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A MODAL APPROACH TO DYNAMIC ONTOLOGY: MODAL MEREOTOPOLOGY

Abstract. In this paper we show how modal logic can be applied in the axiomatizations of some dynamic ontologies. As an example we consider the case of mereotopology, which is an extension of mereology with some relations of topological nature like *contact relation*. We show that in the modal extension of mereotopology we may define some new mereological and mereotopological relations with dynamic nature like *stable part-of* and *stable contact*. In some sense such "stable" relations can be considered as approximations of the "essential relations" in the domain of mereotopology.

Keywords: Ontology, dynamic ontology, mereology, mereotopology, modal logic, essential relations.

Introduction

The difference between static and dynamic ontology is that the later depends on some changing parameters—time, space, environment, etc, which we will consider globally as *situations*. Good examples of static and dynamic ontologies are the *statics* and the *dynamics* from classical physics. There is no unique way in general to make a given static ontology a dynamic one: it depends of what we want to talk about changing objects. For instance in classical dynamics objects are moving in space and time, have a trajectory, velocity, acceleration, and many other characteristics describing moving objects which are not considered in statics. But in some simple cases, when we consider only one changing parameter, say time, one can find a more uniform

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way to make static ontology a dynamic one: just by a suitable combination of the language of the given static ontology with the language of a given temporal logic. Examples of combining spatial ontology and temporal logic are given in [9]. If we consider the vector of all parameters as a situation, then a suitable logic of situations is for instance the modal logic S5. Then we may "dynamise" a given static ontology, combining it in a suitable way with the modal logic S5. Carnap's semantics of S5 considers a non-empty universe U of situations, and each sentence changes its truth from situation to situation. Then the formula $\Box A$ ("necessarily A") is true at a given situation $s \in U$ if A is true in all situations in U. Dually $\Diamond A$ ("possibly A") is true at a given situation $s \in U$ if there is a situation $s' \in U$ in which A is true. Let for instance $P(.)$ be a predicate from a given ontology O and o is a given object from O. If O is a static ontology then either o has the property P or o does not have the property P. But if O depends on different situations, then $P(o)$ may vary its truth from situation to situation. Then the truth of $\Box P(o)$ will mean that $P(o)$ is true in all situations. In some sense this says that P is a "stable" or "essential" property of o. Then $\Diamond\neg P(o)$ will say that P is not a stable or essential property of o. Similar "essentialists" interpretation can be done not only for properties but also for relations. However, there is a long discussion between ontologists whether $\Box P(o)$ says that P is essential property for o. Against this modal interpretation of essence is for instance Kit Fine [7], but we will not go into this discussion. In some sense Fine is right but we may take $\Box P(o)$ as a simple approximation of the statement "P is an essential property of o". Practically we may conclude that P is an essential property of o if we see that for all situations (accessible for us) $P(o)$ is true. Thus, combining a given static ontology O with the modal logic S5, one can obtain a "dynamic version" of O. The extended language can be used to characterize some properties or relations of O as stable (or "essential").

In this paper we will apply this method to combine mereotopology with the modal logic S5 and to obtain in this way a dynamic version of mereotopology which we will call modal mereotopology.

Mereotopology is an extension of mereology with some relations of topological nature. Mereology is an ontologycal discipline which can be characterized shortly as a theory of "Parts and Wholes". Typical in mereology are the relations "part-of" and "overlap". One of the basic mereological systems is Leśniewski's mereology. Its original presentation given by Leśniewski is quite difficult to understand, but as Tarski showed, the mathematical equivalent of mereology are complete Boolean algebras with deleted zero element 0 (see Simons [11] for this fact). However, assuming 0 makes the theory more simple. In Boolean formulation part-off relation coincides with the Boolean ordering $x \leq y$, and the overlap relation $x \mathsf{O} y$ can be defined by $x \cdot y \neq 0$ (where '$\cdot$' is the Boolean multiplication). We will not go into details of mereology and refer for this the book by Simons [11].

Mereology, however, is not capable to describe some relations between individuals as, for instance, one individual to be in a contact with another one. Adding contact-like relations goes back to de Laguna [4] and Whitehead [12]. The intension of de-Laguna and Whitehead was to use mereology for building of a new, pointfree theory of space as an extension of mereology, with the relation of contact (or "connection" in Whitehead terminology). The primitive objects of the new theory of space are called regions and it is "pointfree", because points are not taken as primitives but are definable by means of regions, contact and some mereological relations. As Tarski showed (see Simons [11]) standard point models of the new theory of space are regular open (or regular closed) sets of some topological spaces with topological definition of contact. For regular open sets the contact C has the following definition: $a \mathsf{C} b$ iff $\mathrm{Cl}(a) \cap \mathrm{Cl}(b) \neq \varnothing$ (here Cl is the topological closure operation). This motivates some authors to call the extension of mereology with the contact relation (or some of its derivatives) "mereotopology". Mereotopology is often called also a "region-based theory of space". Since mereology is identified with the theory of Boolean algebras, mereotopology can be identified with the extensions of the language of Boolean algebras with the contact relation C called *contact algebras*. Recent papers on contact algebras are Duentsch and Winter [6] and Dimov and Vakarelov [5]. Survey articles on region-based theory of space are Bennett and Duentsch [3], Pratt [10] and Vakarelov [13].

Logics related to mereotopology, or, region-based theory of space, are presented, for instance, in the papers by Wolter and Zakharyaschev [14] and Balbiani, Tinchev and Vakarelov [2]. In this paper we will combine quantifier-free (propositional) logics related to static mereotopology with the modal logic S5. The preference to use quantifier-free versions of logics is to obtain simpler and possibly decidable systems. They contain the relations $x \leq y$ (x is a part of y) and $x \mathsf{C} y$ (x is in a contact with y) and some of their derivatives like "nontangential part-of", "tangential contact", etc (the formal definitions will be given in the main text). Having in the language the modal operator $\Box$ we may express the relations "x is an essential part of y" by $\Box(x \leq y)$ and "x is in an essential contact with y" by $\Box(x \mathsf{C} y)$, etc. For instance "the head is an essential part of the human body", while "the hands are not essential parts of the human body".

The rest of the parer is structured as follows. Section 1 lists some preliminary information. In Section 1.1 we survey some facts for contact algebras and their topological representation theory, following mainly [5]. Contact algebras here correspond to "static mereotopology". In Section 1.2, following [2], we describe some propositional logics related to contact algebras which we call in this paper contact logics. Some of these logics are equivalent to the logics introduced semantically in [14]. In Section 2 we combine the contact logics from Section 1 with the modal logic S5. The resulting systems are called modal contact logics. The obtained systems correspond to modal mereotopology, which is a certain form of dynamic mereotopology. The main results of this section are several completeness theorems of the new logics. In section 3 we show that the new logics possess finite model property which implies that they are decidable. In the concluding section 4 we show possible extensions of the studied logics with new rules of inference and formulate some open problems.

1. Preliminaries

1.1. Contact algebras

DEFINITION 1.1. Following [5], by a *Contact Algebra* we will mean any system $\mathfrak{B} = (B, 0, 1, \cdot, +, ^*, \mathsf{C})$, where $(B, 0, 1, \cdot, +, ^*)$ is a non-degenerate Boolean algebra with a complement denoted by '*' and C – a binary relation in B, called *contact* and satisfying the following axioms:

(C1) $$x \,\mathsf{C}\, y \to x \neq 0,$$

(C2) $$x \,\mathsf{C}\, y \to y \,\mathsf{C}\, x,$$

(C3) $$x \,\mathsf{C}\, (y + z) \leftrightarrow x \,\mathsf{C}\, y \text{ or } x \,\mathsf{C}\, z,$$

(C4) $$x \cdot y \neq 0 \to x \,\mathsf{C}\, y.$$

We say that $\mathfrak{B}$ is connected if it satisfies the following axiom of connectedness

(Con) $$a \neq 0, 1 \to a \,\mathsf{C}\, a^*.$$

We write $\overline{\mathsf{C}}$ for the complement of C. Obviously, if $\mathfrak{B}'$ is a Boolean subalgebra of $\mathfrak{B}$, then it is also a contact subalgebra of $\mathfrak{B}$. If $\mathfrak{B}'$ as a Boolean algebra is generated by some subset $B_0 \subseteq B$ we say also that $\mathfrak{B}'$ as a contact subalgebra is generated by B_0.

Examples of contact algebras: the contact algebra of regular closed sets. Let X be an arbitrary topological space. We denote by $\mathrm{Cl}(a)$ and $\mathrm{Int}(a)$ the closure and interior of a subset a of X. A subset a of X is *regular closed* if $a = \mathrm{Cl}(\mathrm{Int}(a))$. The set of all regular closed subsets of X will be denoted by $\mathrm{RC}(X)$. It is a well-known fact that regular closed sets with the operations

$$a + b := a \cup b, \qquad a \cdot b := \mathrm{Cl}(\mathrm{Int}(a \cap b)),$$
$$a^* := \mathrm{Cl}(X \setminus a), \qquad 0 := \varnothing, \qquad 1 := X\,.$$

form a Boolean algebra. If we define contact C_X by: $a\,\mathsf{C}_X\,b$ iff $a \cap b \neq \varnothing$ then we have the following fact.

Lemma 1.2. $\mathrm{RC}(X)$ *with the contact* C_X *is a contact algebra. If* X *is a connected space then* $\mathrm{RC}(X)$ *is connected.*

The contact algebra of the above example is said to be *standard contact algebra of regular closed sets.*

The topological space X is called *semiregular* if it has a closed base of regular closed sets.

Theorem 1.3 (Representation theorem for contact algebras [5, 13]). *For each contact algebra* $\mathfrak{B} = (B, \mathsf{C})$ *there exists a compact semiregular* T_0 *space* X *and an embedding* h *of* $\mathfrak{B}$ *into the contact algebra* $\mathrm{RC}(X)$*. Moreover, if* $\mathfrak{B}$ *is connected then* X *is a connected space.*

In the above theorem h satisfies also some additional nice conditions, but in this paper we will not use them.

1.2. Logics related to contact algebras

Following [2] and [13] we present in this section a language for some propositional, quantifier-free logics related to contact algebras, which in this paper are called *contact logics.* We present two kinds of semantics for the contact logics: *algebraic semantics* based on contact algebras and *topological semantics* based on contact algebras of some classes of topological spaces.

1.2.1. The language of contact logics

The *language* $\mathbf{L}(\leq, \mathsf{C})$ of contact logics consists of

- a denumerable set Var of *Boolean variables,*
- *Boolean operations*: $\cdot$ (Boolean meet), $+$ (Boolean join), * (Boolean complement), and 0, 1 (Boolean constants),
- *propositional connectives*: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and propositional constants $\top$ and $\bot$,
- *relational connectives*: $\leq$ (part-of) and C (contact).

The set of *Boolean terms* **B** is defined in a standard way: from Boolean variables and Boolean constants by means of Boolean operations.

Atomic formulas are formulas of the form '$a \leq b$' and '$a \, \mathsf{C} \, b$', where a and b are Boolean terms.

Complex formulas (or simply *formulas*) are defined in a standard way from atomic formulas and propositional constants $\bot$ and $\top$ by means of propositional connectives.

Abbreviations:

$$
\begin{aligned}
a = b &:= (a \leq b) \wedge (b \leq a), \\
a \neq b &:= \neg(a = b), \\
a \overline{\mathsf{C}} b &:= \neg(a \, \mathsf{C} \, b), \\
a \, \mathsf{O} \, b &:= a \cdot b \neq 0\,, \qquad \text{overlap}, \\
a \ll b &:= a \overline{\mathsf{C}} b^*\,, \qquad \text{nontangential part-of}, \\
a \, \mathsf{C}_{\mathrm{t}} \, b &:= a \, \mathsf{C} \, b \wedge \neg(a \, \mathsf{O} \, b)\,. \qquad \text{tangential contact}
\end{aligned}
$$

Substitution. Let α be a Boolean term or a formula, and let $p_1, \ldots, p_n$ be a list of different Boolean variables. We write $\alpha(p_1,..., p_n)$ to indicate that some of $p_1, \ldots, p_n$ occur in α.

If $b_1, \ldots, b_n$ are Boolean terms, then $\alpha(b_1, ..., b_n)$ or, more precisely, $\alpha(p_1/b_1, ..., p_n/b_n)$ means the simultaneous substitution of $b_1, \ldots, b_n$ for $p_1, \ldots, p_n$. The formula $\alpha(b_1, ..., b_n)$ is called a *substitutional instance of* α. If we consider $p_1, \ldots, p_n$ as meta variables for Boolean terms, then $\alpha(p_1, \ldots, p_n)$ is called a "schema". Schemes are usually understood as schemes of axioms of some axiomatic systems.

Let $\alpha = \alpha(q_1, ..., q_n)$ be a formula of the propositional calculus built up by different propositional variables $q_1, \ldots, q_n$ and the propositional connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\bot$ and $\top$, and let $\alpha_1, \ldots, \alpha_n$ be formulas of $\mathbf{L}(\leq, \mathrm{C})$. Then $\alpha(\alpha_1, ..., \alpha_n)$ or, more precisely, $\alpha(q_1/\alpha_1, ..., q_n/\alpha_n)$ is called a *substitutional instance of the propositional formula* α.

1.2.2. Semantics

First of all, we introduce an algebraic semantics of the language $\mathbf{L}(\leq, \mathrm{C})$. Let $\mathfrak{B} = (B, 0, 1, \cdot, +, ^*, C)$ be a contact algebra. A mapping v from Var into B is called a *valuation*. A pair $M = (\mathfrak{B}, v)$, where $\mathfrak{B}$ is a contact algebra and v is a valuation in B, is called an *algebraic model* or an *interpretation* in $\mathfrak{B}$.

Given a model $(\mathfrak{B}, v)$, the valuation v is extended to arbitrary Boolean terms by induction in a standard way: $v(a \cdot b) = v(a) \cdot v(b)$, $v(a+b) = v(a) + v(b)$, $v(a^*) = v(a)^*$, $v(0) = 0$, and $v(1) = 1$.

We define v also for formulas and then the value of $v(\alpha)$ belongs to the set $\{\mathbf{0}, \mathbf{1}\}$, considered as standard truth values false and truth. The definition goes inductively as follows:

$v(a \leq b) = \mathbf{1}$ iff $v(a) \leq v(b)$,
$v(a \mathrel{\mathsf{C}} b) = \mathbf{1}$ iff $v(a) \mathrel{\mathsf{C}} v(b)$,
$v(\neg\alpha) = \mathbf{1}$ iff $v(\alpha) = \mathbf{0}$,
$v(\alpha \vee \beta) = \mathbf{1}$ iff $v(\alpha) = \mathbf{1}$ or $v(\beta) = \mathbf{1}$,
$v(\alpha \wedge \beta) = \mathbf{1}$ iff $v(\alpha) = \mathbf{1}$ and $v(\beta) = \mathbf{1}$.

We say that α is true in the model $M = (\mathfrak{B}, v)$ if $v(\alpha) = \mathbf{1}$ and in this case we say that M is a model of α. We say that α is *true in a contact algebra* $\mathfrak{B}$ if α is true in all interpretations in $\mathfrak{B}$. If Σ is a class of contact algebras, α is said to be *true in* Σ if α is true in all members of Σ. The set of all formulas true in Σ is called the *logic of* Σ and is denoted by $\mathcal{L}(\Sigma)$. This is a semantic definition of logic. If $\Sigma_1 \subseteq \Sigma_2$ then obviously $\mathcal{L}(\Sigma_2) \subseteq \mathcal{L}(\Sigma_1)$. This implies that the logic $\mathcal{L}(\Sigma)$ where Σ is the class of all contact algebras is the smallest contact logic and thus will be denoted by $\mathbb{CL}_{\mathrm{min}}$. The logic corresponding to the class of of all connected contact algebras will be denoted by $\mathbb{LC}_{\mathrm{Con}}$.

Let $\mathcal{T}$ be a class of topological spaces. We consider the class $\Sigma(\mathcal{T}) = \{\mathrm{RC}(X) : X \in \mathcal{T}\}$. The *topological semantics* of $\mathbf{L}(\leq, \mathsf{C})$ in $\mathcal{T}$ consists of interpretations in contact algebras $\mathrm{RC}(X) \in \Sigma(\mathcal{T})$. A pair (X, v), where X is a topological space and v is a valuation in $\mathrm{RC}(X)$ is referred to as a *topological model* or a *topological interpretation*. If α is true in $\mathrm{RC}(X)$, we write "α is true in X" for brevity.

1.2.3. Finite models

The following lemma will be of later use:

Lemma 1.4. *Let α be a formula and $\{b_1, \ldots, b_n\}$ be the set of Boolean variables occurring in α. Let $\mathfrak{B}$ be a contact algebra (connected contact algebra) and v be a valuation in $\mathfrak{B}$. Let $\mathfrak{B}' = \mathfrak{B}(v(b_1), \ldots, v(b_n))$ be the (finite) contact sub-algebra generated by the elements $v(b_1), \ldots, v(b_n)$ and let v' be the restriction of v in $\mathfrak{B}'$. Then the following equality is true: $v(\alpha) = v'(\alpha)$.*

Proof. Since the axioms of contact algebra and connected contact algebra are universal first-order sentences, then the proof follows from the remarks that (1) any Boolean subalgebra of $\mathfrak{B}$ is also a contact subalgebra of the same type, and that (2) Boolean subalgebras generated by finite sets of elements are finite. $\dashv$

The above lemma implies that the logics of all contact algebras and all connected contact algebras have finite model property and are decidable.

1.2.4. Axiomatizations and completeness theorems

Axiomatization of $\mathbb{CL}_{\min}$ and $\mathbb{CL}_{\mathrm{Con}}$. We first introduce the axiomatic system for the minimal logic $\mathbb{CL}_{\min}$. It is a Hilbert-type axiomatic system consisting of axioms and inference rules.

Axioms of $\mathbb{CL}_{\min}$:

I. All formulas of $\mathbf{L}(\leq, \mathrm{C})$ which are substitutional instances of the complete set of axiom schemes of classical propositional logic (or all its tautologies).

II. The set of axiom schemes for Boolean algebra in terms of the part-of $\leq$ (a, b, and c are arbitrary Boolean terms):

$a \leq a$, $a \leq b \wedge b \leq c \Rightarrow a \leq c$, $0 \leq a$, $a \leq 1$,

$c \leq a \cdot b \iff c \leq a \wedge c \leq b$,

$a \cdot (b + c) \leq (a \cdot b) + (a \cdot c)$,

$c \cdot a \leq 0 \Leftrightarrow c \leq a^*$, $a^{**} \leq a$.

III. The set of axiom schemes for the contact C (a, b, and c are arbitrary Boolean terms):

(C1) $\quad a \,\mathrm{C}\, b \Rightarrow a \neq 0$,

(C2) $\quad a \,\mathrm{C}\, b \Rightarrow b \,\mathrm{C}\, a$,

(C3) $\quad a \,\mathrm{C}\, (b + c) \Leftrightarrow a \,\mathrm{C}\, b \vee a \,\mathrm{C}\, c$,

(C4) $\quad a \cdot b \neq 0 \Rightarrow a \,\mathrm{C}\, b$.

Inference rule: *Modus ponens* (MP): $\dfrac{\alpha, \alpha \Rightarrow \beta}{\beta}$

If we add to the above axioms the axiom

(Con) $\quad a \neq 0 \wedge a \neq 1 \Rightarrow a \,\mathrm{C}\, a^*$,

then we obtain the axiomatization of the logic $\mathbb{CL}_{\mathrm{Con}}$. The obtained axiomatic systems will also be denoted correspondingly by $\mathbb{CL}_{\min}$ and $\mathbb{CL}_{\mathrm{Con}}$.

The notion of a *proof* in $\mathbb{CL}_{\text{min}}$ and $\mathbb{CL}_{\text{Con}}$ is a standard one. All provable formulas are called *theorems* of the corresponding logic. It is easy to see that the set of theorems is closed under the *substitution rule*:

> if $\alpha(p_1, ..., p_n)$ is a theorem and $p_1, \ldots, p_n$ is a sequence of different Boolean variables, then for any Boolean terms $b_1, \ldots, b_n$, the formula $\alpha(b_1, ..., b_n)$ is also a theorem.

1.2.5. Canonical models and completeness theorems

In this section, following mainly [2] and [13] we will introduce a canonical-model construction, which is the main tool in the proof of the completeness theorem for $\mathbb{CL}_{\text{min}}$ and $\mathbb{CL}_{\text{Con}}$. We list some definitions and facts about this construction, because they will be used and modified later for the modal extension of contact logics.

Let $\mathbb{L}$ be any of the logics $\mathbb{CL}_{\text{min}}$ and $\mathbb{CL}_{\text{Con}}$. A set Γ of formulas is called an $\mathbb{L}$-*theory* or simply a theory if it contains all theorems of $\mathbb{L}$ and is closed under the rule

(MP) if A and $A \Rightarrow B$ are in Γ, then B in Γ.

For example, the set of all theorems of $\mathbb{L}$ is a theory; moreover, it is the smallest theory. A theory Γ is said to be *consistent* if $\bot \notin \Gamma$ and *maximal* if it is consistent and maximal with respect to the set inclusion. Maximal theories are also referred to as *maximal consistent sets*.

Some well-known properties of theories are listed in the following lemma.

Lemma 1.5. *The following assertions hold.*

(i) *Let Γ be a theory, and let α be a formula. Then the set $\Gamma + \alpha = \{\beta : \alpha \Rightarrow \beta \in \Gamma\}$ is the smallest theory containing Γ and α. The set $\Gamma + \alpha$ is inconsistent if and only if $\neg\alpha \in \Gamma$.*

(ii) *The following conditions are equivalent for any theory Γ:*
- *Γ is maximal,*
- *for any formula α, $\neg\alpha \in \Gamma$ if and only if $\alpha \notin \Gamma$,*
- *for any formulas α and β, $\alpha \vee \beta \in \Gamma$ if and only if $\alpha \in \Gamma$ or $\beta \in \Gamma$.*

(iii) *Any consistent theory can be extended to a maximal theory (the Lindenbaum Lemma).*

The following assertion presents a semantical construction of maximal theories.

LEMMA 1.6. *Let $\mathcal{M}$ be a model. Then the set of formulas $\Gamma = \{\alpha : \mathcal{M}$ is a model of $\alpha\}$ is a maximal $\mathbb{L}_{\min}$-theory. If $\mathcal{M}$ is a model over connected contact algebra, then Γ is a maximal $\mathbb{L}_{Cont}$-theory.*

A set of formulas A is *consistent* in $\mathbb{L}$ if A is contained in an $\mathbb{L}$-consistent theory and, consequently, A is contained in a maximal $\mathbb{L}$-theory in view of the Lindenbaum Lemma.

Let S be a maximal theory in $\mathbb{L}$. Based on the Lindenbaum-algebra construction, we construct in a canonical way a contact algebra associated with S. In the set of Boolean terms $\mathbf{B}$, we introduce the *equivalence relation*: $a \equiv_S b$ if and only if $a = b \in S$. Since $\equiv_S$ is a congruence relation depending on S, it is possible to consider equivalence classes of Boolean terms $|a|_S = \{b : a \equiv_S b\}$ and to define the canonical contact algebra $\mathfrak{B}_S$ over S by setting $|a|_S \cdot |b|_S = |a \cdot b|_S$, $|a|_S + |b|_S = |a + b|_S$, $|a|_S^* = |a^*|_S$, $|a|_S \leq |b|_S$ if and only if $a \leq b \in S$, and $|a|_S C |b|_S$ if and only if $aCb \in S$.

Using the axioms of logic, we can prove that $\mathfrak{B}_S$ is a contact algebra and, if $\mathbb{L}$ contains the axiom (Con) then $\mathfrak{B}_S$ is a connected contact algebra.

We define a canonical valuation for Boolean variables putting $v_S(p) = |p|$. Then the pair $M_S = (\mathfrak{B}_S, v_S)$ is called a *canonical model over* S. We have $v_S(a) = |a|_S$ for any Boolean term a.

The following assertion is proved in a standard way.

LEMMA 1.7. *The following two conditions are equivalent for any formula α in the logic $\mathbb{L}$:*

- *α is a theorem of $\mathbb{L}$,*
- *α is true in all canonical models M_S of $\mathbb{L}$.*

Now, we can state a completeness theorem for the logics $\mathbb{CL}_{\min}$ and $\mathbb{CL}_{\text{Con}}$.

THEOREM 1.8 (Completeness of $\mathbb{CL}_{\min}$ [2]). *The following conditions are equivalent for any formula α:*

- *α is a theorem of $\mathbb{CL}_{\min}$,*
- *α is true in all contact algebras,*
- *α is true in all topological spaces.*

THEOREM 1.9 (Completeness of $\mathbb{CL}_{\text{Con}}$ [2]). *The following conditions are equivalent for any formula α:*

- *α is a theorem of $\mathbb{CL}_{\mathrm{Con}}$,*
- *α is true in all connected contact algebras,*
- *α is true in all connected topological spaces.*

Theorems 1.8 and 1.9 present weak completeness statements. The strong statements are also valid (cf. [2]). For instance, for the logic $\mathbb{CL}_{\mathrm{Con}}$ it can be formulated as follows.

Theorem 1.10 (Strong completeness of $\mathbb{CL}_{\mathrm{Con}}$). *The following conditions are equivalent for any set A of formulas*:

- *A is consistent in $\mathbb{CL}_{\mathrm{Con}}$,*
- *A has an algebraic model,*
- *A has a topological model.*

2. Modal contact logics

We extend the language $\mathbf{L}(\leq, \mathrm{C})$ to the language $\mathbf{L}(\leq, \mathrm{C}, \Box)$ with the modal operation $\Box$ of necessity and introduce $\Diamond$ (possibility) by definition: $\Diamond = \neg\Box\neg$. Now if α is a formula then $\Box\alpha$ and $\Diamond\alpha$ are also formulas. The semantics of $\mathbf{L}(\leq, \mathrm{C}, \Box)$ will be a combination of the algebraic and topological semantics of $\mathbf{L}(\mathrm{C}, \leq)$ with the Carnap's semantics for S5. We want the combination to be quantifier-free in order to obtain simpler and decidable systems. Normally first-order S5 with equality (see [8]) uses constant domain semantics in which the standard axioms of equality are true: these are the axiom (Refl) $x = x$ (Reflexivity) and (Sub) $(x = y) \wedge \varphi[x/z] \Rightarrow \varphi[y/z]$ (substitutivity). These axioms do not contain quantifiers and if we accept constant-domain semantics, we must accept also these axioms. But in combination with S5 axioms, they have some unacceptable consequences, for instance $(x = y) \Leftrightarrow \Box(x = y)$. This formula implies the formula $(x \leq y) \Leftrightarrow \Box(x \leq y)$ which is highly unacceptable, because it says that "x is a part of y iff x is a stable (essential) part of y". So we will use a kind of varying domain semantics.

2.1. Semantics

Let W be a nonempty set and for each $x \in W$ let $\mathfrak{B}(x)$ be a contact algebra. Then the pair $S = (W, \{\mathfrak{B}(x) : x \in W\})$ is called *algebraic modal structure* or shortly a *frame*. If all contact algebras in S are in the form $\mathrm{RC}(X)(x) : x \in$

W for a given topological space $X(x)$, then the frame is called *topological*. The algebras $\mathfrak{B}(x)$ of S are called the contact algebras of S. A frame is connected if the algebras of the frame are connected.

A two-variable function $v(x, b)$ with first argument $x \in W$ and second argument b is a Boolean variable, is called a valuation if for each $x \in W$ and $b \in \mathrm{Var}$, $v(x, b)$ is in $B(x)$. The pair $M = (S, v)$ is called a model.

Given a frame S and a valuation v, we extend v for arbitrary Boolean terms inductively in a standard way: $v(x, a^*) = v(x, a)^*$, $v(x, a \cdot b) = v(x, a) \cdot v(x, b)$, $v(x, a + b) = v(x, a) + v(x, b)$, $v(x, 1) = 1$ and $v(x, 0) = 0$, where the operations $*$, $\cdot$ and $+$ are taken in the contact algebra $\mathfrak{B}(x)$. Further on this will always be assumed. We define v also when the second argument is a formula and the value in this case belongs to the set $\{\mathbf{0}, \mathbf{1}\}$. The definition is by induction as follows:

$$\begin{aligned}
v(x, a \leq b) = \mathbf{1} &\quad \text{iff} \quad v(x, a) \leq v(x, b),\\
v(x, a \mathrel{\mathsf{C}} b) = \mathbf{1} &\quad \text{iff} \quad v(x, a) \mathrel{\mathsf{C}} v(x, b),\\
v(x, \neg\alpha) = \mathbf{1} &\quad \text{iff} \quad v(x, \alpha) = \mathbf{0},\\
v(x, \alpha \wedge \beta) = \mathbf{1} &\quad \text{iff} \quad v(x, \alpha) = \mathbf{1} \text{ and } v(x, \beta) = \mathbf{1},\\
v(x, \alpha \vee \beta) = \mathbf{1} &\quad \text{iff} \quad v(x, \alpha) = \mathbf{1} \text{ or } v(x, \beta) = \mathbf{1},\\
v(x, \Box\alpha) = \mathbf{1} &\quad \text{iff} \quad (\forall y \in W)\, v(y, \alpha) = \mathbf{1}.
\end{aligned}$$

We say that a formula α is true in a model $M = (S, v)$, or that M is a model of α if for all $x \in W$ we have $v(x, \alpha) = \mathbf{1}$; α is true (or valid) in the frame S if it is true in all models over S; α is true in a class of frames Σ if it is true in each frame from Σ. The set $\mathcal{L}(\Sigma)$ of all formulas true in Σ is called a modal contact logic, or simply a logic, of Σ. We denote by $\mathbb{MLC}_{\min}$ the logic of all frames and by $\mathbb{MLC}_{\mathrm{Con}}$ – the logic of all connected frames.

Let a be a Boolean term, considered as a *region*, $S = (W, \{\mathfrak{B}(x) : x \in W\})$ be a frame and v be a valuation. The elements of W will be considered as *situations*. Then $v(x, a)$ will be considered as *the region a at the situation x*. Changing in $v(x, a)$ the parameter x denotes the changes of the region a from one situation to another in the space of situations W. The truth conditions have the following intuitive meaning. For instance $v(x, a \leq b) = \mathbf{1}$ means that the region a is a part of the region b at the situation x. The condition $v(x, a \leq b) = \mathbf{1}$ for all $x \in W$ means that a is a part of b in all situations in W, which says that in this model a is a stable (or in some sense "essential") part of b. The same can be expressed by the truth of the formula $\Box(a \leq b)$ at any $x \in W$. So, indeed $\Box(a \leq b)$ expresses stable (essential) part-of relation. Similar essentialists interpretation can be done for the other mereological or mereotopological relations expressible in our language, for instance $\Box(a \mathrm{C}\, b)$.

2.2. Axiomatizations

In this section we propose a Hilbert style axiomatization of $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$.

Axiom schemes for $\mathbb{MCL}_{\mathrm{min}}$:

I. All formulas of $\mathbf{L}(\leq, C, \Box)$ which are substitution instances of the complete set of axioms of classical propositional logic (or all its tautologies).

II. Axiom schemes for the propositional modal logic S5: $\Box(\alpha \Rightarrow \beta) \Rightarrow (\Box\alpha \Rightarrow \Box\beta)$, $\Box\alpha \Rightarrow \alpha$, $\Box\alpha \Rightarrow \Box\Box\alpha$, $\Diamond\Box\alpha \Rightarrow \alpha$.

III. The set of axiom schemes for Boolean algebra and contact C as in the logic $\mathbb{CL}_{\mathrm{min}}$

Inference rules:

Modus ponens (MP): $\dfrac{\alpha, \alpha \Rightarrow \beta}{\beta}$,

Necessitation (N): $\dfrac{\alpha}{\Box\alpha}$

The above formal system is denoted also by $\mathbb{MCL}_{\mathrm{min}}$. If we add to $\mathbb{MCL}_{\mathrm{min}}$ the axiom (Con) for connectedness we obtain the axiom system for the logic $\mathbb{MCL}_{\mathrm{Con}}$.

It is easy to check that the following lemma is true.

Lemma 2.11 (Soundness of $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$). *The following conditions are true for any formula* α:

(i) *If α is a theorem of $\mathbb{MCL}_{\mathrm{min}}$ then α is true in all frames.*

(ii) *If α is a theorem of $\mathbb{MCL}_{\mathrm{Con}}$ then α is true in all connected frames.*

2.3. Canonical models and completeness theorems

Let $\mathbb{L}$ be any of the logics $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$. The canonical construction for $\mathbb{L}$ which we will follow is similar to that for the non-modal contact logics from Section 2. There to each maximal theory S is associated canonically a contact algebra. The new thing now is that to each maximal theory S we will associate a class of maximal theories W_S (this will give us the set W of the model) and to construct canonically a contact algebra $\mathfrak{B}_S(\Gamma)$, corresponding to each maximal theory $\Gamma \in W_S$.

Let $W_{\mathbb{L}}$ be the set of all maximal theories of $\mathbb{L}$. Let for a set of formulas Γ define $\Box\Gamma = \{\alpha : \Box\alpha \in \Gamma\}$. Define for $\Gamma, \Delta \in W_{\mathbb{L}}$, $\Gamma R \Delta$ iff $\Box\Gamma \subseteq \Delta$.

In the following lemma we list without proof some standard facts for the operation $\Box$ and the relation R.

LEMMA 2.12. (i) *If Γ is a theory then $\Box\Gamma$ is a theory too.*

(ii) *R is an equivalence relation in $W_{\mathbb{L}}$.*

(iii) *For any $\Gamma \in W_{\mathbb{L}}$ and for any formula α we have:*

$$\Box\alpha \in \Gamma \textit{ iff } (\forall \Delta \in W_{\mathbb{L}})(\Gamma R \Delta \rightarrow \alpha \in \Delta).$$

Now for each maximal theory S we define a canonical model associated with S denoted by $M_S = (W_S, \{\mathfrak{B}_S(\Gamma) : \Gamma \in W_S\}, v_S)$.

The definition of W_S. We put $W_S = \{\Gamma \in W_{\mathbb{L}} : \Gamma R S\}$. Since R is an equivalence relation, then W_S is just an R-equivalence class generated by S. We have the following

LEMMA 2.13. (i) *If $\Gamma, \Delta \in W_S$ then $\Gamma R \Delta$.*

(ii) *If $\Gamma \in W_S$ then for any formula α we have:*

$$\Box\alpha \in \Gamma \textit{ iff } (\forall \Delta \in W_S)\ \alpha \in \Delta).$$

PROOF. Condition (i) follows from the fact that R is an equivalence relation and that W_S is an R-equivalence class. Condition (ii) follows from (i) and Lemma 2.12(iii). $\dashv$

The definition of $\mathfrak{B}_S(\Gamma)$. Let $\Gamma \in W_S$. The construction of $\mathfrak{B}_S(\Gamma)$ repeats the construction of the canonical contact algebra $\mathfrak{B}_S$ from Section 1.2 – instead of an arbitrary maximal theory S just take Γ from W_S. The elements of $B_S(\Gamma)$ are the equivalence classes of the form $|a|_\Gamma$. If $\mathbb{L} = \mathbb{MCL}_{\mathrm{Con}}$, then $\mathfrak{B}_S(\Gamma)$ is connected.

The definition of $v_S(\Gamma, a)$. Let $\Gamma \in W_S$ and a be a Boolean variable. Define $v_S(\Gamma, a) := |a|_\Gamma$.

LEMMA 2.14 (Truth Lemma). *Let $M_S = (W_S, \{\mathfrak{B}_S(\Gamma) : \Gamma \in W_S\}, v_S)$ be a canonical model of $\mathbb{L}$. Then the following equivalence is true for any formula α and $\Gamma \in W_S$:*

$$v_S(\Gamma, \alpha) = \mathbf{1} \textit{ iff } \alpha \in \Gamma.$$

Proof. We proceed by induction on the complexity of α. The interesting cases are: $\alpha = a \leq b$, $a \, \mathsf{C} \, b$, $\Box\beta$.

The case $\alpha = a \leq b$. We have the following chain of equivalences: $v_S(\Gamma, a \leq b) = \mathbf{1}$ iff $v_S(\Gamma, a) \leq v_S(\Gamma, b)$ iff $|a|_\Gamma \leq |b|_\Gamma$ iff $a \leq b \in \Gamma$.

The case $\alpha = a \, \mathsf{C} \, b$. The proof is similar to the above.

The case $\alpha = \Box\beta$. We have: $v_S(\Gamma, \Box\beta) = \mathbf{1}$ iff $(\forall\Delta \in W_S)(v_S(\Delta, \beta) = \mathbf{1})$ iff (by induction hypothesis) $(\forall\Delta \in W_S)(\beta \in \Delta)$ iff (by Lemma 2.13(ii)) $\Box\beta \in \Gamma$. $\dashv$

Lemma 2.15. *The following two conditions are equivalent for any formula* α:

(a) α *is a theorem of* $\mathbb{L}$,

(b) α *is true in all canonical models of* $\mathbb{L}$.

Proof. The implication (a)$\to$(b) is just the soundness of $\mathbb{L}$ (For the case $\mathbb{L} = \mathbb{MCL}_{\mathrm{Con}}$ use the fact that all canonical models of $\mathbb{L}$ are connected).

For the implication (b)$\to$(a) suppose that α is not a theorem of $\mathbb{L}$. Then, by the Lindenbaum Lemma there exists a maximal theory S such that $\alpha \notin S$. Consider the canonical model determined by S: $M_S = (W_S, \{\mathfrak{B}_S(\Gamma) : \Gamma \in W_S\}, v_S)$. Since $S \in W_S$ and $\alpha \notin S$ then $v_S(S, \alpha) = \mathbf{0}$, by Lemma 2.14. $\dashv$

Now we are ready to proof the completeness theorem for $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$.

Theorem 2.16 (Completeness theorem for $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$). *The following conditions are equivalent for any formula* α:

(a) α *is a theorem of* $\mathbb{MCL}_{\mathrm{min}}$ ($\mathbb{MCL}_{\mathrm{Con}}$),

(b) α *is true in all frames* (*all connected frames*),

(c) α *is true in all topological frames* (*all connected topological frames*),

(d) α *is true in all topological frames corresponding to the class of* (*connected*) *compact semiregular* T_0 *spaces.*

Proof. We consider only the case for $\mathbb{MCL}_{\mathrm{min}}$, the case for $\mathbb{MCL}_{\mathrm{Con}}$ can be obtained in a similar way.

The implications (a) $\to$ (b) $\to$ (c) $\to$ (d) are obvious by the soundness lemma. The implication (d) $\to$ (b) follows from the representation theorem for contact algebras (Theorem 1.3). For the implication (b) $\to$ (a) suppose that α is true in all frames. Then α is true in all models, and hence α is true in the canonical models of $\mathbb{MCL}_{\mathrm{min}}$. Then by Lemma 2.15, α is a theorem of $\mathbb{MCL}_{\mathrm{min}}$. $\dashv$

The following is the strong version of the completeness theorem.

THEOREM 2.17 (Strong completeness theorem for $\mathbb{MCL}_{\min}$ and $\mathbb{MCL}_{\mathrm{Con}}$). *The following conditions are equivalent for any set A of formulas*:

(a) *A is a consistent set in $\mathbb{MCL}_{\min}$ ($\mathbb{MCL}_{\mathrm{Con}}$)*,

(b) *A has an algebraic model in the class of all frames (in the class of all connected frames)*,

(c) *A has a topological model in the class of topological models (in the class of all connected topological models)*,

(d) *A has a topological model in the class of all (connected) compact semi-regular and T_0 spaces.*

PROOF. Again we consider only the case for $\mathbb{MCL}_{\min}$, the case for $\mathbb{MCL}_{\mathrm{Con}}$ is similar. The equivalence of (b), (c) and (d) follows from the representation theorem for contact algebras (Theorem 1.3). The implication (b) $\rightarrow$ (a) is obvious. For the proof of the implication (a) $\rightarrow$ (b) suppose that A is a consistent set of formulas in $\mathbb{MCL}_{\min}$. Then by the Lindenbaum Lemma there exists a maximal theory S containing A. Consider the canonical model $M_S = (W_S, \{\mathfrak{B}_S(\Gamma) : \Gamma \in W_S\}, v_S)$. Since $A \subseteq \Gamma$ then by the Truth Lemma $v_S(S, \alpha) = \mathbf{1}$ for all $\alpha \in A$. So M_S is an algebraic model of A. $\dashv$

3. Decidability

In this section we will show that the logics $\mathbb{MCL}_{\min}$ and $\mathbb{MCL}_{\mathrm{Con}}$ are decidable, showing that both logics have finite model property. The proof goes trough suitable modification of filtration method from modal logic. First we will describe the relevant constructions only for the logic $\mathbb{MCL}_{\min}$.

The first step is to show that contact algebras in the modal frames can be considered finite. Namely, let $M = (W, \{\underline{\mathbf{B}}(x) : x \in W\}, v)$ be a model and $b_1, \ldots, b_n$ be a set of different Boolean variables. Let for each $x \in W$, $\mathfrak{B}'(x)$ be the the contact subalgebra of $B(x)$ generated by the elements $v(b_1), \ldots,$ $v(b_n)$ and let v' be the restriction of the valuation v on the algebras $\mathfrak{B}'(x)$, considered only for the variables $b_1, \ldots, b_n$. Obviously the number of the elements of $\mathfrak{B}'(x)$ for each $x \in W$ is $\leqslant 2^n$. Then the following lemma is true.

LEMMA 3.18. *Let* $\mathrm{For}(b_1, ..., b_n)$ *be the set of all formulas containing only variables from the list $b_1, \ldots, b_n$. Then the following equivalence is true for any formula $\alpha \in \mathrm{For}(b_1, ..., b_n)$ and $x \in W$: $v(x, \alpha) = 1$ iff $v'(x, \alpha) = 1$.*

Proof. The proof easily follows by induction on the construction of α and Lemma 1.4. ⊣

As a consequence we obtain the following

Corollary 3.19. *The following conditions are equivalent for any formula* α:

- α *is true in the class of all frames,*
- α *is true in the class of all frames of the form* $(W, \{\mathfrak{B}(x) : x \in W\})$ *such that for each* $x \in W$ *the number of the elements of algebra* $\mathfrak{B}(x)$ *is* $\leqslant 2^n$ *where* n *is the number of the Boolean variables in* α.

This corollary restrict the class of frames where a given formula α have to be tested only to frames with finite algebras with upper bound on the number of elements effectively computable from the size of α. This, however, still does not imply decidability. So our second step is to make finite both parts of the frames in which α has to be tested.

Let A be a finite set of formulas, closed under subformulas and let $b_1, \ldots, b_n$ be the list of all Boolean variables occurring in A. Let $M = (W, \{\mathfrak{B}(x), x \in W\}, v)$ be a model such that for all $x \in W$ the cardinality of $\mathfrak{B}(x)$ is $\leqslant 2^n$. Define in W an equivalence relation $\equiv$ (depending on A) as in the filtration construction from the ordinary modal logic: for $x, y \in W$, $x \equiv y$ iff $(\forall \alpha \in A)(v(x, \alpha) = 1 \leftrightarrow v(y, \alpha) = 1)$. For any $x \in W$ let $|x| = \{y \in W : x \equiv y\}$. Define $W' = \{|x| : x \in W\}$. As in the filtration theory from modal logic one can show that W' is a finite set with cardinality $\leqslant 2^m$, where m is the number of the elements of the set A. Now we want to associate with each element from W' a finite contact algebra with cardinality $\leqslant 2^n$. To this end from each equivalence class we choose an element $f(|x|) \in |x|$ (note that in this step the Axiom of choice is applied and that f is the choice function). Using the function f we define a total function $g : W \to W$ as follows: for $x \in W$ put $g(x) = f(|x|)$. Obviously for any $x \in W$ we have $g(x) \in |x|$ and hence $|x| = |g(x)|$. Now for each $|x| \in W'$ we associate a contact algebra $\mathfrak{B}'(|x|)$ putting $\mathfrak{B}(|x|) = B(g(x))$. It is easy to see that if $x_1 \equiv x_2$ then $\mathfrak{B}'(|x_1|) = \mathfrak{B}'(|x_2|)$. In this way we define the frame $(W', \{\mathfrak{B}'(|x|) : |x| \in W'\})$. We define a valuation v' in this frame putting for each Boolean variable from the list $b_1, \ldots, b_n$ and $|x| \in W'$: $v'(|x|, b_i) = 1$ iff $v(g(x), b_i) = 1$, $i = 1, \ldots, n$. Now let a be any Boolean term built from the variables $b_1, \ldots, b_n$. Then, by an easy induction on the complexity of a one can verify the equivalence

$$v'(|x|, a) = 1 \text{ iff } v(g(x), a) = 1\,. \tag{1}$$

LEMMA 3.20 (Filtration Lemma). *The following equivalence holds for any formula $\alpha \in A$ and any $x \in W$: $v(x, \alpha) = 1$ iff $v'(|x|, \alpha) = 1$.*

PROOF. We proceed by induction on the complexity of α. First let us consider the case of atomic α.

Case 1: $\alpha = a \leq b \in A$. We have: $v(x, a \leq b) = 1$ iff $v(g(x), a \leq b) = 1$ (because $x \equiv g(x)$ and $a \leq b \in A$) iff $v(a) \leq v(b)$ in $\mathfrak{B}(g(x))$ iff $v'(a) \leq v'(b)$ in $\mathfrak{B}'(|x|)$ (because of (1)).

Case 2: $\alpha = a \mathrel{\mathsf{C}} b$ can be proved in a similar way.

Now let us consider the case of arbitrary α. The nontrivial case is $\alpha = \Box\beta \in A$ with the inductive hypothesis (i.h.) that for β the statement is true. So we have to prove the equivalence: $v(x, \Box\beta) = 1$ iff $v'(|x|, \Box\beta) = 1$.

($\rightarrow$) Suppose $v(x, \Box\beta) = 1$. Then for any $y \in W$ we have $v(y, \beta) = 1$. Since β is in A we may apply the i.h. for β. Then we obtain: for any $|y| \in W'$, $v'(|y|, \beta) = 1$, which implies that $v'(|x|, \Box\beta) = 1$.

In a similar way we can prove the converse implication ($\leftarrow$). ⊣

Now we are ready to prove the main result in this section.

PROPOSITION 3.21. *The following conditions are equivalent for any formula α:*

(a) *α is true in all frames,*

(b) *α is true in all frames in the form $(W, \{\mathfrak{B}(x) : x \in W\})$ such that for each $x \in W$ the the number of the elements of the algebra $\mathfrak{B}(x)$ is $\leqslant 2^n$ where n is the number of Boolean variables occurring in α,*

(c) *α is true in all frames in the form $(W, \{\mathfrak{B}(x) : x \in W\})$ such that for each $x \in W$ the the number of the elements of the algebra $\mathfrak{B}(x)$ is $\leqslant 2^n$ where n is the number of Boolean variables occurring in α and the number of the elements of W is $\leqslant 2^m$ where m is the number of the subformulas of α.*

PROOF. The equivalence (a) $\leftrightarrow$ (b) is already proved—this is Lemma 3.19. The implication (b) $\leftrightarrow$ (c) is obvious. The implication (c) $\rightarrow$ (b) follows from Lemma 3.20. ⊣

As a corollary of Proposition 3.21 we obtain decidability of $\mathbb{MCL}_{\mathrm{min}}$. Slightly modifying the above constructions we can obtain the same result for the logic $\mathbb{MCL}_{\mathrm{Con}}$.

THEOREM 3.22 (Decidability of $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$). *The logics $\mathbb{MCL}_{\mathrm{min}}$ and $\mathbb{MCL}_{\mathrm{Con}}$ possess finite model property and hence are decidable.*

4. Concluding Remarks

In Section 2 we proved completeness theorems for the logics $\mathbb{MCL}_{\text{min}}$ and its extension with the axiom of connectedness: the logic $\mathbb{MCL}_{\text{Con}}$. Both logics are quantifier-free and in a sense they are propositional. The logic $\mathbb{MCL}_{\text{min}}$ corresponds to mereotopology formalized by contact algebras and $\mathbb{MCL}_{\text{Con}}$ corresponds to the mereotopology formalized by connected contact algebras. Both notions have universal first-order formulations which makes possible to obtain quantifier free formulation of the corresponding logics: $\mathbb{MCL}_{\text{min}}$ and $\mathbb{MCL}_{\text{Con}}$. There are, however, interesting classes of contact algebras characterizing by axioms which are not universal conditions. The following are two examples.

The first is the axiom of *extensionality*:

$$\text{(Ext)} \qquad (\forall a \neq 1)(\exists b \neq 0)\ a\,\overline{\mathsf{C}}\,b\,.$$

This axiom is equivalent on the base of the other axioms of conact algebra to the following condition of extensionality assumed by Whitehead [12]:

$$a = b \leftrightarrow (\forall c)(a\,\mathsf{C}\,c \leftrightarrow b\,\mathsf{C}\,c)\,.$$

By means of this condition the part-of relation $a \leq b$ is definable by the contact relation C as follows:

$$a \leq b \leftrightarrow (\forall c)(a\,\mathsf{C}\,c \rightarrow b\,\mathsf{C}\,c).$$

Extensional contact algebras have models in the class of the so called "weakly regular topological spaces" introduced in [6]. For the representation theory of extensional contact algebras see [6, 5, 3, 13].

Another interesting axiom is the axiom of *normality*:

$$\text{(Nor)} \qquad a\,\overline{\mathsf{C}}\,b \rightarrow (\exists c)(a\,\overline{\mathsf{C}}\,c \wedge b\,\overline{\mathsf{C}}\,c^*).$$

The name (Nor) is adopted, because it is true in contact algebras of normal topological spaces. The combination of (Ext) and (Nor) corresponds to the mereotopology of all compact Hausdorff spaces.

Since the axioms (Ext) and (Nor) are not universal first-order sentences, they cannot be added as additional axioms to the minimal logic $\mathbb{MCL}_{\text{min}}$. But they can be simulated by additional rules of inference in the logic $\mathbb{MCL}_{\text{min}}$, which imitate the quantification rules in the first-order logic. These rules are the following:

The *rule of extensionality*:

EXT $$\frac{\alpha \Rightarrow (p = 0 \vee a\,\mathsf{C}\,p)}{\alpha \Rightarrow (a = 1)}$$

where p is a Boolean variable that does not occur in a and α.

The *rule of normality*:

NOR $$\frac{\alpha \Rightarrow (a\,\mathsf{C}\,p \vee p^*\,\mathsf{C}\,b)}{\alpha \Rightarrow a\,\mathsf{C}\,b}$$

where p is a Boolean variable that does not occur in a, b, and α.

The completeness theorem for extensions of the logic $\mathbb{CL}_{\min}$ with some of the rules EXT and NOR is given in [2]. The completeness techniques is similar to that of the proof of $\mathbb{CL}_{\min}$. Only the notion of a theory is changed: we consider there theories which are closed also with respect to variants of the rules EXT and NOR. The effects of the new rules on the canonical contact algebras is that they satisfy the corresponding first-order axioms (Ext) and (Nor) and consequently imply the completeness with respect to the desired classes of contact algebras, and by applying the corresponding representation theorems—completeness to the corresponding classes of topological spaces.

If we apply this techniques to the extensions of $\mathbb{MCL}_{\min}$ and $\mathbb{MCL}_{\mathrm{Con}}$ with (some of) the rules EXT and NOR we may obtain completeness theorems also for these new logics. This is rather routine technical work similar to the proofs given in Section 2. That is why we left this exercise to the interested reader. There is, however, one interesting question about these new extensions. It is shown in [2] that adding the rules EXT and NOR to the logics $\mathbb{CL}_{\min}$ and $\mathbb{CL}_{\mathrm{Con}}$ they do not produce new theorems and the effect of the new rules is that they give completeness theorems with respect to more special contact algebras and topological spaces. It is an open problem, however, if this is true also for the extensions of the logics $\mathbb{MCL}_{\min}$ and $\mathbb{MCL}_{\mathrm{Con}}$ with EXT and NOR. Another open problem concerns decidability of these extensions. So we conclude the paper with these two open problems.

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