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EXTENSIONS OF THE BASIC CONSTRUCTIVE LOGIC FOR WEAK CONSISTENCY B_{Kc1} DEFINED WITH A FALSITY CONSTANT*

Abstract. The logic B_{Kc1} is the basic constructive logic for weak consistency (i.e., absence of the negation of a theorem) in the ternary relational semantics without a set of designated points. In this paper, a number of extensions of B_{Kc1} defined with a propositional falsity constant are defined. It is also proved that weak consistency is not equivalent to negation-consistency or absolute consistency (i.e., non-triviality) in any logic included in positive contractionless intermediate logic LC plus the constructive negation of B_{Kc1} and the (constructive) contraposition axioms.

Keywords: Weak Consistency, Constructive Falsity, Ternary Relational Semantics, Substructural Logics, Paraconsistent Logics

1. Introduction

A *theory* is a set of formulas closed under adjunction and provable entailment (cf. §2). Then, weak consistency is defined as follows:

Definition 1. Let L be a logic and a be a theory whose underlying logic is L. Then, a is w-inconsistent (weakly inconsistent) iff a contains the negation of a theorem of L (a is w-consistent iff it is not w-inconsistent).

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The logic $\mathrm{B_{Kc1}}$, the basic constructive logic adequate to this sense of consistency is defined in [8]. Next, in the same paper, it is shown how to extend $\mathrm{B_{Kc1}}$ with the strong constructive contraposition axioms

(i) $(A \to \neg B) \to (B \to \neg A)$

and

(ii) $B \to [(A \to \neg B) \to \neg A)$

and with some strong implicative axioms up to positive contractionless intuitionistic logic $\mathrm{JW_+}$ (the logic $\mathrm{B_{Kc1}}$ plus (i) and (ii) is dubbed $\mathrm{B_{Kc2}}$). In [8], it is proved that in $\mathrm{JW_+}$ plus (i) and (ii) (consequently, in all logics included in it), weak consistency is not equivalent to negation-consistency and to absolute consistency (i.e., non-triviality) because the ECQ ('e contradictione quodlibet') axioms

(iii) $(A \wedge \neg A) \to \neg B$

(iv) $(A \wedge \neg A) \to B$

and the EFQ ('e falso quodlibet') axioms

(v) $\neg A \to (A \to B)$

(vi) $A \to (\neg A \to B)$

are not provable in $\mathrm{JW_+}$ plus (i) and (ii). Further, in the same paper, it is proved that if the EFQ axioms (v) and (vi) are added to $\mathrm{JW_+}$ plus (i) and (ii), the ECQ axioms (iii) and (iv) are still unprovable. Consequently, in $\mathrm{JW_+}$ plus (i), (ii), (v), (vi), although weak consistency is equivalent to absolute consistency, it is not equivalent to negation-consistency.

In respect of these results, the aim of this paper is fourfold:

1. It will be proved that the weak constructive contraposition axioms

   (vii) $(A \to B) \to (\neg B \to \neg A)$

   (viii) $\neg B \to [(A \to B) \to \neg A)$

   can be added to $\mathrm{B_{Kc1}}$, the resulting logic being different from $\mathrm{B_{Kc2}}$. This logic is dubbed $\mathrm{B_{Kc1'}}$. Further, it is proved that $\mathrm{B_{Kc1'}}$ can be extended with prefixing,

   (ix) $(B \to C) \to [(A \to B) \to (A \to C)]$

suffixing

(x) $(A \to B) \to [(B \to C) \to (A \to C)]$

and the assertion rule

(xi) $\vdash A \Rightarrow \ \vdash (A \to B) \to B$

the resulting logic being different from that obtainable by adding (ix), (x) and (xi) to $\mathrm{B_{Kc2}}$.

In addition to (1), the results on the independence of w-consistency will be strengthened in the following sense. It will be proved that:

2. The characteristic axiom of Dummett's LC (cf. [3])

   (xii) $(A \to B) \vee (B \to A)$

   can be added to $\mathrm{JW_{K+}}$ plus (i) and (ii), weak consistency still being independent of negation-consistency and absolute consistency.

3. The axiom (xii) can be added to $\mathrm{JW_{K+}}$ plus (i), (ii), (v) and (vi), w-consistency still being independent of negation-consistency.

Last but not least, another aim of this paper is the following (a brief discussion precedes it). Let $\mathrm{S_+}$ be a positive logic. Negation can be introduced in $\mathrm{S_+}$ by adding to the positive language the propositional falsity constant F together with the definition

(xiii) $\neg A \leftrightarrow (A \to F)$

Then, two options are open: either no axioms are added to $\mathrm{S_+}$ and a minimal negation is then defined, or some axioms are added to $\mathrm{S_+}$, thus defining this or that concept of negation. Now, let S_F be the result of introducing a negation with a falsity constant F in $\mathrm{S_+}$ and $\mathrm{S}_\neg$ be the result of adding negation with a negation connective. The question of finding definitionally equivalent logics (the concept is treated in §4) $\mathrm{S}_{F'}$ and $\mathrm{S}_{\neg'}$ definitionally equivalent to $\mathrm{S}_\neg$ and S_F, respectively, depends heavily on the strength of $\mathrm{S_+}$. Thus, for example, if $\mathrm{S_+}$ is $\mathrm{J_+}$ (i.e., positive intuitionistic logic), $\mathrm{J_+}$ plus (i), (ii) and (v) (that is, propositional intuitionistic logic) is definitionally equivalent to $\mathrm{J_+}$ plus the following axioms ((xiv) and (xv) would not be independent):

(xiv) $[A \to (B \to F)] \to [B \to (A \to F)]$

(xv) $B \to [[A \to (B \to F)] \to (A \to F)]$

(xvi) $F \to A$

However, consider the logic $B_{+,F}$ defined in [9]. $B_{+,F}$ is the result of introducing a minimal negation in Routley and Meyer's system B_+, which, as is known, is a weak (but most interesting) logic. The question is, which extension, if any, of B_+ with a negation connective is equivalent to $B_{+,F}$? But let us return to our purpose. Despite that fact that B_{Kc1} is not a strong logic, in [9] it is proved that the logic $B_{Kc1,F}$, in which negation is introduced via a falsity constant, is definitionally equivalent to it. A fourth aim of this paper, therefore, is:

4. To define logics formulated with a falsity constant definitionally equivalent to $B_{Kc1'}$, B_{Kc2} and their extensions.

The structure of the paper is as follows. In §2, the logic B_{K+} along with some well known strong positive extensions of it are defined. The logic B_{K+} is the result of adding the K rule

(xvii) $\vdash A \Rightarrow \ \vdash B \to A$

to Routley and Meyer's B_+. In §3, the logics B_{Kc1} and B_{Kc2} are recalled and the logic $B_{Kc1'}$ is introduced. In §4, logics formulated with F definitionally equivalent to those defined in §3 are introduced, and in §5, the definitional equivalence is proved. In §6, all the logics treated so far are extended with some strong implicative axioms. Finally, in §7 the EFQ axioms are added. All logics are proved sound and complete in respect of a modification of Routley and Meyer's ternary relational semantics for relevance logics (note that all logics defined in this paper have the K rule (xvii)).

We end this introduction by remarking that all logics here introduced are paraconsistent logics in the sense of [7], and that they are paraconsistent in respect of a precisely defined sense of consistency, i.e., w-consistency.

2. The positive logic B_{K+} and its extensions

Firstly, the positive logic B_{K+} is defined. It can be axiomatized with

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B)$ / $B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of inference are

$$\begin{aligned}
\text{Modus ponens (MP):}\ & \vdash (A\ \&\ \vdash A \to B) \Rightarrow\ \vdash B \\
\text{Adjunction (Adj.):}\ & (\vdash A\ \&\ \vdash B) \Rightarrow\ \vdash A \wedge B \\
\text{Suffixing (Suf.):}\ & \vdash A \to B \Rightarrow\ \vdash (B \to C) \to (A \to C) \\
\text{Prefixing (Pref.):}\ & \vdash A \to B \Rightarrow\ \vdash (C \to A) \to (C \to B) \\
\text{K:}\ & \vdash A \Rightarrow\ \vdash B \to A
\end{aligned}$$

Therefore, $\mathrm{B_{K+}}$ is $\mathrm{B_+}$ with the addition of the K rule.

We now define the semantics for $\mathrm{B_{K+}}$. A $\mathrm{B_{K+}}$ *model* is a triple $\langle K, R, \vDash\rangle$ where K is a non-empty set, and R is a ternary relation on K subject to the following definitions and postulates for all a, b, c, $d \in K$ with quantifiers ranging over K:

d1. $a \leq b =_{\mathrm{df}} \exists x Rxab$
d2. $R^2abcd =_{\mathrm{df}} \exists x(Rabx\ \&\ Rxcd)$
P1. $a \leq a$
P2. $(a \leq b\ \&\ Rbcd) \Rightarrow Racd$

Finally, $\vDash$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositional variables p, wff A, B and $a \in K$:

(i) $(a \leq b\ \&\ a \vDash p) \Rightarrow b \vDash p$
(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$
(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$
(iv) $a \vDash A \to B$ iff for all b, $c \in K$, $(Rabc\ \&\ b \vDash A) \Rightarrow c \vDash B$

A formula A is $\mathrm{B_{K+}}$ *valid* ($\vDash_{\mathrm{B_{k+}}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

REMARK 1. The postulates P3 $Rabc \Rightarrow b \leq c$, P4 $(a \leq b\ \&\ b \leq c) \Rightarrow a \leq c$ and P5 $R^2abcd \Rightarrow Rbcd$ hold in all models.

In [5] or in [8], it is proved that $\mathrm{B_{K+}}$ is sound and complete in respect of this semantics.

Remark 2. As is known, in the standard semantics for relevance logics (see, e.g., [10]), there is a set of 'designated points' in terms of which the relation $\leq$ is defined and formulas are determined to be valid. The absence of this set in B_{K+} semantics (and the corresponding changes in d1 and the definition of validity) are the only but crucial differences between B_{+} models and B_{K+} models.

Next, we define some positive extensions of B_{K+}. Consider the following axioms and rule of inference

A7. $(B \to C) \to [(A \to B) \to (A \to C)]$

A8. $(A \to B) \to [(B \to C) \to (A \to C)]$

A9. $\vdash A \Rightarrow\ \vdash (A \to B) \to B$

A10. $A \to [(A \to B) \to B]$

A11. $A \to (B \to A)$

A12. $(A \to B) \vee (B \to A)$

The logic TW_{+} ('Contractionless positive Ticket Entailment') is B_{+} plus A7 and A8; the logic EW_{+} ('Contractionless positive Logic of Entailment') is TW_{+} plus A9; RW_{+} ('Contractionless positive Logic of Relevance') is TW_{+} plus A10 (see, e.g., [10] about these logics), JW_{+} ('Contractionless positive Intuitionistic Logic') is RW_{+} plus A11, and finally, LCW_{+} ('Contractionless superintuitionistic logic LC') is JW_{+} plus A12. Therefore, TW_{K+}, EW_{K+}, RW_{K+}, JW_{K+} and LCW_{K+} are, respectively, TW_{+}, EW_{+}, RW_{+}, JW_{+} and LCW_{+} plus the K rule. Since the K rule is not, of course, independent in JW_{K+} and LCW_{K+}, these logics will be referred to by JW_{+} and LCW_{+}, respectively.

We note:

Proposition 1. 1. RW_{K+} *and* JW_{+} *are deductively equivalent logics.*

2. TW_{K+}, EW_{K+}, RW_{K+} $(= JW_{+})$ *and* LCW_{+} *are different logics.*

Proof. (1) is trivial and (2) follows by well known results on relevance and intuitionistic logics (alternatively, one can use MaGIC, the matrix generator developed by J. Slaney (see [11]). □

We now turn to semantics. Consider the following set of postulates

P6. $R^2abcd \Rightarrow (\exists x \in K)[Rbcx\ \&\ Raxd]$

P7. $R^2abcd \Rightarrow (\exists x \in K)[Racx\ \&\ Rbxd]$

P8. $(\exists x \in K)Raxa$

P9. $Rabc \Rightarrow Rbac$

P10. $Rabc \Rightarrow a \leq c$

P11. $(Rabc \ \& \ Rade) \Rightarrow (b \leq e$ or $d \leq c)$

Now $\mathrm{TW}_{\mathrm{K}+}$ models, $\mathrm{EW}_{\mathrm{K}+}$ models $\mathrm{RW}_{\mathrm{K}+}$ models, JW_{+} models and LCW_{+} models are defined, similarly, as $\mathrm{B}_{\mathrm{K}+}$ models except for the addition of the following postulates:

1. $\mathrm{TW}_{\mathrm{K}+}$ models: P6, P7.
2. $\mathrm{EW}_{\mathrm{K}+}$ models: P6, P7, P8.
3. $\mathrm{RW}_{\mathrm{K}+}$ models: P6, P7, P9.
4. JW_{+} models: P6, P7, P9, P10.
5. LCW_{+} models: P6, P7, P9, P11.

As in $\mathrm{B}_{\mathrm{K}+}$ models, validity is defined in all cases in respect of all points of K.

We next define the canonical models (cf. [5]). We begin by recalling some definitions. A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory a is *regular* iff all the theorems belong to a. Finally, a is *null* iff no wff belong to a. Now, we define the $\mathrm{B}_{\mathrm{K}+}$ canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and a, b, $c \in K^T$, $R^T abc$ iff if $A \to B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime non-null theories and R^C be the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the $\mathrm{B}_{\mathrm{K}+}$ *canonical model* is the triple $\langle K^C, R^C, \vDash^C \rangle$.

Now, let L_{+} be any of the extensions of $\mathrm{B}_{\mathrm{K}+}$ defined above. The L_{+} canonical model is defined, similarly, as the $\mathrm{B}_{\mathrm{K}+}$ canonical models except that its items are referred to L_{+} theories instead of $\mathrm{B}_{\mathrm{K}+}$ theories. Then, we have

Proposition 2. *Given the logic* $\mathrm{B}_{\mathrm{K}+}$ *and* $\mathrm{B}_{\mathrm{K}+}$ *semantics, P6, P7, P8, P9, P10 and P11 are the corresponding postulates to A7, A8, A9, A10, A11 and A12, respectively.*

Proof. Given B_{K+} and B_{K+} semantics, we have to prove that each axiom is proved valid with the corresponding postulate and that the corresponding postulate is proved valid with the axiom. Now, that this is the case for A7 (P6), A8 (P7), A9 (P8), A10 (P9) and A11 (P10) is proved in (or can easily be derived from) e.g., [10]. So, we prove that P11 is the corresponding postulate to A12.

1. *A12 is* LCW_+ *valid*: Suppose $a \nvDash A \to B$, $a \nvDash B \to A$ for wff A, B and $a \in K$ in some model. Then, $b \vDash A$, $d \vDash B$, $c \nvDash B$, $e \nvDash A$ for b, c, d, $e \in K$ such that $Rabc$ and $Rade$. By P11, $b \leq e$ or $d \leq c$. So, either $e \vDash A$ or $c \vDash B$, a contradiction.

2. *P11 holds canonically*: Suppose $R^C abc$, $R^C ade$ for a, b, c, d, $e \in K^C$, and, for reductio, $b \nleq^C e$ and $d \nleq^C c$. Then, $A \in b$, $B \in d$, $A \notin e$, $B \notin c$ for some wff A, B. As a is non-null, it is regular by the K rule. So, $(A \to B) \vee (B \to A) \in a$ by A12. As a is prime, $A \to B \in a$ or $B \to A \in a$. So, either $B \in c$ or $A \in e$, a contradiction. □

Remark 3. The correspondence between postulates and axioms A7 (P6), A8 (P7), A9 (P8) and A10 (P9) stated in Proposition 2 can be proved in respect of B_+ instead of B_{K+}.

Now, it is clear that, given the soundness and completeness of B_{K+}, those of TW_{K+}, EW_{K+}, RW_{K+} ($= JW_+$) and LCW_+ in respect of the corresponding semantics follow immediately by Proposition 2.

3. The logics B_{Kc1}, $B_{Kc1'}$ and B_{Kc2}

We add the unary connective $\neg$ (negation) to the positive language. Consider the following axioms:

A13. $\neg A \to [A \to \neg(A \to A)]$
A14. $[B \to \neg(A \to A)] \to \neg B$
A15. $(A \to B) \to (\neg B \to \neg A)$
A16. $\neg B \to [(A \to B) \to \neg A]$
A17. $(A \to \neg B) \to (B \to \neg A)$
A18. $B \to [(A \to \neg B) \to \neg A]$

Then, the logics are axiomatized as follows:

1. B_{Kc1}: $B_{K+} + A13 + A14$

2. $B_{Kc1'}$: $B_{K+} + A13 + A14 + A15 + A16$
3. B_{Kc2}: $B_{K+} + A17 + A18$

We note the following theorems and rules of inference of B_{Kc1}, $B_{Kc1'}$ and B_{Kc2}:

$T1_{B_{Kc1}}$	$\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$	Pref., Suf., A13, A14
$T2_{B_{Kc1}}$	$\vdash A \Rightarrow \vdash \neg A \to \neg B$	K, $T1_{B_{Kc1}}$
$T3_{B_{Kc1}}$	$\neg A \to (A \to \neg B)$	Pref., A13, $T2_{B_{Kc1}}$
$T4_{B_{Kc1}}$	$\vdash A \Rightarrow \vdash (B \to \neg A) \to \neg B$	$T2_{B_{Kc1}}$, A14
$T1_{B_{Kc1'}}$	$(A \to B) \to \{[B \to \neg(A \to A)] \to [A \to \neg(A \to A)]\}$	A13, A14, A15
$T2_{B_{Kc1'}}$	$[B \to \neg(A \to A)] \to \{(A \to B) \to [A \to \neg(A \to A)]\}$	A13, A14, A16
$T1_{B_{Kc2}}$	$\{A \to [B \to \neg(A \to A)]\} \to \{B \to [A \to \neg(A \to A)]\}$	A13, A14, A17
$T2_{B_{Kc2}}$	$B \to \{[A \to [B \to \neg(A \to A)]] \to [A \to \neg(A \to A)]\}$	A13, A14, A18

We now remark the following

Proposition 3. 1. B_{Kc1} *and* $B_{Kc1'}$ *are deductively included in* B_{Kc2}.

2. B_{Kc1} *and* $B_{Kc1'}$ *are different logics.*
3. B_{Kc1}, $B_{Kc1'}$ *and* B_{Kc2} *are well axiomatized in respect of* B_{K+} (*that is, the negation axioms are, in each case, mutually independent*).

Proof. (1) See [8], §6. (2), (3) by MaGIC. □

We now turn to semantics. Consider the following postulates

P12. $(Rabc \;\&\; c \in S) \Rightarrow a \in S$

P13. $(Rabc \;\&\; c \in S) \Rightarrow (\exists x \in K)(\exists y \in S)Rcxy$

P14. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Racx \;\&\; Rbxy)$

P15. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Rbcx \;\&\; Raxy)$

P16. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in S)R^2acbx$

P17. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in S)R^2bcax$

A B_{Kc1} *model* is a quadruple $\langle K, S, R, \vDash\rangle$ where S is a non-empty subset of K, and K, R and $\vDash$ are defined, in a similar way, as in B_{K+} models, except for the addition of the following clause

(v) $a \vDash \neg A$ iff for all $b, c \in K$, $(Rabc \;\&\; c \in S) \Rightarrow b \nvDash A$

and postulates P12 and P13. Then, $B_{Kc1'}$ *models* (B_{Kc2} *models*) are, similarly, defined as B_{Kc1} models, save for the addition of P14, P15 (P16, P17). In the three cases validity is defined in respect of all points of K.

The B_{Kc1} *canonical model* is the quadruple $\langle K^C, S^C, R^C, \vDash^C\rangle$ where K^C, R^C and $\vDash^C$ are defined in a similar way to which they are defined in the B_{K+} canonical model, and S^C is interpreted as the set of all non-null prime w-consistent theories. A theory a is *w-inconsistent* iff for some theorem A of B_{Kc1}, $\neg A \in a$. A theory a is *w-consistent* iff it is not w-inconsistent (cf. definition 1). The $B_{Kc1'}$ *canonical model* and the B_{Kc2} *canonical model* are defined, similarly, as the B_{Kc1} canonical model, its items being referred now, of course, to $B_{Kc1'}$ and B_{Kc2} theories, respectively.

Remark 4. Clause (v) is an adaptation of the negation clause characteristic of minimal intuitionistic logic in binary relational semantics. The intuitionistic clause reads

$$a \vDash \neg A \text{ iff } (Rab \ \& \ b \in S) \Rightarrow b \nvDash A$$

That is, a formula of the form $\neg A$ is true at point a iff A is false in all consistent points accessible from a –'inconsistent' is here understood in the (minimal) intuitionistic way–. So, in ternary relational semantics, the (minimal) intuitionistic clause would be translated as clause (v). That is, a formula of the form $\neg A$ is true in point a iff A is false in all points b such that $Rabc$ for all consistent points c –consistent' is here understood as w-consistent–.

Now, in [8] it is proved that B_{Kc1} and B_{Kc2} are sound and complete in respect of the corresponding semantics just defined. So, we proceed to prove the soundness and the completeness of $B_{Kc1'}$. We first prove a useful proposition stating that w-consistency of theories is preserved when they are extended to prime theories (this proposition is implicitly used in what follows). Let $B_{+,\neg}$ be any extension of B_+ in which the rule contraposition

$$\text{con. } \vdash A \to B \ \Rightarrow \ \vdash \neg B \to \neg A$$

is provable. We note that the following De Morgan law

$$\text{dm. } \vdash (\neg A \vee \neg B) \to \neg(A \wedge B)$$

is provable in $B_{+,\neg}$ (A2, A5, con.). Note also that con. is provable in B_{Kc1}: it is $T1_{B_{Kc1}}$.

We have

Proposition 4. *Let a be a $B_{+\neg}$ w-consistent theory. Then, there is some prime w-consistent theory x such that $a \subseteq x$.*

PROOF. Define from a a maximal w-consistent theory x such that $a \subseteq x$. If x is not prime, then $A \vee B \in x$, $A \notin x$, $B \notin x$ for some wff A, B. Define the theory $[x, A] = \{C \mid \exists D[D \in x \ \& \vdash_{\mathrm{B}_{+,\neg}} (A \wedge D) \to C]\}$. Define $[x, B]$ similarly. It is not difficult to prove that $[x, A]$ and $[x, B]$ are theories strictly including x. Therefore, they are w-inconsistent. So, $\neg C \in [x, A]$, $\neg D \in [x, B]$ for some theorems of $\mathrm{B}_{+,\neg}$ C and D. By definitions, $\vdash_{\mathrm{B}_{+,\neg}} [(A \vee B) \wedge (G \wedge G')] \to (\neg C \vee \neg D)$ for $G \in x$, $G' \in x$. As $(A \vee B) \wedge (G \wedge G') \in x$, $\neg C \vee \neg D \in x$. Then, $\neg(C \wedge D) \in x$ by dm. But $\vdash_{\mathrm{B}_{+,\neg}} C \wedge D$, by Adj. Consequently, x is w-inconsistent, which is impossible, so x is prime. □

Thus, in any logic including B_+ plus con., w-consistent theories can be extended to prime w-consistent theories.

Next, we prove

PROPOSITION 5. *Given the logic* $\mathrm{B_{Kc1}}$ *and* $\mathrm{B_{Kc1}}$ *semantics,*

1. *P14 is the corresponding postulate to A15, and*
2. *P15 is the corresponding postulate to A16.*

PROOF. We prove case 1. The proof of case 2 is similar and is left to the reader.

A15 is B_{Kc1}*' valid*: Suppose $a \vDash A \to B$, $a \nvDash \neg B \to \neg A$ for wff A, B and $a \in K$ in some model. Then, $b \vDash \neg B$, $d \vDash A$ for b, c, $d \in K$ and $e \in S$ such that $Rabc$ and $Rcde$. By d2, R^2abde, and by P14, $Radz$ and $Rbzu$ for $z \in K$ and $u \in S$. By clause (v), $(Rbxy \ \& \ y \in S) \Rightarrow x \nvDash B$ for all $x \in K$ and $y \in S$. So, $z \nvDash B$ ($Rbzu$, $u \in S$). But, by clause (iv), $z \vDash B$ ($Radz$, $d \vDash A$).

P14 holds canonically: it follows immediately from the following lemma:

LEMMA 1. *Let* a, b, c *be non-null elements in* K^T *and* d *a non-null w-consistent member in* K^T *such that* $R^{T2}abcd$. *Then, there are non-null* x *in* K^T *and some non-null w-consistent* y *in* K^T *such that* R^Tacx *and* R^Tbxy.

Let a, b, c be non-null elements in K^T and d a w-consistent element in K^T such that $R^{T2}abcd$, i.e., by d2, R^Tabz and R^Tzcd for some $z \in K^T$. Define the non-null theories $x = \{B \mid \exists A[A \to B \in a \ \& \ A \in c]\}$, $y = \{B \mid \exists A[A \to B \in b \ \& \ A \in x]\}$ such that R^Tacx and R^Tbxy. We prove that y is w-consistent. Suppose it is not. Then, $\neg A \in y$, A being a theorem. So, $B \to \neg A \in b$, $C \to B \in a$ for some wff B and $C \in c$. As A is a theorem, $\vdash_{\mathrm{B_{Kc1}}} (B \to \neg A) \to \neg B$ by $\mathrm{T4}_{\mathrm{B_{Kc1}}}$. So, $\neg B \in b$. Now, $\neg B \to \neg C \in a$ by A15. Therefore, $\neg C \in z$ (R^Tabz, $\neg B \in b$) whence by A13, $C \to \neg(C \to C) \in z$ and, consequently, $\neg(C \to C) \in d$ (R^Tzcd, $C \in c$), contradicting the w-consistency of d. □

Now, given the soundness and completeness of $\mathrm{B_{Kc1}}$, by Proposition 5, it follows:

THEOREM 1 (soundness and completeness of $\mathrm{B_{Kc1'}}$). $\vdash_{B_{Kc1'}} A$ *iff* $\vDash_{B_{Kc1'}} A$.

4. The logic $\mathrm{B_{Kc1}}_F$ and its extensions

We add the propositional falsity constant F to the positive language together with the definition

D¬: $\neg A \leftrightarrow A \to F$

Now, consider the following axioms:

A19. $F \to (A \to F)$

A20. $\vdash A \Rightarrow \ \vDash (A \to F) \to F$

A21. $(A \to B) \to [(B \to F) \to (A \to F)]$

A22. $(B \to F) \to [(A \to B) \to (A \to F)]$

A23. $[A \to (B \to F)] \to [B \to (A \to F)]$

A24. $B \to [[A \to (B \to F)] \to (A \to F)]$

Then, the following logics are defined:

1. $\mathrm{B_{Kc1\mathit{F}}}$: $\mathrm{B_{K+}}$ + A19 + A20
2. $\mathrm{B_{Kc1\mathit{F'}}}$: $\mathrm{B_{K+}}$ + A19 + A20 + A21 + A22
3. $\mathrm{B_{Kc2\mathit{F}}}$: $\mathrm{B_{K+}}$ + A23 + A24

We shall prove that $\mathrm{B_{Kc1\mathit{F}}}$ and $\mathrm{B_{Kc1}}$, $\mathrm{B_{Kc1\mathit{F'}}}$ and $\mathrm{B_{Kc1'}}$, and $\mathrm{B_{Kc2\mathit{F}}}$ and $\mathrm{B_{Kc2}}$ are definitionally equivalent. So, the relations between the logics stated in Proposition 3 correspondingly hold for the definitionally equivalent logics defined with the falsity constant. Moreover, we remark that $\mathrm{B_{Kc1\mathit{F}}}$, $\mathrm{B_{Kc1\mathit{F'}}}$ and $\mathrm{B_{Kc2\mathit{F}}}$ are well axiomatized in respect of $\mathrm{B_{K+}}$ (MaGIC, cf. Proposition 3).

We note the following theorems of $\mathrm{B_{Kc1\mathit{F}}}$

$\mathrm{T1_{B_{Kc1\mathit{F}}}}$. $\neg A \to [A \to \neg(A \to A)]$ A19, Pref., D¬

$\mathrm{T2_{B_{Kc1\mathit{F}}}}$. $[B \to \neg(A \to A)] \to \neg B$ A20, Pref., D¬

We now define the semantics. Consider the following postulate

P18. $a \in S \Rightarrow (\exists x \in K)(\exists y \in S)Raxy$

A $\mathrm{B_{Kc1\mathit{F}}}$ *model* is a quadruple $\langle K, S, R, \vDash\rangle$, where K, S, R and $\vDash$ are defined, in a similar way, as in a $\mathrm{B_{Kc1}}$ model, except that clause (v) is substituted by the clauses

(vi) $(a \leq b \ \& \ a \vDash F) \Rightarrow b \vDash F$

and

(vii) $a \vDash F$ iff $a \notin S$

and that postulate P13 is substituted by P18.

$\mathrm{B_{Kc1\mathit{F}'}}$ *models* ($\mathrm{B_{Kc2\mathit{F}}}$ *models*) are defined similarly as $\mathrm{B_{Kc1\mathit{F}}}$ models save for the addition of P14 and P15 (P16, P17). In the three cases validity is defined in respect of all points of K.

Now, we introduce the following definition:

Definition 2. Let L_F be a logic whose language has the propositional falsity constant F. Further, let a be a L_F theory. Then, a is inconsistent iff $F \in a$; a is consistent iff a is not inconsistent.

The $\mathrm{B_{Kc1\mathit{F}}}$ *canonical model* is the quadruple $\langle K^C, S^C, R^C, \vDash^C\rangle$, where K^C, R^C and $\vDash^C$ are defined in a similar way to which they are defined in the $\mathrm{B_{Kc1}}$ (or $\mathrm{B_{K+}}$) canonical model, and S^C is the set of all non-null prime consistent theories, 'consistent' being understood as in definition 2. The $\mathrm{B_{Kc1\mathit{F}'}}$ *canonical model* and the $\mathrm{B_{Kc2\mathit{F}}}$ *canonical model* are defined similarly, but with its items referred to $\mathrm{B_{Kc1\mathit{F}'}}$ theories and $\mathrm{B_{Kc2\mathit{F}}}$ theories, respectively.

Now, in [9] it is proved that $\mathrm{B_{Kc1\mathit{F}}}$ is sound and complete in respect of the semantics just defined. So, we shall prove the soundness and completeness of $\mathrm{B_{Kc1\mathit{F}'}}$ and $\mathrm{B_{Kc2\mathit{F}}}$. As in the case of $\mathrm{B_{Kc1}}$, a proposition on the preservation of consistency in building prime theories is provable. Let $\mathrm{B}_{+,F}$ be the result of extending the positive language of B_+ with the propositional falsity constant F, no new axioms, however, being added. We have:

Proposition 6. *Let a be a consistent $\mathrm{B}_{+,F}$ theory. Then, there is some prime consistent theory x such that $a \subseteq x$.*

Proof. Define from a a maximal consistent theory x such that $a \subseteq x$. If x is not prime, then $A \vee B \in x$, $A \notin x$, $B \notin x$ for some wff A, B. Define the theories $[x, A]$ and $[x, B]$ strictly including x, similarly, as in Proposition 4. Then, $[x, A]$ and $[x, B]$ are inconsistent, i.e., $F \in [x, A]$,

$F \in [x, B]$ whence, by definitions, $\vdash_{B_{+,F}} (A \wedge C) \to F$, $\vdash_{B_{+,F}} (B \wedge C') \to F$ for $C \in x$, $C' \in x$. Then, $F \in x$ (cf. Proposition 4), which is impossible. Therefore, x is prime. □

Thus, in any logic including $B_{+,F}$, consistent theories can be extended to prime consistent theories.

We now prove

Proposition 7. *Given the logic* B_{Kc1F} *and* B_{Kc1F} *semantics, P14, P15, P16 and P17 are the corresponding postulates to A21, A22, A23 and A24, respectively.*

Proof. We prove, e.g., that P16 is the corresponding postulate to A23. The rest of the cases are proved similarly and are left to the reader.

A23 is B_{Kc2F} *valid*: suppose $a \vDash A \to (B \to F)$, $a \nvDash B \to (A \to F)$ for wff A, B and $a \in K$ in some models. Then, $b \vDash B$, $d \vDash A$, $e \nvDash F$ for b, c, d, $e \in K$ such that $Rabc$ and $Rcde$. By d2, R^2abde, and as $e \in S$, by P16, $Radx$ and $Rxby$ for $x \in K$ and $y \in S$. So, $x \vDash B \to F$ and then, $y \vDash F$, i.e., $y \notin S$ (clause (vii)), a contradiction.

P16 holds canonically: It follows immediately from the following lemma:

Lemma 2. *Let a, b, c be non-null members in K^T and d a non-null consistent member in K^T such that $R^{T2}abcd$. Then, there are non-null y in K^T and non-null consistent x in K^T such that R^Tacy and R^Tybx, i.e., $R^{T2}acbx$.*

Proof. Suppose non-null a, b, c in K^T and non-null consistent d in K^T such that $R^{T2}abcd$, i.e., R^Tabz and R^Tzcd for some (non-null) $z \in K^T$. Define the non-null theories $y = \{B \mid \exists A[A \to B \in a \ \&\ A \in c]\}$, $x = \{B \mid \exists A[A \to B \in y \ \&\ A \in b]\}$ such that R^Tacy and R^Tybx. We prove that x is consistent. Suppose it is not. Then, $F \in x$. So, $B \to (A \to F) \in a$ for some $A \in b$, $B \in c$. By A23, $A \to (B \to F) \in a$. So, $B \to F \in z$ (R^Tabz) and so, $F \in d$ (R^Tzcd), contradicting the consistency of d. □

Now, given the soundness and completeness of B_{Kc1F}, by Proposition 7, it follows:

Theorem 2 (soundness and completeness of $B_{Kc1F'}$ and B_{Kc2F}).

1. $\vdash_{B_{Kc1F'}} A$ *iff* $\vDash_{B_{Kc1F'}} A$
2. $\vdash_{B_{Kc2F}} A$ *iff* $\vDash_{B_{Kc2F}} A$

We end this section with the following proposition:

Proposition 8. *Let a be a $\mathrm{B}_{\mathrm{Kc1}F}$ theory. Then, a is inconsistent iff a is w-inconsistent.*

Proof. (1) Suppose $F \in a$ and let A be a theorem. By A19, $A \to F \in a$. (2) Let A be a theorem and $A \to F \in a$. Then, $F \in a$ by A20. □

Therefore, in $\mathrm{B}_{\mathrm{Kc1}F}$ (and in all logics included in it) inconsistency (as the presence of F) and w-inconsistency are coextensive.

5. The definitional equivalence between $\mathbf{B}_{\mathbf{Kc1}}$ and $\mathbf{B}_{\mathbf{Kc1}F}$ and their respective extensions

Firstly, we introduce F by definition in $\mathrm{B}_{\mathrm{Kc1}}$. Note that for any formulas A, B, $\neg(A \to A)$ and $\neg(B \to B)$ are equivalent by $\mathrm{T2}_{\mathrm{B}_{\mathrm{Kc1}}}$. Then, we state:

Let A be a wff. Then,

DF: $F \leftrightarrow \neg(A \to A)$

That is, F replaces any wff of the form $\neg(A \to A)$ (note that the defining formula does not depend on the choice of A). We remark:

Proposition 9. *Let a be a $\mathrm{B}_{\mathrm{Kc1}}$ theory. Then, a is w-inconsistent iff for some wff A, $\neg(A \to A) \in a$.*

Proof. By $\mathrm{T2}_{\mathrm{B}_{\mathrm{Kc1}}}$. □

Therefore, in $\mathrm{B}_{\mathrm{Kc1}}$ (and in all logics including it) a theory is w-inconsistent iff it contains F. In fact, this proposition is a corollary of the following:

Proposition 10. *Let a be a $\mathrm{B}_{\mathrm{Kc1}}$ theory. Then, a is w-inconsistent iff a contains the negation of any theorem.*

Proof. By $\mathrm{T2}_{\mathrm{B}_{\mathrm{Kc1}}}$. □

And this proposition is, in turn, a corollary of this one:

Proposition 11. *Let a be a $\mathrm{B}_{\mathrm{Kc1}}$ theory. Then, a is w-inconsistent iff a contains every negative formula.*

Therefore, in B_{Kc1} (and in all logics which include it) w-inconsistency is equivalent to the presence of every negative formula, the presence of the negation of any theorem or, finally, the presence of F (as defined above). Next, we turn to the proof of the definitional equivalence. We shall understand the notion as 'definitional equivalence via translations' (see, e.g., [6]). We have to prove the following two propositions (Proposition 12 is not sufficient: cf. [2]):

Proposition 12. 1. $B_{Kc1F} \subseteq B_{Kc1} \cup \{DF\}$

2. $B_{Kc1} \subseteq B_{Kc1F} \cup \{D\neg\}$

Proposition 13. 1. $D\neg$ *is provable in* $B_{Kc1} \cup \{DF\}$

2. DF *is provable in* $B_{Kc1F} \cup \{D\neg\}$

Propositions 12 and 13 are proved in [9]. So, in order to prove the definitional equivalence between $B_{Kc1'}$ and $B_{Kc1F'}$, B_{Kc2} and B_{Kc2F}, it suffices to prove propositions 14 and 15 that follow:

Proposition 14. 1. $B_{Kc1'} \subseteq B_{Kc1F'} \cup \{D\neg\}$

2. $B_{Kc1F'} \subseteq B_{Kc1'} \cup \{DF\}$

Proof. 1. A21 = A15, A22 = A16, by $D\neg$.

2. $T1_{B_{Kc1'}}$ = A21, $T2_{B_{Kc1'}}$ = A22, by DF. □

Proposition 15. 1. $B_{Kc2} \subseteq B_{Kc2F} \cup \{D\neg\}$

2. $B_{Kc2F} \subseteq B_{Kc2} \cup \{DF\}$

Proof. 1. A23 = A17, A24 = A18, by $D\neg$.

2. $T1_{B_{Kc2}}$ = A23, $T2_{B_{Kc2}}$ = A24, by DF. □

6. Strengthening the positive logics

We take up again the extensions of B_{K+} defined in §2. Now, negation can be introduced in these logics in a similar way to which it was introduced in B_{K+}. Thus, the following logics can be defined:

1. TW_{Kc1}, EW_{Kc1}, RW_{Kc1} (= JW_{c1}), LCW_{c1}

2. $\mathrm{TW_{Kc1'}}$, $\mathrm{EW_{Kc1'}}$, $\mathrm{RW_{Kc1'}}$ (= $\mathrm{JW_{c1'}}$), $\mathrm{LCW_{c1'}}$

3. $\mathrm{TW_{Kc2}}$, $\mathrm{EW_{Kc2}}$, $\mathrm{RW_{Kc2}}$ (= $\mathrm{JW_{c2}}$), $\mathrm{LCW_{c2}}$

It is clear that, given propositions 14 and 15, the logics definitionally equivalent to those in groups 1–3, can be defined:

1′. $\mathrm{TW}_{\mathrm{Kc1}F}$, $\mathrm{EW}_{\mathrm{Kc1}F}$, $\mathrm{RW}_{\mathrm{Kc1}F}$ (= $\mathrm{JW}_{\mathrm{c1}F}$), $\mathrm{LCW}_{\mathrm{c1}F}$

2′. $\mathrm{TW}_{\mathrm{Kc1}F'}$, $\mathrm{EW}_{\mathrm{Kc1}F'}$, $\mathrm{RW}_{\mathrm{Kc1}F'}$ (= $\mathrm{JW}_{\mathrm{c1}F'}$), $\mathrm{LCW}_{\mathrm{c1}F'}$

3′. $\mathrm{TW}_{\mathrm{Kc2}F}$, $\mathrm{EW}_{\mathrm{Kc2}F}$, $\mathrm{RW}_{\mathrm{Kc2}F}$ (= $\mathrm{JW}_{\mathrm{c2}F}$), $\mathrm{LCW}_{\mathrm{c2}F}$

We prove:

Proposition 16. $\mathrm{TW_{Kc1}}$ *and* $\mathrm{TW_{Kc1'}}$ *are deductively equivalent logics. So,* $\mathrm{EW_{Kc1}}$ *and* $\mathrm{EW_{Kc1'}}$*,* $\mathrm{RW_{Kc1}}$ (= $\mathrm{JW_{c1}}$) *and* $\mathrm{RW_{Kc1'}}$ (= $\mathrm{JW_{c1'}}$) *and* $\mathrm{LCW_{c1}}$ *and* $\mathrm{LCW_{c1'}}$ *are deductively equivalent logics.*

Proof. A15 is derivable by A8, A13 and A14; A16 is derivable by A7, A13 and A14. □

Proposition 17. $\mathrm{RW_{Kc1}}$ (= $\mathrm{JW_{c1}}$) *and* $\mathrm{RW_{Kc2}}$ (= $\mathrm{JW_{c2}}$) *and* $\mathrm{LCW_{c1}}$ *and* $\mathrm{LCW_{c2}}$ *are deductively equivalent logics.*

Proof. Firstly, we note that A15 and A16 are derivable. Next, by A11 and A15,

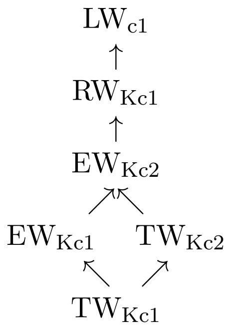
1. $A \to [\neg A \to \neg(A \to A)]$

By 1 and A14

2. $A \to \neg\neg A$

Then, A17 and A18 are easily provable with, respectively, A15 and A16 together with introduction of double negation (2). □

Now, as $\mathrm{EW_{Kc1}}$ and $\mathrm{TW_{Kc2}}$ (so, $\mathrm{TW_{Kc1}}$ and $\mathrm{TW_{Kc2}}$, $\mathrm{EW_{Kc1}}$ and $\mathrm{EW_{Kc2}}$) and $\mathrm{EW_{Kc2}}$ and $\mathrm{RW_{Kc1}}$ are different logics (MaGIC), the relations between these logics can be summarized in the following diagram where the arrow (→) stands for set inclusion.



A similar diagram is, of course, obtained for the definitionally equivalent logics defined with the propositional falsity constant.

Remark 5. Recall that $\mathrm{LCW_{c1}}$, $\mathrm{RW_{Kc1}}$, $\mathrm{EW_{Kc2}}$ and $\mathrm{TW_{Kc2}}$ are the result of adding the strong constructive contraposition axioms A17 and A18 to $\mathrm{LCW_{+}}$, $\mathrm{RW_{K+}}$ (= $\mathrm{JW_{+}}$), $\mathrm{EW_{K+}}$ and $\mathrm{TW_{K+}}$, and that $\mathrm{EW_{Kc1}}$ and $\mathrm{TW_{Kc1}}$ are, respectively, $\mathrm{EW_{K+}}$ and $\mathrm{TW_{Kc+}}$ plus the weak constructive contraposition axioms A15 and A16.

Remark 6. $\mathrm{EW_{Kc2}}$, $\mathrm{EW_{Kc1}}$, $\mathrm{TW_{Kc2}}$ and $\mathrm{TW_{Kc1}}$ are constructive modal logics (the arrow in these logics is some kind of strict implication). But we note that these logics are not included in, e.g., Lewis' modal S5 as axiomatized by Hacking [4] (and, of course, neither do they include it): A13, for example, is not a theorem of S5. On the other hand, we remark that a necessity operator $\Box$ can be introduced (as in [1], §4.3) in $\mathrm{EW_{Kc2}}$ and $\mathrm{EW_{Kc1}}$ via the definition $\Box A =_{\mathrm{df}} (A \to A) \to A$. Generally speaking, the operator thus introduced has the characteristic properties of the necessity operator of Lewis' S4 but with interesting relations with a possibility operator $\Diamond$ definable from it, due to the absence of elimination of double negation and its accompanying theses. The analysis of this question cannot, however, be pursued here.

Regarding soundness and completeness of the logics introduced in this section, it is obvious that they follow immediately from those of the positive logics and $\mathrm{B_{Kc1}}$ ($\mathrm{B_{Kc1\mathit{F}}}$), $\mathrm{B_{Kc1'}}$ ($\mathrm{B_{Kc1\mathit{F}'}}$) and $\mathrm{B_{Kc2}}$ ($\mathrm{B_{Kc2\mathit{F}}}$).

We end this section with the following propositions

Proposition 18. *Let a be a theory of* $\mathrm{B_{Kc1}}$. *Then, if a is w-inconsistent, a contains a contradiction.*

Proof. Suppose $\neg A \in a$, A being a theorem. By the K rule, $\vdash_{\mathrm{B_{Kc1}}} \neg A \to A$. So, $A \in a$ and, consequently, $A \wedge \neg A \in a$. $\Box$

However, the converse of this proposition does not hold because it is proved:

PROPOSITION 19. *The ECQ axioms (iii)* $(A\wedge\neg A)\to B$, *(iv)* $(A\wedge\neg A)\to\neg B$ *and the EFQ axioms (v)* $\neg A\to(A\to B)$, *(vi)* $A\to(\neg A\to B)$ *(cf. §1) are not provable in* LCW_{c1}.

PROOF. By MaGIC. □

Therefore, in $\mathrm{LCW_{c1}}$ (and all logics included in it), w-consistency is not equivalent to negation-consistency or absolute consistency.

Finally, we note:

PROPOSITION 20. *The reductio and contraction axioms cannot be added to* $\mathrm{B_{Kc1}}$ *if we do not want w-consistency to collapse in negation consistency.*

PROOF. 1. Suppose that the principle of non-contradiction

(xviii) $\neg(A\wedge\neg A)$

is added to $\mathrm{B_{Kc1}}$. Then, the ECQ axiom

(iii) $(A\wedge\neg A)\to\neg B$

is derivable by $\mathrm{T3_{B_{Kc1}}}$.

2. Suppose the contraction axiom

(xix) $[A\to(A\to B)]\to(A\to B)$

is added to $\mathrm{B_+}$. Then,

(xx) $[A\to(B\to C)]\to[(A\wedge B)\to C]$

is provable, and so, the ECQ axiom (iii) follows by $\mathrm{T3_{B_{Kc1}}}$.

3. Not even

(xxi) $[A\wedge(A\to B)]\to B$

can be added, because (iii) is again provable by $\mathrm{T3_{B_{Kc1}}}$.

Now, if (iii) is provable, w-consistency collapses in negation-consistency, by Proposition 18. □

7. Introducing the EFQ axioms

In [8], the EFQ axioms are added to $\mathrm{JW}_{\mathrm{Kc1}}$ and it is proved that, though w-consistency is then equivalent to absolute consistency, it is not equivalent to negation-consistency. We shall prove that this result still holds if the EFQ axioms are added to $\mathrm{LCW}_{\mathrm{c1}}$.

Consider the EFQ axioms

A25. $\neg A \to (A \to B)$
A26. $A \to (\neg A \to B)$

and in the form

A27. $(A \to F) \to (A \to B)$
A28. $A \to [(A \to F) \to B]$

The logics are:

1. $\mathrm{LCW}_{\mathrm{c1}}$ + A25 (= $\mathrm{LCW}_{\mathrm{c1}}$ + A26).
2. $\mathrm{LCW}_{\mathrm{c1}F}$ + A27 (= $\mathrm{LCW}_{\mathrm{c1}F}$ + A28).

We note the following theorem of $\mathrm{LCW}_{\mathrm{c1}}$ + A25:

$\mathrm{t1}_{\mathrm{LCW_{c1}+A25}}$. $[A \to \neg(A \to A)] \to (A \to B)$ A14, A25

Remark 7. Semantics for $\mathrm{LCW}_{\mathrm{c1}}$ + A25 (or $\mathrm{LCW}_{\mathrm{c1}F}$ + A27) are considerably different from those of the logics treated so far. The reader is referred to [8] for details.

We prove:

Proposition 21. $\mathrm{LCW}_{\mathrm{c1}}$+A25 *and* $\mathrm{LCW}_{\mathrm{c1}F}$+A27 *are definitionally equivalent logics.*

Proof. Given propositions 14, 15, it follows immediately by $\mathrm{t1}_{\mathrm{LCW_{c1}+A25}}$ with DF and by A27 with D¬. □

Now, we have, of course, by A26:

Proposition 22. *Let a be a* $\mathrm{LCW}_{\mathrm{c1}}$+A25 *theory. Then, a is w-inconsistent iff a contains every wff.*

However, we note (MaGIC):

Proposition 23. *The* ECQ *axioms* (iii) *and* (iv) (*cf.* §1) *are not provable in* $LCW_{c1} + A25$.

Therefore, in $LCW_{c1} + A25$ (and all logics included in it), w-consistency is not equivalent to negation-consistency. So, all logics defined in this paper are paraconsistent logics in the sense of [7]. And, we note, they are paraconsistent in respect of a precisely defined concept of consistency, i.e., w-consistency.

Notes

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