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DECIDABILITY OF MEREOLOGICAL THEORIES

Abstract. Mereological theories are theories based on a binary predicate 'being a part of'. It is believed that such a predicate must at least define a partial ordering. A mereological theory can be obtained by adding on top of the basic axioms of partial orderings some of the other axioms posited based on pertinent philosophical insights. Though mereological theories have aroused quite a few philosophers' interest recently, not much has been said about their meta-logical properties. In this paper, I will look into whether those theories are decidable or not. Besides, since theories of Boolean algebras are in some sense upper bounds of mereological theories which can be found in the literature, I shall also make some observations about the possibility of getting mereological theories beyond Boolean algebras.

Keywords: mereology, mereological theories, part-whole relation, decidability, undecidability.

1. Introduction

A mereological theory is formed with axioms based on a binary relation "being a part of"; hence the formal language of such a theory will then have only one non-logical symbol, a binary predicate 'P', which stands for the aforementioned relation. It is known that the origin of this kind of theory is due to Leśniewski[1]. However, recently there have been other authors, such

[1]It is said that Leśniewski was shocked by Russell's paradox and thought that mereology could offer a way out: in his view the term 'class' is ambiguous and if it is interpreted as 'mereological sum' then Russell's paradox will be solved right away, for everything is a part of itself and hence there is no class which is the sum of things each of which is not a part of itself (see Simons [1987, p. 102] for this story).

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as Simons [1987] or Casati and Varzi [1999], who have tried to reformulate mereological axioms using the formal language with which most logicians nowadays are familiar. In this paper, I shall follow Casati and Varzi's formulation when making my inquiries.

Let's look at those mereological axioms and mereological theories now. As mentioned above, the binary predicate 'P' is the only non-logical symbol in the formal language with which we are concerned. But for the convenience of our saying some complicated things, we shall define the following three predicates:

(Proper Part:) $\mathrm{PP}xy \overset{\mathrm{df}}{=} \mathrm{P}xy \wedge x \neq y\,,$

(Overlap:) $\mathrm{O}xy \overset{\mathrm{df}}{=} \exists z(\mathrm{P}zx \wedge \mathrm{P}zy)\,,$

(Underlap:) $\mathrm{U}xy \overset{\mathrm{df}}{=} \exists z(\mathrm{P}xz \wedge \mathrm{P}yz)\,.$

It would be fair to say that most philosophers believe that 'P' must at least define a partial ordering, that is, it is reflexive, antisymmetric and transitive. Thus we have the following three basic axioms.

(P1) $\forall x \mathrm{P}xx\,,$

(P2) $\forall x\forall y((\mathrm{P}xy \wedge \mathrm{P}yx) \to x = y),$

(P3) $\forall x\forall y\forall z((\mathrm{P}xy \wedge \mathrm{P}yz) \to \mathrm{P}xz).$

The theory axiomatized by these three basic axioms is called *ground mereology* **GM** (this can be conveniently expressed by putting **GM** := (P1) + (P2) + (P3) and henceforth we shall use this kind of notation when defining a theory). There are some other mereological axioms which are arguably still philosophically motivated, such as: *Extensionality Principle* (EP), *Weak Supplementation Principle* (WSP), *Strong Supplementation Principle* (SSP), *Finite Sum* (FS) and *Finite Product* (FP)

(EP) $\forall x\forall y(\exists z\mathrm{PP}zx \to (\forall z(\mathrm{PP}zx \leftrightarrow \mathrm{PP}zy) \to x = y)),$ [2]

(WSP) $\forall x\forall y(\mathrm{PP}xy \to \exists z(\mathrm{PP}zy \wedge \neg \mathrm{O}zx)),$ [3]

[2]Intuitively the mereological counterpart of the extensionality of set theory should be the formula '$\forall x\forall y(\forall z(\mathrm{P}zx \leftrightarrow \mathrm{P}zy) \to x = y)$'. But this is provable from (P1) and (P2), and hence is uninteresting. The proof is easy: from (P1) we have '$\forall x\forall y(\forall z(\mathrm{P}zx \to \mathrm{P}zy) \to \mathrm{P}xy)$', and we use (P2). By the way, it might be worthwhile to note that (P1) and (P3) are jointly equivalent to the formula '$\forall x\forall y(\forall z(\mathrm{P}zx \to \mathrm{P}zy) \leftrightarrow \mathrm{P}xy)$'.

[3]This is Simons' version of (WSP) [1987, p. 28]. In Casati and Varzi's book [1999, p. 39], (WSP) is '$\forall x\forall y(\mathrm{PP}xy \to \exists z(\mathrm{P}zy \wedge \neg \mathrm{O}zx))$'. These two versions are actually equivalent under (P1). But here we shall follow Simons since the term 'supplementation' is due to him.

(SSP) $\forall x \forall y(\neg \mathrm{P}yx \to \exists z(\mathrm{P}zy \wedge \neg \mathrm{O}zx))$,[4]

(FS) $\forall x \forall y(\mathrm{U}xy \to \exists z \forall w(\mathrm{O}wz \leftrightarrow (\mathrm{O}wx \vee \mathrm{O}wy)))$,

(FP) $\forall x \forall y(\mathrm{O}xy \to \exists z \forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \mathrm{P}wy)))$,

(FS) and (FP) are called *closure principles*. Variants of mereological theories can be formed by adding one or more of the foregoing principles on top of **GM**. For example, Casati and Varzi have classified the following mereological theories: Minimal Mereology **MM** := **GM** + (WSP), Extensional Mereology **EM** = **GM** + (SSP),[5] Closure Mereology **CM** := **GM** + (FS) + (FP), Minimal Closure Mereology **CMM** := **MM** + (FS) + (FP) and Extensional Closure Mereology **CEM** := **EM** + (FS) + (FP).

It is not difficult to see that

$$\mathbf{GM} < \mathbf{MM} < \mathbf{EM} < \mathbf{CMM} = \mathbf{CEM}\,,$$

where '$S < T$' means that T is a strictly stronger theory than S. How about **GM** + (EP)? It can be easily shown that **GM** + (EP) < **EM** but **GM** + (EP) and **MM** are independent, that is, **GM** + (EP) $\neq$ **MM** and neither is stronger than the other. Furthermore, it is also not difficult to see that **CM** is independent of any of **EM**, **MM** and **GM** + (EP).

We shall also see in the following that the aforementioned mereological theories are actually in some sense subtheories of the elementary theory of Boolean algebras (henceforth we will abbreviate such a theory as **ETB**).[6]

An algebra of sets is a structure of the form $\langle S, \cup, \cap, -, \emptyset, X\rangle$, where S is a set of subsets of a nonempty set X such that $\emptyset \in S$, $X \in S$ and S is closed under $\cup$, $\cap$ and $-$ (complement with respect to X). Now we'll show how to define a mereological structure inside an algebra of sets. Let's interpret 'P' as set inclusion (this is arguably the most natural interpretation we can come up with for 'P'). It is easy to see that (P1), (P2), (P3) and (EP) are true in any algebra of sets. But (WSP) will be false, for if x is the empty

[4] Though what this principle says looks quite intuitive, Simons thinks that it should be rejected in order to resolve the difficulty which is caused by identifying an object with the sum of its parts: an object could survive the loss of some parts but the sum couldn't (see Simons [1987, pp. 115–116]). However, I shall leave relevant philosophical debates aside here.

[5] Actually, (P1) is redundant. That is to say, (P2), (P3) and (SSP) suffice to axiomatize **EM**, for it can be shown that (P3) and (SSP) entail (P1)(see Pietruszczak [2005, p. 217]).

[6] Indeed, logicians in the past few decades have already discovered the similarity between mereological structures and Boolean algebras (see Grzegorczyk [1955], Tarski [1956] and Clay [1974]). But here I will deal with the matter with a different approach.

set and y is nonempty, then PPxy is the case, but since x is a part of any member in the domain, there is no part of y which does not overlap x. For the same reason, (SSP) will be false in any algebra of sets, and (FS) will be satisfied in a way not welcomed by us: the sum of the empty set and any set will be the whole X or the empty set. So the empty set causes serious problems and this urges us to remove it from the domain. It is not difficult to see that all the axioms listed above are true in any algebra of sets with the empty set removed. By Stone representation theorem, they (actually the translations of those axioms into the language of Boolean algebra) are also true in any Boolean algebra with 0 removed. Hence any theory axiomatized by some of the axioms listed above is in this sense a subtheory of **ETB**. Based on this kind of mereological structures, the strongest theory which we can get is the one which is satisfied by any Boolean algebra with 0 removed, and as we shall see later on, such a theory is axiomatized by exactly all the axioms listed above plus the following two additional axioms:

(C) $\forall x(\neg\forall y\, \mathrm{P}yx \to \exists z\forall y(\mathrm{P}yz \leftrightarrow \neg\mathrm{O}yx)),$

which says that everything which is not the greatest member has a complement, and

(G) $\exists x\forall y\, \mathrm{P}yx,$

which says that there exists the greatest member. We will label this theory as **CEM**+(C)+(G).

Observe that for any algebra of sets $\langle S, \cup, \cap, -, \emptyset, X\rangle$, we can define a structure $\langle S \setminus \{\emptyset\}, \cup, \cap^*, -^*, X\rangle$, where $\cap^*$ and $-^*$ are partial operators such that for any $a, b \in S \setminus \{\emptyset\}$: $a \cap^* b = a \cap b$ if $a \cap b \neq \emptyset$, and $-^*a = -a$ if $a \neq X$; and it is easy to see that **CEM**+(C)+(G) is satisfied by any structure of the aforementioned form if '$\mathrm{P}xy$' is interpreted by '$x \cup y = y$'. Furthermore, any model of **CEM**+(C)+(G) has a definitional expansion (which must be unique) isomorphic to a structure of the aforementioned form (see Theorem 3 below). Hence **CEM**+(C)+(G) can be viewed as the theory which characterizes the class of the structures of the aforementioned form.

Now it is known that **GM** is undecidable but **ETB** is decidable (see next section). Then from the viewpoint of logic, it seems natural that it will be an interesting job to look into whether each mereological theory located in between is decidable or not. My results below will cover a little bit more: some other interesting extensions of **CEM**, for instance, **CEM**+($\neg$C) or **CEM**+($\neg$G) and so on, will also be considered.

Remark 1. (i) I have left out an axiom schema which is called by Casati and Varzi *fusion* [1999, p. 46]: for any formula ϕ,

$$\text{(Fusion)} \qquad \exists x\, \phi \to \exists z \forall y(\mathrm{O}yz \leftrightarrow \exists x(\phi \wedge \mathrm{O}yx)).$$

Fusion will cause some complications with which I have not yet figured out a way to deal. Therefore in this paper I shall confine my research to theories which can be generated by the axioms without fusion. Nonetheless, it is still worthwhile to say something about this axiom schema here. Strictly speaking, in the schema above, ϕ cannot be arbitrary: it should be required that the variables 'z' and 'y' cannot occur free in ϕ. Without loss of generality, we can assume that the variable 'x' occurs free in ϕ (if not, replace ϕ by the equivalent ϕ' which is $\ulcorner \phi \wedge x = x \urcorner$).

Now—assuming the above restrictions on 'y and 'z'—let $\ulcorner z\, \mathtt{Fu}_x\, \phi \urcorner$ abbreviate $\ulcorner \forall y(\mathrm{O}yz \leftrightarrow \exists x(\phi \wedge \mathrm{O}yx)) \urcorner$, which means that z is a fusion of all x such that ϕ. Hence the fusion schema says that if ϕ is fulfilled, then the fusion of all x such that ϕ exists:

$$\exists x\, \phi \to \exists z\; z\, \mathtt{Fu}_x\, \phi\,.$$

But by (P1), it is easy to see that the converse $\ulcorner \exists z\; z\, \mathtt{Fu}_x\, \phi \to \exists x\, \phi \urcorner$ is also the case. Moreover, in all **EM**-theories, if the variable 'u' is not free in ϕ then we obtain:

$$(z\, \mathtt{Fu}_x\, \phi \wedge u\, \mathtt{Fu}_x\, \phi) \to z = u\,,$$

since in these theories we have the following theses:

$$\text{(SSP}'\text{)} \qquad \mathrm{P}xy \leftrightarrow \forall z(\mathrm{O}zx \to \mathrm{O}zy),$$

$$\text{(EO)} \qquad \forall z(\mathrm{O}zx \leftrightarrow \mathrm{O}zy) \to x = y\,.$$

(ii) For $\phi =$ '$x = x$' in (Fusion), by (P1), we obtain '$\exists z \forall y\, Oyz$'. But from (SSP) we have '$\forall y\, \mathrm{O}yz \to \forall y\, \mathrm{P}yz$'. Hence (G) and '$\forall x \forall y\, \mathrm{U}xy$' are theses of **EM** + (Fusion), i.e. **EM** together with the fusion schema.

(iii) Some variables other than 'x' might also occur free in ϕ and this allows us to have new axioms which fall under the schema (Fusion).[7]

For $\phi =$ '$x = u \vee x = v$' we have:

$$\exists x(x = u \vee x = v) \to \exists z \forall y(\mathrm{O}yz \leftrightarrow \exists x((x = u \vee x = v) \wedge \mathrm{O}yx)),$$

[7] Notice that we have '$\forall y(\mathrm{O}yu \leftrightarrow \exists x(x = u \wedge \mathrm{O}yx))$', '$\forall y(\mathrm{O}yu \leftrightarrow \exists x(\mathrm{P}xu \wedge \mathrm{O}yx))$', and '$\exists x\, \mathrm{PP}xu \leftrightarrow \forall y(\mathrm{O}yu \leftrightarrow \exists x(\mathrm{PP}xu \wedge \mathrm{O}yx))$' (the latter two by (P1) and (P3)). Hence: $u\, \mathtt{Fu}_x\, x = u$; $u\, \mathtt{Fu}_x\, \mathrm{P}xu$; and $u\, \mathtt{Fu}_x\, \mathrm{P}xu$ iff u has a proper part. Thus, there is no need to use the schema (Fusion) for formulas: '$x = u$', '$\mathrm{P}xu$' and '$\mathrm{PP}xu$'.

which is logically equivalent to

(FS′) $\exists z \forall y(\mathrm{O}yz \leftrightarrow (\mathrm{O}yu \vee \mathrm{O}yv))$.

So (FS) is a thesis of **EM** + (Fusion) (notice that '$\mathrm{U}uv$' is a thesis of this theory; see (ii)).

For $\phi =$ '$\mathrm{P}xu \wedge \mathrm{P}xv$' we have:

(FP′) $\mathrm{O}uv \rightarrow \exists z \forall y(\mathrm{O}yz \leftrightarrow \exists x(\mathrm{P}xu \wedge \mathrm{P}xv \wedge \mathrm{O}yx))$.

Notice that

$$\forall y(\mathrm{P}yz \leftrightarrow (\mathrm{P}yu \wedge \mathrm{P}yv)) \leftrightarrow \forall y(\mathrm{O}yz \leftrightarrow \exists x(\mathrm{P}xu \wedge \mathrm{P}xv \wedge \mathrm{O}yx))$$

is a thesis of **EM**.[8] Thus, (FP) and (FP′) are equivalent in **EM**.

Now in (Fusion) we put $\phi =$ '$\neg\mathrm{O}xu$'. We obtain:

(C′) $\exists x\, \neg\mathrm{O}xu \rightarrow \exists z \forall y(\mathrm{O}yz \leftrightarrow \exists x(\neg\mathrm{O}xu \wedge \mathrm{O}yx))$.

Notice that

$$\forall y(\mathrm{P}yz \leftrightarrow \neg\mathrm{O}yu) \leftrightarrow \forall y(\mathrm{O}yz \leftrightarrow \exists x(\neg\mathrm{O}xu \wedge \mathrm{O}yx)).$$

is a thesis of **EM**.[9] Thus, formulas (C) and (C′) are equivalent in **EM**, since '$\exists x\, \neg\mathrm{O}xu \leftrightarrow \neg\forall x\, \mathrm{P}xu$' is a thesis of this theory.

Thus, **CEM**+(C)+(G) $\leq$ **EM**+(Fusion), since **EM**+(Fusion) entails (FS), (FP), (C) and (G).

(iv) Now—keeping mentioned restrictions on variables 'y and 'z'—let $\ulcorner z\, \mathtt{CSet}_x\, \phi \urcorner$ abbreviate $\ulcorner \forall x(\phi \rightarrow \mathrm{P}xz) \wedge \forall y(\mathrm{P}yz \rightarrow \exists x(\phi \wedge \mathrm{O}xy) \urcorner$, which means that z is a "collective set" of all x such that ϕ. The term 'collective set' is due to Leśniewski; he once used the following schema to express the existence of a collective set of all x such that ϕ (see Leśniewski [1992, p. 230]):

$$\exists x\, \phi \rightarrow \exists z\, z\, \mathtt{CSet}_x\, \phi\,.$$

In **GM** we obtain the thesis '$\forall z(z\, \mathtt{CSet}_x\, \phi \rightarrow z\, \mathtt{Fu}_x\, \phi)$'. However, it can be shown that in **EM** we obtain the following thesis (see Pietruszczak [2005, pp. 216 and 218])

$$\forall z(z\, \mathtt{CSet}_x\, \phi \leftrightarrow z\, \mathtt{Fu}_x\, \phi).$$

Hence, in **EM** Leśniewski's schema is equivalent to the aforementioned fusion schema.

[8] In the proof of this fact we use (P1), (P3), (SSP), (SSP′) and '$\mathrm{P}zu \leftrightarrow \forall y(\mathrm{P}yz \rightarrow \mathrm{P}yu)$' (see Footnote 2).

[9] We can obtain the proof of this fact by (P1), (P3), (SSP) and (SSP′).

2. Some Useful Meta-logical Theorems

It is worthwhile to note that all the mereological theories from **GM** up to **CEM**+(C)+(G) are recursively axiomatized, but are not complete, for they can be satisfied in finite as well as in infinite Boolean algebras with 0 removed. Therefore, though it is known that a recursively axiomatized complete theory must be decidable, we cannot make use of this fact.

Before introducing the theorems which I will count on, let's first give the definitions needed here.

Two sets A and B of natural numbers are *effectively inseparable* if and only if $A \cap B = \emptyset$ and for all recursively enumerable sets C and D such that $A \subseteq C$, $B \subseteq D$ and $C \cap D = \emptyset$, there is an effective procedure via which we can find a natural number which does not belong to $C \cup D$.

A theory T based on a language L is *inseparable* if and only if $\{\#\alpha : \alpha \text{ is a theorem of } T\}$ and $\{\#\alpha : \alpha \text{ is a sentence whose negation is a theorem of } T\}$ are effectively inseparable, where $\#\alpha$ stands for the Gödel number of α.

A theory T based on a language L is *finitely inseparable* if and only if $\{\#\alpha : \alpha \text{ is a sentence and is true in every structure of } L\}$ and $\{\#\alpha : \alpha \text{ is a sentence whose negation is true in some finite model of } T\}$ are effectively inseparable.

An *interpretation* of a language L into a theory T' of a language L' is a function I whose domain is $\{\forall\} \cup \{\delta : \delta \text{ is a non-logical symbol of } L\}$ such that

(i) $I(\forall) = \alpha_\forall$, a formula of L' with at most one free variable, and $T' \models \exists x\, \alpha_\forall$,

(ii) for any n-placed predicate R of L, $I(R) = \alpha_R$, a formula of L' with at most n free variables, and

(iii) for any n-placed function symbol F of L, $I(F) = \alpha_F$, a formula with at most $n+1$ free variables, and

$$T' \models \forall x_1 x_2 \ldots x_{n+1}((\alpha_\forall(x_1) \wedge \cdots \wedge \alpha_\forall(x_n)) \rightarrow \\ \exists y(\alpha_\forall(y) \wedge \forall x_{n+1}(\alpha_F(x_1, x_2 \ldots x_{n+1}) \leftrightarrow x_{n+1} = y))).$$

It is easy to see that for each model A of T', we can construct a structure A^I of L: the domain of A^I is the set defined on A by $I(\forall)$, the interpretation of an n-placed predicate R is the relation defined in A by $I(R)$ restricted to the domain of A^I and the interpretation of an n-placed function symbol F is the function f such that for all $a_1, \ldots, a_n$ in the domain of A^I, $f(a_1, \ldots, a_n)$ is the unique b such that $A \models I(F)[a_1, \ldots, a_n, b]$. Now an L-theory T can be *interpreted* into T' if and only if L can be interpreted into T' and $T \subseteq \{\alpha : \alpha$ is a sentence which is true in every A^I, where A is a model of $T'\}$ and the interpretation is *faithful* if and only if $T = \{\alpha : \alpha$ is a sentence which is true in every A^I, where A is a model of $T'\}$.

A structure of a language L is *strongly undecidable* if and only if any L-theory which has that structure as a model is undecidable.

Structure A is a *definable substructure* of structure B if and only if $\mathrm{Dom}(A)$ (the domain of A) is a subset of $\mathrm{Dom}(B)$ which is definable in B and each function or predicate of A is the restriction to $\mathrm{Dom}(A)$ of a function or predicate definable in B. Structure A is *definable* in structure B (not necessarily in the same language) if and only if there is a definable substructure of B which is isomorphic to A.[10]

It turns out that the following known theorems are very useful for my purpose.

(1) **GM** is finitely inseparable.

(2) If a theory is finitely inseparable, it is undecidable.

(3) **ETB** is decidable.

(4) Let T and T' be two theories of languages L and L' respectively. If T can be interpreted faithfully into T' and T' is decidable, then T is decidable.

(5) Let T and T' be two theories of languages L and L' respectively. Assume that L has only finitely many function symbols. Suppose L can be interpreted into a finitely axiomatized L'-theory S'. If for each finite model A of T there is a finite model B of $T' \cup S'$ such that $A = B^I$ and T is finitely inseparable, then T' is finitely inseparable.

(6) There is a strongly undecidable model of irreflexive symmetric ordering. Without loss of generality, we can assume that the language has only one non-logical symbol, a binary predicate ‘P’. Then the theorem says that there is a strongly undecidable structure of this language which satisfies ‘$\neg \mathrm{P}xx$’ and ‘$\mathrm{P}xy \to \mathrm{P}yx$’.

[10] All the definitions given here are standard and readers can find them in many prestigious textbooks. But here the formulation of the first three follows Monk [1976, pp. 100 and 266] and I have rephrased them a little bit. The definition of “interpretation” mainly follows Enderton [1972, S. 2.7].

(7) Suppose $L(A)$ (the language of A) has only finitely many non-logical symbols. If A is strongly undecidable and is definable in B, B is strongly undecidable too.

(8) The theory of distributive lattices is finitely inseparable.

(9) If T and S are theories of the same language, T is an extension of S and T is finitely inseparable, then S is finitely inseparable too.

For (1), (8), (5) and (9), see Monk [1976, pp. 280, 272 and 269]. For (3), see Koppelberg [1989, Chapter 7]. For (6) and (7), see Shoenfield [1967, pp. 142 and 136]. (2) and (4) are easy to prove.

3. Results concerning Decidability of Mereological Theories

In the following proofs, I will rely heavily on the fact that models of mereological theories are closely related to partial orderings, lattices or Boolean algebras, depending on how complicated the theory in question is. By the way, both "inseparable" and "finitely inseparable" mentioned above are stronger properties than "undecidable". Naturally, I shall always try to prove a stronger result unless I cannot find a way to carrying it out.

Theorem 1. **EM** *is finitely inseparable and hence undecidable.*

Proof. The idea is to use the aforementioned (1), (2) and (5) to show that **EM** is finitely inseparable and hence undecidable. First we set up an interpretation I of **GM** into some finitely axiomatized theory. Obviously we only have to take care of $I(\forall)$, for $I(\mathrm{P})$ is just '$\mathrm{P}xy$'. Let $I(\forall)$ be '$\exists y\, \mathrm{PP}yx \vee \forall y(\neg \mathrm{PP}yx \wedge \neg \mathrm{PP}xy)$' and S be the theory axiomatized by $\{\exists x(\exists y\, \mathrm{PP}yx \vee \forall y(\neg \mathrm{PP}yx \wedge \neg \mathrm{PP}xy))\}$. It is easy to check that I is an interpretation of L into S. Now we must show that for any finite model A of **GM**, we can find a finite model B of $\mathbf{EM} \cup S$ such that $A = B^I$. Suppose A is a finite model of **GM**. There are four possibilities for any $a \in \mathrm{Dom}(A)$ (the domain of A):

(i) $\forall x \in \mathrm{Dom}(A)(\neg \mathrm{PP}ax \wedge \neg \mathrm{PP}xa)$;

(ii) $(\exists x \in \mathrm{Dom}(A)\ \mathrm{PP}ax) \wedge (\forall x \in \mathrm{Dom}(A)\ \neg \mathrm{PP}xa)$;

(iii) $(\exists x \in \mathrm{Dom}(A)\ \mathrm{PP}ax) \wedge (\exists x \in \mathrm{Dom}(A)\ \mathrm{PP}xa)$;

(iv) $(\exists x \in \mathrm{Dom}(A)\ \mathrm{PP}xa) \wedge (\forall x \in \mathrm{Dom}(A)\ \neg \mathrm{PP}ax)$.

We shall show how to build from A a model B of $\mathbf{EM} \cup S$. For each member of $\mathrm{Dom}(A)$ which meets the condition (ii), we add two new distinct proper parts to it and for each member of $\mathrm{Dom}(A)$ which meets the condition (iii) or (iv), we add one new proper part to it. Such a construction must be such that: if a and b are two distinct members of $\mathrm{Dom}(A)$ which meet the condition (ii) or (iii) or (iv), the new proper parts added to them are also distinct and if a and b are two distinct newly added members, then $\neg\mathrm{PP}ab$, $\neg\mathrm{PP}ba$ and finally for any newly added member a, $\neg\mathrm{P}xa$ for any $x \in \mathrm{Dom}(A)$. Since $\mathrm{Dom}(A)$ is finite, the resultant set after extension is still finite. This set will be $\mathrm{Dom}(B)$. For the interpretation of 'P' in B, P^B, we extend the interpretation of 'P' in A, P^A, by adding new pairs according to reflexivity and transitivity, that is, if a is a newly added member, (a, a) will be added to P^A and if in addition $\mathrm{PP}ax$, for some $x \in \mathrm{Dom}(A)$, then besides (a, x), (a, y) will also be added to P^A, for any y such that $\mathrm{P}xy$. The extended set of pairs is the P^B we need. Next we show that such a B is a model of $\mathbf{EM} \cup S$. It is easy to see that B is a model of **GM** and obviously $B \models \exists x(\exists y\mathrm{PP}yx \vee \forall y(\neg\mathrm{PP}yx \wedge \neg\mathrm{PP}xy))$. It remains to show that $B \models$ (SSP). For any $a, b \in \mathrm{Dom}(B)$, if $\neg\mathrm{P}ab$, then there are two possibilities: (1) either $a \in \mathrm{Dom}(A)$ and meets the condition (i) or a is a newly added member, and in either case a itself witnesses $\exists z(\mathrm{P}za \wedge \neg\mathrm{O}zb)$; (2) $a \in \mathrm{Dom}(A)$ and meets the condition (ii) or (iii) or (iv), in which case a must have a new proper part which witnesses $\exists z(\mathrm{P}za \wedge \neg\mathrm{O}zb)$. Hence $B \models$ (SSP). This shows that B is a model of $\mathbf{EM} \cup S$. Now it is easy to see that $\mathrm{Dom}(B^I)$ by definition is $\{x \in \mathrm{Dom}(B) : \exists y\mathrm{PP}yx \vee \forall y(\neg\mathrm{PP}yx \wedge \neg\mathrm{PP}xy)\}$ and is just $\mathrm{Dom}(A)$ and $P^B|\mathrm{Dom}(A)$ (the restriction of P^B to $\mathrm{Dom}(A)$) $= P^A$. Therefore any finite model A of **GM** is a B^I, for some finite model B of $\mathbf{EM} \cup S$. This completes the proof. ⊣

Corollary. **MM** *and* **GM** + (EP) *are finitely inseparable and hence undecidable.*

Proof. This follows from the fact that **EM** is stronger than **MM** and **GM** + (EP). Hence B constructed in the proof of Theorem 1 is also a model of $\mathbf{MM} \cup S$ and $\mathbf{GM} + (\mathrm{EP}) \cup S$. ⊣

Theorem 2. **CM** *is finitely inseparable and hence undecidable.*

Proof. First observe that **CM** does not imply that there is no least member even when the domain has more than one member. Then it is easy to see that every finite distributive lattice is a model of **CM** if we interpret $\mathrm{P}xy$ as $x \vee y = y$. For under this interpretation, obviously the finite product of x and y is just $x \wedge y$, and anything can be 'a' finite sum of x and y (since a finite lattice must have the least member which is a part of and hence overlaps every member in the domain). Now let $S = \{\forall x\forall y\exists z(\mathrm{P}xz \wedge \mathrm{P}yz \wedge \forall u((\mathrm{P}xu \wedge \mathrm{P}yu) \rightarrow \mathrm{P}zu)), \forall x\forall y\exists z\forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \mathrm{P}wy))\}$. Then the language of lattice can be interpreted into $\mathbf{CM} \cup S$ in the following way. Let $I(\forall)$ be $x = x$, $I(\wedge)$ be $\forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \mathrm{P}wy))$ and $I(\vee)$ be $\mathrm{P}xz \wedge \mathrm{P}yz \wedge \forall u((\mathrm{P}xu \wedge \mathrm{P}yu) \rightarrow \mathrm{P}zu)$. Since every finite distributive lattice is a model of $\mathbf{CM} \cup S$, it is trivial that every finite distributive lattice is some A^I, where A is a finite model of $\mathbf{CM} \cup S$. By (5), **CM** is finitely inseparable. ⊣

Corollary. **GM** + (FS) *and* **GM** + (FP) *are finitely inseparable and hence undecidable.*

Proof. **CM** is their extension. Hence by (9), they are finitely inseparable. ⊣

Theorem 3. **CEM**+(C)+(G) *is decidable.*

Proof. This is owing to the fact that **CEM**+(C)+(G) can be faithfully interpreted into **ETB**. As mentioned earlier, any Boolean algebra with 0 removed will be a model of **CEM**+(C)+(G). Hence we can let $I(\forall)$ be '$x \neq 0$' and let $I(\mathrm{P})$ be $x \leq y$ (that is, $x + y = y$). Then A^I is a model of **CEM**+(C)+(G). Conversely, if $+$, $\times$ and $\sim$ are interpreted by finite sum, finite product and complement respectively, then it is not difficult to see that any model of **CEM**+(C)+(G) can be extended to a Boolean algebra by adding a least member to it. (Note that **CEM**+(C)+(G) does imply that, for the sake of (SSP), there is no least member if the domain contains at least two members; by the way, the only case worth checking is distributivity, but the proof is long and not very challenging, so I shall skip it here). Therefore the interpretation defined above is faithful. Then by (3) and (4), **CEM**+(C)+(G) is decidable. ⊣

Corollary. *All the theories from* **GM** *up to* **CEM**+(C)+(G) *are not inseparable.*

Proof. This is owing to the fact that all of them are subtheories of **CEM** + (C) + (G) and we have just shown that **CEM**+(C)+(G) is decidable. ⊣

Corollary. **CEM**+(C)+(¬G) *is decidable.*

Proof. Let $\sim x$ stand for the complement of x. First we show that **CEM** + (C)+(¬G) has $\forall x\forall y(\neg\mathrm{P}\sim xy \rightarrow \exists z\forall w(\mathrm{O}wz \leftrightarrow (\mathrm{O}wx \vee \mathrm{O}wy)))$ as a theorem, that is, **CEM**+(C)+(¬G) guarantees the existence of the finite sum of any two distinct members one of which does not contain the complement of the other (so that their sum will not be the greatest member). This is vacuously true if the domain contains only two members (any model of **CEM**+(C)+(¬G) must contain at least two members). So let's assume that there are more than two members in the domain. Consider any two distinct x and y ($x = y$ is a trivial case) in the domain such that $\neg\mathrm{P}\sim xy$. Since $\neg\mathrm{P}\sim xy$, by (SSP) there is some u such that $\mathrm{P}u\sim x \wedge \neg\mathrm{O}uy$. So u does not overlap either x or y (this also shows $\neg\mathrm{P}\sim yx$). Therefore both x and y are parts of $\sim u$, that is, $\mathrm{U}xy$, and hence by (FS) $\exists z\forall w(\mathrm{O}wz \leftrightarrow (\mathrm{O}wx \vee \mathrm{O}wy))$. Utilizing the fact just proved, we can easily show that **CEM**+(C)+(¬G) is the mereological theory which is satisfied by any Boolean algebra which has at least three members with 0 and 1 removed, that is to say, following the argument given in Theorem 3, we can show that **CEM**+(C)+(¬G) can be faithfully interpreted into **ETB** plus the axiom saying that there are at least three members. But any finite extension of a decidable theory is also decidable. ⊣

Corollary. **CEM** + (C) *is decidable.*

Proof. Both **CEM**+(C)+(G) and **CEM**+(C)+(¬G) are decidable. Hence **CEM**+(C) must be decidable too, since in general, for any theory T and any sentence α in the language of T, if both $T \cup \{\alpha\}$ and $T \cup \{\neg\alpha\}$ are decidable, T will be decidable too. ⊣

Corollary. **CEM** + (G) *is not finitely inseparable.*

Proof. Observe that every finite model of **CEM** + (G) is also a model of **CEM** + (C). So if **CEM** + (G) is finitely inseparable, **CEM** + (C) will be finitely inseparable too. But **CEM** + (C) is decidable and hence cannot be finitely inseparable. ⊣

Theorem 4. **CEM** *is undecidable.*

Proof. Let A be a strongly undecidable structure of the irreflexive symmetric ordering. Let $\mathrm{Dom}(B) = \{S \in \mathcal{P}(\mathrm{Dom}(A))$ (the Power set of $\mathrm{Dom}(A)$) : S is a singleton or S has exactly two members a and b such that $(a, b) \in P^A\}$. Let $P^B = \{(c, d) \in \mathrm{Dom}(B) \times \mathrm{Dom}(B) : c \subseteq d\}$. Obviously, (P1), (P2) and (P3) are valid in B. Let's check whether (FS) is valid in B. Consider any two members x and y in $\mathrm{Dom}(B)$. If $x = y$ then (FS) is obviously true. Suppose $x \neq y$. There are three possibilities:

- x and y are two distinct pairs. Then Uxy is false in B, for there is no member in Dom(B) which has more than two members.
- x and y are two distinct singletons. If $(\cup x, \cup y) \in P^A$, then $\{\cup x, \cup y\} \in$ Dom(B) and it is the least upper bound of x and y. Otherwise, Uxy is false in B.
- x is a pair and y is a singleton. If y is not a subset of x, then Uxy is false in B. Otherwise, their least upper bound is just x.

Hence (FS) is valid in B. It is easy to see that (FP) and (WSP) are also valid in B. So B is a model of **CEM**. Now we'll show that A is definable in B. Obviously, we can use $\{\{a\} : a \in \mathrm{Dom}(A)\}$ to stand for Dom(A), and this set of singletons is definable in B by $\neg\exists y \mathrm{PP}yx$. As for P^A, we can define it by $x \neq y \wedge \exists z(\mathrm{PP}xz \wedge \mathrm{PP}yz)$. Since A is strongly undecidable, B is strongly undecidable too, and this shows that **CEM** is undecidable. ⊣

COROLLARY. *All the mereological theories from* **GM** *to* **CEM** *are undecidable*[11].

PROOF. The structure B constructed in the proof of Theorem 4 is a model of those theories and B is strongly undecidable. ⊣

COROLLARY. **CEM**+(¬C) *is undecidable and hence incomplete.*

PROOF. **CEM** + (C) is decidable, so **CEM**+(¬C) cannot be decidable, otherwise **CEM** will be decidable. As mentioned earlier, a recursively axiomatizable complete theory will be decidable. Hence **CEM**+(¬C) cannot be complete. ⊣

COROLLARY. **CEM**+(¬G) *and* **CEM** + (¬C) + (¬G) *are undecidable and hence incomplete.*

PROOF. It can actually be shown that there are at least four distinct singletons in the domain of the model B constructed in the proof of Theorem 4 (see Shoenfield [1967, pp. 141–142]). With this additional information about B, we can easily see that B satisfies **CEM** + (¬C) + (¬G). However, since B is strongly undecidable, **CEM** + (¬C) + (¬G) and **CEM**+(¬G) must be undecidable. ⊣

[11] Though we have known that most of them are undecidable owing to the stronger results proved earlier, it is still good to know that just one model is enough to show this fact.

Corollary. **CEM** + (G) *and* **CEM** + (¬C) + (G) *are undecidable and hence incomplete.*

Proof. Consider the strongly undecidable model A in Theorem 4. It can also be requested that the cardinality of the domain of A is exactly ω (see Shoenfield [1967]). Hence there is an isomorphic copy of A on ω and let's label it as A'. A' as a structure of $L(A)$ is of course strongly undecidable. Now we construct a model of **CEM** + (¬C) + (G) and then show that A' can be defined in this model. Since $P^{A'}$ is also countable, let f be a one-to-one function from $P^{A'}$ to ω. Define $C = \{S \subseteq \omega \cup \omega \times \omega : S = \{m, n\} \cup \omega \times \{k\}$ if $(m, n) \in P^{A'}$ and $f((m, n)) = k$, or S is a nonempty finite subset either of ω or of $\omega \times \omega$, or $S = \omega$, or $S = \omega \cup \omega \times \omega$, or $S = \omega \cup \omega \times \omega \setminus \{(m, n)\}$, for $m, n \in \omega\}$. Suppose D is the smallest set including C and closed under finite union and finite nonempty intersection. Consider the structure $M = (D, \subseteq)$ and we interpret $\mathrm{P}xy$ as $x \subseteq y$. Naturally, $\cup$ and $\cap$ will be the interpretations of finite sum and finite product (if $x \cap y \neq \emptyset$). Then it is easy to see that (FS) and (FP) are valid in M. Obviously, $\subseteq$ is a partial ordering, so (P1), (P2) and (P3) are valid in M. G is valid in M since every member in D is a subset of $\omega \cup \omega \times \omega$. (¬C) is also valid in M, for in M every nonempty finite subset of natural numbers has no complement. As for (SSP), if $x, y \in D$ and x is not a subset of y, then there must be some natural number or some ordered pair of natural numbers which belongs to x but not to y, and the singleton of such a number or pair witnesses $\exists z(\mathrm{P}zx \wedge \neg \mathrm{O}zy)$. Therefore, M is a model of **CEM** + (¬C) + (G). Now we want to use the set $\{\{0\}, \{1\}, \{2\}, \dots\}$, i.e., the set of singletons of natural numbers, to stand for the set of natural numbers, and it can be defined in M by $\neg\exists y \mathrm{PP}yx \wedge \neg\exists z \forall w(\mathrm{P}wz \leftrightarrow \neg \mathrm{O}xw)$, that is, x is an atomic member which has no complement. Abbreviate the foregoing formula as $\varphi(x)$. Then $P^{A'}$ can be defined in M by $\varphi(x) \wedge \varphi(y) \wedge x \neq y \wedge \exists z(\mathrm{PP}xz \wedge \mathrm{PP}yz \wedge \forall u((\mathrm{PP}uz \wedge \varphi(u)) \to (u = x \vee u = y)) \wedge \exists u(\mathrm{PP}uz \wedge \neg\exists y(\mathrm{P}yu \wedge \varphi(y))) \wedge \forall u((\mathrm{PP}uz \wedge \neg\exists y(\mathrm{P}yu \wedge \phi(y))) \to \exists t(\mathrm{PP}tz \wedge \neg\exists y(\mathrm{P}yt \wedge \varphi(y)) \wedge \mathrm{PP}ut)))$, for in D if an infinite set is of the form $\{m, n, (k, j), \dots\}$, that is, it has exactly two distinct natural numbers m and n and infinitely many ordered pairs, then $(m, n) \in P^{A'}$. This shows that A' is definable in M, so M is strongly undecidable and hence **CEM** + (G) and **CEM** + (¬C) + (G) are undecidable. ⊣

To squeeze more information, also observe that **CEM** + (¬C) + (G) has no finite models, hence it is finitely separable. I shall summarize in the following chart the foregoing results as well as questions that remain open.

Theory	Inseparable?	Finitely Inseparable?	Undecidable?
GM := (P1) + (P2) + (P3)	No	Yes	Yes
GM + (EP)	No	Yes	Yes
MM := **GM** + (WSP)	No	Yes	Yes
EM := **GM** + (SSP)	No	Yes	Yes
GM + (FS)	No	Yes	Yes
GM + (FP)	No	Yes	Yes
CM := **GM** + (FS) + (FP)	No	Yes	Yes
CM + (EP)	No	Unknown	Yes
CEM := **EM** + (FS) + (FP) $\shortparallel$ **CMM** := **MM** + (FS) + (FP)	No	Unknown	Yes
CEM + (C)	No	No	No
CEM + (¬C)	Unknown	Unknown	Yes
CEM + (G)	No	No	Yes
CEM + (¬G)	Unknown	Unknown	Yes
CEM + (C) + (G)	No	No	No
CEM + (C) + (¬G)	No	No	No
CEM + (¬C) + (G)	Unknown	No	Yes
CEM + (¬C) + (¬G)	Unknown	Unknown	Yes

Remark 2. An anonymous referee draws my attention to the following axiom called by Simons [1987, p. 28] *Proper Parts Principle*:

(PPP) $\quad \forall x \forall y(\exists z \mathrm{PP}zx \to (\forall z(\mathrm{PP}zx \to \mathrm{PP}zy) \to \mathrm{P}xy)).$

Obviously (EP) follows from (P2) and (PPP). And it is easy to see that **GM**+(EP) does not imply (PPP). Hence we may get some stronger theories after replacing (EP) by (PPP) in the theories listed above. The same referee also points out a theory named by Simons [1987, p. 119] *Minimal Extensional Mereology*: **MEM** := **MM**+(FP), as well as the following two possible axioms which are theorems of **MM**.

(∃E) $\quad \exists x \exists y\, x \neq y \leftrightarrow \exists x \exists y \neg \mathrm{O}xy,$

(∄0) $\quad \exists x \exists y\, x \neq y \leftrightarrow \neg \exists x \forall y \mathrm{P}xy.$

Notice that (∃E) follows from (P1) and (WSP), and (∄0) follows from (P1) and (∃E).

Now several strictly increasing chains of theories are noted by the referee:

1. $\mathbf{GM} + (\mathrm{EP}) < \mathbf{GM} + (\mathrm{EP}) + (\nexists 0) < \mathbf{GM} + (\mathrm{EP}) + (\exists \mathrm{E}) < \mathbf{MM} + (\mathrm{EP}) < \mathbf{MM} + (\mathrm{PPP})$;
2. $\mathbf{GM} + (\nexists 0) < \mathbf{GM} + (\mathrm{EP}) + (\nexists 0) < \mathbf{GM} + (\mathrm{EP}) + (\exists \mathrm{E}) < \mathbf{MM} + (\mathrm{EP}) < \mathbf{MM} + (\mathrm{PPP})$;
3. $\mathbf{GM} + (\nexists 0) < \mathbf{GM} + (\exists \mathrm{E}) < \mathbf{GM} + (\mathrm{EP}) + (\exists \mathrm{E}) < \mathbf{MM} + (\mathrm{EP}) < \mathbf{MM} + (\mathrm{PPP})$;
4. $\mathbf{GM} + (\nexists 0) < \mathbf{GM} + (\exists \mathrm{E}) < \mathbf{MM} < \mathbf{MM} + (\mathrm{EP}) < \mathbf{MM} + (\mathrm{PPP})$;
5. $\mathbf{GM} + (\mathrm{EP}) < \mathbf{GM} + (\mathrm{PPP}) < \mathbf{MM} + (\mathrm{PPP}) < \mathbf{EM} < \mathbf{MEM} < \mathbf{CMM} = \mathbf{CEM}$.

I would like to point out the following two facts. First, all theories in the first four chains are finitely inseparable and hence undecidable, for (PPP) is provable from (P1) and (SSP)[12] and therefore $\mathbf{MM} + (\mathrm{PPP}) \leq \mathbf{EM}$, so by Theorem 1 and (9), any theory $\leq \mathbf{MM} + (\mathrm{PPP})$ is finitely inseparable. Second, **MEM** is undecidable and separable, for it has a finite extension, e.g. **CEM**, which is undecidable as well as a finite extension, e.g. $\mathbf{CEM} + (\mathrm{C})$, which is decidable. The only question still open to inquiry is whether **MEM** is finitely inseparable or not.

4. Further Observations

Both set theory and mereological theories are based on a binary predicate, and the predicate 'being a member of' used in set theory looks akin to 'being a part of'. Actually, at the early stage of their development, it might be envisaged that set theory would be replaced by mereological theories[13]. However, mereological theories are (as we have seen, most of them are in some sense subtheories of **ETB**[14]) quite weak compared with set theory which has proven to be so powerful that all kinds of mathematical objects can

[12]The proof is easy. Suppose toward contradiction $\exists z \mathrm{PP}zx$, $\forall z(\mathrm{PP}zx \to \mathrm{PP}zy)$ and $\neg \mathrm{P}xy$. By (SSP), $\exists z(\mathrm{P}zx \wedge \neg \mathrm{O}zy)$. Fix that z. It cannot be x itself since $\mathrm{O}xy$ follows from $\exists z \mathrm{PP}zx$ and $\forall z(\mathrm{PP}zx \to \mathrm{PP}zy)$. So $\mathrm{PP}zx$ and hence $\mathrm{PP}zy$, but then by (P1), $\mathrm{O}zy$, which contradicts $\neg \mathrm{O}zy$.

[13]It is also said that Leśniewski's intention was to use his mereological theory to provide a new foundation for mathematics (see Simons [1987, p. 60]).

[14]Note that even with fusion axiom schema (see Remark 1), the mereological theories classified by Casati and Varzi will still be bounded by the theory of complete Boolean algebras.

be defined within it. I think this huge difference mainly results from the difference between the ways in which they generate new objects respectively. An example will be in order here. Suppose we start with two atomic objects (which are presumably not sets), say, x and y. How many sets can we generate from them? Obviously, there are infinitely many, for x is not equal to $\{x\}$, and by the same token, $\{x\}$ is not equal to $\{\{x\}\}$ and such a process can go on forever. But there are only three mereological sums that can be generated in this case: x, y and xy (the thing generated by putting x and y together). To see this, observe that the mereological sum of x and xy is again xy (for these two sums share the same parts) and hence does not add anything new. In general, if we start with n atomic objects, there are only $2^n - 1$ mereological sums. Ontologically speaking, set theory is committed to the abstract things generated by adding that "magic" pair of curly brackets, while mereological construction has no resources other than the objects already there.

Is it possible to get new mereological axioms to generate some more powerful theories? It would be fair to request that such new axioms, if they can be found, have to be philosophically well-motivated. One might suggest that they have to be some true statements of a reasonable mereological representation of the real world. A likely model is to have a domain with infinite atoms and all objects in that domain are composed of atoms. So an object can be identified with a set of atoms and 'P' can be interpreted as the set inclusion. But then the model thus constructed will be an infinite atomic Boolean algebra, and it is known that the theory of infinite atomic Boolean algebras is complete[15] (and hence decidable), which means that whatever sentences which are true in that model are theorems of the theory of infinite atomic Boolean algebras. As a matter of fact, the mereological theory of this model will be exactly the theory of infinite atomic Boolean algebras (translated into the language of mereology) plus the axiom saying that there is no least element. It is of course axiomatizable, but not in an interesting way we are expecting. Well, maybe there are only finitely many atoms. But this case is even easier, for the theory of a finite Boolean algebra is axiomatized by the original axioms plus the axiom specifying a finite cardinality. Thus it seems not very promising to try to fix an intended model along the foregoing line.

On a different line, one might want to put some restrictions on the domain of a mereological structure. For example, some philosophers did argue that

[15] See Monk [1976, pp. 360–361].

arbitrarily putting some objects together does not necessarily form a new object[16]. Therefore, one could modify those closure principles such as (FS) and (FP) according to some additional conditions which one believes are required for composition. Note that the same consideration might lead to adding (¬C) or (¬G) as new axioms. However, whether some restrictions should be imposed on composition is a profoundly controversial philosophical issue still under debate.

Though pointing out some possibilities of ways to locate new mereological axioms, here I shall not pursue them any further.

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[16] For example, Peter van Inwagen claims that a collection of objects compose an object if and only if they compose a life or there is only one atom in that collection. See van Inwagen [1990, pp.81–97].

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