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ON BÉZIAU'S LOGIC Z

Abstract. In [1] Béziau developed the paraconsistent logic **Z**, which is definitionally equivalent to the modal logic **S5** (cf. Remark 2.3), and gave an axiomatization of the logic **Z**: the system `HZ`. In the present paper, we prove that some axioms of `HZ` are not independent and then propose another axiomatization of **Z**. We also discuss a new perspective on the relation between **S5** and classical propositional logic (**CPL**) with the help of the new axiomatization of **Z**. Then we conclude the paper by making a remark on the paraconsistency of `HZ`.

Keywords: paraconsistent logic **Z**, modal logic **S5**, classical propositional logic.

1. Introduction

In [1], Béziau offered a possible solution to the Jaśkowski's problem by developing the paraconsistent logic **Z**. This logic is built in the set $\mathrm{For}_{\mathbf{Z}}$ of formulas which are formed in a standard way from propositional letters: 'p', 'q', 'p_0', 'p_1', 'p_2', ...; truth-value operators: 'N', '$\vee$', '$\wedge$', and '$\supset$' (connectives of negation, disjunction, conjunction, and implication, respectively).

Béziau gives an axiomatization of the logic **Z**, the Hilbertian system `HZ` (see [1, Definition 3.1]). The system `HZ` contains axioms for positive classical propositional logic (which corresponds to $\supset$-$\wedge$-$\vee$-fragment of **CPL**), i.e., for all $A, B, C \in \mathrm{For}_{\mathbf{Z}}$ the following formulas:

(AP1) $$A \supset (B \supset A)$$

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(AP2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

(AP3) $((A \supset B) \supset A) \supset A$

(AP4) $(A \wedge B) \supset A$

(AP5) $(A \wedge B) \supset B$

(AP6) $A \supset (B \supset (A \wedge B))$

(AP7) $A \supset (A \vee B)$

(AP8) $B \supset (A \vee B)$

(AP9) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

and the following negation-related axioms for all $A, B \in \mathrm{For}_{\mathbf{Z}}$:

(AZ1) $A \vee \mathsf{N}\, A$

(AZ2) $(A \wedge \mathsf{N}\, B \wedge \mathsf{N}(A \wedge \mathsf{N}\, B)) \supset (A \wedge \mathsf{N}\, A)$

(AZ3) $\mathsf{N}(A \wedge B) \supset (\mathsf{N}\, A \vee \mathsf{N}\, B)$

(AZ4) $\mathsf{N}\,\mathsf{N}\, A \supset A$

The system HZ contains the rules:

(MP) $$\frac{A \supset B \qquad A}{B}$$

(RZ) $$\frac{A \supset B}{\mathsf{N}(A \wedge \mathsf{N}\, B)}$$

Of course, (MP) is *modus ponens* for '$\supset$'. Moreover, we shall refer to the rule (RZ) as *negation-related rule*.

The logic **Z** is the set of all formulas in $\mathrm{For}_{\mathbf{Z}}$ which are provable in HZ. All members of **Z** are called *theses* of **Z**.

Remark 1.1. According to [1, pp. 101–102], the logic **Z** (and also the system HZ) was designed by the idea to define the paraconsistent negation 'N' by '$\neg\Box$' (or '$\neg\,\mathsf{L}$'), where '$\neg$' is the classical negation and '$\Box$' (or 'L') is the necessity operator in **S5**. It should be noted that Waragai and Shidori, in [5], developed a system of paraconsistent logic based on the fact that '$\Diamond\neg$' (or '$\mathsf{M}\,\neg$'), where '$\Diamond$' (or 'M') is the possibility operator in **S5**, has some of the properties satisfied by the classical negation. Their system, called PCL1, is not the same system as that of Béziau but a subsystem of it (cf. Remark 5.5). ⊣

Remark 1.2. In [1] the logic **Z** is introduced semantically. *Bivaluations* are functions from $\text{For}_{\mathbf{Z}}$ to $\{0,1\}$. A **Z**-*cosmos* is any non-empty set $\mathbb{C}$ of bivaluations defined by the condition: $v \in \mathbb{C}$ iff it obeys the classical conditions for '$\wedge$', '$\vee$' and '$\supset$', and moreover obeys the following condition for 'N' ("intended to be a paraconsistent negation"):

$$v(\mathsf{N}\, A) = 1 \quad \text{iff} \quad \exists_{u \in \mathbb{C}}\, u(A) = 0.$$

A formula A is **Z**-*valid* iff the value of A is one in any **Z**-cosmos $\mathbb{C}$ for all bivaluations of $\mathbb{C}$, i.e. $\forall_{\mathbb{C}} \forall_{v \in \mathbb{C}}\, v(a) = 1$.

In [1], it is proved that a formula A is provable in HZ iff A is **Z**-*valid*. ⊣

Now, as it is mentioned in the Postscript of [1], the system HZ seems to be of great interest, since it gives an axiomatization of **S5** using not the necessity operator or possibility operator explicitly but a specific negation-like operator as its primitive connective. Therefore, we might be able to reach a new point of view in seeing the system of modal logic **S5**. But at the same time, some questions seem to arise out of the axiomatization of **Z**:

Q1. How can we derive the rule corresponding to the rule of necessitation in the system HZ?

Q2. How can we prove the replacement theorem for negation, which is mentioned in [1, Corollary 2.2], in the system HZ syntactically?

Q3. What is the bottom particle of the system HZ?

Q4. Is it possible to make the role of the rule (RZ) clear?

In the following sections, we shall give some answers to questions from Q1 to Q3 raised above by showing some syntactical proofs, and as for the answer to Q4, we will propose another axiomatization of **Z**, the system HZ′.

2. The system HZ

Beginning with a preliminary on the "positive" part of the system HZ, (AP1)–(AP9), some answers to the questions Q1, Q2 and Q3 will be given in this section. It will also be proved that the negation-related axioms are *not* independent in the system HZ.

2.1. Answers to the questions Q1, Q2 and Q3

Firstly, we shall see some theses and the rule (R1) within the "positive" part of HZ, which we shall make use of in this paper. For any $A, B, C \in \text{For}_{\mathbf{Z}}$, we can prove the following formulas of **Z**:

(1) $$A \supset A$$

(2) $$(A \supset B) \supset ((B \supset C) \supset (A \supset C))$$

(3) $$(A \supset (B \supset C)) \supset (B \supset (A \supset C))$$

(4) $$(A \vee B) \supset ((B \supset C) \supset (A \vee C))$$

(5) $$((A \wedge B) \supset C) \supset (A \supset (B \supset C))$$

(6) $$(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$$

(7) $$(A \supset (A \wedge B)) \supset (A \supset B)$$

(8) $$(A \supset B) \supset ((A \supset C) \supset (A \supset (B \wedge C)))$$

(9) $$(A \supset C) \supset ((B \supset D) \supset ((A \wedge B) \supset (C \wedge D)))$$

(10) $$(A \wedge B) \supset (B \wedge A)$$

(11) $$(A \vee B) \supset (B \vee A)$$

Therefore we can easily see that the following rule can be derived:

(R1) $$\frac{A \supset B \qquad B \supset C}{A \supset C}$$

Now we shall pass on to giving some answers to the questions we raised.

Ad Q1. Notice that the following rule is derivable in HZ:

(R2) $$\frac{A}{\mathrm{N}\,\mathrm{N}\,A}$$

Proof:

1. A sup.
2. $\mathrm{N}\,A \supset A$ 1, (AP1), (MP)
3. $\mathrm{N}(\mathrm{N}\,A \wedge \mathrm{N}\,A)$ 2, (RZ)
4. $\mathrm{N}\,\mathrm{N}\,A \vee \mathrm{N}\,\mathrm{N}\,A$ 3, (AZ3), (MP)
5. $\mathrm{N}\,\mathrm{N}\,A$ 4, (1), (AP9)

This shows an answer to the question Q1.

Ad Q2. Firstly, notice that by (1), (AZ1), (AP9), (3), (MP), for any $A \in \mathrm{For}_{\mathbf{Z}}$ the following formula is provable in HZ:

(12) $$(A \supset \mathrm{N}\,A) \supset \mathrm{N}\,A$$

Hence, by (7) and (R1) we obtain:

(13) $$(A \supset (A \wedge \mathrm{N}\, A)) \supset \mathrm{N}\, A$$

Secondly, for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formula is provable in HZ:

(AZ2′) $$\mathrm{N}(A \wedge \mathrm{N}\, B) \supset (\mathrm{N}\, B \supset \mathrm{N}\, A)$$

Proof:
1. $(A \wedge \mathrm{N}\, B \wedge \mathrm{N}(A \wedge \mathrm{N}\, B)) \supset (A \wedge \mathrm{N}\, A)$ (AZ2)
2. $(\mathrm{N}\, B \wedge \mathrm{N}(A \wedge \mathrm{N}\, B)) \supset (A \supset (A \wedge \mathrm{N}\, A))$ 1, (5), (10), (MP)
3. $(\mathrm{N}\, B \wedge \mathrm{N}(A \wedge \mathrm{N}\, B)) \supset \mathrm{N}\, A$ 2, (13), (R1)
4. $\mathrm{N}(A \wedge \mathrm{N}\, B) \supset (\mathrm{N}\, B \supset \mathrm{N}\, A)$ 3, (5), (3), (MP)

This shows that the axiom (AZ2), which seems to be quite difficult to grasp, is actually equivalent to a rather simple formula (AZ2′) in the system HZ.[1]

Thus, the following rule is derivable in HZ:

(R3) $$\frac{A \supset B}{\mathrm{N}\, B \supset \mathrm{N}\, A}$$

Proof:
1. $A \supset B$ sup.
2. $\mathrm{N}(A \wedge \mathrm{N}\, B)$ 1, (RZ)
3. $\mathrm{N}\, B \supset \mathrm{N}\, A$ 2, (AZ2′), (MP)

This shows an answer to the question Q2.

Ad Q3. Notice that for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formula is provable in HZ:

(14) $$(\mathrm{N}\, A \wedge \mathrm{N}\,\mathrm{N}\, A) \supset B$$

Proof:
1. $(\mathrm{N}\, B \wedge \mathrm{N}\, A) \supset \mathrm{N}\, A$ (AP5)
2. $\mathrm{N}\,\mathrm{N}\, A \supset \mathrm{N}(\mathrm{N}\, B \wedge \mathrm{N}\, A)$ 1, (R3)
3. $\mathrm{N}\,\mathrm{N}\, A \supset (\mathrm{N}\, A \supset \mathrm{N}\,\mathrm{N}\, B)$ 2, (AZ2′), (R1)
4. $(\mathrm{N}\,\mathrm{N}\, A \wedge \mathrm{N}\, A) \supset \mathrm{N}\,\mathrm{N}\, B$ 3, (5), (MP)
5. $(\mathrm{N}\, A \wedge \mathrm{N}\,\mathrm{N}\, A) \supset B$ 4, (AZ4), (10) (R1)

According to (14), $\ulcorner \mathrm{N}\, A \wedge \mathrm{N}\,\mathrm{N}\, A \urcorner$ will be the bottom particle in the system HZ, which gives an answer to the question Q3.

[1] It should also be noted that we did not make use of the axiom (AZ3) in this proof.

Remark 2.3. (i) This bottom particle enables us to define the classical negation in HZ. In order to see this, note that for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formula is provable in HZ:

$$(\mathsf{N}\, A \wedge \mathsf{N}\,\mathsf{N}\, A) \equiv (\mathsf{N}\, B \wedge \mathsf{N}\,\mathsf{N}\, B)$$

where the bi-implication '≡' standardly defined as an abbreviation with '⊃' and '∧'.

Thus, we can define, as an abbreviation, a new logical constant f:

$$\ulcorner \mathsf{f} \urcorner \qquad \text{abbreviates} \qquad \ulcorner \mathsf{N}\, p \wedge \mathsf{N}\,\mathsf{N}\, p \urcorner .$$

(ii) We can define the classical negation as follows:

$$\ulcorner \neg A \urcorner \qquad \text{abbreviates} \qquad \ulcorner A \supset \mathsf{f} \urcorner .$$

We can say that this negation '¬' is classical, since—by the intended interpretation of 'N' (we mentioned about it in Remark 1.1)—the formula $A \supset \mathsf{f}$ is the same as the formula $A \supset (\neg\Box p \wedge \neg\Box\neg\Box p)$ in **S5**. The last formula is equivalent in **S5** to each of formulas: $A \supset (\neg\Box p \wedge \Box p)$ and $\neg A$, and '¬' is classical in **S5**.

Moreover, semantically (cf. Remark 1.2), for any **Z**-cosmos $\mathbb{C}$, for any $v \in \mathbb{C}$, and for any $A \in \mathrm{For}_{\mathbf{Z}}$ we obtain the classical condition:

$$v(\neg A) = 1 \quad \text{iff} \quad v(A) = 0 .$$

Indeed, $v(\neg A) = 1$ iff $v(A \supset \mathsf{f}) = 1$ iff $v(A) = 0$ or $v(\mathsf{f}) = 1$, but $v(\mathsf{f}) = 0$.

By the above fact, for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formulas

$$A \vee \neg A \tag{15}$$

$$A \supset (\neg A \supset B) \tag{16}$$

$$(A \wedge \neg A) \supset B \tag{17}$$

$$(A \supset (A \wedge \neg A)) \supset \neg A \tag{18}$$

$$(A \supset B) \supset ((A \supset \neg B) \supset \neg A) \tag{19}$$

$$(\neg B \supset \neg A) \supset (A \supset B) \tag{20}$$

$$A \equiv \neg\neg A \tag{21}$$

are **Z**-valid; so they are provable in HZ.

For the ¬-∧-∨-⊃-language we have e.g. the following axiomatizations of **CPL**: (AP1), (AP2), (AP4)–(AP9), (15), (16), (19) and (MP); (AP1)–(AP9), (20) and (MP); (AP3), (AP4), (AP5), (AP7), (AP8), (2), (5), (6), (8), (15), (17), (18) and (MP) [4, pp. 188–189]. Thus, all ¬-∧-∨-⊃-theses of **CPL** are provable in HZ.

(iii) We can define the necessity operator as follows:

$$\ulcorner \Box A \urcorner \qquad \text{abbreviates} \qquad \ulcorner \neg\, \mathrm{N}\, A \urcorner .$$

Semantically, for any **Z**-cosmos $\mathbb{C}$, for any $v \in \mathbb{C}$, and for any $A \in \mathrm{For}_{\mathbf{Z}}$ we obtain the following condition:

$$v(\Box A) = 1 \quad \text{iff} \quad \forall_{u \in \mathbb{C}}\, u(A) = 1 .$$

Indeed, $v(\Box A) = 1$ iff $v(\neg\, \mathrm{N}\, A) = 1$ iff $v(\mathrm{N}\, A) = 0$ iff $\nexists_{u \in \mathbb{C}}\, u(A) = 0$. So we obtain an interpretation of '□' in **S5**.

By the above fact, for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formulas

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(T) $\Box A \supset A$

(E) $\neg\Box A \supset \Box\neg\Box A$

are **Z**-valid; so they are theses in HZ. Moreover,

(RG) if A is **Z**-valid (a thesis of HZ), then
$\Box A$ is **Z**-valid (a thesis of HZ).

Thus, in ¬-∧-∨-⊃-□-reformulation of HZ we obtain the modal logic **S5**.

(iv) As it is mentioned in Remark 1.1, we can reproduce **Z** (in the N-∧-∨-⊃-language) in **S5** (in the ¬-∧-∨-⊃-□-language) if 'N' abbreviates '¬□'. ⊣

2.2. Redundancy of negation-related axioms in HZ

We shall see in this subsection that negation-related axioms are *not* independent in the system HZ.

Fact 2.1. (a) *The formula* (AZ1) *is provable from other axioms of* HZ.

(b) *The formula* (AZ3) *is provable from other axioms of* HZ.

Proof. (a) For (AZ1):

1.	$A \supset A$	(1)
2.	$\mathrm{N}(A \wedge \mathrm{N}\, A)$	1, (RZ)
3.	$\mathrm{N}\, A \vee \mathrm{N}\,\mathrm{N}\, A$	2, (AZ3), (MP)
4.	$(\mathrm{N}\, A \vee \mathrm{N}\,\mathrm{N}\, A) \supset ((\mathrm{N}\,\mathrm{N}\, A \supset A) \supset (\mathrm{N}\, A \vee A))$	(4)
5.	$\mathrm{N}\, A \vee A$	3, 4, (AZ4), (MP)
6.	$A \vee \mathrm{N}\, A$	5, (11), (MP)

(b) For (AZ3):

1. $(A \wedge \mathrm{N}\,\mathrm{N}\,B) \supset (A \wedge B)$ (1), (AZ4), (9), (MP)
2. $\mathrm{N}(A \wedge B) \supset \mathrm{N}(A \wedge \mathrm{N}\,\mathrm{N}\,B)$ 1, (R3)
3. $\mathrm{N}(A \wedge \mathrm{N}\,\mathrm{N}\,B) \supset (\mathrm{N}\,\mathrm{N}\,B \supset \mathrm{N}\,A)$ (AZ2′)[2]
4. $(\mathrm{N}\,B \vee \mathrm{N}\,\mathrm{N}\,B) \supset ((\mathrm{N}\,\mathrm{N}\,B \supset \mathrm{N}\,A) \supset (\mathrm{N}\,B \vee \mathrm{N}\,A))$ (4)
5. $(\mathrm{N}\,\mathrm{N}\,B \supset \mathrm{N}\,A) \supset (\mathrm{N}\,B \vee \mathrm{N}\,A)$ 4, (AZ1), (MP)
6. $\mathrm{N}(A \wedge B) \supset (\mathrm{N}\,A \vee \mathrm{N}\,B)$ 2, 3, 5, (11), (R1) ⊣

Therefore, in order to make the logical content of the system HZ more clear, it is necessary to give an axiomatization of **Z** in which the axioms are independent of others. This problem will be discussed in the next section.

3. The system HZ′

In this section, for the logic **Z**, we shall consider an axiomatic system HZ′ which contains “positive axioms” (AP1)–(AP9), negation-related axioms (AZ2′) and (AZ4), and the rules (MP) and (RZ).

It should be noted that the formulas (1)–(11) are provable in HZ′. Moreover, the rule (R3) can be easily proved in HZ′ using (AZ2′) and (RZ).

Theorem 3.1. *The systems* HZ *and* HZ′ *are inferentially equivalent.*

Proof. HZ′ is a subsystem of HZ: As we stated on p. 309, (AZ2′) is provable in HZ. Moreover, (AZ4) is taken as an axiom in HZ. Therefore, HZ′ is a subsystem of HZ.

HZ is a subsystem of HZ′: Notice that, by Fact 2.1b, the axiom (AZ3) is provable from other axioms of HZ. So, this time we have to prove that two axioms (AZ1) and (AZ2) of HZ are provable in HZ′. Before giving the proofs, note that the following formula is provable in HZ′ using (AP7), (AP8), (R3), and (8):[3]

$$\mathrm{N}(A \vee B) \supset (\mathrm{N}\,A \wedge \mathrm{N}\,B) \tag{22}$$

Now, the proof runs as follows:

For (AZ1):

1. $\mathrm{N}(A \vee \mathrm{N}\,A) \supset (\mathrm{N}\,A \wedge \mathrm{N}\,\mathrm{N}\,A)$ (22)
2. $(\mathrm{N}\,A \wedge \mathrm{N}\,\mathrm{N}\,A) \supset (A \wedge \mathrm{N}\,A)$ (AZ4), (10), (R1)

[2] Cf. Footnote 1; we did not make use of (AZ3) in the proof of (AZ2′).

[3] Notice that also the formulas (1)–(11) are provable in HZ′, and the rules (R1) and (R3) are derivable in HZ′.

3. $\mathsf{N}(A \vee \mathsf{N}\,A) \supset (A \wedge \mathsf{N}\,A)$ 1, 2, (R1)
4. $\mathsf{N}(A \wedge \mathsf{N}\,A) \supset \mathsf{N}\,\mathsf{N}(A \vee \mathsf{N}\,A)$ 3, (R3)
5. $\mathsf{N}(A \wedge \mathsf{N}\,A) \supset (A \vee \mathsf{N}\,A)$ 4, (AZ4), (R1)
6. $\mathsf{N}(A \wedge \mathsf{N}\,A)$ (1), (RZ)
7. $A \vee \mathsf{N}\,A$ 5, 6, (MP)

For (AZ2):

1. $\mathsf{N}(A \wedge \mathsf{N}\,B) \supset (\mathsf{N}\,B \supset \mathsf{N}\,A)$ (AZ2′)
2. $(\mathsf{N}\,B \wedge \mathsf{N}(A \wedge \mathsf{N}\,B)) \supset \mathsf{N}\,A$ 1, (6), (10), (MP)
3. $\mathsf{N}\,A \supset (A \supset (A \wedge \mathsf{N}\,A))$ (AP6), (3), (MP)
4. $(\mathsf{N}\,B \wedge \mathsf{N}(A \wedge \mathsf{N}\,B)) \supset (A \supset (A \wedge \mathsf{N}\,A))$ 2, 3, (R1)
5. $(A \wedge \mathsf{N}\,B \wedge \mathsf{N}(A \wedge \mathsf{N}\,B)) \supset (A \wedge \mathsf{N}\,A)$ 4, (6), (10), (MP) ⊣

Fact 3.2. *The negation-related axioms* (AZ2′) *and* (AZ4)*, and the negation-related rule* (RZ) *are independent in the system* HZ′*.*

Proof. This can be proved by using matrices. A *matrix* $\mathfrak{M}$ for $\mathsf{For}_{\mathbf{Z}}$ is a structure $\langle V_{\mathfrak{M}}, D_{\mathfrak{M}}, \mathsf{N}_{\mathfrak{M}}, \wedge_{\mathfrak{M}}, \vee_{\mathfrak{M}}, \supset_{\mathfrak{M}}\rangle$, where $V_{\mathfrak{M}}$ is a nonempty set of values, $D_{\mathfrak{M}}$, which is a subset of $V_{\mathfrak{M}}$, is a set of designated values, $\mathsf{N}_{\mathfrak{M}}\colon V_{\mathfrak{M}} \to V_{\mathfrak{M}}$ and $\wedge_{\mathfrak{M}}, \vee_{\mathfrak{M}}, \supset_{\mathfrak{M}}\colon V_{\mathfrak{M}}^2 \to V_{\mathfrak{M}}$. A *homomorphism* from $\mathsf{For}_{\mathbf{Z}}$ into $\mathfrak{M}$ is a mapping $h\colon \mathsf{For}_{\mathbf{Z}} \to V_{\mathfrak{M}}$ which preserve functions $\mathsf{N}_{\mathfrak{M}}$, $\wedge_{\mathfrak{M}}$, $\vee_{\mathfrak{M}}$ and $\supset_{\mathfrak{M}}$, i.e. for any $A, B \in \mathsf{For}_{\mathbf{Z}}$:

$$h(\mathsf{N}\,A) = \mathsf{N}_{\mathfrak{M}}(A)\,,$$
$$h(A \circ B) = \circ_{\mathfrak{M}}(A, B), \ \text{for } \circ \in \{\wedge, \vee, \supset\}\,.$$

Let $\mathrm{Hom}(\mathsf{For}_{\mathbf{Z}}, \mathfrak{M})$ be the set of all homomorphisms from $\mathsf{For}_{\mathbf{Z}}$ into $\mathfrak{M}$.

For (AZ2′): We consider the following three-valued matrix $\mathfrak{M}$ which can be reached through two modifications on Heyting's three-valued matrix. One is to replace the mapping for negation as follows and the other is to take not only 1 but also ½ as designated value, i.e., $V_{\mathfrak{M}} := \{0, ½, 1\}$, $D_{\mathfrak{M}} := \{½, 1\}$ and:

$\supset_{\mathfrak{M}}$	1	½	0
1	1	½	0
½	1	1	0
0	1	1	1

$\wedge_{\mathfrak{M}}$	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

$\vee_{\mathfrak{M}}$	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

	$\mathsf{N}_{\mathfrak{M}}$
1	0
½	½
0	1

We have:

(a) values of all instances of (AP1)–(AP9), (AZ4) belong to $D_{\mathfrak{M}}$, for all homomorphisms in $\mathrm{Hom}(\mathsf{For}_{\mathbf{Z}}, \mathfrak{M})$,

(b) for any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ and $A, B \in \mathrm{For}_{\mathbf{Z}}$:

$$\text{if } h(A \supset B), h(A) \in D_{\mathfrak{M}}, \text{ then } h(B) \in D_{\mathfrak{M}}, \text{ and}$$
$$\text{if } h(A \supset B) \in D_{\mathfrak{M}}, \text{ then } h(\mathsf{N}(A \wedge \mathsf{N}\, B)) \in D_{\mathfrak{M}}.$$

Thus, values of all formulas, which are provable from (AP1)–(AP9), (AZ4) by the rules (MP) and (RZ) belong to $D_{\mathfrak{M}}$, for all homomorphisms in $\mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$.

Finally, for any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ such that $h(p) = 1$ and $h(q) = ½$ we have $h(\mathsf{N}(p \wedge \mathsf{N}\, q) \supset (\mathsf{N}\, q \supset \mathsf{N}\, p)) = 0$. Therefore not all instances of (AZ2′) are provable from schemata (AP1)–(AP9), (AZ4) by the rules (MP) and (RZ).

For (AZ4): Take the ordinary two-valued matrix for **CPL** (i.e. $V_{\mathfrak{M}} := \{0, 1\}$, $D_{\mathfrak{M}} := \{1\}$) and replace the mapping for the negation with the following:

	$\mathsf{N}_{\mathfrak{M}}$
1	1
0	1

We have:

(a) values of all instances of (AP1)–(AP9), (AZ2′) belong to $\{1\}$, for all homomorphisms in $\mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$,

(b) for any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ and $A, B \in \mathrm{For}_{\mathbf{Z}}$:

$$\text{if } h(A \supset B) = 1 = h(A), \text{ then } h(B) = 1, \text{ and}$$
$$h(\mathsf{N}(A \wedge \mathsf{N}\, B)) = 1.$$

So values of all formulas provable from (AP1)–(AP9), (AZ2′) by the rules (MP) and (RZ) belong to $\{1\}$, for all homomorphisms in $\mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$.

Finally, for any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ such that $h(p) = 0$ we have $h(\mathsf{N}\,\mathsf{N}\, p \supset p) = 0$. So not all instances of (AZ4) are provable from schemata (AP1)–(AP9), (AZ2′) by the rules (MP) and (RZ).

For (RZ): Also take the ordinary two-valued matrix for **CPL**, but replace the matrix for the negation with the following:

	$\mathsf{N}_{\mathfrak{M}}$
1	0
0	0

We have:

(a) values of all instances of (AP1)–(AP9), (AZ2′), (AZ4) belong to $\{1\}$, for all homomorphisms in $\mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$,

(b) for any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ and $A, B \in \mathrm{For}_{\mathbf{Z}}$:

$$\text{if } h(A \supset B) = 1 = h(A), \text{ then } h(B) = 1.$$

Thus, values of all formulas provable from (AP1)–(AP9), (AZ2′), (AZ4) by the rule (MP) belong to $\{1\}$, for all homomorphisms in $\mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$. Now notice that for any $A \in \mathrm{For}_{\mathbf{Z}}$ the following formula

$$\mathsf{N}(A \wedge \mathsf{N}\, A) \tag{23}$$

is provable in HZ by (1) and (RZ). However, $h(23) = 0$ for any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$. Hence (23) is not provable from (AP1)–(AP9), (AZ2′), (AZ4) by the rule (MP). ⊣

4. Some observations on HZ′ and CPL

Since the system HZ′ and the logic **CPL** have the same kind of connectives, we can now offer some observations on HZ′ and **CPL** from a certain point of view.

4.1. A relation between HZ′ and CPL

Notice that for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formula:

$$\mathsf{N}(A \wedge \mathsf{N}\, B) \equiv (\mathsf{N}\, B \supset \mathsf{N}\, A) \tag{24}$$

is **Z**-valid. So it is a thesis of **Z** (it is provable in HZ and HZ′; see Remark 1.2 and Theorem 3.1).

Seeing the negation-related rule and axioms of HZ′ from the point of view of classical propositional logic, i.e. if we regard the negation 'N' as the classical negation, we can see that for any $A, B \in \mathrm{For}_{\mathbf{Z}}$ the following formulas (AZ4), (24) and :

$$(A \supset B) \supset \mathsf{N}(A \wedge \mathsf{N}\, B) \tag{†}$$

$$\mathsf{N}(A \wedge \mathsf{N}\, B) \supset (A \supset B) \tag{‡}$$

$$A \supset \mathsf{N}\,\mathsf{N}\, A \tag{$\star$}$$

are theses of **CPL**. Thus in **CPL** we of course have that:

- the formulas $\mathsf{N}(A \wedge \mathsf{N}\, B)$, $A \supset B$ and $\mathsf{N}\, B \supset \mathsf{N}\, A$ are equivalent,
- the formulas A and $\mathsf{N}\,\mathsf{N}\, A$ are equivalent.

Of course, not all instances of (†), (‡) and (⋆) are theses of **Z**. In $\mathtt{HZ}'$ instead of formulas (†) and (⋆) we have only the rules (RZ) and (R2), respectively.

4.2. The system $\mathtt{HZ}^{\mathsf{c}}$

We shall here examine a system which can be reached by replacing '/' with '⊃' in the rule (RZ) of $\mathtt{HZ}'$. We shall refer to the system as $\mathtt{HZ}^{\mathsf{c}}$. Thus, the system $\mathtt{HZ}^{\mathsf{c}}$ has the following axiom (AP1)–(AP9), (AZ2′), (AZ4) and

(AZ5) $$(A \supset B) \supset \mathsf{N}(A \wedge \mathsf{N}\, B)$$

and only one rule (MP).

Fact 4.3. *The negation-related axioms* (AZ2′), (AZ4) *and* (AZ5) *are independent in the system* $\mathtt{HZ}^{\mathsf{c}}$.

Proof. We use, respectively, the same matrices as in the proof of Fact 3.2.

For (AZ2′) and (AZ4): Notice that values of all instances of (AZ5) belong to $D_{\mathfrak{M}}$, for all homomorphisms in $\mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ (in both cases).

For (AZ5): For any $h \in \mathrm{Hom}(\mathrm{For}_{\mathbf{Z}}, \mathfrak{M})$ such that either $h(p) = 0$ or $h(q) = 1$, we have $h((p \supset q) \supset \mathsf{N}(p \wedge \mathsf{N}\, q)) = 0$. Thus, not all instances of (AZ5) are provable from (AP1)–(AP9), (AZ2′), (AZ4) by the rule (MP). ⊣

Now, it can actually be proved that the system $\mathtt{HZ}^{\mathsf{c}}$ is inferentially equivalent to classical propositional logic. The proof runs as follows.

Lemma 4.1. *All formulas provable in* $\mathtt{HZ}^{\mathsf{c}}$ *are theses of* **CPL**.

Proof. All three negation-related axioms of $\mathtt{HZ}^{\mathsf{c}}$ are theses (i.e. tautologies) of **CPL** and the rule (MP) preserves tautologies. ⊣

Lemma 4.2. *All formulas provable in* $\mathtt{HZ}'$ *are provable in* $\mathtt{HZ}^{\mathsf{c}}$.

Proof. By (AZ5) and (MP) in $\mathtt{HZ}^{\mathsf{c}}$ we have the rule (RZ) ⊣

Since we have the above lemma, Theorem 3.1 and a fact given in p. 309, we can prove:

Lemma 4.3. (AZ1) *and* (13) *are provable in* $\mathtt{HZ}^{\mathsf{c}}$.

Lemma 4.4. *For any* $A, B \in \mathrm{For}_{\mathbf{Z}}$ *the following formula*

$$(\P) \qquad (A \wedge \mathsf{N}\, A) \supset B$$

is provable in $\mathtt{HZ}^{\mathsf{c}}$.

Proof.
1. $(\mathsf{N}\, B \supset A) \supset \mathsf{N}(\mathsf{N}\, B \wedge \mathsf{N}\, A)$ (AZ5)
2. $\mathsf{N}(\mathsf{N}\, B \wedge \mathsf{N}\, A) \supset (\mathsf{N}\, A \supset \mathsf{N}\,\mathsf{N}\, B)$ (AZ2′)
3. $(\mathsf{N}\, B \supset A) \supset (\mathsf{N}\, A \supset \mathsf{N}\,\mathsf{N}\, B)$ 1, 2, (R1)
4. $A \supset (\mathsf{N}\, B \supset A)$ (AP1)
5. $A \supset (\mathsf{N}\, A \supset \mathsf{N}\,\mathsf{N}\, B)$ 3, 4, (R1)
6. $(A \wedge \mathsf{N}\, A) \supset \mathsf{N}\,\mathsf{N}\, B$ 5, (6), (MP)
7. $\mathsf{N}\,\mathsf{N}\, B \supset B$ (AZ4)
8. $(A \wedge \mathsf{N}\, A) \supset B$ 6, 7, (R1) ⊣

Lemma 4.5. *All theses* (*i.e. tautologies*) *of* **CPL** *are provable in* $\mathtt{HZ}^{\mathsf{c}}$.

Proof. In [4, pp. 188–189], the following axiomatization of **CPL** is given for the language $\{\mathsf{N}, \wedge, \vee, \supset\}$: (AP3), (AP4), (AP5), (AP7), (AP8), (2), (5), (6), (8), (¶), (13), (AZ1) and (MP). So, all theses of **CPL** are provable in this system. By lemmas 4.2, 4.3 and 4.4, all axioms of this system are provable in $\mathtt{HZ}^{\mathsf{c}}$. Thus, all theses of **CPL** are provable in $\mathtt{HZ}^{\mathsf{c}}$. ⊣

By lemmas 4.1 and 4.5 we obtain:

Theorem 4.2. *The system* $\mathtt{HZ}^{\mathsf{c}}$ *is an axiomatization of* **CPL**.

4.3. Getting Z and S5 from CPL

Now, as it is known, there are many systems of non-classical logics and some of them are developed by *regulating* some axioms or theses of classical propositional logic. For example, intuitionistic propositional logic can be obtained from classical propositional logic by eliminating $\ulcorner A \vee \mathsf{N}\, A \urcorner$ in Rasiowa–Sikorski's axiomatization of **CPL** (see [4, pp. 188–189]). It is thus not a new but a rather common idea to view some of the systems of non-classical logic as "regulated" classical propositional logic. However, there seems to be no systematic method to treat non-classical logics with the spirit of regulation.

In this part, we will sketch an idea which might enable us to treat the "regulated" CPLs in a systematic way through a certain relation between $\mathtt{HZ}'$ and **CPL**.

To begin with, recall here the Remark 2.3 which shows that the system HZ is definitionally equivalent to any system of modal logic **S5**. Then, together with the result proved in the previous section that HZ′ is inferentially equivalent to HZ, we reach the fact that **S5** is an "obtainable" logic, i.e. **S5** can be obtained from **CPL** by "specifying" and "splitting" in the following sense:

- firstly, "specify" in **CPL** the formulas of forms $\ulcorner \mathsf{N}\,\mathsf{N}\,A \equiv A \urcorner$ and $\ulcorner (A \supset B) \equiv \mathsf{N}(A \wedge \mathsf{N}\,B) \equiv (\mathsf{N}\,B \supset \mathsf{N}\,A) \urcorner$;
- secondly, "split" the above formulas in a way so that an axiomatization of the concerned system will be given; instead of the first form of formulas in HZ′ we take the axiom (AZ4) (and the derivable rule (R2)); instead of the second form of formulas we take the axiom (AZ2′) (and provable (24) in HZ′) and the rule (RZ).

The result of "splitting" can be presented in a diagram as follows:

$$
\begin{array}{ccc}
A \supset B & \overset{(\mathrm{RZ})}{\Longrightarrow} & \mathsf{N}(A \wedge \mathsf{N}\,B) \\
 & \searrow & {\scriptstyle (\mathrm{AZ2}')}\downarrow\uparrow{\scriptstyle (24)} \\
 & & \mathsf{N}\,B \supset \mathsf{N}\,A
\end{array}
\qquad\qquad
\begin{array}{c}
\mathsf{N}\,\mathsf{N}\,A \\
{\scriptstyle (\mathrm{AZ4})}\downarrow\Uparrow{\scriptstyle (\mathrm{R2})} \\
A
\end{array}
$$

where $X \longrightarrow Y$ stands for: $\ulcorner X \supset Y \urcorner$ is a form of theses of HZ′; and $X \Longrightarrow Y$ for: $\ulcorner X \supset Y \urcorner$ is not a form of theses of HZ′, but the rule $X \;/\; Y$ holds in HZ′.

The perspective on the relation between **CPL** and **S5** we elaborated above seems to be a new kind since it shows that "modality" represented in the system of modal logic **S5** can be reproduced by cutting some rather simple properties of classical negation off and collecting them (cf. Remark 2.3). In this result, there seems to be a clue for us to re-examine what a modality is and what kind of modality is expressed in the modal logic **S5**. It is from this point of view that Béziau's logic **Z** seems to be of great interest and also the new axiomatization of **Z**, given as HZ′, seems to be quite informative.

5. A remark on the paraconsistency of HZ

In this final section, we shall make a remark on the paraconsistency of HZ.

In general, there seem to be two kinds of necessary conditions for a logical system to be called paraconsistent. Let us here assume that N is intended to be a paraconsistent negation. Then, the two conditions can be stated as follows:

(A) For some formulas A and B, $A \supset (\mathsf{N}\,A \supset B)$ is not provable in the system.

(B) For some formulas A and B, B cannot be inferred from A and $\mathsf{N}\,A$ in the system.

Remark 5.4. These two conditions are not independent if the rule *Modus Ponens* is assumed, since the condition (B) implies the condition (A). However, the converse does not hold in general. ⊣

Remark 5.5. The condition (B) is important if we accept the Jaśkowski's original idea to make a distinction between the two notions of a deductive system being *inconsistent* and *overfilled* (cf. [3, p. 38]). This is because if a system of a propositional logic which does not satisfy the condition (B) is applied to any inconsistent system, then the inconsistent system would turn to be overfilled. It should be noted that the system PCL1 satisfies the condition (B) (cf. Remark 1.1). ⊣

Now as for the system HZ, the condition (A) is satisfied but the condition (B) is not. Indeed, the following fact can be proved.

Fact 5.4. *For any $A, B \in \mathrm{For}_{\mathbf{Z}}$, B can be inferred from A and $\mathsf{N}\,A$, i.e., the following rule can be derived in* HZ*:*

$$\frac{A \qquad \mathsf{N}\,A}{B}$$

Proof.

1.	A	sup.
2.	$\mathsf{N}\,A$	sup.
3.	$\mathsf{N}\,\mathsf{N}\,A$	1, (R2)
4.	$\mathsf{N}\,A \wedge \mathsf{N}\,\mathsf{N}\,A$	2, 3, (AP6), (MP)
5.	B	4, (14), (MP) ⊣

Based on the observations given above, Béziau's system HZ is paraconsistent in the sense that it satisfies the condition (A). So, also for some formulas $A, B \in \mathrm{For}_{\mathbf{Z}}$, the formula $A \supset (\mathsf{N}\,A \supset B)$ is not a thesis of **Z**. Thus, it is in this sense that we called **Z** paraconsistent logic in the present paper. At the same time, however, the system HZ is not paraconsistent in the sense that it does not satisfy the condition (B). Therefore, there might be several points of view on the paraconsistency of HZ; nevertheless, as we pointed out in the last paragraph of the previous section, Béziau's logic **Z** is of great importance in the light of the relation between negation and modality.

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References

[1] Béziau, J.-Y., "The paraconsistent logic Z. A possible solution to Jaśkowski's problem", *Logic and Logical Philosophy* 15 (2006), 2, 99–111.

[2] Jaśkowski, S., "Rachunek zdań dla systemów dedukcyjnych sprzecznych", *Studia Societatis Scientiarum Torunensis*, Sectio A, Vol. I, No. 5 (1948), 57–77.

[3] Jaśkowski, S., "A Propositional calculus for inconsistent deductive systems", *Logic and Logical Philosophy* 7 (1999), 35–56; translation of [2] by O. Wojtasiewicz with corrections and notes by J. Perzanowski.

[4] Rasiowa, H., and R. Sikorski, *The Mathematics of Metamathematics*, Monografie Matematyczne, tom 41, Warszawa, Polish Scientific Publishers, 1963.

[5] Waragai, T., and T. Shidori, "A system of paraconsistent logic that has the notion of 'behaving classically' in terms of the law of double negation and its relation to S5", pp. 177–187 in: *Handbook of Paraconsistency*, J.-Y. Béziau, W. A. Carnielli and D. Gabbay (eds.), College Publications, 2007.

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