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FORMULAS OF ONE PROPOSITIONAL VARIABLE IN INTUITIONISTIC LOGIC WITH THE SOLOVAY MODALITY

Abstract. A description of the free cyclic algebra over the variety of Solovay algebras, as well as over its pyramid locally finite subvarieties is given.

Keywords: Heyting algebras, Kripke frames, free algebras.

MSC: 06B30, 06F20, 54F05

1. Introduction

Before entering into a technical description of the free cyclic algebras with the Solovay modality let us indicate some motivation for bringing up this topic.

In [8] Robert Solovay, among other things, presented a set-theoretical translation of modal formulas by putting $\Box p$ to mean "p is true in every transitive model of Zermelo-Fraenkel Set Theory **ZF**". By defining an interpretation as a function s sending modal formulas to sentences of **ZF** which commutes with the Boolean connectives and putting $s(\Box p)$ to be equal to the statement "$s(p)$ is true in every transitive model of **ZF**", Solovay formulated a modal system, which we call here *SOL*, and announced its **ZF**-completeness.

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SOL is the classical modal system which results from the Gödel-Löb system *GL* (alias, the provability logic) by adding the formula

$$\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p \wedge p)$$

as a new axiom.

ZF-completeness: *For any modal formula p, $SOL \vdash p$ iff* **ZF** $\vdash s(p)$ *for any Solovay's interpretation s.* (For a proof, see [2, Chapt. 13, p. 166].)

Now we shall formulate a simple system $I.SOL$, which is an intuitionistic "companion" of SOL: the composition of the well-known Gödel's modal translation of Heyting Calculus and split-map (= splitting a formula $\Box p$ into the formula $p \wedge \Box p$) provides the needed embedding of $I.SOL$ into SOL.

Definition 1. An intuitionistic modal system $I.SOL$ is an extension of the proof-intuitionistic logic KM obtained by postulating the formula

$$(\Box p \to \Box q) \vee (\Box q \to p)$$

as a new axiom [4].

We recall that the proof-intuitionistic logic KM (=Kuznetsov-Muravitsky [7]) is the Heyting propositional calculus HC enriched by $\Box$ as Prov modality satisfying the following conditions:

$$p \to \Box p, \qquad \Box p \to (q \vee (q \to p)), \qquad (\Box p \to p) \to p\,.$$

The purpose of this paper is to investigate the set of formulas of one propositional variable of the system $I.SOL$, that is in algebraic terms it means the description of the free cyclic algebra in the variety corresponding to the system $I.SOL$. We introduce some algebraic and relational background. We choose the formulations below because they seem to be appropriate to our aim. The algebraic semantics for the system $I.SOL$ is based on the notion of Heyting algebra with an operator.

Definition 2. A Heyting algebra with an operator $\Box$ is called *Solovay algebra*, if the following conditions are satisfied:

$$p \le \Box p, \qquad \Box p \le q \vee (q \to p), \qquad \Box p \to p = p\,,$$
$$(\Box p \to \Box q) \vee (\Box q \to p) = \top\,.$$

The class of all Solovay algebras forms a variety, which we denote by **SA**. It is known that the variety **SA** is finitely approximated and that if $(H, \Box) \in \mathbf{SA}$ then the Heyting algebra H is cascade Heyting algebra [4]. A Heyting algebra H is called a *cascade Heyting algebra*, if H belongs to the variety generated by the class of all finite Boolean cascades. A finite Heyting algebra H is a *Boolean cascade*, if there exist Boolean lattices $B_1, \ldots, B_k$ such that $H = B_1 + \cdots + B_k$, where each B_i is a convex sublattice of H and $B_i + B_{i+1}$ denotes the ordinal sum of B_i and B_{i+1} in which the smallest element of B_i and the largest element of B_{i+1} are identified.

2. Preliminaries

Let (P, R) be a partially ordered set (for short, a *poset*) and $Q \subseteq P$. Then we say that Q is a *down cone* if, whenever $x \in Q$, $y \in P$ and $R(y, x)$, it follows that $y \in Q$. Dually, $W \subseteq P$ is called an *upper cone* (or simply *cone*) if whenever $x \in W$, $y \in P$ and $R(x, y)$, then $y \in W$. The smallest down cone (upper cone) containing a given subset U of P we denote by

$$R^{-1}(U) := \{y \in P : R(y, x) \ \&\ x \in U\},$$
$$R(U) := \{y \in P : R(x, y) \ \&\ x \in U\}.$$

Instead of $R^{-1}(\{x\})$ ($R(\{x\})$) we write $R^{-1}(x)$ ($R(x)$). The same notation we have if binary relation R, defined on P, is transitive and irreflexive. We say that x *covers* y, and write $y \prec x$, if $R(y, x)$ and $y \neq x$ and there is no z such that $z \neq y$, $z \neq x$ and $R(y, z)$ and $R(z, x)$. In the case when R is transitive we denote its reflexive closure by $R_\circ$. Often, we shall use the symbol $<$ for transitive and irreflexive relation, and do $\leq$ for reflexive closure of $<$.

Let $(X; R)$ be a poset and $x \in X$. A *chain out* of x is a lineary ordered subset (i.e. for every y, z from the subset either yRz or zRy) of X with the least element x; $d(x)$ (the *depth of* x) denotes the largest cardinality of chains out of x.

A topological space X with binary relation R is said to be *GL-frame* if:

(1) X is a Stone space (i.e. 0-dimensional, Hausdorf and compact topological space);

(2) $R(x)$ and $R^{-1}(x)$ are closed sets for every $x \in X$ and $R^{-1}(A)$ is a clopen for every clopen A of X;

(3) for every clopen A of X and every element $x \in A$ there is an element $y \in A \setminus R^{-1}(A)$ such that either xRy or $x \in A \setminus R^{-1}(A)$.

A map $f\colon X_1 \to X_2$ from a GL-frame X_1 to a GL-frame X_2 is said to be *strongly isotone* if

$$f(x)R_2y \Leftrightarrow (\exists z \in X_1)(xR_1z \;\&\; f(z) = y).$$

Let us denote by $\mathbf{G}$ the category of GL-frames and continuous strongly isotone maps.

An algebra $(A; \vee, \wedge, \Diamond, -, 0, 1)$ is said to be *diagonalizable algebra* if $(A; \vee,$
$\wedge, -, 0, 1)$ is Boolean algebra and $\Diamond$ satisfies the following conditions:

(1) $\Diamond(a \vee b) = \Diamond(a) \vee \Diamond(b)$,

(2) $\Diamond(0) = 0$,

(3) $\Diamond(a) \leq \Diamond(a \vee -\Diamond(a))$.

Let us denote by $\mathbf{D}$ the category of diagonalizable algebras with diagonalizable algebra homomorphisms.

Let $X \in \mathbf{G}$ and $A \in \mathbf{D}$. The set $\mathcal{D}(X)$ of all clopen subsets of X is closed under the set union, intersection, complementation and the operator R^{-1}. So $\mathcal{D}((X;R)) = (\mathcal{D}(X); \cup, \cap, -, R^{-1}, \emptyset, X)$ is an object of $\mathbf{D}$; the set $\mathcal{G}(A)$ of all ultrafilters of $A \in \mathbf{D}$ with a relation $xRy \Leftrightarrow (\forall a \in A)(a \in y \Rightarrow \Diamond(a) \in x)$, topologized by taking the family of sets $h(a) = \{F \in \mathcal{G}(A) : a \in F\}$ as a base, is an object of $\mathbf{G}$. Furthermore, setting $\mathcal{G}(h) = h^{-1}\colon \mathcal{G}(B) \to \mathcal{G}(A)$ for every morphism $h\colon A \to B$ of $\mathbf{D}$, and $\mathcal{D}(f) = f^{-1}\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ for every morphism $f\colon X \to Y$ of $\mathbf{G}$, we obtain contravariant functors $\mathcal{D}\colon \mathbf{D} \to \mathbf{G}$ and $\mathcal{G}\colon \mathbf{G} \to \mathbf{D}$. In [1] it is proved, that these functors establish a dual equivalence between the categories $\mathbf{D}$ and $\mathbf{G}$.

On every diagonalizable algebra A is defined a unary operator $\Box$ which is dual to $\Diamond$: $\Box(x) = -\Diamond-(x)$. The sublattice $H = \{\Box(a) \wedge a : a \in A\}$ forms Heyting algebra, where $a \to b = \Box(-a \vee b) \wedge (-a \vee b)$. Let us note, that the set H is closed under the operation $\Box$. Moreover the operation $\Box$ on elements of H satisfies the following conditions: (a) $p \leq \Box p$, (b) $\Box p \leq (q \vee (q \to p))$, (c) $(\Box p \to p) = p$. As we see the class of all algebras $(H, \vee, \wedge, \to, \Box, \bot, \top)$, where $(H, \vee, \wedge, \to, \bot, \top)$ is Heyting algebra and $\Box$ satisfies the conditions (a), (b), (c), forms a variety which we denote by $\mathbf{H}_\Box$. The variety of Solovay algebras $\mathbf{SA}$ is a subvariety of $\mathbf{H}_\Box$ determined by the additional identity $(\Box p \to \Box q) \vee (\Box q \to p) = \top$.

A pair $(X;R)$ is said to be *S-frame* if:

(1) $(X;R)$ is GL-frame;

(2) $(X, R_\circ)$ is a poset;

(3) for every $x, y, z, u \in X$ if uRx, uRz, xRy and $\neg(xRz)$, then zRy.

Let **S** be the category of S-frames and continuous strongly isotone maps. The duality between the category of Solovay algebras and the category of S-frames is obtained by specialization of the duality between the categories **D** and **G** on the case of Solovay algebras. For any S-frame $(X;R)$ and $U, V \in \mathcal{SA}(X)$ (= the set of all clopen cones of X) define:

$$U \to V = X \setminus (R^{-1}(U \setminus V) \cup (U \setminus V)), \qquad \Box U = X \setminus R^{-1}(X \setminus U).$$

Then the algebra $\mathcal{SA}((X;R)) = (\mathcal{SA}(X); \vee, \wedge, \to, \Box, 0, 1)$ is a Solovay algebra. Furthemore, for any morphism $f\colon (X_1;R_1) \to (X_2;R_2)$ in **S**, $\mathcal{SA}(f) = \mathcal{G}(f)$ is a Solovay algebra homomorphism from $\mathcal{SA}(X_2;R_2)$ into $\mathcal{SA}(X_1;R_1)$. On the other hand, for each Solovay algebra **A**, the set $\mathcal{S}(A)$ of all prime filters of **A** with the binary relation R on it, defined by the following way: $xRy \Leftrightarrow (\forall a \in A)(\Box a \in x \Rightarrow a \in y)$, and topologised by taking the family of $\beta(a) = \{F \in \mathcal{S}(A) : a \in F\}$, for $a \in A$, and their complements as a subbase, denote it by $\mathcal{S}(\mathbf{A})$, is an object of **S**; and for each Solovay algebra homomorphism $h\colon A \to B$, $\mathcal{S}(h) = \mathcal{G}(h)$ is a morphism of **S**. Therefore we have two contravariant functors $\mathcal{S}\colon \mathbf{SA} \to \mathbf{S}$ and $\mathcal{SA}\colon \mathbf{S} \to \mathbf{SA}$. These functors establish a dual equivalence between the categories **SA** and **S**.

As follows from a duality there is one-to-one correspondence between homomorphic images of a Solovay algebra A and closed cones of $\mathcal{S}(A)$, and between subalgebras of a SA-algebra A and correct partitions of $\mathcal{S}(A)$, where *a correct partition* [6] of a $(X;R) \in \mathbf{S}$ is such an equivalence relation E on X that

- E is a closed equivalence relation, i.e. E-saturation[1] of any closed subset is closed;
- E-saturation of any upper cone is an upper cone;
- $(\forall x \in X)(E(x) \cap R^{-1}(E(x)) \neq \emptyset \Rightarrow E(x) \subseteq R^{-1}(E(x))$;
- there is S-frame $(Y;Q)$ and strongly isotone map $f\colon X \to Y$ such that $\operatorname{Ker} f = E$.

Suppose $(X;R)$ is an S-frame, $A = \mathcal{SA}((X;R))$ and $g_1, \ldots, g_n \in A$. Now we will present a criterion deciding whether or not A is generated by $g_1, \ldots, g_n$. Our criterion extends the analogous one for descriptive intuitionistic frames from [5] to S-frames.

[1]If $V \subseteq X$ then E-saturation of V is $E(V) = \bigcup_{x \in X} E(x)$.

Denote by $\mathbf{n}$ the set $\{1, \ldots, n\}$. Let $G_p = g_1^{\varepsilon_1} \cap \cdots \cap g_n^{\varepsilon_n}$, where $\varepsilon_i \in \{0, 1\}$, $p = \{i : \varepsilon_i = 1\}$ and

$$g_i^{\varepsilon_i} = \begin{cases} g_i, & \varepsilon_i = 1\,, \\ -g_i, & \varepsilon_i = 0\,. \end{cases}$$

It is obvious that $\{G_p\}_{p \subseteq \mathbf{n}}$ is a partition of X which we call the *colouring* of X.[2] A point $x \in G_p$ is said to have the *colour* p, written as $\mathrm{Col}(x) = p$. Let us remark that $g_i = \bigcup_{i \in p} G_p$.

Lemma 3. *Suppose E is a correct partition of X. The following two conditions are mutually equivalent*:

(i) *every g_i is E-saturated, that is $E(g_i) = g_i$ $(1 \leq i \leq n)$,*

(ii) *every class G_p is E-saturated, that is $E(G_p) = G_p$ $(p \subseteq \mathbf{n})$.*

Proof. Easy. $\dashv$

Theorem 4 (Coloring theorem). *A Solovay algebra A is generated by g_1, $\ldots$, g_n iff for every non-trivial correct partition E of X $(= \mathcal{S}(A))$, there exists an equivalence class of E containing points of different colors.*

Proof. Suppose A is generated by $g_1, \ldots, g_n$ and E is a non-trivial correct partition of X. Consider the set A_E of E-saturated elements of A. Since $g_1, \ldots, g_n$ generate A, there exists $i \leq n$ such that $g_i \notin A_E$. Therefore there exists $p \subseteq \mathbf{n}$ such that G_p is not E-saturated. But then there exists $x \in G_p$ such that $E(x) \cap G_p \neq \emptyset$ and $E(x) \cap -G_p \neq \emptyset$. Hence $E(x)$ contains points of different colour.

Conversely, suppose A is not generated by $g_1, \ldots, g_n$. Denote by A_0 the least subalgebra of A containing $g_1, \ldots, g_n$. Obviously A_0 is a proper subalgebra of A and the correct partition E of X corresponding to A_0 is non-trivial[3]. Moreover, since $g_1, \ldots, g_n \in A_0$, $E(g_i) = g_i$ $(1 \leqslant i \leqslant n)$ and hence $E(G_p) = G_p$ $(p \subseteq \mathbf{n})$. But then every equivalence class of E contains points of the same colour. $\dashv$

3. Cyclic Free Solovay Algebra

Now we are going to describe the free Solovay algebra $H(G)$ of one generator G. For description of cyclic free Solovay algebra we shall use a sequence of

[2] It is also clear that G_p are the atoms of the Boolean algebra $\mathcal{B}(g_1, \ldots, g_n)$ generated (in the set of all subsets of X) by $g_1, \ldots, g_n$.

[3] E is defined on X by putting: xEy iff $x \in U \Leftrightarrow y \in U$, for every $U \in A_0$.

finite S-frames $(X_i, <_i)$, $i = 1, 2, \ldots,$ directed by inclusion. In other words, such S-frames $(X_i, <_i)$, that $X_i \subseteq X_{i+1}$, $<_i \subseteq <_{i+1}$. As we mentioned above *color* is defined as any subset of $\{1, \ldots, n\}$. In our case $n = 1$, so there exist exactly 2 colors: $\{1\}$ and $\emptyset$. We shall define X_i recurrently by levels (i.e. by elements of fixed depth), every element of which is colored with one of the 2 colors.

Let $X_1 = \{(\Delta_0, t) : \Delta_0 = \emptyset, t \subseteq \{1\}\}$. On X_1 we define a relation $<_1$: $x \not<_1 y$ for every $x, y \in X_1$. Thus we have constructed S-frame $(X_1, <_1)$ of the first level (i.e. the elements of X_1 have the depth 1 with respect to the relation $\leq_1$), which will be contained in the universal frame (X, R) that is constructed by induction. X_1 (the elements of depth 1) consists of 2 different incomparable elements (two pairs), the second components of which are the colors of the elements: $(\emptyset, \emptyset)$ and $(\emptyset, \{1\})$, and $\mathrm{Col}((\emptyset, \emptyset)) = \emptyset$ and $\mathrm{Col}((\emptyset, \{1\})) = \{1\}$. Let $X_2 = X_1 \cup \{(\Delta_1, t) : \Delta_1 \subseteq X_1, |\Delta_1| = 1, t \subseteq \mathrm{Col}(\Delta_1)\} \cup \{(\Delta_1, \emptyset) : \Delta_1 = \{(a, \emptyset), (a, \{1\})\} \subseteq X_1\}$. In our notation $X_2 = \{(\emptyset, \emptyset),\ (\emptyset, \{1\}),\ (\{(\emptyset, \emptyset), (\emptyset, \{1\})\}, \emptyset),\ (\{(\emptyset, \emptyset)\}, \emptyset),\ (\{(\emptyset, \{1\})\}, \{1\}),\ (\{(\emptyset, \{1\})\}, \emptyset)\}$. On X_2 define the relation $<_2$ as follows: for every $x, y \in X_1$ $x <_2 y$ iff $x <_1 y$; $(\Delta_1, t) <_2 x$ for every $x \in \Delta_1 \subseteq X_1$; $x \not<_2 x$ for every $x \in X_2 \setminus X_1$. Let $(X_i, <_i)$ have already been constructed for $i \geq 2$. Then $X_{i+1} = X_i \cup \{(\Delta_i, t) : \Delta_i \subseteq X_i \setminus X_{i-1}, |\Delta_i| = 1, t \subseteq \mathrm{Col}(\Delta_i)\} \cup \{(\Delta_i, \emptyset) : \Delta_i = \{(a, \emptyset), (a, \{1\})\} \subseteq X_i \setminus X_{i-1}\}$. X_i consists of elements of depth $\leq i$.Let us note that for every element $(a, b) \in X_i$ b is its color, i.e. $\mathrm{Col}((a, b)) = b$. On X_{i+1} define binary the relation $<_{i+1}$ as follows: for every $x, y \in X_i$ $x <'_{i+1} y$ iff $x <_i y$; $(\Delta_i, t) <'_{i+1} x$ for every $x \in \Delta_i \subseteq X_i \setminus X_{i-1}$; $x \not<'_{i+1} x$ for every $x \in X_{i+1} \setminus X_i$. Let $<_{i+1}$ be the transitive closure of $<'_{i+1}$. The elements of $i + 1$ depth are exhausted by these described above elements. X_{i+1} is the set of all elements of depth $j \leq i + 1$. $(X_{i+1}, <_{i+1})$ is S-frame determined by the construction.

Let $(X, R) = \bigcup_{i=1}^{\infty} (X_i, <_i)$, $g_1 = \{x \in X_1 : \mathrm{Col}(x) = \{1\}\} = \{(\emptyset, \{1\})\}$ and $g_i = \{x \in X_i \setminus X_{i-1} : \mathrm{Col}(x) = \{1\}\}$ for $i > 1$.

It is possible to represent Kripke frames graphically by the following device. Represent the points of the Kripke frame (X, R) by small circles (or nodes) in such a way that if $R(x, y)$, then the node representing x is lower (in the Figure) than that representing y. Below is depicted the Figure "Pine-tree" of the Kripke frame $(\kappa X, \kappa R)$ which will be of subsequent interest. The Figure requires some explanation. First of all notice that the sets g_i are singletons and are identified with their elements.

A black-node z of our Pine-tree is an irreflexive point ($\kappa R(z, z)$ is false). A white-node z is a reflexive point ($\kappa R(z, z)$ is true). The whole Pine-tree represents the Kripke frame $(\kappa X, \kappa R)$, whereas the frame (X, R) represents some "subtree"; namely, X is the set of black-points $x \in \kappa X$ such that there is no white-point y with $\kappa R(x, y)$. The relation R on X is the restriction of the relation κR to X (i.e. $R = \kappa R \cap X^2$). An inspection of Figure 1 shows that κR and R are transitive relations satisfying the ascending chain condition. Moreover, the relation R is irreflexive. Recall that we use $R_\circ$ (resp., $\kappa R_\circ$) to denote the reflexive closure of R (resp. κR). It is well-know that the ring $\mathrm{Con}(X)$ of $R_\circ$-cones of X is a Heyting algebra in which $\wedge$ and $\vee$ are the set-theoretical intersection and union and for $A, B \in \mathrm{Con}(X)$

$$A \to B = \{x \in X : (\forall y)(xR_\circ y \& y \in A \Rightarrow y \in B\}.$$

Define the operation $\Box$ on the Heyting algebra $\mathrm{Con}(X)$ by putting

$$\Box A = \{x \in X : (\forall y)(xRy \Rightarrow y \in A)\}.$$

OBSERVATION. *The algebra* $(\mathrm{Con}(X), \Box)$ *is a Solovay algebra.*

One more notation: we denote $\Box(\Box^k p)$ by $\Box^{k+1} p$. Given subsets $A, B \subseteq X$, denote by $A - B$ their set-theoretical difference. We shall consider a ring of cones $K_0(X)$ of X obtained from the finite sets $\Box^k \emptyset$ (k-storied pyramid of the figure, $k \in \omega$) and the sets $X - R_\circ^{-1}(x)$ $(x \in X)$ by applying the operations of union and intersection. We remark that the cones $X - R_\circ^{-1}(x)$ may be of two types: 1)$X - R_\circ^{-1}(x)$ for $x \in \mathrm{Trunk}(= \bigcup_{i=1}^{\infty}\{g_i\})$, i.e. $x = g_i$ for some $i \geq 1$, and 2) $X - R_\circ^{-1}(x)$ for $x \notin \mathrm{Trunk}$, i.e. $x \neq g_i, i \in \omega$. A set of the first type contains a finite number of branches, whereas a set of the second type contains infinite number of branches.

THEOREM 5. *The ring of cones* $K_0(X)$ *is closed under the implication* $\to$ *and the box-operation* $\Box$ *of the Solovay algebra* $(\mathrm{Con}(X), \Box)$. *Thus the ring* $K_0(X)$ *is itself a Solovay algebra.*

PROOF. By direct inspection. $\dashv$

Denote by G the cone $\{g_i \in X : i \in \omega\}(= \mathrm{Trunk})$ and by G_k the cone $\{g_i \in X : i \leqslant k \in \omega\}$. Denote by $K(X)$ the smallest subring of $\mathrm{Con}(X)$ which contains the ring $K_0(X)$ and the cones $\Box^k G$ $(k \in \omega)$. It should be noted that $\Box G = \{a_i : i \in \omega\} \cup G$, where the elements of $\{a_i : i \in \omega\}$ are maximal elements of the set $X \setminus G$ with respect to the relation $R_\circ$. In our notation $a_1 = (\emptyset, \emptyset)$.

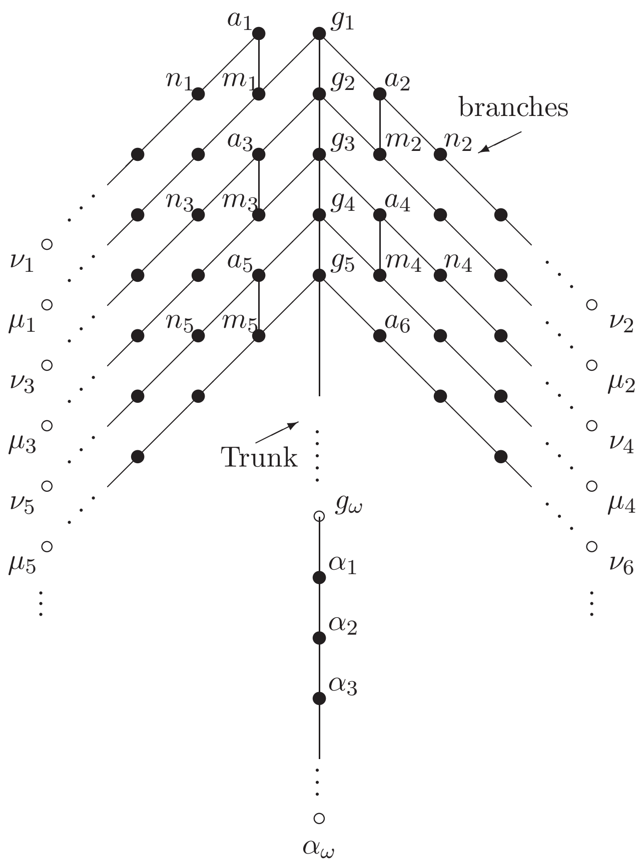

Figure 1. Pine-tree of $(\kappa X, \kappa R)$

An analysis of the Pine-tree confirms (by direct inspection)

Theorem 6. *The ring $K(X)$ is also closed under operations $\to$ and $\Box$ of the algebra* $\mathrm{Con}(X)$ *and hence is a Solovay algebra, which we denote by $H(G)$.*

Now we can formulate the following

Theorem 7. *The Solovay algebra $H(G)$ is the free cyclic algebra with generator G over the variety* $\mathbf{SA}$.

For the proof of the theorem we need some auxiliary assertions.

Let us define a sequence $(\mathbf{SA}_k; k \in \omega)$ of subvarieties of $\mathbf{SA}$: the pyramid variety $\mathbf{SA}_k$ is described by equation: $\Box^k \bot = \top$. Obviously $\mathbf{SA}_k \subseteq \mathbf{SA_{k+1}}$, for an arbitrary $k \geq 1$ and $k \in \omega$. Say that a Solovay algebra has *height* k if k is the least integer such that $\Box^k \bot = \top$.

It is well known that every interval $[p, q]$ of a Heyting algebra is a Heyting algebra. Let $H(G_k)$ is the interval $[\emptyset, \Box^k \emptyset]$ of the algebra $H(G)$ and define an operator $\Box_k$ on $H(G_k)$ to be $\Box_k A = \Box A \wedge \Box^k \emptyset$. Observe that $H(G_k)$ coincides with the Solovay algebra of all cones of S-frame $(X_k, <_k)$.

Let $(Z_1, <_1), (Z_2, <_2)$ be finite S-frames such that $x \not<_i y$ for every $x, y \in Z_i$, $1 \leqslant i \leqslant 2$. By *the ordinal sum* $Z_1 \oplus Z_2$ of S-frames Z_1 and Z_2 we mean S-frame consisting of all elements $x \in Z_1$ and all elements $y \in Z_2$ in which the relation R is defined as follows: xRy if and only if $x \in Z_1$ and $y \in Z_2$. It is clear that the ordinal sum of S-frames can be easily extended to the case of more than two S-frames. We shall consider ordinal sums $Z_k^{(i,i')} = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$ such that $1 < k \in \omega$ and $|Z_i| = 2$ at most for one $i > 1$, where $Z_j = \{j\}$ for $j \neq i$ (> 1) and $Z_i = \{i, i'\}$. Let us denote by $C_k^{(i,i')}$, $1 < i \leqslant k$, the Solovay algebra (of all cones) of the ordinal sum $Z_k^{(i,i')} = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$, and by $C_k^{(i)}$, $0 \leqslant i \leqslant k$, Solovay algebras of all cones $Z_k^{(i)} = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$ with the generator $R(i)$, where $Z_i = \{i\}$ for all $i \in \{1, \ldots, k\}$ and $R(0) = \emptyset$.

LEMMA 8. *$C_k^{(0)}$, $C_k^{(1)}$, $\ldots$, $C_k^{(k)}$, $C_k^{(2,2')}$, $\ldots$, $C_k^{(k,k')}$ are the only* (*up to isomorphism*) *cyclic subdirect irreducible Solovay algebras of height k, where $\emptyset$ $(= \bot)$ is a generator of $C_k^{(0)}$ and $R(i)$ is a generator of $C_k^{(i)}$ for $1 \leqslant i \leqslant k$, and $R(i)$ is a generator of $C_k^{(i,i')}$ for $1 < i \leqslant k$.*

PROOF. Since the condition $p \leftrightarrow q \leq \Box p \leftrightarrow \Box q$ holds in Solovay algebras, we have that the lattice of congruences of Heyting algebra H coincides with the lattice of congruences of Solovay algebra $(H, \Box)$. So, since the S-frames of the algebras contain the least element, it is evident that the algebras are subdirect irreducible. According to Theorem 4 (Coloring theorem) it is easy to check the algebras are cyclic. Let us suppose that A is a finite cyclic subdirect irreducible Solovay algebra of height k which are differ from the ones mentioned in the lemma. Then the S-frame of A coincides with an ordinal sum $Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$, where either (1) $|Z_i| > 2$ for some $i > 1$ or (2) there are $i_1, i_2 > 1$ such that $|Z_{i_1}| = |Z_{i_2}| = 2$. Let $Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$ and $a \subseteq Z$ be a generator of A which is identified with the set of all cones of Z. Since A is cyclic, Z is colored with two colors: the elements which belong to a and the elements which belong to $Z \setminus a$. In the case (1) at least two elements, say z_1 and z_2, of Z_i have the same color. Then the partition, which has the only non-trivial class $\{z_1, z_2\}$, is correct. Therefore, according to Coloring theorem, a does not generate A. In the case (2), supposing that $i_1 < i_2$, either i_1 or i'_1 belong to a or $i_1, i'_1 \notin a$. In the first case the partition, the only non-trivial class of which is $\{i_2, i'_2\}$, is correct and $i_2, i'_2 \in a$. In the second case the partition, the only non-trivial class of which is $\{i_1, i'_1\}$, is correct. In both cases according to Coloring theorem A is not generated by a. ⊣

Lemma 9. *Let x be arbitrary element of $X_{k+1} \setminus X_k$. The Solovay algebra A_x (which is a homomorphic image of $(H(G_k))$ of all cones of S-frame $R_\circ(x)$ with generator $G_k \cap R_\circ(x)$ is isomorphic to the one of the algebras $C_k^{(0)}$, $C_k^{(1)}$, $\dots$, $C_k^{(k)}$, $C_k^{(2,2')}$, $\dots$, $C_k^{(k,k')}$, where the isomorphism sends the generator of A_x to the corresponding generator of the algebra from this list of algebras.*

Proof. By direct inspection. It is also easy to see from the Fig. 1. ⊣

Theorem 10. *Every pyramid variety $\mathbf{SA}_k$ $(k \in \omega)$ is locally finite and the algebra $H(G_k)$ $(k \in \omega)$ is the free cyclic Solovay algebra with the generator G_k over the variety $\mathbf{SA}_k$.*

Proof. First, we shall show that the free Solovay algebra $F_{\mathbf{SA}}(n)$ on n generators over the variety $\mathbf{SA}_k$ is finite. Let us consider n-generated subdirectly irreducible algebra from $\mathbf{SA}_k$. Then $\mathcal{K}(A)$ can have not more than 2^n elements every of which has color $t \subseteq \{1, \dots, n\}$ (according to Coloring Theorem 4). Then $\mathcal{K}(A)$ has only one element of depth 2. Let us suppose that $\mathcal{K}(A)$ has less than 2^n maximal elements, say $a_1, \dots, a_p$, with corresponding colours. Then the number of elements of $\mathcal{K}(A)$ having depth 2 is not more than $|\{t : t \subseteq \bigcap_{i=1}^{p} \mathrm{Col}(a_i)|$. Continuing this process we observe that the cardinality of the set of elements of depth $i \leq k$ is bounded. In consequence of this observation we conclude that the number of non-isomorphic n-generated subdirectly irreducible Solovay algebras from $\mathbf{SA}_k$ is finite. Consequently the free n-generated Solovay algebra $F_{\mathbf{SA}}(n)$ is finite and therefore $\mathbf{SA}_k$ is locally finite.

Now we shall show that $H(G_k)$ is the free cyclic Solovay algebra with the free generator G_k over the variety $\mathbf{SA}_k$. It is evident that $(H_1, \Box_1)$ is a free cyclic diagonalizable algebra over $\mathbf{D}^{(\mathbf{1})} (= \mathbf{SA_1})$. Let $k > 1$. It is sufficient to show that if an identity $\mathbf{p} = \mathbf{q}$, containing only one variable, does not hold in $\mathbf{SA}_k$ then $\mathbf{p} = \mathbf{q}$ does not hold in $H(G_k)$ on the generator G_k. Let us suppose that an identity $\mathbf{p} = \mathbf{q}$, containing only one variable, does not hold in $\mathbf{SA}_k$. Then there is a finite subdirectly irreducible one generated Solovay algebra $A \in \mathbf{SA}_k$ such that $P(a) \neq Q(a)$, where P, Q are polynomials corresponding to polynomial symbols $\mathbf{p}$, $\mathbf{q}$ respectively and $a \in A$ is a generator of A. Then, according to Lemma 8, there is an isomorphism $\varphi\colon C \to A$ and $\varphi(c) = a$, where $C \in \{C_k^{(0)}, C_k^{(1)}, \dots, C_k^{(k)}, C_k^{(2,2')}, \dots, C_k^{(k,k')}\}$ and $c \in C$ is the generator of C. But, according to Lemma 9, there exists homomorphism $h\colon H(G_k) \to C$ such that $h(G_k) = c$. Therefore there is a homomorphism $u = h\varphi\colon H_k \to A$ such that $u(G_k) = a$ and, consequently, $P(G_k) \neq Q(G_k)$ in $H(G_k)$. $\dashv$

LEMMA 11 ([4]). *The variety* $\mathbf{SA}$ *is generated by their finite members. Moreover* $\mathbf{SA}$ *is generated by* $\bigcup_{k=1}^{\infty} \mathbf{SA}_k$.

PROOF. Here we give a sketch of a proof. First of all observe that the variety of all cascade Heyting algebras is locally finite. Indeed, observe that in any finite subdirectly irreducible Boolean cascade $H = B_1 + \cdots + B_k$ hold the formulas $(\exists x)(x \prec \top \;\&\; (\forall y)(y \prec \top \Rightarrow x = y))$, $(\forall x, y, z)(x \not\leq z \;\&\; z \not\leq x \;\&\; x \leq y \Rightarrow y \leq z)$, $(\forall x, y, z)(x \not\leq z \;\&\; z \not\leq x \Rightarrow ((x \to z) \vee x) \leq y \bigvee y \leq ((x \to z) \vee x)$. The first formula points out that the algebra is subdirectly irreducible, the second one that the algebra is cascade type and the third that there exist nodes in the algebra. Since every subdirectly irreducible cascade Heyting algebra is a homomorphic image of a subalgebra of a ultraproduct of subdirectly irreducible finite Boolean cascades, any subdirectly irreducible cascade Heyting algebra contains Boolean lattices as a convex sublattices (according to second formula). Let us observe that if subdirectly irreducible cascade Heyting algebra is n-generated, then the number of convex Boolean sublattices not more that n. For any n-generated subdirect irreducible cascade Heyting algebra $H = B_1 + \cdots + B_k$ $(k \leq n)$ if $a_1, \dots, a_n \in B_i$ then either $P(a_1, \dots, a_n) \in B_i$ or $P(a_1, \dots, a_n) \in \{\top, \bot\}$, where $P(x_1, \dots, x_n)$ is arbitrary n-ary Heyting polynomial. Moreover, the set of elements $\{\bot_{B_k}, x \in H : x = \top_{B_i}, 1 \leq i \leq k\}$ forms a Heyting subalgebra of H which is a chain. Therefore the cardinality of B_i is at least 2^{2^n}. From here we conclude that the cardinality of subdirect irreducible cascade Heyting algebras which are generated by n generators is bounded. Therefore the variety of Heyting algebras generated by cascade Heyting algebras is locally finite.

Now suppose that $\mathbf{p} = \mathbf{q}$ does not hold in $\mathbf{SA}$. Then there is a Solovay algebra A such that $P(a_1, \dots, a_n) \neq Q(a_1, \dots, a_n)$ for some elements $a_1, \dots, a_n \in A$, where P and Q are polynomials corresponding to the polynomial symbols $\mathbf{p}$ and $\mathbf{q}$ respectively. Let $P_1(x_1, \dots, x_n), \dots, P_m(x_1, \dots, x_n)$ be all subpolynomials of $P(x_1, \dots, x_n)$. Then, since A, as a Heyting algebra, belongs to the variety of cascade Heyting algebras, the Heyting subalgebra of A generated by $P_1(a_1,\dots,a_n)$, $\dots$, $P_m(a_1,\dots,a_n)$ will be finite. Let us denote the subalgebra by A_{fin}.On the Heyting algebra A_{fin} we can define unique operator $\Box$ as follows:

$$\Box p = \bigwedge_{q\in A_{\mathrm{fin}}} (q \vee (q \to p)).$$

Then $(A_{\mathrm{fin}}, \Box)$ will be a Solovay algebra such that $P(a_1,\dots,a_n) \neq Q(a_1,\dots, a_n)$ in $(A_{\mathrm{fin}}, \Box)$. It is evident that since $A_{\mathrm{fin}} \in \mathbf{SA}_k$ for some $k \in \omega$, $\mathbf{p} = \mathbf{q}$ does not hold in $\mathbf{SA}_k$. From here we conclude that $\mathbf{SA}$ is generated by $\bigcup_{k=1}^{\infty} \mathbf{SA}_k$. ⊣

Now we can give

PROOF OF THEOREM 7. It is enough to show that if an identity $\mathbf{p} = \mathbf{q}$, containing only one variable, does not hold in $\mathbf{SA}$ then $\mathbf{p} = \mathbf{q}$ does not hold in $H(G)$ on the generator G. Let us suppose that an identity $\mathbf{p} = \mathbf{q}$, containing only one variable, does not hold in $\mathbf{SA}$. Then there is a finite subdirectly irreducible one generated Solovay algebra $A \in \mathbf{SA}_k$ such that $P(a) \neq Q(a)$, where P, Q are polynomials corresponding to polynomial symbols $\mathbf{p}$, $\mathbf{q}$ respectively and $a \in A$ is a generator of A. Then there is homomorphism $h\colon H(G_k) \to A$, where $h(G_k) = a$. But $H(G_k)$ is a homomorphic image of $H(G)$, where the homomorphism $\varphi\colon H(G) \to H_k$ sends the generator G to the generator G_k. Therefore there is a homomorphism $h\varphi\colon H(G) \to A$ such that $h(\varphi(G)) = a$. It means that $P(G) \neq Q(G)$. ⊣

THEOREM 12. *The algebra $H(G)$ is isomorphic to the subalgebra of the inverse limit $\lim_{\leftarrow}\{H(G_k)\}_{k\in\omega}$ generated by the generator $(G_1, G_2, \dots) = G$.*

PROOF. Observe that the family $\{H(G_k), \pi_{nm} : n \geqslant m \geqslant 1\}$ forms an inverse system. Indeed, $\pi_{nm}\colon H(G_n) \to H(G_m)$ is homomorphism onto for $n \geqslant m$. In the dual picture we have corresponding embedding $\mathcal{S}(\pi_{nm})\colon (X_m, <_m) \to (X_n, <_n)$ $(n \geqslant m)$, where $(X_m, <_m) \cong \mathcal{S}(H(G_m))$ and $(X_n, <_n) \cong \mathcal{S}(H(G_n))$. Identifying the elements of an algebras with the corresponding upper cones the homomorphism π_{nm} is defined as follows: $\pi_{nm}(a) = a \cap X_m$ for every $a \in H(G_n)$. Then $\pi_{ml} \circ \pi_{nm} = \pi_{nl}$, for $n \geqslant m \geq l \geqslant 1$. Recall that the inverse limit

$$\lim_{\leftarrow}\{H(G_k)\}_{k\in\omega} = \{(x_k)_k \in \prod_{k=1}^{\infty} H(G_k) : \pi_{lm}(x_l) = x_m, l \geq m \geq 1\}.$$

Let $\pi_n\colon \prod_{k=1}^{\infty} H(G_k) \to H(G_n)$ be the canonical projection. Let $G = (G_1, G_2, \dots)$. G is an element of the inverse limit and let $F(G)$ be the subalgebra of the inverse limit generated by G. It is well known that because of the fact that $\mathbf{SA}$ is generated by $\bigcup_{k=1}^{\infty} \mathbf{SA}_k$ we only need to show that every map $G \to a$ to an algebra A, generated by a, belonging to some $\mathbf{SA}_k$ can be extended to a homomorphism from $F(G)$ to that algebra. Since $A \in \mathbf{SA}_k$, there exists a homomorphism $h\colon H(G_k) \to A$ such that $h(G_k) = a$ and $h\pi_k\colon H(G) \to A$ is needed homomorphism extending the map $G \to a$. It means that $F(G)$ is the free cyclic Solovay algebra whence it follows that $F(G) \cong H(G)$. Indeed, the map

$$f\colon (a_1, a_2, \dots) \to \bigcup_{i=1}^{\infty} a_i$$

establishes the isomorphism between $F(G)$ and $H(G)$. $\dashv$

THEOREM 13. *Finite upper cones of the kind $R_\circ(x) \subseteq X$, for every $x \in X$, is join irreducible elements of the free cyclic algebra $H(G)$.*

PROOF. Let $x \in X$ be an element of some finite depth $k \geqslant 1$. Since $R_\circ(x)$ is an upper cone of (X, R), $R_\circ(x)$ is an element of $H(G_k)$ and, consequently, $H(G_{k+1})$. Therefore there is a polynomial $P(x)$ such that $R_\circ(x) = P(G_{k+1}) \in H(G_{k+1})$. Then $\pi_{k+1}(P(G)) = P(\pi_{k+1}(G)) = P(G_{k+1}) = R_\circ(x)$, where $\pi_{k+1}\colon H(G) \to H(G_{k+1})$ is the canonical homomorphism which is defined as the map $V \to V \cap X_{k+1}$, for every $V \in H(G)$. Since $R_\circ(x) \cap (X_{k+1} \setminus X_k) = \emptyset$ and $R_\circ(x)$ is an upper cone, $P(G) = R_\circ(x)$. $\dashv$

4. Analysis of S-frame $(\kappa X, \kappa R)$

In conclusion, we outline a portion of the duality theory of the relationship between the free algebra $H(G)$ and the S-frame $\mathcal{S}(H(G)) = (\kappa X, \kappa R)$ (see Pine-tree). Based on the duality between Solovay algebras and S-frames the algebra $H(G)$ can (and henceforth will) be identified with the lattice of clopen cones of its dual topological Kripke frame $(\kappa X, \kappa R)$. We want to associate with the algebra $H(G)$ our Pine-tree. To this end define κX to be a topological space whose points are prime filters of $H(G)$ with the topology determined by the subbase $\{x \in \kappa X : A \in x\}$ and $\{x \in \kappa X : A \notin x\}$ ($A \in H(G)$). Then κX is a Stone space (=a compact, Hausdorff, zero-dimensional space). The dual of $\Box$-operation is the relation κR on X defined by $\kappa R(x, y)$ iff for every $A \in H(G)$, if $\Box A \in x$ then $A \in y$. Moreover, $(\kappa X, \kappa R_\circ)$ is a poset and $\kappa R_\circ$ coincides with the inclusion relation between prime filters of $H(G)$.

Let U be the set of all principal prime filters of the algebra $H(G)$ generated by $R_\circ(x)$ for $x \in X$. It is evident that, according to Theorem 13, since $R_\circ(x)$ is join irreducible, the filter generated by $R_\circ(x)$ will be prime. Let us identify U with X: identify x with the prime filter generated by $R_\circ(x)$. Then $X \subseteq \kappa X$ and $(X, \kappa R) \cong (X, R)$. Since $R_\circ(x) \in H(G)$ for every $x \in X$, $R_\circ(x)$ is a clopen subset of the space κX. Arbitrary element $x \in X$ is covered at least by two-element set, say $\{y_1, y_2\} \subseteq X$, i.e. $x \prec \{y_1, y_2\}$, and $R_\circ(y_i)$ is a clopen subset of κX $(i = 1, 2)$. Therefore $\{x\} = R_\circ(x) \cap (\kappa X \setminus (R_\circ(y_1) \cup R_\circ(y_2)))$ is a clopen, i.e. arbitrary element $x \in X \subseteq \kappa X$ is an isolated point of the space κX. Let us remark that $(X, R_\circ)$ has only two maximal elements a_1 and g_1, and $\{a_1, g_1\} \subseteq X \subseteq \kappa X$ is a clopen of κX and it is an element of $H(G)$. Then $-R_\circ^{-1}\{a_1, g_1\} = \emptyset$. Therefore $-\kappa R_\circ^{-1}\{a_1, g_1\} = \emptyset$. Consequently the limit,i.e. not isolated, points of κX are "below" (with respect to the order $\kappa R_\circ$) the elements of X. From these observations we can prove the following theorems.

Theorem 14. *X is an increasing open dense subset of κX and thus κX is an order compactification of X.*

Proof. The family $\{R_\circ(x)\}_{x\in X}$ has the least upper bound $\top \in H(G)$. It means that $\mathrm{cl}\bigcup_{x\in X} R_\circ(x) = \kappa X$ and $\bigcup_{x\in X} R_\circ(x)$ $(= X)$ is an upper open cone of κX. ⊣

Theorem 15. *A point x is an isolated point of the topological space κX iff $x \in X$.*

Proof. As we have shown above any point $x \in X$ is an isolated point of κX. But X is a dense subset of κX (Theorem 14). Hence any isolated point belongs to X. ⊣

Now we shall describe the prime filters of $H(G)$ which belong to the growth $\kappa X - X$. Let us denote by g_ω the prime filter generated by $G \in H(G)$. g_ω is a prime because G is a chain. It is evident that g_ω is contained in all prime filters g_k $(k \in \omega)$ generated by G_k, which are linearly ordered by inclusion. Notice that if $A \in H(G)$ is a proper subset of G then A is finite, i.e. coincides with G_k for some $k \in \omega$. Therefore for any $A \in H(G)$ if $\Box A \in g_\omega$ then $A \in g_\omega$. Indeed, suppose that for some $A \in H(G)$ $\Box A \in g_\omega$ and $A \notin g_\omega$. It means $A \cap G = G_k$ for some $k \in \omega$. Then g_{k+1} is a maximal element of $X \setminus A$ and therefore $g_{k+2} \notin \Box A$ which contradicts to the supposition $\Box A \supseteq G$. Consequently g_ω is a reflexive point of κX, i.e. $\kappa R(g_\omega, g_\omega)$.

Recall that $\Box G = G \cup \{a_i : i \in \omega\}$ where $\{a_i : i \in \omega\}$ is the set of all maximal elements (with respect to $R_\circ$) of $X \setminus G$. $\Box^2 G = \Box G \cup \{n_i, m_i : i \in \omega\}$ where $\{n_i, m_i : i \in \omega\}$ is the set of all maximal elements (with respect to $R_\circ$) of $X \setminus \Box G$. In our notations $n_1 = (\{(\emptyset, \emptyset)\}, \emptyset)$ and $m_1 = (\{(\emptyset, \emptyset), (\emptyset, \{1\}\}, \emptyset)$. It is easy to see that $R(n_i, a_i)$, $R(m_i, a_i)$ and $R(m_i, g_i)$ $(i \in \omega)$(see Pine-tree). It is evident that $C_{n_i} = R_\circ^{-1}(n_i) \cup R_\circ(n_i)$ and $C_{m_i} = R_\circ^{-1}(m_i) \cup R_\circ(m_i)$ are elements of $H(G)$ and C_{n_i} and $R_\circ^{-1}(m_i)$ are infinite chains. Moreover C_{n_i} and C_{m_i} are join irreducible. Therefore the filters generated by C_{n_i}, C_{m_i}, denote them by ν_i, μ_i $(i \in \omega)$ respectively, will be prime (see Pine-tree). By the same reasons as in the case of the prime filter g_ω we conclude that ν_i, μ_i are reflexive points of κX, i.e. $\kappa R(\nu_i, \nu_i)$ and $\kappa R(\mu_i, \mu_i)$.

Further, analogically as in the case of the principal prime filter g_ω,we can show that the filters generated by $\{\Box^k G, -R_\circ^{-1}(x) : x \in X \setminus G\}$, for some $k \geqslant 1$, are prime which we denote by α_k (see Pine-tree). Let us observe that for the element $\Box(\Box^{k-1}G) \in \alpha_k$ $(k \geqslant 1)$ $\Box^{k-1}G \notin \alpha_k$, whence it follows that α_k is irreflexive point of κX, i.e. $\neg\kappa R(\alpha_k, \alpha_k)$. The filter generated by the set of elements $-R_\circ^{-1}(x)$, where $x \notin G$, is a non-principal prime filter which we denote by α_ω. It is enough to observe that α_ω contains the elements of $H(G)$ with co-finite number of branches. It is evident that $\alpha_\omega \subseteq \ldots \alpha_2 \subseteq \alpha_1$ and for any $i, j, t \in \omega$ all ν_i, μ_j, α_t are incomparable. Let us note that the elements of the kind $-R_\circ^{-1} - (V)$ is a finite set for any finite set $V \subseteq X \setminus G$. Therefore $\Box V \setminus V$ is a finite set. Consequently for any $A \in H(G)$ if $\Box A \in \alpha_\omega$ then $A \in \alpha_\omega$, whence it follows that α_ω is a reflexive point of κX, i.e. $\kappa R(\alpha_\omega, \alpha_\omega)$.

Proceeding from the structure of $H(G)$ which coincides with the ring of cones $K(X)$ and the fact that the intersection of all elements of $H(G)$, represented as clopen cones of κX, belonging to some prime filter F of $H(G)$ coincides with a closed cone, which is either chain or contains no more than two incomparable elements, with the least element, which is isolated if $F \in X$ and is not isolated if $F \notin X$. Therefore we have

Theorem 16. *The set κX of all prime filters of $H(G)$ coincides with the set $X \cup \{\nu_i, \mu_i : i \in \omega\} \cup \{g_i : i \in \omega + 1\} \cup \{\alpha_i : i \in \omega + 1\}$. Moreover, the set $X \cup \{\alpha_i : i \in \omega\}$ coincides with the set of all irreflexive points of κX and $\{g_\omega, \nu_i, \mu_i : i \in \omega\}$ coincides with the set of all reflexive points of κX.*

We conclude with

Theorem 17. *The Heyting lattice of clopen cones of the topological Kripke frame $(\kappa X, \kappa R)$ is isomorphic to the free cyclic Solovay algebra $H(G)$.*

Remark. We associate with every clopen cone A of κX the trace $\varphi A = X \cap A$ of A on X. The map φ establishes the necessary isomorphism. Moreover, the trace of the clopen cone $\kappa R(g_\omega)$ on X is equal to the generator $G = \{g_i \in X : i \in \omega\}$ of $H(G)$.

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