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PEIRCE AND SCHRÖDER ON THE AUFLÖSUNGSPROBLEM

Abstract. The aim of this article is Schröder's treatment of the so called *solution problem* [Auflösungsproblem]. First, I will introduce Schröder's ideas; then I will discuss them taking into account Peirce's considerations in *The Logic of Relatives* ([13, pp. 161–217] now republished in [9, pp. 288–345]).

Keywords: solution problem, elimination problem, algebra of logic, calculus of relations.

1. Short Historical Background

The *Solution Problem* in the Algebra of Logic dates back to the work of George Boole, [5], where given a set of premises it is required to infer their possible consequences and to reveal the relation obtaining among them. For example, let

$$A, B, C, D$$

be our premises. Firstly, we must rewrite each of them in equational terms; so each premise is an equation from a formal point of view. Let us collect these equations in a unique identity $\mathfrak{A}$.[1] What can we deduce from $\mathfrak{A}$?

[1] We can reformulate every equation in the form $f(x) = 0$ or $f(x) = 1$. We know that $f(x) + f(y) = 0$ iff $f(x) = 0$ or $f(y) = 0$. So, if we know that $f(x) = 0$ and $g(y) = 0$, then $f(x) + g(x) = 0$. In this way we have reduced two formulas to one. This process is always feasible because any equation is rewritable in the form $f(x) = 0$. So we can reduce many equations to only one. For example, given $f_1(x) = 0, \ldots, f_n(x) = 0$, it is $f_1(x) + ... + f_n(x) = 0$.

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To answer this question, it is necessary to solve $\mathfrak{A}$ in a purely algebraical way; i.e. it is necessary to pick out the values of the unknowns of $\mathfrak{A}$, because a solution is a consequence of our premises.

1.1. In this sense we can say that the *Solution Problem* is connected to the scope and power of the algebraical calculus settled by our premises. In fact, it answers the question: *What consequences can I deduce from the premises?*

> Mathematical logic developed in the 19th century primarily in the form of logical algebra. The analogy leading to the creation of logical algebra lay in the fact that *each solution of a problem,* via *the setting up and solution of an equation, is in essence the derivation of consequences from the statement of the problem.* [2, p. 33; the italic is mine]

So, the *Solution Problem* for Boole requires three steps:

1. transforming every premise into an equational form,
2. collecting all premises in a unique equation,
3. solving this equation by means of purely algebraic tools.

All this without reference to a possible interpretation of the symbols involved in the process. In this context, it must be stressed that Boole was the first one to consider the process of computation as *independent* from the meanings attached to the symbols used and it was Boole again to expand the usual meaning of *interpretation* in terms of *quantity*, accepting also *qualitative* interpretations.

> The idea, developed by Boole, was to extend algebraic methods beyond quantitative problems. [2, ivi]

1.2. In the *Operationskreis*[2] Schröder adhered to this equational view of the *Solution Problem*, but later he found Peirce's approach in terms of subsumptions more natural. Peirce's approach is preferred because it relies on subsumptions instead of identities. Identity is not considered to be primitive in this context, as it is defined in the following way:

$$A = B \quad \longleftrightarrow \quad A \subseteq B \;\wedge\; B \subseteq A\,.$$

Using Peirce's own words:

[2] [14]; now reprinted as [16].

> There is a difference of opinion among logicians as to whether $\subseteq$ [3] or $=$ is the simple relation. But [...] I have strictly demonstrated that the preference must be given to $\subseteq$ in this respect. The term *simpler* has an exact meaning in logic; it means that whose logical depth is smaller; that is, if one conception implies another, but not the reverse, then the latter is said to be simpler. Now to say that $A = B$ implies that $A \subseteq B$, but not conversely.[4]

Moreover, the use of subsumption is preferable also for another aspect: it makes manifest the *subject-predicate* structure of the algebraic statements.[5]

Nevertheless, the Schröder of the third volume of the *Vorlesungen*[6] returned to an equational view of the *Auflösungsproblem*. This doesn't mean, of course, that Schröder fully agreed with Boole. For example, from a formal point of view, following Jevons, Schröder adopted the *inclusive* interpretation of sum. But this is not all. Schröder went back to an equational *stance* driven also by aesthetical reasons. In fact, it is customary in the tradition beginning with Schröder to regard the algebraic calculus as a source of aesthetical delight. We notice this also in Löwenheim and Tarski.[7]

But now, let us introduce the *Solution Problem* as depicted in the fifth *Lecture* of the third volume of the *Vorlesungen*.

2. The Solution Problem

We will understand *binary relations* in a modern way, as sets of ordered pairs and denote them by means of capital letters 'R' and 'S'. Let V be an arbitrary set and let $x, y, z \in V$. We introduce the following definitions.

$$V^2 \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid x \in V \wedge y \in V\,] \qquad \text{(total relation)}$$

$$\Lambda^2 \stackrel{\text{df}}{=} -V^2 \qquad \text{(empty relation)}$$

$$\mathsf{Id} \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid \langle x, y\rangle \in V^2 \wedge x = y\,] \qquad \text{(identity)}$$

$$\mathsf{Di} \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid \langle x, y\rangle \in V^2 \wedge x \neq y\,] \qquad \text{(diversity)}$$

$$-R \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid \langle x, y\rangle \notin R\,] \qquad \text{(complement)}$$

[3] I substitute Peirce's original sign of subsumption for thypographical reasons.

[4] [12]; now republished in [9]. See [9, p. 111].

[5] In a subsumption $A \subseteq B$, A is the sign for the subject and B the sign for the predicate.

[6] [17]; now reprinted as [15].

[7] I refer the reader to [3]. In particular, see [3, pp. 74–77] and [3, pp. 84–85].

$$R \circ S \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid \exists z(\langle x, z\rangle \in R \wedge \langle z, y\rangle \in S)\,] \qquad \text{(peircean product)}$$

$$R \bullet S \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid \forall z(\langle x, z\rangle \in R \vee \langle z, y\rangle \in S)\,] \qquad \text{(peircean sum)}$$

$$R^{-1} \stackrel{\text{df}}{=} [\,\langle x, y\rangle \mid \langle y, x\rangle \in R\,] \qquad \text{(converse)}$$

The first four out of the above eight relations are called by Schröder *modules*.

2.1. Well, in virtue of *Schröder's Law*[8] we know that every *finite* set of propositions which belong to the calculus of relatives (where any proposition can be formulated in equational terms) is equivalent to a single equation $f = 0$, where f is a *polynomial*, that is it is build up from the six operations of the relative calculus (union, intersection, complement, Peircean sum and product, and converse) and constants (determinate relatives, together with the four *modules* V^2, Λ^2, Id, Di). There are two possible cases:

(1) f doesn't contain any variable and then it's true or false,

(2) f contains some variable.

A *solution* for $f = 0$ is a *system* of relatives, which satisfy f once they are put in it as the values of the unknowns. If f contains only one variable, i.e. it has the form $g(x) = 0$, its solution is simply a relative t such that $g(t) = 0$; if f contains n variables, i.e. it has the form $g(x_1, \ldots, x_n) = 0$, then its solution is a system of relatives $t_1, \ldots, t_n$ such that $g(t_1, \ldots, t_n) = 0$.

Obviously, the interesting case is (2). It admits three further subcases:

(2a) $f = 0$ does *not* have a solution, i.e., $\neg\exists x_1, \ldots, x_n\ f(x_1, \ldots, x_n) = 0$;

(2b) *Any* system of relatives satisfies $f = 0$, i.e., $\forall x_1, \ldots, x_n\ f(x_1, \ldots, x_n) = 0$;

(2c) *Some*, but *not* every system of relatives is a solution for $f = 0$, i.e., $\exists x_1, ..., x_n\ f(x_1, ..., x_n) = 0\ \wedge\ \neg\forall x_1, ..., x_n\ f(x_1, ..., x_n) = 0$.

The cases (2a) and (2b) are border-line: the equation is *always true*, or *always false*. Thus, the proper Auflösungsproblem is given by the case (2c). Let us focus on it.

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[8] See below p. 7 of this paper. "[...] in unsrer Algebra [der Relative] jede *Ungleichung* mit der rechten Seite 0 oder 1 sich umformen lässt in eine *Gleichung* von ähnlichem Charakter" [17, p. 151]. "[...] in our Algebra [of relatives] we can transform any *disequality* with the right side 0 or 1 in an *equation* of similar character [i.e. with the right side 0 or 1]" (translations from this book are mine).

Let $f = 0$ be of the form $g(x_1, \ldots, x_n, y) = 0$; then:

(2c1) $$\forall x_1, \ldots, x_n \exists y\ g(x_1, \ldots, x_n, y) = 0$$

or

(2c2) $$\neg\forall x_1, \ldots, x_n \exists y\ g(x_1, \ldots, x_n, y) = 0, \\ \text{i.e., } \exists x_1, \ldots, x_n \forall y\ g(x_1, \ldots, x_n, y) \neq 0.$$

In the case (2c2), for any system of values of x_1, ..., x_n such that

$$\forall y\ g(x_1, \ldots, x_n, y) \neq 0\,,$$

there is a function h, such that

(3) $$h(x_1, \ldots, x_n) = 0 \quad \text{iff} \quad \forall y\ g(x_1, \ldots, x_n, y) \neq 0\,.$$

We define $h(x_1, \ldots, x_n) = 0$, the *resultant* of the elimination of y in $g(x_1, \ldots, x_n, y) \neq 0$. In other words, h is a relation obtaining among those relatives $t_1, \ldots, t_n$ which *don't* satisfy $g(x_1, \ldots, x_n, y) = 0$, i.e. $f = 0$. Therefore the resultant answers the following question: given an equation $f = 0$, what is the relation fulfilled by any relative which is *not* a solution of this equation?

2.1.1. This point deserves a pause of reflexion. Geraldine Brady, in her book on Peirce and Schröder, [7], states that Schröder in this context foresaw Skolem functions:

> This is where he [Schröder] introduces a precursor of Skolem functions, replacing existential quantifiers by function symbols that witness them. [7, p. 258]

This is a great misunderstanding. Schröder introduces something like Skolem functions in another occasion, but not here. In the fifth *Lecture*, devoted to the Auflösungsproblem, Schröder aimed only to find the relation obtaining among all relations which are *not* solutions of a given equation of the form $F(x) = 0$. Moreover, according to Brady's reading, Schröder would have said something like:

(4) $$\forall x\ F(x, f(x)) = 0 \leftrightarrow \forall x \exists y\ F(x, y) = 0\,.$$

Taken for granted (4), Brady's conclusion is straightforward. However, Schröder doesn't state (4), but (see above (3))

$$\forall x\ F(x, f(x)) = 0 \;\longleftrightarrow\; \neg\forall x \exists y\ F(x, y) = 0\,.$$

What Schröder is searching for is the relation obtaining among all values *not* satisfying a given equation. It is only in the eleventh *Lecture* that Schröder tries to eliminate the existential quantifiers, producing something *similar* to Skolem functions. In fact, Löwenheim refers to this *Lecture* proving his famous theorem (see [10, p. 451] and [4, p. 357]).

2.2. Thanks to Schröder's Law, we can obtain the *complete resultant* from the union of all resultants, which, as I have just shown, have an equational form; i.e. the complete resultant $H(x_1, \ldots, x_n) = 0$ of the elimination of y in $g(x_1, \ldots, x_n, y) \neq 0$ is defined as follows:

$$H(x_1, \ldots, x_n) = 0 \leftrightarrow \bigcup_{i \leqslant n} h_i(x_1, \ldots, x_n) = 0,$$

where any $h_i(x_1, \ldots, x_n) = 0$ is a single resultant. So the *elimination problem* [*Eliminationsproblem*] is the problem of finding this complete resultant:

> He [i.e., Schröder] rightly remarks that such problem [the solution problem] often involves problems of elimination. [9, p. 321]

It is clear that the solution problem involves an elimination problem every time we find ourselves in a case like (2c2), i.e. everytime an equation is solved by *some* system of relatives but not by *any* system of relatives. If a system of relatives is a solution, or a root for $f = 0$, a *general* solution is a *set* of systems of relatives for that equation, i.e. the set of all systems which are solutions of $f = 0$. Now, the general root of $f(x) = 0$ is always reformulable as $x = g(u)$, with g being a relative polynomial, and u an indeterminate relative. In other words, if x is a root of $f(x) = 0$, then there *always* exists a relative u such that $x = g(u)$. In symbols,

$$f(x) = 0 \quad \longleftrightarrow \quad \exists u \; x = g(u)\,.$$

A relative polynomial g in the equivalence above must fulfill an important requirement:

FIRST ADVENTIVE CONDITION. $f(x) = 0 \quad \longleftrightarrow \quad g(x) = x.$

According to this condition, g is such that if we know that x is a solution for $f(x) = 0$ and we apply g to x, then we obtain x *again*. Now, let us suppose that a is a solution of $f(x) = 0$; that is $f(a) = 0$. Then, there exists a relative u such that $a = g(u)$. So

$$g(u) = (a \cap [\, V^2 \circ f(u) \circ V^2\,]) \cup (u \cap [\, \Lambda^2 \bullet -f(u) \bullet \Lambda^2\,]) \tag{5}$$

represents the *general solution* of $f(x) = 0$. In Schröder's words:

> [The formulation of the general solution] [...] gave us a hint, in what form we ought to search for the general solution of an equation; it taught us, first of all, that the complete solution of the equation $f(x) = 0$ exists in the form $x = g(u)$.[9]

Well, the point is to calculate the value of the right side of (5); to do it, we need *Schröder's Law*:

SCHRÖDER'S LAW ([17, p. 147]).

$$\text{(i)} \qquad V^2 \circ R \circ V^2 = \begin{cases} V^2, & \textit{if } R \neq \Lambda^2 \\ \Lambda^2, & \textit{otherwise} \end{cases}$$

$$\text{(ii)} \qquad \Lambda^2 \bullet R \bullet \Lambda^2 = \begin{cases} \Lambda^2, & \textit{if } R \neq V^2 \\ V^2, & \textit{otherwise} \end{cases}$$

Applying Schröder's law to (5), we obtain:

$$(5') \qquad V^2 \circ f(u) \circ V^2 = \begin{cases} V^2, & \text{if } f(u) \neq \Lambda^2 \\ \Lambda^2, & \text{otherwise} \end{cases}$$

$$(5'') \qquad \Lambda^2 \bullet -f(u) \bullet \Lambda^2 = \begin{cases} V^2, & \text{if } f(u) \neq \Lambda^2 \\ \Lambda^2, & \text{otherwise} \end{cases}$$

Then, if $f(u) = V^2$, (5) becomes:

$$(5''') \qquad g(u) = (a \cap V^2) \cup (u \cap \Lambda^2) = a\,.$$

On the contrary, if $f(u) = \Lambda^2$, we have:

$$(5'''') \qquad g(u) = (a \cap \Lambda^2) \cup (u \cap V^2) = u\,.$$

Therefore, if u is a root of $f(x) = 0$, then $g(u) = u$ (it satisfies Schröder's Law); otherwise, $g(u) = a$.

[9] "Immerhin gabe ihre Aufstellung [i.e. der allgemeinen Lösung] einen Fingerzeig, in welcher Form überhaupt wir auf die allgemeine Lösung einer Gleichung zu fahnden haben werden, lehrte sie uns vor allem, dass die vollständige Lösung der Gleichung 1) $F(x) = 0$ existirt in der Form 2) $x = g(u)$" [17, p. 168].

3. Peirce vs Schröder: the real Meaning of general Solution

Now the following problem arises: if u is not a solution, then $g(u)$ gives us once more the already known a. Note that Schröder relies on the existence of a *particular* solution (in this case, a) to find the general one.

> The rigorous solution will be completely determined with reference to a particular solution a of the problem [. . .].[10]

Earlier in the text, Schröder, solving the *Problem 3*, assumes a in place of Id and Di (see [17, p. 194]). As a matter of fact, Peirce rightly remarks:

> But that [particular solution] is seldom difficult to find. Either Λ^2, or V^2, or some other trivial solution commonly offers itself. [9, p. 325]

In my opinion, to clarify the interpretation of Schröder's work on the solution problem it is necessary to value *individually* two facets:

1. The meaning of the Auflösungsproblem in itself.
2. Its use for the creation of a calculus of relations.

3.1. Peirce, in *The Logic of Relatives*, summarizes Schröder's point of view this way:

> Professor Schröder chiefly occupies himself with what he calls "solution-problem", in which it is required to deduce from a given proposition an *equation* of which one member consists in a certain relative determined in advance, while the other member shall not contain that relative.
> [9, p. 321]

Peirce is referring here to the general solution:

$$g(u) = (a \cap [\, V^2 \circ f(u) \circ V^2 \,]) \cup (u \cap [\, \Lambda^2 \bullet -f(u) \bullet \Lambda^2 \,])\,,$$

where a is the *relative determined in advance*. Schröder had already occupied himself with this problem before,[11] but it is only in the *Vorlesungen* that the solution problem acquires a fundamental rôle. It is such centrality that

[10] "Die rigorose Lösung wird hienach völlig bestimmt sein durch den Hinweis auf eine partikulare Lösung a des Problemes, welche, als a priori erkannt, ihr zugrunde zu legen wäre" [17, p. 195]. Schröder means that we must assume a particular solution *a priori* for building up the general solution. In fact, above we assumed that a was a particular solution of $f(x) = 0$ and with a we found the general solution.

[11] For example in [16, pp. 25–28].

the American philosopher reproves to Schröder. Peirce, in fact, *doesn't* deny the importance of the Auflösungsproblem, but refuses only to reduce all the logic to it:

> While I am not at all disposed to deny that the so-called solution-problems [...] are often of *considerable importance*, I cannot admit that the interest of logical study centres in them.
> [9, ivi; the italic is mine]

It is manifest that Peirce has as a background for his logic a different and more articulated point of view compared to Schröder's one. For the German mathematician the Auflösungsproblem is tightly tied to the building a calculus. Peirce could agree also on this point; what he does not approve is the extreme generality which Schröder associates with his technique of solution:

> Professor Schröder attaches great importance to the *generality* of solutions. In my opinion, this is a mistake. [9, ivi; the italic is mine]

According to Peirce, it is a mistake first of all because the general solutions just for their generality are for the most part trivial, in the sense that they do not contain a specific information, but are abstract:

> As for general solutions, they are for the most part trivial.
> [9, p. 322]

Here, Peirce seems to emphasize a procedure, which, according to his point of view, leads to a pure and unnecessary formal play. It was not by chance, that he previously refuted to admit the importance of rewriting *every* sentence in equational terms thanks to Schröder's Law:

> When the information contained in a proposition is not of the nature of an equation, why should we [...] insist upon expressing it in the form of an equation? [9, p. 321]

Note that Peirce, even in this quotation, doesn't understimate Schröder's work: it is right to express *some* proposition as identities, but not *every* one. According to Peirce there is neither reason nor necessity to express *every* proposition as an equation, when that is not implied directly from the information contained in it.

3.2. So far, we have reached the heart of Peirce's critique. Exactly as it is the *specific nature* of a proposition that implies its equational reformulation, in the same way it is a *specific context* to determine a problem or a question.

In Peirce's opinion, Schröder seems to have forgotten these relations and to have isolated a problem from its specific needs.

If we limit ourselves to finding the most general solution which contains every possible solution, it is not possible to disagree with Peirce that such generality leads to an extreme grade of abstraction and therefore lacks real epistemological value. In other words, does it make sense in science to discuss general problems (that is, abstract), when what we deal with in everyday practice are only *specific* problems, i.e. relating to a given context? It is a scientific and a *precise context* to determine the problematic status of a question:

> [...] every question is prompted by some need that is, by some unsatisfactory condition of things [...]. [9, p. 323]

The way Schröder deals with the Auflösungsproblem is, on the contrary, such that the interest is shifted away from the necessity to find a remedy (i.e. in algebraic terms, a solution for a problem) for a problematic state of affairs, to the possibility of finding the most general formulation of a problem whatever:

> Professor Schröder endeavours to give the most general formula of a logical problem. [...] [But] To seek a formula for all logical problems is to ask what it is [...] that men inquire [...], what the essence of a question, in general, is. [9, p. 322]

In other words, while Peirce expects by the solution of a problem the answer to a particular necessity, Schröder inquires the meaning in itself of what is expected by the solution of a problem.

It is here where we find the difference between Peirce and Schröder: both attend to the same solution problem, but while the first brings it back to specific situations and, therefore, he looks for some specific answers (algebraically, *particular solutions*), the second takes it out from any context, and give value only to answers which are the most general ones (algebraically, *general* solutions). Peirce, of course, doesn't deny the interest in general solutions, but he thinks that they can at most pave the way to finding particular solutions; as a scientist, he exploits a *universal* law only to account for a specific fact. One can object that in such cases Peirce too relies on universal solutions, but they are for him only means to reach a particular goal. They are not ends in themselves, because in themselves they are not of interest for a Peircean epistemologist:

> Only in those cases in which a general solution points the way to the particular solutions is it valuable; for it is only the particular solutions which picture to the mind the solution of a problem; and a form of words which fails to produce a definite picture in the mind is meaningless. [9, ivi].

Obviously, it is meaningless only if we don't consider the value of the general answers in themselves; i.e. if we use general solutions only as tools to reach the answer to a particular problem.

3.2.1. In defense of Schröder, it may be observed that his general solution, also in the unfortunate case when it gives again the particular solution which has been assumed for hypothesis, it shows at least the form, under which one can seek a possible general solution. Therefore, it allows for speaking about a solution, only referring to its formal structure, and therefore even in the case in which one has not found a concrete solution.

What Peirce doesn't understand is the analysis of the various forms of relations which Schröder is able to obtain, shifting himself to a more abstract level. It is only thanks to the use of his solution, for example, that he is able to prove that every *reflexive* relative R is such that $R \subseteq \mathsf{Id}$. (see [17], p. 199). Moreover, such solution, showing the general *form* of a solution, allows to classify the problems in terms of their formal solutions.

3.2.2. But, *what* is a problem for Peirce?

> [...] a question is a rational contrivance or device, and in order to understand any rational contrivance, experience shows that the best way is to begin by considering what circumstances of need prompted the contrivance, and then upon what general principle its action is designed to fill that need. [9, pp. 322–323].

In other words, in Peirce's view, a problem requires for its solution an examination of those conditions that determined it, that is, its contextualization. Therefore, the right question to ask is: *why did this particular problem arise?*. A satisfactory answer is obtainable by referring to a *general* law; so, we must take into consideration every possible law under which it is possible to subsume our case. In fact, our problematic situation is a particular case of a general one, and it is solvable only by means of an appropriate principle. In this sense, a general solution, or a general principle, are not answers in themselves, but are seen as ways to find the solution of a very specific problem:

> [. . .] special solutions are the only ones which directly mean anything or embody any [*genuine*] knowledge; [. . .] general solutions are only useful when they happen to suggest what the special solutions will be. [9, p. 324–325]

Peirce reckons the indubitable value of general solutions, but only if they are not answers in themselves. But this is just what Schröder expect of a solution. According to Schröder the importance of a general solution doesn't consist in the fact that it can help to solve a specific problem, but only in its generality. On the contrary, Peirce *doesn't* accept to isolate the general solutions from a specific context:

> [. . .] I do not find much virtue in general formulæ. [9, p. 323]

I insist upon the fact that Peirce doesn't deny the value of the general solutions, but he refuses only the *non*-contextualized use of them that Schröder assumes. Peirce, in fact, points out that any problem forces us to seek the causes which determined it and the principle which allows us to solve it. Indeed, the principle to exploit, notwithstanding that it has to be *general*, has to serve the *particularity* of the problem. The American philosopher doesn't question the importance of the general solutions, but only the use that Schröder makes of them.

3.2.3. For Peirce,

> A question, I say, is an indication suggestive (in the hypnotic sense) of what has to be thought about in order to satisfy some more or less pressing want. [9, ivi]

For him, a problem to be meaningful has to arise from a mental puzzling, determined from the embarrassment caused by the incapacity of giving an answer to some situation. This embarrassment pushes the mind to seek solutions. In a fortunate case, these solutions solve directly its worry: they are *particular* solutions, as is the problem to be resolved. Nevertheless, in many cases it is necessary to scrutinize a lot of laws, until we find that under which our problem can be subsumed. In this way, we discard all the laws but the ones that imply a (particular) solution to our problem. It is in this sense that Peirce may state that *only in those cases in which a general solution points the way to the particular solutions is it valuable* (see [9], p. 322). That is, only a general solution which is helpful in finding a particular one is of value, useful. Only a law which can *concretely* help us is of use.

Of course, to understand what it is meant by this *being of use*, we have to agree with Peirce in his definition of what a problem is. Peirce treats the matter from a purely epistemological point of view. In this perspective, to formulate a question apart from any contextual reference, contradicts the scientist's everyday experience: the scientist devises *ad hoc* algorithms to solve particular problems; in other words, it is the problematic state of an elusive situation that pushes the scientist to find a strategy, and therefore, a solution to master it. It is clear that there is not any *a priori* strategy which is valid for every situation. At most, one can ponder on the meaning of a strategy; but this is another matter. If we adopt Peirce's scheme, that is the perspective of an average scientist, what Peirce affirms can be accepted easily, despite the possible merits of Schröder's work. The point is that Schröder had no intention of playing the part of an average epistemologist. The fact that Schröder "was a great [...] step forward toward the Skolem-Löwenheim theorem and modern model theory" [7, p. 155], in this context, can awaken admiration or curiosity, but not interest. Such statement, in fact, presupposes an algebraic view of the problem, not an epistemological one.

3.2.4. Peirce himself, however, observes that a problem can arise not only from a specific situation, but also from a purely mental necessity:

> [In this case] [...] the need to which the question relates is nothing but the intellectual need of having the question answered. [9, p. 324]

The allusion to Schröder as someone who tried to answer some question only because it was put forth is clear:

> [The solution problem according Schröder is] [...] to find that form of relative which necessarily fulfils a given condition and in which every relative that fulfils that condition can be expressed. [9, p. 325]

In other words, according to Peirce's point of view, Schröder shows that every solution problem amounts to stating that:

$$\alpha \quad \text{iff} \quad f = 0\,.$$

That is, an equation is solvable iff the relative which constitutes its solution fulfils a given condition α. Therefore, we must find the general form in which it is possible to rewrite any relative which satisfies α. More clearly, what is the *general* form of a relative which is a solution of $f = 0$, and which fulfils α? Here, the interest is shifted from the *effective* discovery of relatives which are solutions of the equation $f = 0$, to their most general formulation in terms of the condition α. It is the same as asking the following question: what do all solutions of $f = 0$ have in common, as regards the condition α?

This amounts to classify the problems according to the form of the relatives which fulfil a given condition. In other words, it is equivalent to saying that a problem is individuated by an opportune classification of its solutions, according to a given principle α.

3.3. Let us proceed with the analysis of Schröder's position. Putting oneself in place of Schröder, as usually, is more difficult. What he does not do is rather clear: to establish a procedure to find particular solutions of an equation. Schröder aimed to build up an equational calculus whose main law of inference was the Auflösungsproblem. Let a certain number of equations A, B, C, D, etc. be given, and let us collect them in an unique identity $\mathfrak{A}$ using *Schröder's Law*. Assuming a particular solution a, we obtain easily the general solution of $\mathfrak{A}$ introducing a generic relative u, which either gives us again a, or u. In any case, such a general solution will include every possible solution for $\mathfrak{A}$. Now, if A, B, C, D, ... were our premises and a a given conclusion, the general solution would be the set of all possible conclusions. Thus, we have brought back the problem to find a solution for a set of equations to that to find the conclusions of a set of premises.

Schröder doesn't want to find a unique solution, but all of them. For this reason he writes down the general solution, which is the set of all possible solutions. Under this intepretation, the Auflösungsproblem reduces itself to the question: given a set of premises, by which algorithm can every conclusion be generated? Such an algorithm is the general solution. This is in harmony with Schröder's conception of a *logical calculus*:

> A "logical calculus" is the set of formulas which can be produced in a circle of operations with logical connecting operations.[12] Schröder calls it a characteristic mark of "*mathematical logic*", or the "*logical calculus*" that its derivations and inferences can be done in the form of calculations [...]. [11, p. 16 of the *preprint*]

Here, Peckhaus refers to the *Operationskreis*, but this point of view is present also in the *Vorlesungen*.

[12]Note that the *connecting* operations of Schröder are *not* our connectives, otherwise the passage of Peckhaus doesn't make sense. In fact, the connectives correspond to that operational sphere [Operationskreis] in which the formulas are derived.

With regard to this topic, note Peirce's classification of the various typologies of problem:

> Every problem [...] is either a problem of consequences, a problem of generalization, or a problem of theory. [9, p. 324]

The *problem of consequences* is no other than the one that Schröder formulates with extreme generality; i.e, what consequences can be derived from a given number of premises?

4. Conclusion

To briefly sum up, I showed that in the fifth *Lecture* of the *Vorlesungen* Schröder introduces the *solution problem* [Auflösungsproblem] giving it, according to Peirce, an exaggerate importance. I have argued that the question ought to be examined on two levels, one in relation to the meaning of the problem in itself, the other in relation to its interpretation as a deductive method to obtain a certain number of consequences from some premises. From the point of view of its intrinsic meaning, I supported Peirce's remarks, stressing his epistemological reading: in everyday experience of a scientist, a problem arises only from a specific urgency.

Nevertheless, on the other hand, one could notice in Schröder's treatment of the Auflösungsproblem the tentative to build the theory of relatives as a calculus; the general solution is for him the rule, the algorithm, by which to generate, given a certain number of premises, all possible consequences from them.

4.1. But, it is better not force too much the dichotomy Peirce-Schröder, because the difference between them doesn't lie in the fact that the first was a scientist and the second a mathematician, but in the respective philosophical stances. Schröder saw himself as a discoverer who marked on a notebook everything he encountered in an area which had never been explored, not forgetting to stress every facets of the flora and the fauna, the likenesses, the differences, the types, the sub-types, etc. In this way, he was able to draw a *cartography*, a *map*, a *classification* of the objects he encountered. Out of metaphor, in the third volume of the *Lectures* Schröder gives this classification of the relations, not omitting any facet, with a matchless precision and order.

This work of classifying is made also for the Auflösunsgsproblem, with respect to a given condition α. It is impossible to deny the importance and the *scientific* value of such classification. Peirce did, because he had behind him a *pragmatic* point of view. It is in this sense that is was necessary to read the matter on two levels, one tied to philosophical and epistemological considerations, the other to scientific and mathematical ones.

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References

[1] Kolmogorov, A. N., A. P. Yushkevich, *Matematika XIX veka: Matematicheskaya logika, algebra, teoriya chisel, teoriya veroyatnostei*, lzdatel'stvo Nauka, Moskva, 1978.

[2] Kolmogorov, A. N., A. P. Yushkevich (eds.), *Mathematics of the 19th Century, Mathematical Logic, Algebra, Number Theory, Probability Theory*, Birkhäuser Verlag, Basel – Boston – Berlin, 2001. Second revised edition, translated from the Russian by A. Shenitzer, H. Grant and O. B. Sheinin.

[3] Bondoni, Davide, *La teoria delle relazioni nell'algebra della logica schröderiana*, Led, Edizioni Universitarie di Lettere Economia Diritto, Milano, 2007.

[4] Bondoni, Davide, Review of [7], *Logic and Logical Philosophy* 17, 4 (2008), 355–360.

[5] Boole, George, *An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities*, Walton & Maberly, London, 1854. Reprinted Dover, New York, ca. 1958.

[6] Boole, George, *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*, Dover Publications, Inc., New York, 1958. First American printing.

[7] Brady, Geraldine, *From Peirce to Skolem. A Neglected Chapter* in the History of Logic, Elsevier, 2000.

[8] Fenstad, Jens Erik (ed.), *Selected Works in Logic by Th. Skolem*, Universitetsforlaget, Oslo – Bergen – Tromsö, 1970.

[9] Hartshorne, Charles, and Paul Weiss (eds.), *Collected Papers of Charles Sanders Peirce*, vol. 3, Exact Logic, and vol. 4, The Simplest Mathematics, Belknap Press of Harvard University Press, 1960.

[10] Löwenheim, Leopold, "Über Möglichkeiten im Relativkalkül", *Math. Ann.* 76 (1915), 447–470.

[11] Peckhaus, Volker, "Schröders Logic", in Dov M. Gabbay and John Woods (eds.), *Handbook of the History and Philosophy of Logic*, vol. 3, Elsevier, 2004.

[12] Peirce, Charles Sanders, "On the algebra of logic", *American Journal of Mathematics* 3 (1880), 15–57.

[13] Peirce, Charles Sanders, "The logic of relatives", *The Monist* (1897), 161–217.

[14] Schröder, Ernst, *Der Operationskreis des Logikkalkuls*, Teubner, 1877.

[15] Schröder, Ernst, *Vorlesungen über die Algebra der Logik, Exakte Logik*, vol. 3, "Algebra und Logik der Relative", B. G. Teubner, 1895.

[16] Schröder, Ernst, *Der Operationskreis des Logikkalkuls*, Wissenschaftliche Buchgesellschaft, 1966.

[17] Schröder, Ernst, *Vorlesungen über die Algebra der Logik, Exakte Logik*, vol. 3, "Algebra und Logik der Relative", Chelsea Publishing Company, 1966.

[18] Skolem, Thoralf, "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen", *Videnskaps. skrift., I. Matem.-natur. klasse 4* (1920).

[19] van Heijenoort, Jean, *From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, 1967.

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