj is on b, and that we apply Tc1 to give an extended branch, b', of b containing A, j. Since b is satisfiable in M, $f(i)\, R_A\, f(j)$. Hence, A is true at $f(j)$, since M satisfies condition Cc1. Consequently, Tc1 produces at least one extension, b', of b such that b' is satisfiable in M.

(Tc3): Suppose that A, i and j are on b, and that we apply Tc3 to give an extended branch, b', of b containing $j\, r_A\, k$ where k is new. Since b is satisfiable in M and M fulfills condition Cc3, for all i on b $f(i)\, R_A\, w$ for some w in W. Let f' be the same as f except that $f'(k) = w$. f' shows that b is satisfiable in M; for k is not on b. By construction, $f'(j)\, R_A\, f(k)$. So, f' also shows that b' is satisfiable in M. It follows that, Tc3 produces at least one extension, b', of b such that b' is satisfiable in M.

(Tc6): Assume that A, i and $i\, r_A\, j$ are on b, and that we apply Tc6 to give an extended branch, b', of b containing $i = j$. Then A is true at $f(i)$ and $f(i)\, r_A\, f(j)$, since b is satisfiable in M. Since, M satisfies condition Cc6, A is true at $f(i)$ and $f(i)\, r_A\, f(j)$, $f(i) = f(j)$. Consequently, Tc6 produces at least one extension, b', of b such that b' is satisfiable in M.

(Tc7): Suppose that $i\, r_A\, j$ and $i\, r_A\, k$ are on b, and that we apply Tc7 to obtain an extended branch, b', of b including $j = k$. Then $f(i)\, r_A\, f(j)$ and $f(i)\, r_A\, f(k)$, since b is satisfiable in M. Since, M satisfies condition

Cc7, $f(i)\, r_A\, f(j)$ and $f(i)\, r_A\, f(k)$, $f(j) = f(k)$. Accordingly, at least one extension, b', of b such that b' is satisfiable in M is produced by Tc7.

Remaining rules are left to the reader. ⊣

THEOREM 10 (Soundness Theorem). *Let S be any of the tableau systems we discuss in this essay. Then S is sound with respect to the class of models $\boldsymbol{M}$ that corresponds to S. For finite Σ, if $\Sigma \vdash_S B$, then $\Sigma \Vdash_{\boldsymbol{M}} B$.* (For an extension to the infinite case, see Section 5.3.)

PROOF. This proof is essentially the same as the proof that certain normal modal systems are sound. (See e.g. [21], especially chapters 1 and 2.) ⊣

5.2. Completeness theorems

Let b be an open branch of a tableau and I the set of numbers on b. We shall say that $i \rightleftharpoons j$ just in case $i = j$, or "$i = j$" or "$j = i$" occur on b. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. The model, $M = \langle W, \{R_A : A \in L\}, V\rangle$, induced by b is defined as follows. $W = \{w[i] : i \in I\}$, $w[i]\, R_A\, w[j]$ iff $i\, r_A\, j$ occurs on b. If p, i occurs on b, then p is true in $w[i]$, $Vw[i](p) = T$; and if $\neg p, i$ occurs on b, then p is false in $w[i]$, $Vw[i](p) = F$.

LEMMA 11 (Completeness Lemma). *Let b be any open branch of a complete tableau and let $M = \langle W, \{R_A : A \in L\}, V\rangle$ be the model induced by b. Then:* (i) *A is true at $w[i]$, if A, i is on b, and* (ii) *A is false at $w[i]$, if $\neg A, i$ is on b.*

PROOF. The proof is by induction on the complexity of A.

Basis. If A is atomic, the result is true by definition.

Induction step. If A is complex, it is of the form $B \vee C$, $B \wedge C$, $B \to C$, $B \leftrightarrow C$, $\neg B$, $B \mathrel{\Box\!\!\rightarrow} C$, $B \mathrel{\Diamond\!\!\rightarrow} C$, $\Box B$, $\Diamond B$ or $\diamondminus B$. The proofs for the propositional connectives are standard and the proofs for the alethic operators are similar to proofs found in [21]. Let us consider the case when $A = B \mathrel{\Diamond\!\!\rightarrow} C$ to illustrate the method. The remaining cases are proved similarly.

$A = B \mathrel{\Diamond\!\!\rightarrow} C$. Suppose that A occurs on b, i.e. $B \mathrel{\Diamond\!\!\rightarrow} C, i$ is on b. Since the tableau is complete $\mathrel{\Diamond\!\!\rightarrow}$ has been applied to $B \mathrel{\Diamond\!\!\rightarrow} C, i$. Thus, for some j, $i\, r_B\, j$ and C, j are on b. By induction hypothesis, $w[i]\, R_B\, w[j]$ and C is true at $w[j]$. Hence, $B \mathrel{\Diamond\!\!\rightarrow} C$ is true at $w[i]$. Thus, if $B \mathrel{\Diamond\!\!\rightarrow} C, i$ is on b, then $B \mathrel{\Diamond\!\!\rightarrow} C$ is true in $w[i]$. Suppose $\neg A$ occurs on b, i.e. $\neg(B \mathrel{\Diamond\!\!\rightarrow} C), i$ is on b. Since the tableau is complete $\neg \mathrel{\Diamond\!\!\rightarrow}$ has been applied to $\neg(B \mathrel{\Diamond\!\!\rightarrow} C), i$ and

$B \;\Box\!\!\rightarrow \neg C, i$ is on b, and $\Box\!\!\rightarrow$ has been applied to $B \;\Box\!\!\rightarrow \neg C, i$. Thus, $\neg C, j$ is on b, for all j such that $i\, r_B\, j$ is on b. By the induction hypothesis, $\neg C$ is true at $w[j]$, for all $w[j]$ such that $w[i]\; R_B\; w[j]$. Accordingly, $B \;\Diamond\!\!\rightarrow C$ is false at $w[i]$. Thus, if $\neg(B \;\Diamond\!\!\rightarrow C), i$ is on b, then $B \;\Diamond\!\!\rightarrow C$ is false in $w[i]$. Consequently, the lemma holds for $A = B \;\Diamond\!\!\rightarrow C$.

Conclusion. We can now conclude that the lemma holds for a sentence A of any complexity. ⊣

THEOREM 12 (Completeness Theorem I). *For finite Σ, if $\Sigma \Vdash_{\boldsymbol{M}(V)} B$, then $\Sigma \vdash_{\mathrm{T}} B$.* (For an extension to the infinite case, see Section 5.3.)

PROOF. Details are left to the reader. (Similar proofs can be found in [21], especially chapters 1 and 2.) ⊣

THEOREM 13 (Completeness Theorem II). *Let S be any of the remaining 255 systems we discuss in this essay, not including* Tc0 *or* Tc0′, *and let $\boldsymbol{M}$ in each case be the corresponding class of models. For finite Σ, if $\Sigma \Vdash_{\boldsymbol{M}} B$, then $\Sigma \vdash_S B$.* (For an extension to the infinite case, see Section 5.3.)

PROOF. The proof is as for T. In addition, we just have to check that the model induced by the open branch, b, is of the right kind. To do this we first check that this is true for every single condition on M. Then we combine each of the individual arguments. We look at Cc1, Cc2, Cc3 and Cc6. The remaining cases are left to the reader.

(Cc1): Suppose $w[i]\; R_A\; w[j]$, where $w[i], w[j] \in W$. Then $i\, r_A\, j$ occurs on b [by the definition of an induced model]. Since the tableau is complete Tc1 has been applied and A, j is on b. So, [by the completeness lemma] A is true at $w[j]$, as required.

(Cc2): Suppose that $w[i]\; R_A\; w[j]$ and B is true at $w[j]$, where $w[i]$, $w[j]$ $\in W$. Then $i\, r_A\, j$ [by the definition of an induced model]. Since the tableau is complete CUT has been applied and either B, j or $\neg B, j$ is on b. Suppose $\neg B, j$ is on b. Then B is false at $w[j]$ [by the completeness lemma]. But this is impossible. Hence, B, j is on b. Since b is complete Tc2 has been applied and $ir_{A\wedge B}j$ is on b. Accordingly, $w[i]R_{A\wedge B}w[j]$ as required [by the definition of an induced model].

(Cc3): Let $w[i]$, $w[j]$, $w[k] \in W$. Suppose A is true at $w[i]$. Since the tableau is complete CUT has been applied and either A, i or $\neg A, i$ is on b. Suppose $\neg A, i$ is on b. Then A is false at $w[i]$ [by the completeness lemma]. But this is not possible. Hence, A, i is on b. Since b is complete Tc3 has been applied and $j\; r_A\; k$ occurs on b for every j on b. Accordingly, for all

$w[j]$ there is a $w[k]$ such that $w[j]\ R_A\ w[k]$, as required [by the definition of an induced model].

(Cc6): Suppose that A is true at wi and that $w[i]\ R_A\ w[j]$, where $w[i]$, $w[j] \in W$. Then $i\ r_A\ j$ [by the definition of an induced model]. Since the tableau is complete CUT has been applied and either A, i or $\neg A, i$ is on b. Suppose $\neg A, i$ is on b. Then A is false at $w[i]$ [by the completeness lemma]. But this is impossible. Accordingly, A, i is on b. Since the tableau is complete Tc6 has been applied and $i = j$ is on b. Hence, $i \rightleftharpoons j$. So, $[i] = [j]$. It follows that $w[i] = w[j]$, as required. $\dashv$

5.3. Infinite premise sets

So far we have assumed that the set Σ of premises in a tableau derivation of B from Σ is finite. But we can say that B is a logical consequence of Σ not only when Σ is finite but also when Σ is a (denumerably) infinite set of sentences. So, we want to extend our tableau technique to be able to deal with (denumerably) infinite sets of premises. To do this we can adapt a method mentioned by Smullyan in [25, p. 64] to our systems. First we arrange all premises in Σ in a denumerable sequence $A_1, A_2, \ldots, A_n$. Then we construct a tableau for the negation of B, i.e. a tableau whose root is $\neg B, 0$. We proceed as in the finite case (see section 4), but for every cycle n before we conclude n, we add $A_n, 0$ to the end of every open branch in the tree. Then sooner or later every premise in Σ gets added to every open branch.

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