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A MIND OF A NON-COUNTABLE SET OF IDEAS

For the 80th birthday anniversary of my late dear teacher and mentor, A. V. Kuznetsov

Abstract. The paper is dedicated to the 80^{th} birthday of the outstanding Russian logician A. V. Kuznetsov. It is addressing a history of the ideas and research conducted by him in non-classical and intermediate logics.

Keywords: Propositional logic, superintuitionistic logic, intermediate logic, admissible rule, inference rule, modal logic, multi-valued logic, functional completeness, Boolean function.

This is not exactly a scientific article or a review. Rather, it is an attempt to revisit and retrospectively understand the origin and development of some ideas that I have been a witness to. This approach was suggested to me by Professor A. Yu. Muravitsky, to whom I am grateful.

This article is dedicated to Alexander Vladimirovich Kuznetsov, the scientist who has, perhaps, accomplished more than anyone else in the studies of non-classical propositional logics. He was very broad-minded and conducted research in many areas of non-classical logics. Always in a main stream of research, he very often defined what the main stream was. Unfortunately, not all of his results were published in easily accessible scientific journals and thus they are not known enough and recognized in the scientific community.

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Sometimes these results were independently rediscovered and published by others. Often his results with the original proofs were presented at various seminars and published only much later or never were published. To a certain degree, they were part of the Russian logical folklore. No one of his students, myself including, has enough knowledge in all these areas in order to write a comprehensive survey about Kuznetsov's research.

When I started getting interested in mathematical logic in 1968, I was doing research in polynomial representations of k-valued functions, being unaware at the time of Kuznetsov's 1957 results on this topic (see [52]). Later, when I started my research in propositional logics I was impressed by his paper [24]. Since it was hard for me to imagine that the same person accomplished so much in so many different areas of logic and because the family name *Kuznetsov* is quite common in Russia, I thought that there were at least two different individuals named *Kuznetsov*. Only much later, when we met and I began my doctorate in superintutionistc logics under his direction, did I realize my confusion. In this paper, I will focus merely on certain aspects of research in superintuitionistic propositional logics and some precursors of this research.

I start with two definitions very well known to all of Kuznetsov's students. *Philosophical Encyclopedia*, one of the most authoritative sources on philosophy and logic in Russia in the 1950s, defines *Logic* as a science about acceptable means of reasoning. Kuznetsov liked this definition and convinced me to like it too. He often quoted P. S. Poretsky's definition of *Mathematical Logic* as a science, which is logic by subject matter and mathematics by means. This short, elegant and quite exact definition captures two main roots of mathematical logic: philosophical logic and mathematics.

The discovery of paradoxes at the beginning of the 20$^{\text{th}}$ century led to revision of what is admissible in scientific reasoning and what is not. As a result of this revision, the concept of algorithm and notions of computable (i.e. recursive, primitive recursive, etc.) functions were introduced. All this had led to emerging of new systems of logic and analysis of admissibility of rules of inference. The monopoly of classical logic came to an end. In order to support intuitionism, in 1930 A. Heyting had suggested the basis for intuitionistic way of reasoning, which is known to-day as *Intuitionistic Logic*. This logic immediately attracted attention of researchers.

No matter what Kuznetsov was researching, a connection to logic as a means of reasoning was always very important to him, as well as a *finitist approach*, that is, reasoning in the scope of limited means outlined by D. Hilbert. He never completely accepted the approach to propositional logic through closure operator because the means of defining such an operator are often too undetermined or not finitististic enough. This is why he was always more interested in the logic defined by calculus or at least by recursive algebra. This is why he preferred to use the terms "modus rule", "modus completeness" instead of "structural rule", "structural completeness"—in order to emphasize that such a rule is defined by finite lists of formulas. Also, he valued very highly the results on intermediate logics obtained by A. Wroński who has more often presented them in terms of pseudo-Boolean algebras rather than in terms of closure operators. It was very desirable for Kuznetsov to know, whether a particular logic is decidable and how complex the decision procedure is. This is why he had never been interested in *Kleene-Rose Logic* of recursive realizability and, on the contrary, was very interested in Wajsberg's approach to Heyting's propositional calculus, [48] where along with a set of axioms and rules of inference, Wajsberg suggested a decision procedure, the proof of the correctness of which seemed to be finitary. He was disappointed when he learned that the Wajsberg's proof of the main lemma appeared to be incorrect (see [36, 1, 20]).

In the second half of the 20^{th} century, a new twist was added to the research of propositional logics—applications. In 1938 C. Shannon discovered connection between Boolean functions and switching circuits. Since a switch can be in two states, a switching circuit represents a Boolean function. Connection of switching circuits in "function world" corresponds to superposition of corresponding functions. This discovery led to the following important questions:

1. If switching circuit represents a Boolean function, how can a circuit be built with the same functionality by using smallest possible number of basic elements (*minimization of Boolean functions*)? Only when the notion of NP-completeness was introduced, it was proven that the problem of minimization is NP-complete. Therefore, implementation of algorithms of minimization is difficult.
2. How can a circuit with a required functionality be built by using a given set of elements (*functional expressiveness, realization*)?
3. What are the basic elements, by using which the circuit of any required functionality can be built (*functional completeness*)?

The development of new types of logical elements (multi-state elements) required a new logical apparatus that would "upgrade" the Boolean logic to multi-valued logic. The transition to multi-valued logics was natural, since if the Boolean logic, the apparatus of Boolean functions, suits the regular switching circuit needs, the apparatus of multi-valued logics is convenient for needs of circuits of multi-state elements. The same issues of functional expressiveness and completeness for $\boldsymbol{k}$-valued logics attracted attention of researchers.

In 1941 E. Post noticed that there were five functionally closed classes of Boolean functions (now known as *Post classes*) that can be used to determine whether a set of Boolean functions is functionally complete.[1] Taking into account that every Post class is decidable, there is a deciding procedure to determine the functional completeness of any given finite set of Boolean functions. In addition to that, Kuznetsov noticed something else (see [23]):

> *The Post classes are the maximal elements in partially ordered (respective to inclusion) set of all non-trivial functionally closed sets of Boolean functions.*

Thus, the Post classes are (functionally) pre-complete. In other words, adding a new element to a Post class will make it functionally complete; that is, the closure of this enriched class by superposition of functions equals the set of all functions. A. V. Kuznetsov introduced this notion of pre-completeness in 1955, when reviewing Yablonsky's results in (see [52]).

The concept of pre-completeness made a profound impact on many branches of logic and algebra. Kuznetsov generalized the notion of pre-completeness, applying it to any algebraic systems, and proved that there is a finite set of functionally pre-complete classes of multi-valued functions. Realizing that finiteness of the set of functionally pre-complete classes is an astonishing result, one can see that the possibility of applying the notion of pre-completeness to other branches of logic might be even more important. Indeed, it inspired many researchers to use the notion of pre-completeness for different types of closure operations and they started investigating pre-tabular logics, pre-locally tabular logics, pre-finitely approximable logics, pre-structural complete logics etc.

The generalization of notions of the functional expressibility and completeness from $\mathbf{2}$- to $\boldsymbol{k}$-valued logics was natural and relatively simple. The definition of functional expressibility or completeness for a logic without finite characteristic matrix like the intuitionistic logic is much harder. The problem is that certain functions cannot be defined in a finitistic manner.

[1] A set of functions is *functionally complete* if and only if for each of the Post classes, this set contains at least one function that does not belong to this class.

Kuznetsov solved this problem, using syntactical means to define expressibility. I believe, his first result on functional completeness in intuitionistic logic was presented in [25]. He defined the expressibility of a formula F by formulas $F_1, F_2, \ldots, F_n$ as an ability to infer the formula F from $F_1, F_2, \ldots, F_n$ and propositional variables, by using the rule of *weak substitution* (when formula being substituted must be already inferred) and the rule of *replacement by equivalent*. Article [25] gave a jump start to researchers working on expressibility and completeness in the field of propositional logics (e.g., see [40, 41, 10, 27]).

As mentioned above, the *inferential* side of logic was always important to Kuznetsov. So in [3], he introduced the notion of a *Regular Propositional Calculus* (RPC). This notion is general enough and yet is not a pure mathematical abstraction but rather carries a logical sense. Namely, according to Kuznetsov, a RPC is a finite list of axioms and two inference rules, *modus ponens* and *substitution*, in the ordinary propositional language with connectives: $\&, \vee, \supset, \neg$. Thus two RPCs can be different only with their axiom sets. As we see, a calculus for him was just a way to finitarily define a logic as a set of deducible formulas. He always paid a special attention to the extreme cases: the empty calculus, that is, one whose axiom set is empty, and the absolutely contradictory calculus, the axiom set of which contains one axiom, p. He distinguished the terms *superintuitionistic* and *intermediate* as applied to extensions of the intuitionistic propositional logic, favoring the former.[2] In [24] Kuznetsov formulated three algorithmic problems what in a way changed a logical paradigm (in Kuhn's sense):

1. General problem of equivalence: to construct an algorithm that by 2 RPCs I_1 and I_2 recognizes whether I_1 and I_2 are equivalent: define the same logics
2. General problem of decidability: to construct and algorithm that for RPC I and formula A tells whether A is deducible in I.
3. General problem of completeness: to construct an algorithm that by 2 RPCs I_1 and I_2 such that $I_1 \leq I_2$ recognizes whether I_1 and I_2 are equivalent, i.e. I_1 is complete in I_2.

The algorithmic problems were raised up to the next level. In addition to the question, whether a particular logic is decidable, Kuznetsov raised

[2]The distinction is that the former includes the logic of the absolutely contradictory calculus but the latter does not.

the question, whether the property of decidability itself is decidable. In fact, Kuznetsov was a great master of introducing generalizations that would include everything possible, but he never crossed the line of a common logical sense. The notion of a pre-complete class for similar algebraic systems, that of a RPC, and that of *means of separability* [30] are good examples of his approach.

In [24] Kuznetsov proved that the problems 1–3 above are undecidable. Furthermore, the problem of recognition, whether any given RPC is superintuitionistic, is also undecidable. In his proof, the ability to use a non-superintuitionistic calculi was essential. It was natural to ask, whether this problem is decidable for any *Superintuitionistic Propositional Calculus* (SPC). In [3, p. 52] Kuznetsov formulated the following conjecture (despite the fact that the set of all SPCs as a poset is rather complex; see [46]): Every SPC is finitely approximable (which means that every SPC is decidable). The problem, whether every SPC is finitely approximable, became known as *Kuznetzov problem.* In [17] R. Harrop noted that it was unknown, whether there were undecidable SPCs. In the footnote on p. 288, R. Harrop mentioned that paper [24] was unavailable to him and he is merely familiar with the review of it in the *Mathematical Reviews.* The questions asked by Kuznetsov and Harrop prompted a lot of further research on SPCs.

The notion of a *finitely approximable logic* is well known. The idea to use a set of finite matrices to validate formulas when logic is not tabular goes back to [19] (see also [35, 15, 42]). The notion was introduced in [24] and put in the center of stage. The term *finite model property* is more popular nowadays. It was introduced in 1958 by R. Harrop [14]. Kuznetsov published his paper in 1963 and it does not contain a reference to the Harrop's paper. I believe that at the time of publication, not to say at the time when the Kuznetsov's research had been conducted and presented at the Moscow seminar on mathematical logic, he was not familiar with the Harrop's paper. But, in any case, despite the fact that the terms *finite model property* and *finite approximability* became synonyms today, these notions are different. The difference is that the term *model*, as R. Harrop used it, is equivalent to the notion of *logical matrix* whose essential part is a set of designated elements. Obviously, for any superintuitionistic logic, its model in the above sense can be replaced with a pseudo-Boolean algebra. Also, Kuznetsov mentioned to me once that the term *model* had already been used in several different meanings which makes the term *finite model property* ambiguous. I think that he might have borrowed the term *finite approximability* from algebra, specifically from group theory, in particular, from A. I. Mal'cev (see [34]). I believe that one of the reasons that *finite model property* won the 'term contest' is that it is more comprehensible in English than its counterpart.

It is interesting that a notion equivalent to *approximability*, was introduced much earlier by J. C. C. McKinsey. In his paper [35] published in 1943, J. C. C. McKinsey defined the notion of *finite reducibility* as follows:

> "The set of formulas **A** is *finitely reducible* with respect to class of algebras $\mathfrak{A}$ if every formula $\boldsymbol{\alpha}$ of **A**, which is true of every finite algebra of $\mathfrak{A}$, is true of $\mathfrak{A}$." [35, p. 69]

Two paragraphs further, McKinsey states the following theorem:

> "If **A** is class of sentences, that is finitely reducible with respect to axiomatizable class $\mathfrak{A}$ of algebras, then there is a decision method for **A**."

Kuznetsov told me once that the McKinsey's paper became available to him much later after the work on [24] had been completed, partly because during the World War II the delivery of the issues of *Journal of Symbolic Logic* was discontinued. On the other hand, when he was working on his address to the 17$^{\text{th}}$ International Congress of Mathematicians, I pointed out to him that the term *finite model property* might, perhaps, sound better than *finite approximability* for the English speaking audience. But he was skeptical about that and favored the latter term.

In 1965 A. S. Troelstra published a very exciting result: Every intermediate logic is a limit of some monotonic decreasing sequence of tabular logics. This obviously meant that every intermediate logic is finitely approximable and, hence, all SPCs are decidable. The problem was that this statement turned out to be wrong: later in [49] V. A. Yankov[3] proved that there are intermediate logics which are not finitely approximable, but since these logics are not finitely axiomatizable, i.e., are not defined by SPC, the question about finite approximability of SPCs remained open. He also proved [50, 51] that there is a continuum of intermediate logics that are not finitely approximable, by building a set of strongly independent formulas: no one formula of this set can be deduced from the rest. It also meant that the structure of the intermediate logics is much richer than it had been anticipated.

In order to build a strongly independent set of formulas, Yankov used (previously introduced by him [49]) the notion of a *characteristic formula*.

[3] Or "Jankov"—there are two English spellings of his name in literature.

These formulas are linked to pseudo-Boolean algebras, which became commonly accepted algebraic semantics for superintuitionistic logics. Among all pseudo-Boolean algebras, one subclass plays a special role. They, called *Gödelean algebras*, are pseudo-Boolean algebras that satisfy the following condition (discovered by Gödel in his work on the intuitionistic propositional calculus): if disjunction of any two elements is equal to the unit[4], then at least one of these two elements equals the unit. In [49] Yankov introduced the notion of a *Gödelean implicative structure*, that is to say, of a Gödelean pseudo-Boolean algebra. He noticed that a finite pseudo-Boolean algebra is Gödelean if and only if it has a pre-top element. For each finite algebra with a pre-top element, one can build a characteristic formula so that refutability of any formula F on this algebra is equivalent to deducibility of the characteristic formula from the formula F. It is very interesting because typically a deduction is used for proving the validity of formulas, while here deduction is used to refute a formula by deducing something that is not valid on a particular pseudo-Boolean algebra.[5]

The negative answer to the hypothesis in [24] was given in [26] by constructing an example of SPC without the finite approximability. Despite the fact that this discovery was important in its own right, there is something else that is, perhaps, even more important: a link between superintuitionistic logics and varieties of pseudo-Boolean algebras.[6] With each superintuitionistic logic, one can associate a variety of all pseudo-Boolean algebras such that all formulas from this logic are valid on each algebra in this variety. Conversely, each variety of pseudo-Boolean algebras corresponds to the superintuitionistic logic, the formulas of which are valid on the free algebras of this variety. In fact, the free algebra with the countable set of free generators is isomorphic to the Lindenbaum algebra of the corresponding superintuitionistic logic. Thus, finite approximability of the logic is the same as finite approximability of free algebras of the corresponding variety. This idea about dualism between propositional logics and varieties of their models had a profound influence on researchers probably in all classes of propositional logics. It allowed them to have at their disposal both notions and techniques developed in logic and universal algebra.

[4]The unit of a pseudo-Boolean algebra is its greatest element.

[5]In [31] J. Łukasiewicz defined a calculus for the non-theorems of the classical propositional logic. The Łukasiewicz's calculus was formulated not purely in terms of the new deducibility, but also included the deducibility in the classical logic.

[6]To some extent, the idea is traced back to McKinsey and Tarski.

The problem of decidability of SPCs is equivalent to the problem of deducibility in IPC (with substitution rule): Is there a decision procedure determining for any two propositional formulas, whether one of them is deducible from the other?

The hope to answer this question positively by using finite approximability failed, since not every SPC is finitely approximable. The question had remained open until the negative answer was given by V. B. Shehtman in [38] by constructing an undecidable SPC.

In [12] V. Ya. Gerčiu proved that the non-finitely approximable SPC found in [26] is not an exception. He constructed an infinite segment (in the poset of all SPCs), no calculus from which is finitely approximable. In contrast to the "negative" results that showed that class of all SPCs is much more complex than it had been anticipated, L.L. Maksimova proved [33] that there are just three pre-tabular superintuitionistic logics. This means that the problem of recognition, whether SPC is tabular, is decidable.

In [13] it was suggested to approach investigation of SPLs by studying finite slices. A slice is defined by the cardinality of a longest chain in a pseudo-Boolean algebra on which all the formulas of this logic are valid. In [28] Kuznetsov announced that every finitely generated pseudo-Boolean algebra such that not every finite chain algebra is embedded in it is finite.[7] This implies that the logics of finite slices are finitely approximable. Another notion in superintuitionistic logics, adapted from algebra, was that of local tabularity. A logic is called *locally tabular* if for each finite set of propositional variables any set of non-equivalent formulas containing only these variables is also finite. In other words, the corresponding variety of pseudo-Boolean algebras is locally finite, when all its free algebras with finite number of generators are finite. Some results on locally tabular SILs can be found in [8]. One important property of the intuitionistic calculus is the disjunction property: if disjunction of two formulas is provable in the intuitionistic calculus then at least one of these formulas is provable. Obviously, this property pertains to intuitionistic criticism of the foundations of mathematics. From the algebraic standpoint this property being considered for a particular logic means that any free algebra of the corresponding variety of pseudo-Boolean algebras is Gödelian. In [32] the algebraic criteria of disjunctive property was discovered and in [21] it was proven that there is no maximal superintuitionistic logic with the disjunction property.

[7] Another proof of this theorem can be found in [6].

Usually, a propositional calculus is defined as a set of axioms along with a set of rules of inference. An inference rule that preserves the set of provable (in this calculus) formulas is called admissible (in this calculus). Let IPC be an intuitionistic propositional calculus with two postulated rules of inference, modus ponens and substitution. In [16] R. Harrop noticed that besides the postulated rules the rule

$$\neg A \supset (B \vee C)/(\neg A \supset B) \vee (\neg A \supset C)$$

preserves the set of formulas provable in IPC, that is, is *admissible*, but the corresponding implication

$$(\neg A \supset (B \vee C)) \supset ((\neg A \supset B) \vee (\neg A \supset C))$$

is not provable in IPC, that is, the rule above is not *derivable*. It seemed to be an isolated example. However, in 1972 during my discussion with Kuznetsov regarding the problem of deducibility in IPC, he mentioned that there is perhaps another example of admissible non-derivable in IPC rule; namely, one that corresponds to the characteristic formula of the 7-element cyclic pseudo-Boolean algebra,[8] which is:

$$(\neg\neg A \supset A) \supset (A \vee \neg A)/\neg A \vee \neg\neg A$$

Then, he linked this rule to the quasi-equality

$$(\neg\neg A \supset A) \supset (A \vee \neg A) = 1 \Rightarrow \neg A \vee \neg\neg A = 1,$$

which is valid on the free pseudo-Boolean algebras. Naturally, the following question emerged: Is there an algorithm to find whether a given rule is admissible in IPC? Since not so many admissible non-derivable rules had been known, Kuznetsov thought at that time that the variety of rules admissible in IPC should have had a simple description. For instance, there might exist a finite basis for the admissible rules. And, if this hypothesis were true, then the problem of the algorithmic recognition of admissibility would have been resolved positively. He set me a task to try to prove the existence of a finite basis. Several years later after this discussion, when Friedman's paper [11] was published, the problem of the recognition of admissibility in IPC became known as the *40th Problem of Friedman*. The fact that IPC, as

[8]This formula is also known as Scott Formula (see [22]). The rule corresponding to it is also known as Lemmon-Scott Rule (see [45]).

it follows from the above examples, is not structurally complete, that is to say, not every structural rule admissible in IPC is also derivable, while the classical propositional logic is structurally complete, made it interesting to study, which superintuitionistic logics are structurally complete and whether there is a decision procedure to determine by IPC whether it is structurally complete.[9] And the idea of pre-completeness helped one more time. In [4] the complete list of all structurally pre-complete superintuitionistic logics was presented and, since there are only 5 pre-complete logics, the problem of the structural completeness is decidable.

Let us call a *modus variation* a pair $\langle L, R \rangle$, where L is a propositional logic and R is a set of admissible in L structural rules. Two modus variations $\langle L, R_1 \rangle$ and $\langle L, R_2 \rangle$ are equivalent if for any formulas A and B, B can be deduced from A using rules from R_1 if and only if B can be deduced from A using rules from R_2. The lattice of all modus variations of superintuitionistic logics is dual isomorphic to the lattice of all quasi-varieties of pseudo-Boolean algebras. Logic L is structurally complete if there is only one modus variation of this logic. The correspondence between modus variations and quasi-varieties works in the same way as the correspondence between logics and varieties.

Characteristic formulas introduced by Yankov in [49] has been extremely helpful in the exploration of superintuitionistic logics. If we have a finite irreducible algebra, the characteristic formula is the implication, the antecedent of which represents the tables that define all operations for the elements of this algebra and the consequent of which is the pre-top element.[10] If instead of implication we use a rule we will get the characteristic rule of the algebra. In other words, if we have a finitely defined irreducible algebra where equalities

$$A_1 = 1,\ A_2 = 1,\ \ldots, A_n = 1$$

define this algebra and B represents the unique pre-top element (which exists since algebra is irreducible) then

$$A_1 = 1,\ A_2 = 1,\ \ldots, A_n = 1 \Rightarrow B = 1$$

is a characteristic quasi-equality of this algebra. It turned out that every finitely presented pseudo-Boolean algebra is finitely approximable and,

[9] The notion of structural completeness was introduced in [37].

[10] It is a well-known fact that a pseudo-Boolean algebra is irreducible if and only if it has a pre-top element.

therefore, every irreducible finitely presented pseudo-Boolean algebra is finite (see [6]).

The idea of using characteristic rules allowed to prove not only that IPC has many modus variations, but that it is the case for much simpler logic such as the logic L**Z** of infinite one-generated pseudo-Boolean algebra [3]. There are continuum different modus variations of this logic, even though all the rules that are valid in L**Z** are consequences of axioms of L**Z** (see [3]) and generalized Mints rule (see [36]):

$$((A \supset B) \supset (A \vee C)) \vee D/((A \supset B) \supset A) \vee ((A \supset B) \supset C) \vee D.$$

There is another approach to the investigation of structural rules that uses a generalized notion of logical matrix. Traditionally, a logical matrix is a triplet $\mathsf{M} = \langle \mathbb{A}, \Sigma, \Delta \rangle$, where $\mathbb{A}$ is a set of elements, Σ is a set of operations on $\mathbb{A}$ that represent connectives, and Δ is a set of designated elements. Formula A is valid on M if the result of each interpretation of A belongs to Δ. In order to use matrices for modus variations the definition of a matrix should be generalized as follows: instead of the subset Δ, a closure operator Cn should be used with the condition $\Delta = \mathrm{Cn}\varnothing$.[11] Using generalized matrices one can prove that there is an algorithm deciding for two finite generalized matrices, whether they define the same modus variation ([2, 53]).

The third way of looking at modus variations is to introduce [5, 7] an additional modal-like operator $\Box$ on Lindenbaum algebra $\mathfrak{L}$ of IPC, which would represent in a way deducibility in IPC. This operator can de defined as follows:

$$\Box x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following axioms

$$\Box A \supset A$$
$$\Box A \supset \Box\Box A$$
$$\Box(A \supset B) \supset (\Box A \supset \Box B)$$
$$\neg\Box\neg\Box A \supset \Box A$$
$$\Box(A \vee B) \supset (\Box A \vee \Box B)$$

and additional rule $A/\Box A$ are valid. Let $\mathfrak{L}^\Box$ denote this algebra and $\mathrm{L}\mathfrak{L}^\Box$ is the logic of the algebra $\mathfrak{L}^\Box$.

[11] This type of matrices was introduced in [39].

The axioms look fairly familiar. The last one represents Gödel disjunction theorem. The rule A/B is represented in $\mathrm{L}\mathfrak{L}^{\Box}$ by the formula $\Box A \supset \Box B$. Logic $\mathrm{L}\mathfrak{L}^{\Box}$ represents admissibility in the following sense:

> There is an algorithm that for each formula A builds a deductive equivalent in $\mathrm{L}\mathfrak{L}^{\Box}$, a formula of the following form:
>
> $$\mathop{\&}_{i=1}^{n} (\Box A_i \supset \Box B_i)$$

This means that every formula is deductively equivalent in $\mathrm{L}\mathfrak{L}^{\Box}$ to a conjunction of formulas representing rules. The main question was whether $\mathrm{L}\mathfrak{L}^{\Box}$ is decidable (H. Friedman) or, in other words, whether $\mathrm{L}\mathfrak{L}^{\Box}$ is finitely axiomatizable (which is related to the A. V. Kuznetsov's hypothesis about existence of finite basis for admissible rules).

In 1979 at the *2nd Soviet-Finnish Symposium on Modal Logics*, I formulated the following conjecture:

> Logic $\mathrm{L}\mathfrak{L}^{\Box}$ can be defined by the above five axioms and by the formulas:
>
> $$\Box(A_n \supset (s \vee t)) \supset \Box(\bigvee_{j=1}^{n} (A_n \supset p_j) \vee (A_n \supset s) \vee (A_n \supset t)) \quad (*)$$
>
> where $A_n = \mathop{\&}_{i=1}^{n} (p_i \supset q_i)$.

The positive answer to Friedman's problem was given by V. V. Rybakov [44] (see also [43]). The fact that the rules corresponding to formulas (∗) constitute the basis of admissible in IPC rules was proved in [18], where these rules are called Visser's rules (since they were reintroduced by A. Visser [47]). But since the means of inference in $\mathrm{L}\mathfrak{L}^{\Box}$ are stronger than regular, the conjecture that $\mathrm{L}\mathfrak{L}^{\Box}$ may be finite axiomatizable remains open.

Kuznetsov presented a summary his school had conducted by 1974 in [29].

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