Logic and Logical Philosophy
Volume 17 (2008), 287–303
DOI: 10.12775/LLP.2008.016

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THE NOTION OF PROBLEM, INTUITIONISM AND PARTIALITY

Abstract. *Problems* are defined as abstract procedures. An explication of procedures as used in Transparent Intensional Logic (TIL) and called *constructions* is presented and the subclass of constructions called *concepts* is defined. Concepts as closed constructions modulo α- and η-conversion can be associated with meaningful expressions of a natural or professional language in harmony with Church's conception. Thus every meaningful expression expresses a concept. Since every problem can be unambiguously determined by a concept we can state that every problem is a concept and every concept can be viewed as a problem.

Kolmogorov's idea of a connection between problems and Heyting's calculus is examined and the non-classical features of the latter are shown to be compatible with realistic logic using partial functions.

Keywords: abstract procedures, constructions, effective procedures, concepts, partiality.

Introduction

In 1932 Kolmogorov has drawn our attention to the notion of problem. He has shown that this notion can be logically handled and that an attempt of doing it has been made by Heyting in his intuitionistic calculus. In the present paper I try to show that a rather universal tool for logical analysis of the notion of problem is offered by Tichý's Transparent intensional logic, where the key notion of *constructions* has been introduced. Further, an anti-realistic (because intuitionistic) interpretation of Heyting's calculus is not

Received September 24, 2008
 ISSN: 1425-3305

the only one possible: it can be compatible with realistic philosophy as soon as the critical deviations from classical logic are explained as consequences of partiality.

1. Examples

Compare following expressions

A.

$2^3 < 3^2$

$\lambda n\, \lambda abc\, (a^n + b^n = c^n)$

$\forall nabc\, (n > 2 \supset \neg(a^n + b^n = c^n))$

$\neg\exists K(\mathrm{Card}\, \aleph_0 < \mathrm{Card}\, K < \mathrm{Card}\, \aleph_1)$

$\lambda r(r \in \mathfrak{R}^2 \wedge \forall x\, \neg r(x,x) \wedge \forall xy(r(x,y) \supset r(y,x)) \wedge \forall xyz(r(x,y) \supset (r(y,z) \supset r(x.z)))),\ x, y, z \to \mathbf{N}$

There are organisms in Mars.

Mountains over 8000m.

B.

Is it the case that $2^3 < 3^2$?

For any integer n find such integers a, b, c that $(a^n + b^n = c^n)$

Is it the case that for all integers n, a, b, c it holds that
$(n > 2 \supset \neg(a^n + b^n = c^n)$? (*Fermat's Last Theorem*)

Is *Continuum Hypothesis* true?

Which symmetric and transitive relations of naturals are irreflexive?

Are there organisms in Mars?

Which mountains are higher than 8000m?

Our claim is: The expressions in the group **B** possess the same semantics as the respective expressions in the group **A**. The distinction consists in a *pragmatic factor*: If the members of **A** are connected with an attitude at all (and this is not at all necessary) then something like 'stating' could be a characteristic of such an attitude. On the other hand, the members of **B** could be characterized as 'seeing a problem'. (Cf. [Kolmogorov 1932] for his examples of problems and [Tichý 1978] for a similar conception of *questions*.)

2. Abstract procedures

What I intend to claim in the present paper is that

I. *Every problem can be treated as an abstract procedure.*

II. *Accepting some assumptions concerning logical analysis of natural language we can state that every problem is a concept and every concept can be viewed as a problem.*

In this section I have to say some relevant points clarifying the point I.

(An important source is here [Tichý 1995], where the principal (and for many philosophers surprising) distinction between such *simple* entities as cars or sets and *complexes* like melodies or *constructions* is argued for. I will not repeat here the fundamental arguments; instead I will try to characterize abstract procedures in a most general way, accompanying the characteristics by some examples.)

A *procedure* consists of at least one *step*. It determines an object which can be called *result*. A *real procedure* is a time consuming event. An *abstract procedure* is a procedure that is not real; in other words, it cannot be localized in time or space. All the same, it is well definable and intelligible.

Example. Let us come to a terminological agreement: Let a computer *program* (type, not token) be an expression, whose meaning is the respective *algorithm* and whose denotation is the result of the algorithm (if any). (See [Moschovakis 1994].) The algorithm is an example of abstract procedures: *it is a sequence of steps ('instructions'), being itself an executable instruction.* It cannot be reduced to a set of instructions. Indeed, let $\mathbf{A}$ be the algorithm and $\{\mathbf{i_1}, \ldots, \mathbf{i_k}\}$ be the set of its instructions. Obviously $\mathbf{A} \neq \{\mathbf{i_1}, \ldots, \mathbf{i_k}\}$. □

(As soon as we define meaning of an expression E as an abstract procedure we will see that identifying an abstract procedure with a *tuple* of particular steps (meanings of the subexpressions of E) is a mistake as well: it holds that $\mathbf{A} \neq \langle \mathbf{i_1}, \ldots, \mathbf{i_k} \rangle$. This is Cresswell's mistake, see his [1975], in particular p. 30; see also [Tichý 1994, 2004] and [Jespersen 2003].)

To make our characterization more precise we have to ask what kind of object the *result* of a procedure is. In general we can state that the result (if any) of an abstract procedure is an *n-ary function* for $n \geqslant 0$.

Examples. Consider the procedure that consists in applying the operation adding to the pair of numbers 3, 4. The result is the 0-ary function 7.

The procedure consisting in adding 1 to the given natural number, so

$$\lambda x(x+1), x \to \mathbf{N},$$

results in the unary function Successor.

The procedure $\forall nabc\,(n > 2 \supset \neg(a^n + b^n = c^n)$, $n, a, b, c \to \mathbf{N}$, results—as we know today—in **true**, which is a 0-ary function.

A procedure, which for any possible world and time determines those objects that are at that world-time higher than 800m, results in a property of individuals, which is a unary function[1], whose range is a unary function, whose range is a unary function.

A procedure deciding whether the greatest prime number is odd or even does not have any result. (Better: There is *no* effective procedure deciding whether the greatest prime is odd.)

Any procedure defined over the wffs of a 1st order predicate calculus C that consists in outputting **true** iff the given wff is a theorem of C results in the set of all theorems of C. This set is again a unary (characteristic) *function*.

A procedure defined over the wffs of a 1st order predicate calculus C that consists in outputting **true** if the given wff is a theorem of C and **false** otherwise does not have any result (or: is not effective (algorithmic)). □

3. Structured meanings

Now we try to argue that the best explication of *meaning* is an abstract procedure. A thoroughgoing argumentation can be found in various writings by the followers of TIL, in particular [Tichý 1988] and [Materna 1998, 2004].

This explication shows that meaning is not an obscure object[2] and solves some problems that necessarily arise if meaning gets a set-theoretical explication. That set-theoretical explications of meaning are essentially unsatisfactory has been suspected by more philosophers. Remarkable hints can be found in [Bolzano 1837]; Tichý's [1968] and [1969] are probably the first explicit declarations of a procedural character of meaning, David Lewis' "General semantics" [1972], Cresswell's attempts from [1975, 1985], Bealer's [1982], Chierchia's [1989], Moschovakis' [1994] are examples of exhibiting

[1] Using the typing of Transparent Intensional Logic (TIL)—see below—we get the type $(((o\iota)\tau)\omega)$, similar to Montague's $(s \to (e \to t))$.

[2] See [Materna 2007], where Quine's position is attacked.

dissatisfaction with set-theoretical conceptions, which do not enable us to explain how a structured expression can express[3] an unstructured meaning. The most commonly accepted (cf. [Kirkham 1997]) set-theoretical explication of meaning consists in the claim that meaning—as a heir to Fregean *Sinn*—should be conceived of as an *intension*, obviously in the P(ossible)-W(orld)-S(emantics). Which means that meaning would be a function from possible worlds. But functions are sets, i.e., they are simples that do not possess components which could be confronted with particular subexpressions of the given expression; so applying the desirable Principle of Compositionality would be at least imperfect and in some cases impossible. (The Principle of Substitutability, which is implied by the Principle of Compositionality, does not work for attitudinal contexts if meaning is not structured.)

It is evident that the Principle of Compositionality could not be followed in all the contexts if semantics were reduced to set-theoretical denotational semantics. That the given sentence denotes a proposition is not extremely interesting; besides imagine that your logical analysis would culminate by presenting a table where the left most column would be the set of all possible worlds (!), the next one will associate each member of the first column with the infinite (NB not denumerable) list of time moments and the final column would contain a distribution of truth-values. Not only impossible, also totally useless. As soon as the set-theoretical standpoint is abandoned and the procedural view is accepted our problem disappears. Now the meaning is complex: it is a procedure containing, in general, subprocedures. Moreover, the idea characteristic of TIL, viz. that expressions of a language encode procedures, gets now a particular specification that makes it possible to realize *logical* analyses of natural language.

I will only briefly suggest the way abstract procedures are treated in TIL. Exact definitions and more detailed description can be found in the TIL literature mentioned above.

Abstract procedures are defined as *constructions*.[4] They are defined in a typed environment. The most frequently applied *basis* for types involves 'non-functional' ('atomic', if you like) types (= types of 0-ary functions) o (truth-values **T**, **F**), ι (individuals, members of the universe), τ (time moments, also real numbers) and ω (possible worlds), and functional types, i.e., sets of partial functions with tuples of arguments of types $\beta_1, \ldots, \beta_m$,

[3] Using a Fregean terminology.

[4] Another notion of constructions, sharing some essential points with ours, can be found in the excellent monograph *Truth, Proof and Infinity* by Peter Fletcher [1998].

respectively, and value of the type α, denoted by $(\alpha\beta_1 \ldots \beta_m)$. Thus the *types of order* 1 are defined.

Examples. Propositions are of the type $((o\tau)\omega)$, abbreviated as $o_{\tau\omega}$[5]: given a world W and time T the *proposition* returns either **T**, or **F**, or nothing (partiality!).

Every class/relation is represented by the respective characteristic function. So the type of *properties of individuals* is $(o\iota)_{\tau\omega}$; in general, properties of objects of type α are of the type $(o\alpha)_{\tau\omega}$. □

Constructions: The most important ones are

variables, i.e., special constructions that construct objects of the given type dependently on *valuation*; we write "v-construct", where v is the parameter of valuation; any construction containing free variables v-constructs. (We will omit v in the following text.)

trivialization, which constructs an object by mentioning it. So we have ${}^0X$, which constructs the object / construction denoted by X,

composition: where X constructs a function and $X_1, \ldots, X_m$ construct objects that are arguments of the function, composition constructs the object (if any) that is the value of the function on $X_1, \ldots, X_m$. We write $[XX_1 \ldots X_n]$. Composition may be (v-)*improper*, i.e. (v-)construct nothing (if a constructed function is not defined on its arguments).

closure: where $x_1, \ldots, x_m$ are distinct variables ranging over the (not necessarily distinct) types $\beta_1, \ldots, \beta_m$ and X is a construction constructing objects of the type α, the closure constructs a function of the type $(\alpha\beta_1 \ldots \beta_m)$ in the way well-known from λ-calculi. We write $[\lambda x_1 \ldots x_m X]$. (Observe that closure is never improper: always a function is constructed, even, in the worst case, an 'ugly' (= nowhere defined) one.)

The expressivity of TIL grows up essentially as soon as the procedures/constructions can be not only *used* but also *mentioned*, i.e., as soon as constructions become objects *sui generis*. This is made possible *via* defining *ramified hierarchy of types*. The principle is simple: Types of order 1 have been already defined. Then *constructions of order n* are defined (they construct objects of lower order) and the set of all constructions of order n is denoted $*_n$. This set (and also the types of order n) is the type of order $n+1$.

Finally having a type $(\alpha\beta_1 \ldots \beta_m)$, where some of α, β_i is of the type of order $n+1$ we ascribe this order to $(\alpha\beta_1 \ldots \beta_m)$ as well.

[5] Every *intension* is an object of a type $((\alpha\tau)\omega)$. We abbreviate it as $\alpha_{\tau\omega}$.

After ramified hierarchy of types has been defined we have got the possibility to distinguish between the type of a construction C and the type of the object (if any) constructed by C. The former case: C/α, the latter case: $C \rightarrow \beta$.

It is not by chance that Church's λ-calculus has been used also by Montague [1974]. Advantages connected with functional (rather than relational) approach are obvious. In TIL, besides, we must be aware of the fact that what looks like a formula / λ-term is, actually, a record of an *abstract procedure*. To illustrate, considering

$$\lambda x(x+1), x \rightarrow \mathbf{N},$$

as a formal λ-term it is interpreted as the *function* Successor, while as a record of a construction[6] it denotes an abstract *procedure* that constructs the function Successor.

Logical analysis of an expression E consists—roughly—in assigning typed objects to particular subexpressions of E and creating a construction, where the subconstructions assigned to the former will be interconnected in the way suggested by the grammar of the respective language.

Now we will show some (imperfect) analyses.

Examples. (a) Mathematical expressions. Here the analyses are relatively easy, since the expressions of mathematical languages reflect very directly the interplay of procedures and functions.

i) $$a, b, c, n \rightarrow \mathbf{N}\colon \{\langle a, b, c, n\rangle : a^n + b^n = c^n\}$$

For the sake of simplicity let us interpret τ as the set of *natural numbers.* Types:

$$a, b, c, n \rightarrow \tau,\ \exp^y a/(\tau\tau\tau)\ (\exp^y x = x^y),\ = /(o\tau\tau), +/(\tau\tau\tau).$$

$$\lambda abcn[{}^0{=}[{}^0{+}[{}^0\exp a\, n][{}^0\exp b\, n]][{}^0\exp c\, n]]$$

ii) $$a, b, c, n \rightarrow \mathbf{N}\colon \forall abcn(n > 2 \supset a^n + b^n \neq c^n)$$

$$\forall/(o(o\tau\tau\tau\tau)), > /(o\tau\tau)$$

$$[{}^0\forall\, \lambda abcn[{}^0{\supset}[{}^0{>}n\ {}^02][{}^0\neg[{}^0{=}[{}^0{+}[{}^0\exp a\, n][{}^0\exp b\, n]][{}^0\exp c\, n]]]]]$$

[6] As a standard TIL record: $\lambda x[{}^0{+}\, x\ {}^01]$

The examples i) and ii) illustrate the well-known classification of procedures[7]. The procedure i) is effective: the constructed function is primitive recursive. The procedure ii) is not effective because of the universal quantifier. (But the Fermat's riddle has been solved all the same. How come? See below.)

(b) Empirical expressions.

There are organisms in Mars.

Types:

$$\exists/(o(o\iota)), \wedge/(ooo), \mathrm{Organ}/(o\iota)_{\tau\omega}, \mathrm{Be\_}in/(o\iota\iota)_{\tau\omega}, \mathrm{Mars}/\iota,$$
$$w \to \omega, t \to \tau, x \to \iota.$$

Instead of $[[Xw]t]$ we write X_{wt}.

$$\lambda w\lambda t[{}^0\exists\, \lambda x[{}^0\wedge[{}^0\mathrm{Organ}_{wt}x][{}^0\mathrm{Be\_}in_{wt}x\,{}^0\mathrm{Mars}]]$$

This procedure would decide for every world W and time T whether the class of individuals that are in W at T organisms and are in Mars is empty. Observe that this procedure—and this holds for all meanings of empirical expressions—cannot determine the truth-value in the actual world-time: In general, no meaning of an empirical expression can determine the value of the intension denoted by the expression in the actual world-time. The meaning of *the Pope* cannot determine who the Pope is in the real world-time. It determines only the conditions under which an object can satisfy the conditions given by the constructed intension. □

Since we know that to determine the actual values of intensions we need *experience* we can immediately understand that *no procedure which is the meaning of an empirical expression can be effective.*

4. Solutions

Every (meaningful) expression that does not involve indexicals can be viewed as a *formulation of a problem*. Let P be a problem. Every expression whose

[7] mostly of *functions*. We can however say that if a function is 'computable' (i.e., partial or general recursive or Turing computable etc.) then so is the respective procedure. We will however not speak about 'computable *procedures*': instead we will use the customary term 'effective procedures'.

meaning can be viewed as P will be called *a formulation of the problem* P, briefly F_P.

A *reformulation of* F_P is an expression $F'_{P'}$ distinct from and equivalent with F_P.[8]

Let P be a non-empirical (typically, a mathematical) problem and let it be a non-effective procedure. In some cases there is a reformulation $F'_{P'}$ of F_P such that the meaning of $F'_{P'}$ is an effective procedure P' equivalent with P (in that it constructs the same values as P for each argument).

Let P be a problem. If F_P is a formulation of a not effective procedure then any reformulation of F_P such that the meaning of $F'_{P'}$ is an effective procedure will be called a *solving reformulation* (of F_P) (see [Kleene 1952, p. 317]).

Let P be a non-empirical (typically a mathematical) problem. The *Solution*$_1$ of P is the object (if any) constructed by P. The *Solution*$_2$ (if feasible) of P is a solving reformulation of P.

Example. Let F_P be

> *For any diophantine equation decide whether it is solvable in rational numbers.*

The procedure that is the meaning of this formulation is surely not effective. Hilbert's 10$^{\text{th}}$ problem from 1900 is properly speaking the task to find a solving procedure, so a Solution$_2$: "*To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*" (Here it has been proved in 1970 that no Solution$_2$ existed.)

Another example:

The meaning of any original formulation of Fermat's Last Theorem, for example

$$[^0\forall\, \lambda abcn[^0\supset[^0> n\, ^02][^0\neg[^0=[^0+[^0\exp a\, n][^0\exp b\, n]][^0\exp c\, n]]]]],$$

is a not *an effective* procedure. The reformulation—in this case a very complicated one—enabled us to know that Solution$_1$ is **T**. □

From our viewpoint problems are abstract procedures and their formulations are any meaningful expression of a (natural or professional) language.

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[8] An interesting kind of reformulation is *refinement*: Let P_1, P_2 be problems. Let C_{P_1}, C_{P_2} be the set of constituents of P_1, P_2, respectively. If cardinality of C_{P_2} is greater than cardinality of C_{P_1} and C_{P_2} is not a subset of C_{P_2}, then if P_1 is equivalent to P_2 then P_2 is a *refinement of* P_1 and, derivatively, F_{P_2} is a *refinement of* F_{P_1}.

If E is an *empirical expression*, then E is an F_P of an *empirical problem.* Logical analysis of E should fix the procedure. A specific feature of an empirical problem is that what it constructs is not its solution. Since empirical expressions denote intensions[9] we get only a *criterion* of seeking the solution. The solution itself is no more achievable by means of logical procedures. Experience is necessary. We could define:

Let P be an *empirical problem.* The *Solution* of P is the object (if any) that is the value of P in the actual world-time.

Example. Consider the meaning of the sentence *The Moon is smaller than the Earth.* Assuming that the Moon as well as the Earth is an *individual role* (a function from worlds-times to individuals) and knowing that smaller than is an empirical relation we get the construction (concept)[10]

$$\lambda w \lambda t [{}^0\mathrm{Smaller}_{wt} {}^0\mathrm{Moon}_{wt} {}^0\mathrm{Earth}_{wt}]$$

and we can view it either as simply constructing the respective proposition or as an empirical problem whether it is true. □

From our definitions it follows that Solutions of empirical problems cannot be acquired by logical analysis alone.

Remark. In our workaday communication, when chatting, shopping etc. we, of course, do not view the used expressions as problems. This does not mean that we should somehow narrow down our definitions. □

5. Concepts and problems

CLAIM. *Every problem P can be formulated in such a way that the meaning of the F_P does not contain free variables.*

Example. The problem to *find all natural numbers x, y, z, n such that $x^n + y^n = z^n$* is the procedure $\lambda xyzn\, [{}^0{=}[{}^0{+}[{}^0\exp x\, n][{}^0\exp y\, n]][{}^0\exp z\, n]]$. We can formulate a similar problem as follows: *for any natural number n find the natural numbers x, y, z such that $x^n + y^n = z^n$.* Here n is a 'parameter' and x, y, z are 'unknowns'. We have

$$\lambda n \lambda xyz [{}^0{=}[{}^0{+}[{}^0\exp x\, n][{}^0\exp y\, n]][{}^0\exp z\, n]]. \qquad \square$$

[9] The TIL constructions that model the meaning of the given expression construct intensions in the case of empirical expressions. Typical form of such a construction is $\lambda w\, \lambda t\, X$, where w, t range over possible worlds and times, respectively, and X is a construction.

[10] Even the meaning of an empirical *sentence* is a concept (identifying the respective proposition), which is in harmony with Church's viewing concepts in his (1956).

PROOF. Let C be a construction containing n free variables. If $n = 1$ then the respective problem is $\lambda x C$. Otherwise more similar problems can be determined by C (cf. [Church 1956]). □

Example. Consider construction $[^0 > x\, y]$. The problems determined by this construction are $\lambda xy\, [^0 > x\, y]$, $\lambda yx\, [^0 > x\, y]$, $\lambda x \lambda y\, [^0 > x\, y]$, $\lambda y \lambda x\, [^0 > x\, y]$. □

Concepts in TIL are defined as abstract procedures not containing free variables: they are *closed constructions modulo α- and η-conversion.*[11]

Example. A concept of those concrete buildings in Prague that are older than 100 years can be—viz. if the following simple concepts are at our disposal: ^{0}Concrete, ^{0}Building, ^{0}Be_in, ^{0}Prague, ^{0}Age_of, $^0>$, ^{0}100—the construction (or its α- or η variant)

$$\lambda w \lambda t \lambda x\, [^0\wedge\, [[^0\mathrm{Concrete}\, ^0\mathrm{Building}]_{wt} x]$$
$$[[^0\wedge [^0\mathrm{Be\_in}_{wt} x\, ^0\mathrm{Prague}][^0 > [^0\mathrm{Age\_of}_{wt} x]\, ^0 100]]]$$
$$(\mathrm{Concrete}/((o\iota)_{\tau\omega}(o\iota)_{\tau\omega}), \mathrm{Building}/(o\iota)_{\tau\omega}, \mathrm{Be\_in}\, /\, (o\iota\iota)_{\tau\omega},$$
$$\mathrm{Prague}/\iota, \mathrm{Age\_of}/(\tau\iota)_{\tau\omega}, 100/\tau, >\, /(o\tau\tau))$$

The same concept can be viewed as the *problem of finding* those concrete building in Prague that are older than 100 years. This can be generalized: *Every concept can be viewed as a problem, and every problem is a concept.* □

6. Problems, intuitionism and partiality

Kolmogorov (in [1932]) interprets Heyting's calculus as a calculus of *problems (Aufgaben).*

Kolmogorov's idea in short: If the variables in the axioms and rules range over statements (*Aussagen*) then it holds that every proved statement has to be valid (*richtig*) while we do not possess a corresponding notion for problems. Reading the formulas of Heyting's calculus intuitionistically ("if the solution of a is given to solve b" in the case of $a \supset b$ etc.) the axioms mean that the solutions of the problems should be recognized as *necessary* Solutions. □

As we know the intuitionists build up their logic starting from an antirealistic philosophy while the classical logics are realistic. Our question is now:

Q *Can we have a procedural theory of problems such that it would not be necessarily connected with intuitionism?*

[11] So that, e.g., $\lambda xy\, [^0 > x\, y]$ represents the same concept as $\lambda yz\, [^0 > y\, z]$ or $^0>$.

To be able to answer this question let us consider three characteristic examples of classical rules / theorems not valid in intuitionistic logic.

(a) *Excluded middle*, i.e.,

$$\overline{A \vee \neg A}$$

(b) *De Morgan*:

$$\vdash \neg\forall x : \mathbf{N}.\varphi(x) \supset \exists x : N.\neg\varphi(x)$$

(c) *Double Negation*:

$$\vdash \neg\neg A \supset A$$

Ad (a) Let A be a problem (a procedure, a construction). Is it necessary to either solve A or derive an absurdity from the solution of A?

The intuitionist answers in a well-known way. Here we quote G. Sundholm [2000, p 7]:

> We consider, with Kronecker, a classical function $f \in \mathbf{N} \to \mathbf{N}$ that is defined by a non-decidable separation of cases:
>
> $$f(k) =_{\text{def}} \begin{cases} 1 & \text{if the Riemann Hypothesis is true} \\ 0 & \text{if the Riemann Hypothesis is false} \end{cases}$$
>
> According to Kronecker, and I agree, f is *not* well-defined, that is, the rule does not give a function from $\mathbf{N}$ to $\mathbf{N}$. [...]

Thus (a), accepted classically, is not accepted intuitionistically.

Ad (b) The intuitionistic answer is also clear:

If the assumption of solving $\varphi(x)$ for any x leads to contradiction then it does not mean that an x has been constructed for which solving $\varphi(x)$ would lead to contradiction.

Ad (c) Again: If the assumption that solving A would lead to contradiction led to contradiction then it does not mean that A is solved.

By contrast, if A and $\varphi(x)$ are interpreted classically, i.e., as denoting truth-values (so that they are statements, *Aussagen*, not problems, *Aufgaben*), we get a valid rule and tautologies.

With one exception, however: if truth-gaps are admitted, i.e., if our analysis uses partial functions. Then neither a) nor b) or c) does hold.

Indeed:

A let be the statement *the greatest prime is odd.* Since this statement is neither true nor false a) does not hold.

Further, let $\varphi(x)$ be *x is **the** natural number smaller than* 0. The expression *the natural number smaller than* 0 does not denote anything, so the respective procedure does not construct anything on any valuation and the class of the numbers satisfying $\varphi(x)$ is not the class of all natural numbers, thus the antecedent of b) is true. However the class of the numbers satisfying $\neg\varphi(x)$ is not non-empty so that the consequent of b) is false.

Finally, let A be truthless (as in a)). Then the antecedent as well as consequent of c) is truthless and c) does not hold either.

Now it could seem that intuitionists use simply a logic of partial functions. A following distinction can be stated.

For a *realistic partial logic* to be partial is independent of whether the given problem has been solved. The partiality of a function is 'absolute', so that, e.g., the function *the only x such that* (TIL type $\tau(o\tau)$) returns nothing if applied to an empty class, which is independent of such contingent facts as whether somebody did or did not solve a problem.

By contrast consider the problem G known as Goldbach conjecture. Let T be the time interval within which the problem has not been solved (we are still within T). For a realist there is no reason to say that this problem contains some partiality, and therefore (s)he believes that the law of excluded middle is applicable to it. Given that once this problem will get its solution, be it **T** or **F**, we must state that such an event is something extrinsic w.r.t. the problem. For an intuitionist something changed: (s)he claimed inadmissibility of using, e.g., existential quantifier in connection with G within T and now, after the solution has been achieved, existential quantifier is permitted. From the realistic viewpoint this is a strange situation; Kolmogorov says:

> So entsteht diese ganz besondere Art von Aussagen, welche zwar einen mit der Zeit nicht veränderlichen Inhalt haben sollen und doch nur unter speziellen Bedingungen ausgesprochen werden können.
>
> [1932, p. 64]

Let us now return to our question **Q**.

Partiality is not very popular in the logical community, but especially intuitionists are philosophically interested in refuting partial functions (see, e.g., Sundholm 2000, 11–12). Building up a procedural theory of problems in the spirit of the preceding text we are not bound to share this position with intuitionists. If procedures / problems are meanings of non-empirical expressions the partiality is given *a priori*. Once more: the contingent fact that the problem has been / has not yet been solved cannot influence acceptance or refusal of a rule / tautology. As for empirical problems, what is *a priori* is that the results of the respective procedures are non-trivial intensions, i.e., intensions whose values differ in at least two possible worlds. And we can expect (rightly, as we know) that some non-trivial (= non-constant) intensions will be in some possible worlds undefined. (For example all intensions whose type is $\alpha\tau_\omega$ for $\alpha \neq o$ or $(o\beta)$ or $(o\beta_1 \ldots \beta_m)$.)

First, what does it mean *to be necessarily connected with intuitionism*?

If the question concerns a system of formulas/propositions then the necessary connection with intuitionism means that two measures are taken:

a) The logical constants are intuitionistically reinterpreted (thus '$\neg$', '$\wedge$', '$\vee$' are no more truth-functional etc.).

b) Some intuitionistic calculus is accepted.

To accept a) means however to reinterpret formulas: '$A \wedge B$' is no more interpreted as conjunction of two claims; we no more ask whether A and B are both *true*: instead we ask whether they are *proved*. Here we have to emphasize that most philosophical problems around intuitionism concern the relation between truth and proof. As Fletcher [1998, pp. 73–74] sums up

> [T]here seem to be four views on this:
>
> (1) [...] once a formula has been proved then it *becomes* true, [...]
> (2) [...] when a formula is proved this shows that it was true all along, [...]
> (3) [...] an assertion 'A is true' [...] is simply an abbreviation for a certain assertion of the form 'P is a proof of A' [...]
> (4) [...] to drop the concept of truth altogether [...]

The option (2) is evidently one acceptable by a realist. But now it is not important which alternative we take sides with: what is of interest w.r.t. our question is what happens when we interpret (from the very beginning) 'formulas' as *problems* and accept our definition of problems (as concepts).

The non-empirical case:

We have seen that Kolmogorov interpreted rules and theorems of Heyting's calculus in terms of *solutions* of the problems represented by the particular formulas: these solutions should be recognized as *necessary*. This interpretation narrows, of course, the area of problems to the set of *decision problems*, otherwise no rational interpretation of logical connectives could take place. Also, only our Solution$_1$ comes into question: for there is obviously no *logical* possibility of foreseeing whether there is a solving reformulation of the given F_P. On these assumptions, and setting meanwhile aside the problem of partiality, what would mean to follow intuitionism and not accept, e.g., rule of Excluded Middle? Thereby we would admit that some decision problems are neither solvable nor unsolvable, or, as the case may be, neither provable nor disprovable. Since the intuitionistic notion of proof is (not definite enough but in any case) not restricted to a particular formal system this would imply that there are absolutely undecidable problems, which is only one side of the famous Gödel's dilemma. On the other hand, accepting (and intuitionistically interpreting) the rule of Excluded Middle would also be a premature answer to this dilemma.

Thus it looks like as if our theory of problems—if ever comparable with Heyting *et alii* in the Kolmogorov style—were neutral w.r.t. the choice between classical and intuitionistic logic. There is however another moment present, as already suggested: partiality. We have identified problems with concepts (as a kind of constructions), so we have to take into account cases like

$$[{}^0\mathrm{Odd}\ [{}^0\mathrm{The\_only\_}x\mathrm{\_such\_that}[{}^0\wedge[{}^0\mathrm{Prime}\,x][{}^0\forall\lambda y[{}^0\supset[{}^0\mathrm{Prime}\,y][{}^0\leqslant y\,x]]]]]].$$

Concepts like this one are improper constructions: Therefore we cannot accept the Rule of Excluded Middle.

Thus if a calculus of problems (capturing, of course, only a part of what a theory of problems has to say) should not accept some principles of classical logic and be so similar to some version of intuitionistic logic it would not be because of some anti-realistic version of intuitionistic philosophy: partiality would be the culprit.

As for *empirical problems*, true, the solutions of those problems are dependent on the instantaneous state of the world, but we can see that a logical treatment of empirical problems is possible in virtue of necessary relations between empirical problems. To illustrate this claim observe the following example. Considering the empirical problems

1. $\lambda w\lambda t[{}^0\exists\lambda x[{}^0\wedge[{}^0\mathrm{Married}_{wt}x][{}^0\mathrm{Brother}x[{}^0\mathrm{Father}_{wt}\,{}^0\mathrm{Charles}]]]]$
2. $\lambda w\lambda t[{}^0\exists\lambda x[{}^0\mathrm{Aunt}_{wt}x\,{}^0\mathrm{Charles}]]$

we can immediately see that the problem 2 is somehow a necessary consequence of the problem 1; it can be proved as soon as the particular empirical objects are replaced by definitional reductions based on some conceptual systems (for details see Materna 2004) and the resulting constructions are viewed as constructing propositions rather than being empirical problems. Thus there is no *principal* distinction between empirical and non-empirical problems in this respect: both can be logically handled and differ from classical case only due to taking into account partiality.

Acknowledgments. The present paper has been supported by the Grant Agency of Czech Republic, project No 401/07/0451.

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