Logic and Logical Philosophy
Volume 16 (2007), 147–257
DOI: 10.12775/LLP.2007.006

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MODAL HYBRID LOGIC

Abstract. This is an extended version of the lectures given during the 12-th Conference on Applications of Logic in Philosophy and in the Foundations of Mathematics in Szklarska Poręba (7–11 May 2007). It contains a survey of modal hybrid logic, one of the branches of contemporary modal logic. In the first part a variety of hybrid languages and logics is presented with a discussion of expressivity matters. The second part is devoted to thorough exposition of proof methods for hybrid logics. The main point is to show that application of hybrid logics may remarkably improve the situation in modal proof theory.

Keywords: modal logics, tense logics, hybrid logics, correspondence theory, proof methods

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Received August 23, 2007; *Revised* September 17, 2007
 ISSN: 1425-3305

0. Preface

Modal hybrid logic (MHL) is an extension of the standard modal logic (ML) obtained by some modifications of the language. The fundamental change, forming the basis of the whole family of hybrid languages, involves addition of special symbols called *nominals*, that enable explicit reference to states in Kripke models. The name of this approach reflects the fact that nominals are at the same time names of states in a model, and sentences of modal language. Although the first attempts in this field are quite old and can be traced back to Arthur Prior's work on modal and tense logics in late 1950's, the serious and systematic studies started in 1990's. Contemporary MHL seems to be one of the most dynamic branch of modern modal logic and offer a lot of improvements over classical results. I hope that this elementary introduction to the subject will show some of the advantages offered by MHL for researchers on modal logics. If so, you should visit MHL homepage where you can find most of the papers including results described in these notes.

The text consists of the short introduction containing also brief historical sketch, and two more elaborate parts dealing with two main advantages offered by hybrid logic. The first part of this survey is devoted mainly to the presentation of the problem of hybrid languages expressivity. After short recollection of the basics of ordinary modal logics, we present a survey of the most important hybrid languages, logics and their hierarchy. In every case the weakest logic is defined and complete axiomatization is presented. We conclude this part with an exposition of first-order modal hybrid logic, with short remarks on decidability and complexity of hybrid logics, and results on interpolation property. It must be stressed that this part has very rudimentary character; it is merely a collection of results with references. Usually no proofs are offered since interested reader may find them in referred papers. Sketchy character of this part is justified because one can find comprehensive survey of these matters in [10], also Ph.D thesis of Balder ten Cate [115] may serve as a good introduction.

The second part covers proof theory of hybrid logics and offers more detailed exposition. I've tried to present all deductive systems constructed so far for hybrid logics and describe their most interesting features. This panorama of systems is embedded in the wider context of the application of labels in proof systems. Even if it is not an exhaustive presentation it is certainly fuller than those present in other surveys of hybrid logics. There are at least three reasons that I've decided to focus on the proof methods for MHL. First of all, there are already mentioned good surveys of MHL concerned mainly with model theory but they do not deal with proof theory in systematic manner ([10] contains rather brief exposition). Secondly, I've an impression that investigations on proof methods for modal logics are still rather neglected area, in comparison to model theoretic research. But using MHL instead of standard ML may offer a breakthrough in this field, so careful analysis is really needed. I hope that the second part of this survey may serve as a first (very imperfect) step in this direction. Last, but not least, this domain is closer to my interests.

These notes are based on the lectures I gave on the 12-th Conference on Applications of Logic in Philosophy and in the Foundations of Mathematics in Szklarska Poręba (7–11 May 2007). This is the main reason that the language of the text is rather informal and the technicalities are not treated in an exact way. I had no enough time to prepare more satisfying exposition. I'd like to express my gratitude to the Organizers of this Conference: Jan Zygmunt, Janusz Czelakowski and Tomasz Połacik for inviting me and giving an impact to more serious study on MHL. I am also greatly indebted to Andrzej Pietruszczak and Marek Nasieniewski for encouraging me to transform crude conference slides into this text, and giving an opportunity for publication in LLP. I hope that the resulting survey will not disappoint their expectations. Finally I apologize for possible mistakes and omissions of many interesting and important results and their authors in this survey. Some of them are briefly mentioned in the historical part of the Introduction.

INTRODUCTION

1. Hybrid Logic in a Nutshell

Before we present MHL we should say a few words on motivation for the introduction of this variant of modal logic. No doubts, the main breakthrough in the development of modal logic was the invention of relational semantics, often called Kripke semantics (although it has many fathers—see e.g. historical notes in [22]), in 1950's and 1960's. Simple, natural, philosophically motivated semantics is still considered as a basic tool for model theoretic investigations on modal logic, but four decades of research has shown that the correspondence with old syntactical tradition is far from being perfect. Carlos Areces has pointed out (as mentioned in [20]) that the very source of the problem is an asymmetry between local perspective of relational semantics and global perspective of standard modal language. Namely, states in a model which are essential in relational semantics, are not represented in modal syntax. But what's wrong with this? We can mention at least two undesirable results of this situation:

- the lack of adequate representation for many semantic features
- problems with suitable modal proof theory

The first item is quite easy to explain. Standard modal languages have no mechanisms for naming particular *states* (*worlds*) in a model, asserting or denying equality of states, talking about accessibility of one state from another. All these things lie at the heart of modal model theory but there is no way of representing them in standard modal syntax. The situation is striking; especially if we compare it with the situation in classical first order logic, where elements of a model have direct representation in a language. In effect, many important properties of relational frames are expressible in a very roundabout way, while many others are not expressible at all in the standard modal language.

The second item is harder to describe and it will be treated in detail in the second part of this survey. Here we just notice that standard proof theory for modal logics is very limited in the scope of applications. The problem with application of ordinary proof methods to standard modal logics is connected mainly with the fact that it is difficult to handle the information which is under the scope of modal operators. There is a lot of non-axiomatic proof systems for many modal logics, but in many cases they represent ad hoc artificial solutions of the problem of their formalization. Systems which seem to be natural (in what sense?—we will discuss this problem later) are rather formalizations of some particular logics and their generalizations are often not easy to provide. So, for the time being, it must be stated that in standard modal logic there is no uniform syntactic frame comparable to successful semantic framework provided by relational models.

Hence the natural question arises how to find a remedy for the problem of discrepancy between a syntax and a semantics. One possibility is to introduce an explicit syntactic representation of states of a model in a language. Such an extension is needed to get enough flexibility of expression. But it leads to the next question: In what way we can realize this task?

We can distinguish at least two approaches:

- external: e.g. Gabbay's Labelled Deductive Systems (LDS)
- internal = HYBRID LOGICS!

In external approach we use additional metatheoretic apparatus connected to the language in question. In case of modal logics the most popular solution was the addition of the machinery of prefixes to formulae, due to Fitting [45]. The best advocate of this approach in its mature form is Dov Gabbay with his general theory of formalization of logics as labelled deductive systems [50]. We will say more about several applications of labels in logic in the second part.

Internal approach consists of the enrichment of the object language obtained via sorting (of the atoms) and addition of the new operators and/or modalities. It is the way of doing hybrid logic.

What do we get with the help of such an enrichment? In particular, do we have some substantial advantages over standard modal languages? This question is particularly interesting in the context of sorting. It is well known that in the case of first order languages we do not get more expressiveness if we use many sorts of variables—we may only obtain more compact and simpler formulation of things already expressible in standard one-sorted language. However, in the context of modal languages the use of several sorts of (propositional) variables leads to real changes in expressive power and in consequence to further improvements. So hybrid modal languages are constructed mainly as tools for repairing the situation of asymmetry between elements of relational structures and language abilities. In short, an introduction of hybrid languages give us the following advantages:

- more expressive language,
- better behavior in completeness theory,
- more natural and simpler proof theory,
- good behavior in decidability, complexity, interpolation and other important features.

The first item in the most literal sense means that we have more validities in the logic formulated in enriched language. But more important fact is that hybrid languages allow definability of many frame properties which are not expressible in standard modal languages.

These improved expressive capabilities lead to more straightforward, and in fact complete, theory of frame definability. General completeness theorems obtained in MHL are also simpler than respective results in standard ML, like famous Sahlqvist completeness theorem.

In what sense proof systems for MHL are more natural and simpler, we will explain in the second part of these notes. But a few words of explanation are in order. We have mentioned that application of standard proof methods to modal logics is complicated because of the difficulties with handling the sentences which are under the scope of modal operators. As we will see, in modal hybrid logics there are natural tools, namely nominals and sat-operators, to deal with this problem. Every modalised sentence in MHL may be broken into separate parts; some of them carry information on the structure of a model, whereas one gives us directly the sentence being previously in the scope of modal operators. This natural way of decomposing complex information into simpler parts, makes easier the transfer of non-axiomatic methods from classical logics to modal logics. Hence, richer languages of MHL offer more general and uniform syntactical setting for modal proof theory.

Last thing worth mentioning is that in many cases (but not all!) we do not need to pay for the improved expressive power of the language. One of the very important features of logics is their decidability and complexity of decision procedures. As we will see, hybrid counterparts of decidable modal logics are still decidable and usually complexity is also untouched (for example sat-problem for basic hybrid logic is PSPACE-complete as in standard modal logics **K**). Moreover, in many respects hybrid logics behave better than standard modal logics—it is evident, for example, in the case of interpolation theorems.

2. Historical Remarks

Although MHL is quite fresh branch of modal logic it has origins in late 1950's. But the importance of hybrid logic was not recognized properly until 1990's. I'm not going to enter into historical details (one should consult [10] and [94] for Prior's ideas), but few words are in order.

All the sources agree that the name of the inventor of MHL belongs to Arthur Prior. He is well known as a father of standard tense logic, but some of his later contributions passed unnoticed. Prior devised two different calculi formally related to McTaggart's analysis of time in terms of A- and B-series. Standard tense logic (T-calculus) using tense constants F and P (see the next section for an explanation) corresponds to A-series (time expressed in terms of past, present and future). I-calculus (later called U-calculus), using binary I-relation over instants of time, corresponds to B-series (earlier/later).

Although I-calculus is more expressive than T-calculus, Prior was convinced that tenses are metaphysically more fundamental. I-calculus provides only a convenient, but indirect way of speaking. So the Prior's problem was: how to show the primacy of T-calculus over I-calculus?

The solution he finally proposed was to develop I-calculus inside T-calculus via extension of the language, and this led him to invention of strong hybrid logic with instant-variables and $\forall$. In [104], inspired by Quine's famous considerations on modality, he introduced the concept of four grades of tense-logical involvement. Whereas in the first grade, tenses are regarded just as handy definitions added to I-calculus, further grades offer essentially hybrid ideas. In the second grade Prior introduced formulae of the form $T(a, \varphi)$ meaning "φ is true at time a" and, moreover, in the second grade he admitted that instant variables a, b, c should also represent propositions. So, two essential ideas of contemporary MHL were introduced: internalization of satisfaction relation (here relative to time instants) and sorting of propositions into ordinary and nominals (as they are commonly called nowadays).

The first idea, of using some syntactical operators which encode semantical satisfaction relation, was quite popular. One may recall at least three early well known constructions that make use of such operators: the situation calculus of J. McCarthy and P. Hayes [90], topological logic of N. Rescher, A. Urquhart [105], and "Holds" operator of J. Allen [3] in his language for temporal representation in AI. Independent line of thought leading to similar ideas is present in the work of J. Perzanowski [98, 99, 100] introducing the general theory of modal operators ("makings") in formal ontology.

Similar concept, but developed on the metalevel, was inherent in the idea of labeled deduction, mentioned above as the external approach to representation of states. The origin of this approach is due to Fitting [44, 45], and the general development is present in Gabbay's theory of labelled deductive systems [50]. But there is a lot of interesting papers on several forms of labelled deduction worth mentioning; we will return to this point in the second part of these notes.

The second idea, although more fundamental for MHL (there are hybrid languages with only nominals), was for a long time forgotten. The early work of Prior's student R. Bull [31] introduces "history variables" for representing paths in branching tense logic, but it was unnoticed and for a long time there are no traces of interest in using nominals. This idea of using nominals comes back in the number of papers (e.g. [96, 97, 53, 58]) written by logicians from Sofia school (Gargov, Tinchev, Passy, Goranko) and devoted particularly to the development of CPDL (Combinatory Propositional Dynamic Logic). By the way, except the return of nominals, in the aforementioned works we have also the development of hybrid binders, see e.g. [58].

Genuine hybrid logic movement started with the works of P. Blackburn [16, 17] devoted to nominal tense logic, and with the works of J. Seligman

[109] devoted to proof methods for situation theory. Since then many researchers, including M. Tzakova, M. Marx, C. Areces, Balder ten Cate and many others, took part in the development of strong and versatile theory of MHL which will be the subject of our presentation.

PART I

3. Basic Modal and Tense Logic

Before we start with the presentation of modal hybrid logics we recall the most basic and the most important (for our interests) facts concerning standard modal languages and logics. In particular, we restrict the presentation to relational semantics and normal modal logics, since investigations on hybrid modal logics are mainly concerned with this area. Most of the information from this section is just to fix a notation and to keep the text self-sufficient. The reader who needs deeper knowledge of the subject should consult some textbooks e.g. [22, 56].

3.1. Languages and Logics

Let $\mathbf{L_M}$ denote standard modal propositional language, i.e., abstract algebra of formulae $\langle \mathrm{FOR}, \neg, \wedge, \vee, \rightarrow, \Box, \Diamond \rangle$, with denumerable set of propositional variables: $\mathrm{VAR} := \{ p, q, r, \ldots, p_1, q_1, \ldots \} \subseteq \mathrm{FOR}$, where $\neg$, $\wedge$, $\vee$, $\rightarrow$ denote boolean negation, conjunction, disjunction and implication. $\Box$, $\Diamond$ denote unary modal operators of (*alethic*) necessity and possibility, but of course many other interpretations of epistemic or deontic character may be applied.

To represent elements of FOR we will use φ, ψ, χ; Γ, Δ will denote subsets of FOR. Propositional variables are the only *atomic formulae* (shortly: atoms) of standard language. Formulae are defined in ordinary way, i.e.,

- $\mathrm{VAR} \subseteq \mathrm{FOR}$;
- if $\varphi \in \mathrm{FOR}$ and $\odot \in \{\neg, \Box, \Diamond\}$, then $\odot\varphi \in \mathrm{FOR}$;
- if $\varphi, \psi \in \mathrm{FOR}$ and $\odot \in \{\wedge, \vee, \rightarrow\}$, then $(\varphi \odot \psi) \in \mathrm{FOR}$.

Languages (and logics) with one pair of modalities are commonly called *monomodal*. Bur for many purposes the generalization to multimodal languages with many modalities is quite natural. In such cases we will use $\Box_i$, $\Diamond_i$ to denote respective modalities ($i \geqslant 1$) in contrast to $\Box^n$, $\Diamond^n$ which means that suitable modal constant is put n-times before some formula. If different modalities are independent of themselves (no interaction), the results which hold for monomodal logics are straightforward to extend to multimodal case. The situation is more interesting and more difficult for multimodal logics with interactive modalities but their expressive power is usually considerably stronger. The best, up-today exposition of problems connected with combining logics may be find in [51].

In particular, $\mathbf{L_T}$ is the bimodal variant of $\mathbf{L_M}$ with interactive Priorean operators G, F, H, P designed for dealing with temporal interpretation of modalities. These operators are interpreted respectively as: always in the future, sometimes in the future, always in the past, sometimes in the past.

From the technical point of view $\mathbf{L_T}$ is very important modal language since except ordinary (forward-looking) modalities (G corresponding to $\Box$ and F to $\Diamond$) it has a pair of backward-looking operators H and P. It gives us extra expressive power which is evident in the context of MHL.

Every modal logic may be defined as a class of formulae in $\mathbf{L_M}$ containing all tautologies of **CPL** (classical propositional logic) and closed under (MP) (modus ponens) and substitution. Every *normal modal logic* satisfies additional conditions:

1) it includes the following formulae:

$$\Box(p \to q) \to (\Box p \to \Box q) \tag{K}$$

$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{Pos}$$

where $(\varphi \leftrightarrow \psi) := ((\varphi \to \psi) \wedge (\varphi \leftarrow \psi))$.

2) for any $\varphi \in \mathrm{FOR}$, if it contains φ then it contains $\Box\varphi$.

The class of Priorean tense logics is defined similarly in $\mathbf{L_T}$, but we must double (K) and (Pos) by putting G (or H) instead of $\Box$, and F (or P) instead of $\Diamond$. We need also an inclusion of every pair of formulae concerning interrelation between future and past:

$$p \to \mathsf{G}\,\mathsf{P}\,p \tag{GP}$$

$$p \to \mathsf{H}\,\mathsf{F}\,p \tag{HF}$$

Instead of clause 2, we have:

2′) for any $\varphi \in \mathrm{FOR}$, if it contains φ then it contains $\mathsf{G}\,\varphi$ and $\mathsf{H}\,\varphi$.

The minimal normal modal logic is **K**, whereas its tense counterpart is **Kt**.

3.2. Relational Semantics

The standard semantic approach to modal logic is based on the use of relational frames (models) often called Kripke frames. Although this approach has serious limitations—it is not suitable not only for weak modal logics (like classical or monotonic) but also for many normal and regular ones—it is still the most popular and simple way of interpreting normal modal logics.

DEFINITION. The Modal Frame $\mathfrak{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, where $\mathcal{W} \neq \varnothing$ is the set of states (*worlds*), and $\mathcal{R}$ is a binary relation on $\mathcal{W}$, called *accessibility relation.*

In multimodal case we have a family of accessibility relations, each for one (pair of) modalities. But in temporal case we can still use one relation since the intended meaning of the second one is the converse of the first. So it is simpler to define:

DEFINITION. The Temporal Frame $\mathfrak{T} = \langle \mathcal{T}, < \rangle$, where $\mathcal{T} \neq \varnothing$ is the set of time-instants and $<$ is a binary relation on $\mathcal{T}$—*the flow of time relation.*

DEFINITION. A model on the frame $\mathfrak{F}$ (or $\mathfrak{T}$) is any structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ (or $\mathfrak{M} = \langle \mathfrak{T}, V \rangle$), where V is a valuation function on propositional variables $V\colon \mathrm{VAR} \longrightarrow \mathcal{P}(\mathcal{W})$ (or $V\colon \mathrm{VAR} \longrightarrow \mathcal{P}(\mathcal{T})$).

In what follows, we will usually state facts in general for modal logic and only in cases where the use of temporal language leads to different results we will point out the differences.

Interpretation. Due to the more complicated character of a semantics, the notion of an interpretation of a formula (and related semantical concepts) may be defined on different levels. The most basic is the notion of satisfaction of a formula in a state of a model, which is defined as follows:

$$
\begin{array}{lll}
\mathfrak{M}, w \vDash \varphi & \text{iff} & w \in V(\varphi), \text{ for any } \varphi \in \mathrm{VAR} \\
\mathfrak{M}, w \vDash \neg\varphi & \text{iff} & \mathfrak{M}, w \nvDash \varphi \\
\mathfrak{M}, w \vDash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \vDash \varphi \text{ and } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \varphi \vee \psi & \text{iff} & \mathfrak{M}, w \vDash \varphi \text{ or } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \varphi \rightarrow \psi & \text{iff} & \mathfrak{M}, w \nvDash \varphi \text{ or } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \Box\varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ for any } w' \text{ such that } \mathcal{R}ww' \\
\mathfrak{M}, w \vDash \Diamond\varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ for some } w' \text{ such that } \mathcal{R}ww'
\end{array}
$$

and for temporal operators:

$$
\begin{array}{lll}
\mathfrak{M}, t \vDash \mathsf{G}\,\varphi & \text{iff} & \mathfrak{M}, t' \vDash \varphi \text{ for any } t' \text{ such that } t < t' \\
\mathfrak{M}, t \vDash \mathsf{F}\,\varphi & \text{iff} & \mathfrak{M}, t' \vDash \varphi \text{ for some } t' \text{ such that } t < t' \\
\mathfrak{M}, t \vDash \mathsf{H}\,\varphi & \text{iff} & \mathfrak{M}, t' \vDash \varphi \text{ for any } t' \text{ such that } t' < t \\
\mathfrak{M}, t \vDash \mathsf{P}\,\varphi & \text{iff} & \mathfrak{M}, t' \vDash \varphi \text{ for some } t' \text{ such that } t' < t
\end{array}
$$

The set of all states (time-instants) where φ is satisfied in a model, often called a *proposition expressed by* φ, will be denoted as $\|\varphi\|_{\mathfrak{M}}$.

The preceding definition states conditions for local (at a state in a model) satisfiability. We have also the concept of global satisfiability in a model, defined as follows:

- $\mathfrak{M} \models \varphi$ iff $\forall_{w \in \mathcal{W}_{\mathfrak{M}}} \mathfrak{M}, w \models \varphi$ (or $\|\varphi\|_{\mathfrak{M}} = \mathcal{W}$).

Formulae globally satisfiable are often called universally true in a model. Both notions of satisfiability may be extended to sets of formulae, namely:

- $\mathfrak{M}, w \models \Gamma$ iff $\forall_{\varphi \in \Gamma} \mathfrak{M}, w \models \varphi$,
- $\mathfrak{M} \models \Gamma$ iff $\forall_{\varphi \in \Gamma} \mathfrak{M} \models \varphi$.

We say that Γ is simply *satisfiable* iff there is some model and a state which locally satisfies Γ, otherwise Γ is *unsatisfiable*.

Semantical characterization of modal logics is connected not with particular models but with frames and their sets, otherwise we do not secure the closure under substitution[1]. This leads to further generalization of the notion of interpretation, namely validity at a state on a frame and validity on a frame. Both relations are defined as follows:

- $\mathfrak{F}, w \models \varphi$ iff $\forall_{\mathfrak{M} \in \mathrm{MOD}(\mathfrak{F})} \mathfrak{M}, w \models \varphi$
- $\mathfrak{F} \models \varphi$ iff $\forall_{\mathfrak{M} \in \mathrm{MOD}(\mathfrak{F})} \mathfrak{M} \models \varphi$

where $\mathrm{MOD}(\mathfrak{F})$ is the set of all models built on the basis of $\mathfrak{F}$. These relations may be generalized in a natural way to classes of frames (denoted by $\mathcal{F}$) which is of great importance for defining modal logics. If we take the class of all frames we obtain the concept of validity of a formula:

- $\models \varphi$ iff $\forall_{\mathfrak{F}} \mathfrak{F} \models \varphi$ iff $\forall_{\mathfrak{M}} \mathfrak{M} \models \varphi$.

It is well known fact that the set of all valid formulae in $\mathbf{L_M}$ coincides with $\mathbf{K}$, and the set of all valid formulae in $\mathbf{L_T}$ coincides with $\mathbf{Kt}$. Stronger logics over $\mathbf{K}$ or $\mathbf{Kt}$ are modelled by restricting the class of frames to those that satisfy some conditions on accessibility relation. This leads to the concept of restricted validity on the suitable class of structures:

- $\models_{\mathcal{F}} \varphi$ iff $\forall_{\mathfrak{F} \in \mathcal{F}} \mathfrak{F} \models \varphi$

We say that Γ is $\mathcal{F}$-satisfiable ($\mathcal{F}$-unsatisfiable) if we restrict ourselves only to models belonging to $\mathrm{MOD}(\mathcal{F})$.

[1] Of course, if we drop this condition from the definition of modal logic, we may characterize logics in terms of models, see e.g. [56].

In fact, the set of validities of any $\mathcal{F}$ is normal modal logic, although not every normal modal logic is characterized by a class of frames. Since, in what follows we will be dealing only with logics that posses such a characterization, we will usually identify logics with suitable sets of validities, but distinguish their several syntactic formalizations.

Entailment. The concept of an entailment (consequence relation) may be defined in two nonequivalent ways:

1) φ *follows locally* in $\mathcal{F}$ from Γ:

- $\Gamma \models_{\mathcal{F}} \varphi$ iff $\forall_{\mathfrak{M}\in \mathrm{MOD}(\mathcal{F})} \|\Gamma\|_{\mathfrak{M}} \subseteq \|\varphi\|_{\mathfrak{M}}$
  or $\forall_{\mathfrak{M}\in \mathrm{MOD}(\mathcal{F})}\forall_{w\in \mathcal{W}_{\mathfrak{M}}}$ if $\mathfrak{M}, w \vDash \Gamma$ then $\mathfrak{M}, w \vDash \varphi$

2) φ *follows globally* in $\mathcal{F}$ from Γ:

- $\Gamma \mid\models_{\mathcal{F}} \varphi$ iff $\mathrm{Mod}_{\mathcal{F}}(\Gamma) \subseteq \mathrm{Mod}_{\mathcal{F}}(\varphi)$
  or $\forall_{\mathfrak{M}\in \mathrm{MOD}(\mathcal{F})}$ if $\mathfrak{M} \vDash \Gamma$ then $\mathfrak{M} \vDash \varphi$

where $\mathrm{Mod}_{\mathcal{F}}(\varphi) = \{\mathfrak{M} \in \mathrm{MOD}(\mathcal{F}) : \mathfrak{M} \vDash \varphi\}$.

Note the following:

Fact 1. (1) *if* $\Gamma \models_{\mathcal{F}} \varphi$ *then* $\Gamma \mid\models_{\mathcal{F}} \varphi$.

(2) $\Gamma \mid\models_{\mathcal{F}} \varphi$ *iff* $\Box^n\Gamma \models_{\mathcal{F}} \varphi$, *where* $\Box^n\Gamma = \{\Box^n\varphi : \varphi \in \Gamma\}$.

(3) $\Gamma \models_{\mathcal{F}} \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is* $\mathcal{F}$*-unsatisfiable.*

3.3. Axiomatic Approach to Normal Modal Logics

The earliest and still the most popular syntactic style of defining modal logics was axiomatic. In particular, popular axiomatic (or Hilbert) formalization of the weakest normal modal logic **K** denoted by **HK** consists of:

1. axioms of **CPL** (any complete set is suitable),
2. axioms of **K**, i.e., (`K`) and (`Pos`),
3. rules:

$$\vdash \varphi \to \psi, \vdash \varphi \;/\; \vdash \psi \qquad (\mathtt{MP})$$

$$\vdash \varphi \;/\; \vdash \Box\varphi \qquad (\mathtt{RG})$$

$$\vdash \varphi \;/\; \vdash e(\varphi) \qquad \text{where } e\colon \mathrm{VAR} \longrightarrow \mathrm{FOR} \qquad (\mathtt{Sub})$$

$\vdash \varphi$ means of course that φ is a thesis of **HK**, i.e., has a proof in **HK** being a sequence of formulae deduced from axioms by the three rules displayed above. Moreover, $\vdash \Gamma$ means that all formulae in Γ are theses.

For axiomatic characterization of **Kt** we must double (K), (Pos) and (RG) by putting G (or H) instead of $\Box$ and F (or P) instead of $\Diamond$. We need also a pair of interactive axioms concerning interrelation between future and past: (GP) and (HF).

All other normal modal (tense) logics are obtained by addition of some further axioms to **HK** (**HKt**). In this case we write $\vdash_{\boldsymbol{L}} \varphi$ to denote that φ is a thesis of a logic $\boldsymbol{L}$. Let $\mathrm{Th}(\boldsymbol{L})$ be the set of theorems of the logic $\boldsymbol{L}$. So syntactically every normal modal logic may be defined as a class of formulae of $\mathbf{L_M}$ containing **K** and closed under primitive rules displayed above. It is evident that axiomatic characterization is closely related to abstract definition of normal modal logics as sets satisfying some closure conditions.

Deducibility. The relation of deducibility (provability) may be defined in two nonequivalent ways:

- $\Gamma \vdash_{\boldsymbol{L}} \varphi$ iff for some where $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$: $\vdash_{\boldsymbol{L}} \psi_1 \wedge \dots \wedge \psi_n \to \varphi$
- $\Gamma \Vdash_{\boldsymbol{L}} \varphi$ iff there is a proof of φ in $\boldsymbol{L}$, where formulae from Γ are also used as premises for the application of rules

We have: $\Gamma \vdash_{\boldsymbol{L}} \varphi$ iff there is a sequence of formulae which deduced φ from $\Gamma \cup \mathrm{Th}(\boldsymbol{L})$ by only one rule, the rule (MP).

In what follows we will be rather interested in the first (weaker) notion of deducibility. In particular, we define Γ as $\boldsymbol{L}$-inconsistent iff $\Gamma \vdash_{\boldsymbol{L}} \bot$, where $\bot := p \wedge \neg p$. Otherwise, Γ is $\boldsymbol{L}$-consistent.

Three primitive rules of axiomatic system are theoretically sufficient, but in practice one can use many others in order to obtain shorter proofs. We divide secondary (or additional) rules on two groups:

- $\Gamma \;/\; \varphi$ is $\boldsymbol{L}$-derivable iff $\Gamma \vdash_{\boldsymbol{L}} \varphi$,
- $\Gamma \;/\; \varphi$ is $\boldsymbol{L}$-admissible iff $\vdash_{\boldsymbol{L}} \Gamma$ implies $\vdash_{\boldsymbol{L}} \varphi$.

Clearly, every $\boldsymbol{L}$-derivable rule is also $\boldsymbol{L}$-admissible, but the opposite usually does not hold for logics we consider. The syntactic proofs of admissibility of rules are sometimes hard to obtain. Note also that the set of derivable rules is hereditary with respect to stronger logics, but it does not hold for admissible rules in general. These two classes of rules will be of interest for us not in the context of axiomatic systems but rather in the second part, where we consider other kinds of deductive systems. Despite the differences of formulation between several formalizations, these two concepts may be easily adapted to other notions of a proof and provability. For considerations on their interrelations it will be crucial to show that some rules primitive in one system may be shown secondary in the other.

Completeness. We've already mentioned that stronger logics (in semantic sense) are modelled by classes of frames, where relation of accessibility satisfies some conditions. It was a great success of relational semantics that many well known (in axiomatic sense) modal logics like Feys' **T** or Lewis' **S4** and **S5** obtained simple semantic characterizations. The link between syntactic formalizations and classes of frames is obtained via soundness and completeness theorems of the form:

- (Soundness) if $\Gamma \vdash_{\boldsymbol{L}} \varphi$, then $\Gamma \models_{\mathcal{F}} \varphi$
- (Completeness) if $\Gamma \models_{\mathcal{F}} \varphi$, then $\Gamma \vdash_{\boldsymbol{L}} \varphi$

The last one is often formulated equivalently:

- if Γ is $\boldsymbol{L}$-consistent, then Γ is $\mathcal{F}$-satisfiable.

If the first theorem holds, then $\boldsymbol{L}$ is sound with respect to $\mathcal{F}$, if the second holds, then $\boldsymbol{L}$ is (strongly) complete with respect to $\mathcal{F}$. If $\boldsymbol{L}$ is both sound and complete, then $\mathcal{F}$ characterizes $\boldsymbol{L}$ or $\boldsymbol{L}$ is determined by $\mathcal{F}$.

Note that if Γ is empty (in the first formulation) or finite, we have weak completeness, otherwise we have strong form (i.e., admitting infinite Γ). There are modal logics which are weakly complete, but not strongly complete, with respect to some class of frames.

Rules of the axiomatic system (primitive or secondary) may be divided from the semantic point of view on two groups:

- $\Gamma \;/\; \varphi$ is $\mathcal{F}$-normal iff $\Gamma \models_{\mathcal{F}} \varphi$
- $\Gamma \;/\; \varphi$ is $\mathcal{F}$-valid iff $\models_{\mathcal{F}} \Gamma$ implies $\models_{\mathcal{F}} \varphi$

For example (MP) is normal (and hence also valid) rule in every logic we consider, but (RG) is only valid for most of them. This semantic characterization of rules gives useful semantic criterion for admissibility of rules, namely:

Fact 2. *$\Gamma \;/\; \varphi$ is $\boldsymbol{L}$-admissible, if $\boldsymbol{L}$ is determined by some $\mathcal{F}$ and $\Gamma \;/\; \varphi$ is $\mathcal{F}$-valid.*

3.4. Expressive Strength of Ordinary Modal Language

Explosion of completeness results in 1960's and 1970's, and discovery of some limitations, led to more systematic research on the expressive power of modal languages. Among others, serious investigations started on the applicability of modal languages as description languages for several relational structures used in AI, and Van Bentham laid the foundations of so called correspondence theory. But why modal languages may be used for talking about relational structures, and how much can they express? It is possible because formulae of modal languages correspond to some relational conditions; more precisely:

Definition. A formula φ defines the class of structures $\mathcal{F}$ iff

$$\forall_{\mathfrak{F}}\ (\mathfrak{F} \vDash \varphi \text{ iff } \mathfrak{F} \in \mathcal{F})$$

For example well known axioms:

$$\Box p \to p \tag{T}$$

$$\Box p \to \Box\Box p \tag{4}$$

$$p \to \Box\Diamond p \tag{B}$$

$$\Box p \to \Diamond p \tag{D}$$

define reflexivity, transitivity, symmetry and seriality (successors), respectively. Moreover, standard modal language is expressive enough to define not only elementary (i.e., first-order) conditions but also many important conditions which are expressible in second-order language, e.g. McKinsey axiom $\Box\Diamond\varphi \to \Diamond\Box\varphi$. We will come back to the question of frame definability in a more systematic manner later, for hybrid languages. Here we must recall basic facts concerning standard modal language, mainly for showing that their expressive power has serious limitations.

The main tool for investigations in correspondence theory is the standard translation function $\mathrm{ST}_{\boldsymbol{x}}$ which translates modal formulae into first-order formulae with one free variable $\boldsymbol{x}$ ($\boldsymbol{x} \in \{x, y, z, x_1, x_2, \ldots\}$), in accordance with the definition of satisfaction relation. It may be defined as follows:

$$\begin{aligned}
\mathrm{ST}_{\boldsymbol{x}}(p_k) &= P_k\boldsymbol{x}\\
\mathrm{ST}_{\boldsymbol{x}}(\neg\varphi) &= \neg\mathrm{ST}_{\boldsymbol{x}}(\varphi)\\
\mathrm{ST}_{\boldsymbol{x}}(\varphi \odot \psi) &= \mathrm{ST}_{\boldsymbol{x}}(\varphi) \odot \mathrm{ST}_{\boldsymbol{x}}(\psi)\\
\mathrm{ST}_{\boldsymbol{x}}(\Diamond\varphi) &= \exists_{\boldsymbol{y}}(\mathcal{R}\boldsymbol{xy} \wedge \mathrm{ST}_{\boldsymbol{x}/\boldsymbol{y}}(\varphi))\\
\mathrm{ST}_{\boldsymbol{x}}(\Box\varphi) &= \forall_{\boldsymbol{y}}(\mathcal{R}\boldsymbol{xy} \to \mathrm{ST}_{\boldsymbol{x}/\boldsymbol{y}}(\varphi))
\end{aligned}$$

where $\odot \in \{\wedge, \vee, \to\}$ and $\boldsymbol{y}$ is a variable not occurring in $\mathrm{ST}_{\boldsymbol{x}}(\varphi)$.

Since relational models may be treated as models for first-order correspondence language, it may be shown that:

Lemma 1. *For all φ, w, $\mathfrak{M}$, $\mathfrak{F}$ the following holds:*

$$\begin{aligned}
\mathfrak{M}, w \vDash \varphi \quad &\textit{iff} \quad \mathfrak{M} \vDash \mathrm{ST}_{\boldsymbol{x}}(\varphi)\left[\tfrac{\boldsymbol{x}}{w}\right],\\
\mathfrak{M} \vDash \varphi \quad &\textit{iff} \quad \mathfrak{M} \vDash \forall_{\boldsymbol{x}}\, \mathrm{ST}_{\boldsymbol{x}}(\varphi),
\end{aligned}$$

$$\mathfrak{F}, w \vDash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \forall_{P_1} \ldots \forall_{P_n} \, \mathrm{ST}_{\boldsymbol{x}}(\varphi) \left[\begin{smallmatrix} \boldsymbol{x} \\ w \end{smallmatrix}\right],$$
$$\mathfrak{F} \vDash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \forall_{P_1} \ldots \forall_{P_n} \forall_{\boldsymbol{x}} \, \mathrm{ST}_{\boldsymbol{x}}(\varphi).$$

where P_1, ..., P_n are standard translations of all propositional variables in φ, and $\mathfrak{M} \vDash \mathrm{ST}_{\boldsymbol{x}}(\varphi) \left[\begin{smallmatrix} \boldsymbol{x} \\ w \end{smallmatrix}\right]$ means that $\mathrm{ST}_{\boldsymbol{x}}(\varphi)$ is satisfied in $\mathfrak{M}$ under the assignment of w to the free variable $\boldsymbol{x}$ in $\mathrm{ST}_{\boldsymbol{x}}(\varphi)$.

This lemma shows that on the level of models modal language corresponds to first-order language, whereas on the level of frames it corresponds to second-order language. But this is only general result; in fact there is a lot of elementary (first-order) frame conditions equivalent to second-order standard translations of modal formulae. Conditions mentioned above, like reflexivity, symmetry or transitivity may serve as good examples. The most general result showing which modal formulae define first-order conditions is due to Sahlqvist.

DEFINITION. Let boxed formula be any formula of the form $\Box^n\varphi$, $n \geqslant 0$ (called boxed atom, if $\varphi \in \mathrm{VAR}$), negative formulae be any formulae where each occurrence of an atom is in the scope of odd number of negation (otherwise it is positive). $\varphi \to \psi$ is *Sahlqvist implication* iff φ is built up from $\top$ (where $\top := p \vee \neg p$), $\bot$, boxed atoms and negative formulae with the help of $\vee$, $\wedge$ and $\Diamond$, and ψ is positive formula. Finally, *Sahlqvist formula* is any boxed Sahlqvist implication, boxed conjunction of them, and a disjunction of Sahlqvist formulae that have no atoms in common.

The definition is quite complicated but it covers large class of modal formulae and it will be an important point of reference for discussion on hybrid language expressivity. We have two results based on this concept:

THEOREM 1 (Sahlqvist correspondence). *Every Sahlqvist formula φ is equivalent on frames to some first-order condition effectively computable from φ by so called Sahlqvist-van Benthem algorithm.*

THEOREM 2 (Sahlqvist completeness). *Let Γ be any set of Sahlqvist formulae, then* **HK** $+ \Gamma$ *is strongly complete for the class of frames defined by Γ.*

HK $+ \Gamma$ is an axiom system obtained from **HK** by addition of Γ as the set of additional axioms. The last theorem is very important since we obtain automatically the completeness result for any logic which is axiomatizable by Sahlqvist formulae only.

Although expressive abilities of modal language exceeds first-order language (e.g. McKinsey axiom is an example), there is a lot of first-order conditions, often very simple, that are not modally definable. Below we list some of the more important:

name	condition
irreflexivity	$\forall_x \neg \mathcal{R}xx$
asymmetry	$\forall_{xy}(\mathcal{R}xy \rightarrow \neg \mathcal{R}yx)$
antisymmetry	$\forall_{xy}(\mathcal{R}xy \wedge x \neq y \rightarrow \neg \mathcal{R}yx)$
intransitivity	$\forall_{xyz}(\mathcal{R}xy \wedge \mathcal{R}yz \rightarrow \neg \mathcal{R}xz)$
right directedness	$\forall_{xy}\exists_z(\mathcal{R}xz \wedge \mathcal{R}yz)$
dichotomy	$\forall_{xy}(\mathcal{R}xy \vee \mathcal{R}yx)$
trichotomy	$\forall_{xy}(\mathcal{R}xy \vee \mathcal{R}yx \vee y = z)$
right discreteness	$\forall_{xy}(\mathcal{R}xy \rightarrow \exists_z(\mathcal{R}xz \wedge \neg \exists_v(\mathcal{R}xv \wedge \mathcal{R}vz)))$

Famous Goldblatt-Thomason theorem establishes model theoretic criteria for definability of first-order conditions (for details consult e.g. [22]):

Theorem 3. *Elementary class of frames is definable by the set of modal formulae iff it is closed under construction of generated frames, disjoint unions and bounded morphic images, and reflects ultrafilter extensions.*

Which means that first-order property of such a class of frames is preserved under taking one of these three operations, whereas its negation is preserved under taking ultrafilter extensions. All the classes of frames that satisfy some conditions from the table, break at least one of the four requirements, e.g. irreflexivity and asymmetry are not preserved under bounded morphic images. There are some ways to overcome such limitations, e.g. the use of nonstructural rules of Gabbay [49]. But it seems that passing to hybrid languages offers one of the simplest solution, as we shall see.

4. Basic Hybrid Logic

4.1. Basic Hybrid Language

We get basic hybrid propositional modal language $\mathbf{L_{H@}}$ by adding to $\mathbf{L_M}$ (or $\mathbf{L_T}$):

a) The second sort of symbols called *nominals*. We assume denumerable set NOM $:= \{i, j, k, \ldots\}$ such that VAR $\cap$ NOM $= \varnothing$. Members of NOM are introduced for *naming states of a model domain*.

b) Denumerable collection of unary satisfaction operators indexed by nominals $@_\imath$, for $\imath \in \mathrm{NOM}$.

In the basic hybrid propositional modal language $\mathbf{L_{H@}}$ the nominals from NOM play double role:

(i) As propositional symbols – they represent propositions of the form 'the actual state = the state i (j, k, ...)'.

(ii) As indexes of unary satisfaction operators – they represent names of states. Formulae of the shape $@_\imath\varphi$ reads "In the state i it is the case that φ". Thus, for '$@_i j$' we have: "In the state i it is the case that the actual state = the state j", i.e., "$i = j$".

Note some important features of $\mathbf{L_{H@}}$:

- Both nominals and satisfaction operators are genuine language elements not an extra metalinguistic machinery. This is what we've called internal approach in contrast to external approach present in Fitting's or Gabbay's solutions.
- Although nominals are terms they are treated as ordinary sentences. In particular, they can be connected with the help of boolean operators and combined with modal and tense operators.

Let $\mathrm{AT} := \mathrm{VAR} \cup \mathrm{NOM}$ be the set of *atomic formulae*. The set of formulae $\mathrm{FOR_{H@}}$ is defined in ordinary way, i.e.,:

- $\mathrm{AT} \subseteq \mathrm{FOR_{H@}}$;
- if $\varphi \in \mathrm{FOR_{H@}}$ and $\odot \in \{\neg, \Box, \Diamond\}$, then $\odot\varphi \in \mathrm{FOR_{H@}}$;
- if $\varphi \in \mathrm{FOR_{H@}}$ and $\imath \in \mathrm{NOM}$, then $@_\imath\varphi \in \mathrm{FOR_{H@}}$;
- if $\varphi, \psi \in \mathrm{FOR_{H@}}$ and $\odot \in \{\wedge, \vee, \rightarrow\}$, then $(\varphi \odot \psi) \in \mathrm{FOR_{H@}}$.

It is convenient to distinguish some classes of formulae. Formulae built up from nominals and logical constants only are called *pure*; formulae of the shape $@_\imath\varphi$ or $\neg @_\imath\varphi$ are called *sat-formulae*. Some examples:

$\Diamond(i \wedge p)$	–	neither pure nor sat-formula
$i \rightarrow \Diamond j$	–	pure but not sat-formula
$@_i p$, $@_j(p \rightarrow \Diamond q)$	–	sat- but not pure formulae
$@_i j$, $@_i \Diamond j$	–	both pure and sat-formulae

It should be observed that both examples of pure sat-formulae play very important roles. The first one expresses identity of states named i and j, and the second one expresses accessibility of j from i (see p. 167).

Some authors (e.g. Blackburn [17], Tzakova [117], Demri [38]) prefer to have weaker language, with only nominals added but without satisfaction operators, as a basic hybrid language. In what follows, we use $\mathbf{L_H}$ to denote such a language and we will call it *the weak hybrid propositional language*.

4.2. Hybrid Models

What is nice with MHL is the fact that changes in the language are sometimes so small that they do not affect seriously the rest of the machinery applied in ML. In particular, modifications in the relational semantics are minimal. The concept of a frame is the same as in ordinary normal modal (or tense) logics, only on the level of models we have some changes. A model on the frame $\mathfrak{F}$ is any structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where V is valuation function on atoms such that: $V\colon \mathrm{VAR} \longrightarrow \mathcal{P}(\mathcal{W})$ and $V\colon \mathrm{NOM} \longrightarrow \mathcal{W}$.

Let's note certain difference in interpretation. $V(p)$ picks up the set of all *states* (*worlds*) in which the variable p is true in a model, whereas $V(i)$ is the unique *state* (the *world*) assigned to the nominal i in a model.

Satisfaction of new formulae in states of a model is defined as follows for any $\imath \in \mathrm{NOM}$ and $\varphi \in \mathrm{FOR}_{\mathrm{H@}}$ (see (i) and (ii) on p. 165):

$$\begin{array}{lll} \mathfrak{M}, w \vDash \imath & \text{iff} & w = V(\imath) \\ \mathfrak{M}, w \vDash @_\imath\varphi & \text{iff} & \mathfrak{M}, V(\imath) \vDash \varphi \end{array}$$

The concepts of global satisfiability and of validity are the same as for ordinary modal language. Also definitions of consequence relations remain intact. The only difference is that if we say “model” we mean a model in a hybrid sense with a constraint on valuation of nominals.

Note that sat-operators enable to jump to the named state, so in consequence we have:

Fact 3. $\mathfrak{M}, w \vDash @_\imath\varphi$ *iff* $\mathfrak{M} \vDash @_\imath\varphi$.

Let's focus on some consequences of the above definitions. The most important features of $\mathbf{L_{H@}}$ seem to be:

1. Internalization of local discourse—nominals give direct representation of states in a language (we have an object-language mechanism for storing model data).
2. Possible jumping to already specified states in a model (we have a mechanism for retrieving model data).
3. Internalization of $\vDash$ by sat-formulae $@_\imath\varphi$.

4. Representation of identity theory (for states) by pure-formulae $@_\imath \jmath$. We have: $\mathfrak{M}, w \models @_\imath \jmath$ iff $V(\imath) = V(\jmath)$.
5. Internalization of accessibility relation by pure-formulae $@_\imath \Diamond \jmath$. We have: $\mathfrak{M}, w \models @_\imath \Diamond \jmath$ iff $\mathcal{R}V(\imath)V(\jmath)$.

One should note that points 2–5 are due to the presence of satisfaction operators, so in $\mathbf{L_H}$ we have only the first property.

4.3. The Logic

Let's look at some syntactic properties of our new language elements. First of all, note that satisfaction operators are indeed modal—in fact normal modal—constants. One can easily check that for any $\varphi, \psi \in \mathrm{FOR}_{\mathbf{H@}}$ and $\imath \in \mathrm{NOM}$ the following formula

$$@_\imath(\varphi \to \psi) \leftrightarrow (@_\imath\varphi \to @_\imath\psi)$$

is valid and moreover $@_\imath\varphi$ is valid, whenever φ is valid.

Let $\mathbf{K_{H@}}$ denote the set of all valid formulae in $\mathbf{L_{H@}}$. It is easy to check that $\mathbf{K} \subseteq \mathbf{K_{H@}}$ and that $\mathbf{K_{H@}}$ satisfies closure conditions of normal modal logic. $\mathbf{K_{H@}}$ is indeed the weakest normal modal logic in $\mathbf{L_{H@}}$. Analogously we will use $\mathbf{K_H}$ as a name of the basic hybrid logic in $\mathbf{L_H}$ and $\mathbf{Kt_{H@}}$, $\mathbf{Kt_H}$ as names of respective logics in suitable hybrid versions of $\mathbf{L_T}$. All these logics are also normal modal logics.

Clearly, due to richer language, $\mathbf{K_{H@}}$ contains denumerably many new tautologies e.g.

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \to \Diamond(p \wedge q)$$

One can easily check that if we change i with some propositional variable we obtain non-valid formula in $\mathbf{K}$—it is valid on frames with functional accessibility relation. But if we check it in any state w of any hybrid model we can see that both states w' and w'' that must be accessible from w in order to satisfy an antecedent are denotations of i, so they are the same state which guaranties that consequent is satisfied too.

As we shall see in the next section the expressivity of hybrid language has more serious character than just the presence of new tautologies. We may state new frame-defining formulae, e.g.: $i \to \neg\Diamond i$ defines irreflexivity and $i \to \neg\Diamond\Diamond i$ defines asymmetry. Moreover, $\mathbf{K_{H@}}$ is decidable and PSPACE-complete as ordinary $\mathbf{K}$ (see [6]).

5. Complete Hilbert Calculi for $\mathbf{K_{H@}}$ and $\mathbf{K_H}$

We will focus on proof theory for MHL in the second part, but our considerations will be connected with the practically useful formalizations. Axiomatic formulations of suitable hybrid logics will be stated in this part since they are useful for considerations on expressiveness, in the context of completeness results. The axiomatic (or Hilbert) formalization of the basic hybrid logic $\mathbf{K_{H@}}$ is denoted by $\mathbf{HK_{H@}}$ and, in addition to axioms of **HK**, contains:

1. axioms of **CPL** (any complete set is suitable),
2. specific hybrid axioms:

$$@_i(p \to q) \to (@_i p \to @_i q) \tag{K@}$$

$$@_i p \leftrightarrow \neg @_i \neg p \tag{Selfdual@}$$

$$@_i i \tag{Ref@}$$

$$@_i @_j\, p \leftrightarrow @_j\, p \tag{Agree}$$

$$i \wedge p \to @_i p \tag{Intro@}$$

$$\Diamond @_i p \to @_i p \tag{Back}$$

3. rules: (MP), (RG) and

$$\vdash \varphi \;/\; \vdash @_i \varphi \tag{RG@}$$

$$\vdash \varphi \;/\; \vdash e(\varphi) \quad \text{where } e\colon \mathrm{VAR} \longrightarrow \mathrm{FOR_{H@}},\ e\colon \mathrm{NOM} \longrightarrow \mathrm{NOM} \tag{Sub@}$$

Note that this axiomatization is not in a sense structural since we have an important constraint on the substitution rule. Moreover notice that—by (Sub@) and specific hybrid axioms—we obtain the following theses:

$$@_i(j \to k) \to (@_i j \to @_i k)$$

$$@_i j \leftrightarrow \neg @_i \neg j$$

$$@_i @_j\, k \leftrightarrow @_j\, k$$

$$i \wedge j \to @_i j$$

$$\Diamond @_i j \to @_i j$$

Our axiomatization is sufficient for completeness but the full character of @ is not evident from it. One can learn more from the following lemma.

LEMMA 2. *The following formulae are* $\mathbf{HK_{H@}}$*-theses:*

$$@_i j \leftrightarrow @_j i \tag{Sym@}$$

$$@_i j \wedge @_j k \to @_i k \tag{Tran@}$$

$$@_j p \wedge @_j i \to @_i p \tag{Nom1}$$

$$@_j p \wedge @_i j \to @_i p \qquad \texttt{(Nom2)}$$

$$\Diamond i \wedge @_i p \to \Diamond p \qquad \texttt{(Bridge)}$$

$$(@_i p \to @_i q) \to @_i (p \to q) \qquad \texttt{(ConvK@)}$$

Now we can read off the identity theory of @ from these theorems. As an illustration we give a proof of the (`Bridge`).

1. $\vdash i \wedge \neg p \to @_i \neg p$ (`Intro@`)
2. $\vdash \Diamond(i \wedge \neg p) \to \Diamond @_i \neg p$ 1, by K-admissible rule (`RM`)
3. $\vdash \Diamond i \wedge \Box \neg p \to \Diamond(i \wedge \neg p)$ K-thesis
4. $\vdash \Diamond i \wedge \Box \neg p \to \Diamond @_i \neg p$ 2, 3, by CPL
5. $\vdash \Diamond @_i \neg p \to @_i \neg p$ (`Back`)
6. $\vdash \Diamond i \wedge \Box \neg p \to @_i \neg p$ 4, 5, by CPL
7. $\vdash \Diamond i \wedge \neg @_i \neg p \to \neg \Box \neg p$ 6, by CPL
8. $\vdash \Diamond i \wedge @_i p \to \Diamond p$ 7, (`Selfdual@`), (`Pos`)

Theorem 4 (Completeness). *The above axiomatic system is strongly complete for* $\mathbf{K_{H@}}$.

Soundness of the $\mathbf{HK_{H@}}$ is easy to prove, the proof of completeness is by standard canonical model construction applied in modal logics. But something more is needed for extensions of $\mathbf{K_{H@}}$ if we want to obtain some general completeness theorem. Let $\mathbf{HK^+_{H@}}$ be $\mathbf{HK_{H@}}$ with 2 additional rules:

$$\vdash @_\imath \varphi \;/\vdash \varphi \qquad \text{provided } \imath \notin \varphi \qquad \texttt{(NAME)}$$

$$\vdash @_\imath \Diamond \jmath \to @_\jmath \varphi \;/\; \vdash @_\imath \Box \varphi \qquad \text{provided } \imath \neq \jmath \text{ and } \jmath \notin \varphi \qquad \texttt{(BG)}$$

Both rules are admissible in $\mathbf{HK_{H@}}$, so we have:

Lemma 3. $\mathrm{Th}(\mathbf{HK_{H@}}) = \mathrm{Th}(\mathbf{HK^+_{H@}})$.

Note again that both additional rules are not standard because they have provisos. In this respect they are similar to famous Gabby-style nonstructural rules applied for defining frame conditions undefinable by standard modal formulae. Let's look at the rule (`BG`). The premise says that if the denotation of $\jmath$ is accessible from the denotation of $\imath$, then φ is satisfied in $\jmath$. But $\jmath$ is arbitrary which is guaranteed by the proviso, so it means that φ is satisfied in every accessible (from $\imath$) state. This justifies the assertion that $\Box\varphi$ is satisfied in (the denotation of) $\imath$. The name (`BG`) comes from *Bounded*

Generalization because it is a modal analog of *Universal Generalization* from first-order logic. But it is bounded because the premise is conditional ($\jmath$ is not simply arbitrary but arbitrary $\imath$-accessible state). In that it is more like respective rule from free logic. The sense of (NAME) is clear: if φ is satisfied in arbitrary state (again by syntactical proviso), then it is simply valid. Despite its simplicity the rule plays important role in general completeness theorem stated below. As we shall see in the second part it is also the theoretical basis for many proof systems called there sat-calculi.

As we've noticed, both rules—being admissible—have no impact on the set of theses of $\mathbf{HK_{H@}}$. But they have strong influence on the redundancy of the set of primitive rules. For example ordinary (RG) is derivable in $\mathbf{HK^{+}_{H@}}$. Sometimes (see e.g. [22]) different nonstandard rules are applied, particularly useful for completeness proof and when @ is not present.

Lemma 4. *The following rules are admissible in* $\mathbf{HK_{H@}}$ *(or derivable in* $\mathbf{HK^{+}_{H@}}$*):*

$\vdash \imath \to \varphi \;/\; \vdash \varphi$	*provided* $\imath \notin \varphi$	(NAME′)
$\vdash @_\imath\Diamond\jmath \wedge @_\jmath\varphi \to \psi \;/\; \vdash @_\imath\Diamond\varphi \to \psi$	*provided* $\imath \neq \jmath$ *and* $\jmath \notin \varphi, \psi$	(PASTE)

For the sake of illustration we put the proof of derivability of (NAME′) in $\mathbf{HK^{+}_{H@}}$ (by (NAME))

1.	$\vdash \imath \to \varphi$	Premise, $\imath \notin \varphi$
2.	$\vdash @_\imath(\imath \to \varphi)$	1, (RG@)
3.	$\vdash @_\imath\imath \to @_\imath\varphi$	2, (K@)
4.	$\vdash @_\imath\imath$	(Ref@)
5.	$\vdash @_\imath\varphi$	3, 4, (MP)
6.	$\vdash \varphi$	1, 5, (NAME)

These rules may be used instead of (BG) and (NAME). Moreover, (PASTE) is deductively stronger than (BG) because we may not only show the derivability of this rule by (PASTE), but also deduce one of the axioms from our basis, namely (Back).

It is also possible to axiomatize $\mathbf{K_H}$—the set of all valid formulae in $\mathbf{L_H}$. We should add to axioms of $\mathbf{HK}$ only the following formulae for any $n, m \geqslant 0$:

$$\Diamond^n(i \wedge p) \to \Box^m(i \to p) \tag{Nom}$$

Instead of (RG@) we have (in addition to (MP) and (RG)) the following rule:

$$\vdash \neg\imath \;/\; \vdash \bot \tag{NAMELITE}$$

This rule has special character. It is admissible in every consistent extension of $\mathbf{HK_H}$. Note however that $\neg\imath$ is not valid on any frame. So the function of this rule is only to make inconsistent every logic with $\neg\imath$ added as an axiom.

If we want an axiomatization of $\mathbf{HK_H}$ which is an analogon of $\mathbf{HK^+_{H@}}$ we should add (NAME′) and the following @-free version of (PASTE) for any $n \geqslant 0$:

$$\vdash \Diamond^n(\imath \wedge \Diamond(\jmath \wedge \varphi)) \rightarrow \psi \;/\; \vdash \Diamond^n(\imath \wedge \Diamond\varphi) \rightarrow \psi, \text{provided } \imath \neq \jmath \text{ and } \jmath \notin \varphi, \psi \tag{PASTE′}$$

In fact (NAMELITE) is a special case of (NAME′) with $\varphi := \bot$, so we can get rid of this rule in extended axiomatization.

6. General Completeness Results

Now we are able to state rather general completeness theorem for considerable number of extensions of $\mathbf{HK^+_{H@}}$ (or $\mathbf{HK^+_H}$) obtained with the help of pure axioms.

THEOREM 5 (Pure completeness). *Let Γ be any set of pure formulae. Then* $\mathbf{HK^+_{H@}} + \Gamma$ *is strongly complete for the class of frames defined by* Γ.

We will sketch completeness proof. It is a mix of modal and first-order ideas—essentially a combination of canonical model construction and witnessed Henkin method. In addition to usual concepts of consistent and maximal sets we need:

DEFINITION.
- Γ is *named* iff it contains at least one nominal (it is the name of Γ)
- Γ is $\Diamond$-*saturated* iff for all $@_\imath\Diamond\varphi \in \Gamma$: there is a nominal $\jmath$ such that $@_\imath\Diamond\jmath \in \Gamma$ and $@_\jmath\varphi \in \Gamma$

These additional concepts play important role in suitable modification of Lindenbaum construction.

LEMMA 5 (Lindenbaum). *Every* $\mathbf{HK^+_{H@}} + \Gamma$*-consistent set can be extended to a named,* $\Diamond$*-saturated, maximal,* $\mathbf{HK^+_{H@}} + \Gamma$*-consistent set.*

Sketch of a proof: Similarly as in Henkin proof for first-order logic we must supply countably infinite set of new nominals, its arbitrary enumeration and some enumeration of all formulae in the extended (by new nominals) language. The procedure of extending our consistent set is mostly standard, by addition of each new formula which does not lead to inconsistency. Two points should be noticed:

- In order to get a named set in the first step of construction we add the first new nominal. By (NAME′) this set must be consistent.
- In order to get $\Diamond$-saturated set, every time we add in a consistent way a formula of the type $@_\imath\Diamond\psi$ we add also $@_\imath\Diamond\jmath$ and $@_\jmath\psi$, where $\jmath$ is a new nominal (witness). By (PASTE) such extended set must be also consistent.

Obviously the union of all so generated sets satisfies postulated conditions.

We do not use canonical model construction from ordinary modal logic, where states are simply (all) maximal consistent sets. Here one set is enough and the states of this model are built up from equivalence classes of nominals from this maximal consistent set. Formally:

DEFINITION. *Henkin Model* for $\mathbf{HK}^+_{\mathbf{H@}} + \Gamma$-maximal, consistent set Δ is defined as $\mathfrak{M}_\Delta = \langle \mathcal{W}_\Delta, \mathcal{R}_\Delta, V_\Delta\rangle$ in where:
$\mathcal{W}_\Delta = \{|\imath| : \imath \text{ is a nominal}\}$, where $|\imath| = \{\jmath : \ @_\imath\jmath \in \Delta\}$,
$\mathcal{R}_\Delta(|\imath|, |\jmath|)$ iff $@_\imath\Diamond\jmath \in \Delta$,
$V_\Delta(\varphi) = \{|\imath| : @_\imath\varphi \in \Delta\}$, for $\varphi \in$ VAR,
$V_\Delta(\imath) = \{|\imath|\}$, for $\imath \in$ NOM.

By (almost) ordinary inductive argument we obtain:

LEMMA 6 (Truth Lemma). $@_\imath\varphi \in \Delta$ *iff* $\mathfrak{M}_\Delta, |\imath| \models \varphi$.

SKETCH OF THE PROOF. One should note that because we make an induction on the formula which on one side of the equivalence is changed into sat-formula we must apply suitable axioms or theses. For example if φ is negation we must use (Selfdual@), if it is implication we must use (K@) and (ConvK@), if it is sat-formula we need (Agree) and if it is a diamond-formula we need (Bridge). $\square$

As a result of this construction we obtain a lemma which gives us automatically general completeness for every set of pure formulae that defines some frame conditions.

LEMMA 7 (Frame Lemma). *If Δ is $\Diamond$-saturated $\mathbf{HK}^+_{\mathbf{H@}} + \Gamma$-maximal, consistent set, then the frame of $\mathfrak{M}_\Delta$ satisfies all properties defined by Γ.*

It is obvious since Δ is named and contains all instances of Γ, so on the frame of this model all elements of Γ are valid. Pure completeness theorem follows from this in a standard way. This result leads to better completeness theory due to more general theory of frame definability than standard modal logic provides. The following lists some examples:

$\Box i \to \Diamond i$	seriality (successors)	($\mathtt{D}'$)
$\Diamond i \to \Box i$	partial functionality	($\mathtt{ConvD}'$)
$\Box i \to i$	reflexivity	($\mathtt{T}'$)
$\Box(\Box i \to i)$	almost-reflexivity	($\mathtt{BoxT}'$)
$i \to \Box\neg i$	irreflexivity	($\mathtt{Irr}$)
$\Box i \to \Box\Box i$	transitivity	($\mathtt{4}'$)
$\Box\Box i \to \Box i$	density	($\mathtt{Conv4}'$)
$\neg\Box i \to \Box\Box i$	intransitivity	($\mathtt{Intr}$)
$i \to \Box\Diamond i$	symmetry	($\mathtt{B}'$)
$i \to \Box\Box\neg i$	asymmetry	($\mathtt{As}$)
$i \to \Box(\Diamond i \to i)$	antisymmetry	($\mathtt{Ant}$)
$\Diamond i \to \Box\Diamond i$	Euclideaness	($\mathtt{5}'$)
$\Diamond i$	universality	($\mathtt{Un}$)
$\Box(\Box i \to j) \vee \Box(\Box j \to i)$	strong connectedness	($\mathtt{3}'$)
$\Box(\Box i \wedge i \to j) \vee \Box(\Box j \wedge j \to i)$	weak connectedness	($\mathtt{L}'$)
$@_i\Diamond j \vee @_j\Diamond i$	dichotomy	($\mathtt{Dich}$)
$@_i\Diamond j \vee @_j\Diamond i \vee @_i j$	trichotomy	($\mathtt{Tri}$)

Note in particular that:

1. Many conditions from the table are not definable in $\mathbf{L_M}$, e.g.: irreflectivity, intransitivity, asymmetry, antisymmetry, universality, dichotomy and trichotomy.
2. All conditions except dichotomy and trichotomy are definable in $\mathbf{L_H}$.

For the sake of illustration we will show that ($\mathtt{Irr}$) defines irreflexivity. Assume that $\mathcal{R}$ is irreflexive but ($\mathtt{Irr}$) is not valid, so in some w we have $w \vDash i$ but $w \nvDash \Box\neg i$. So in some accessible w' we have $w' \vDash i$ but then $w = V(i) = w'$ which contradicts the assumption of irreflexivity. Now assume that ($\mathtt{Irr}$) is valid in the frame where for some w, $\mathcal{R}ww$. Let $V(i) = w$ (recall that canonical model is named!), so $w \vDash i$ and $w \vDash i \to \Box\neg i$. But if $w \vDash \Box\neg i$, then $w \nvDash i$—contradiction.

One should note that this result is also in a sense simpler than celebrated Sahlqvist completeness theorem. The criteria for being Sahlqvist formula are rather complicated whereas the requirement of purity is extremely simple. But there are also some considerable limitations—pure-formulae define only first-order properties but still not all of them!

That second-order properties are not definable by pure formulae should be clear if we look at how standard translation works. For $\mathbf{L_{H@}}$ we add two clauses to the definition of $\mathrm{ST}_{\boldsymbol{x}}$:

$$\begin{array}{rcl} \mathrm{ST}_{\boldsymbol{x}}(\imath) & = & \boldsymbol{x} = c_\imath \\ \mathrm{ST}_{\boldsymbol{x}}(@_\imath\varphi) & = & \exists_{\boldsymbol{y}}(\boldsymbol{y} = c_\imath \wedge \mathrm{ST}_{\boldsymbol{x}/\boldsymbol{y}}(\varphi)) \end{array}$$

where $c_\imath$ is individual constant for $\imath$ and $\boldsymbol{y}$ is a variable not occurring in $\mathrm{ST}_{\boldsymbol{x}}(\varphi)$.

Lemma 1 still holds but recall that second-order quantification deals only with monadic predicates being standard translation of propositional variables from translated formula. But there are no such variables in pure formulae; nominals, despite their syntactic category, play the role of names and in enriched standard translation are mapped onto first-order individual constants. In consequence, every condition definable by pure formula must be elementary.Of course, second-order properties definable in standard modal languages are also expressible in hybrid languages, since trivially the former are contained in the latter, but they are not expressible by pure formulae.

But which first order properties are definable and which are not definable in basic hybrid language? In particular, which are definable by pure formulae? [115] contains the following characterization theorems:

THEOREM 6. *An elementary class of frames is definable by formulae of* $\mathbf{L_{H@}}$ *iff it is closed under ultrafilter morphic images and generated subframes.*

THEOREM 7. *A class of frames is definable by pure formulae of* $\mathbf{L_{H@}}$ *iff it is elementary and closed under images of bisimulation systems.*

For suitable definitions and details of proofs one should consult [115]. Here we discuss some concrete negative examples of definability in $\mathbf{L_{H@}}$. For instance not all Sahlqvist formulae have pure formulae equivalents (e.g. Church-Rosser property), predecessors, right- (left)-directedness.

- Church-Rosser property – $\forall_{xyz}(\mathcal{R}xy \wedge \mathcal{R}xz \rightarrow \exists_v(\mathcal{R}yv \wedge \mathcal{R}zv))$ is defined in $\mathbf{L_M}$ by
$$\Diamond\Box p \rightarrow \Box\Diamond p \tag{CR}$$
but $\Diamond\Box i \rightarrow \Box\Diamond i$ doesn't work.
- Predecessors – $\forall_x\exists_y\mathcal{R}yx$ which is not defined in $\mathbf{L_M}$ either (although the converse, namely seriality, is defined by (D)).
- Right-directedness – $\forall_{xy}\exists_z(\mathcal{R}xz \wedge \mathcal{R}yz)$ is not definable in $\mathbf{L_M}$. Note that it is definable in $\mathbf{L_{H@}}$ by $@_i\Box p \rightarrow @_j\Diamond p$, but it is not pure formula so pure completeness theorem does not apply. Left-directedness is undefinable in $\mathbf{L_{H@}}$ too.

Remark. (CR) is a special case of a Geach Axiom: $\Diamond^m\Box^n p \rightarrow \Box^s\Diamond^t p$ which defines frame properties expressed in short by:

$$\forall_{xyz}\exists_v(\mathcal{R}^m xy \wedge \mathcal{R}^s xz \rightarrow \mathcal{R}^n yv \wedge \mathcal{R}^t zv) \tag{$\star$}$$

where m, n, s and t denote the lengths of $\mathcal{R}$-paths in each case. Obviously, every instance of Geach Axiom is also an instance of Sahlqvist formula.

As a result we have a strange situation. For hybrid logics in $\mathbf{L_{H@}}$ we have both: pure completeness, and

THEOREM 8 (Hybrid Sahlqvist completeness). *Let Γ be any set of Sahlqvist-formulae, then $\mathbf{HK}^+_{\mathbf{H@}}+\Gamma$ is strongly complete for the class of frames defined by Γ.*

But completeness fails for some combinations of pure and Sahlqvist formulae, e.g. $\mathbf{HK}^+_{\mathbf{H@}}$ + (CR) + (NG) is incomplete, where

$$\Diamond(i \wedge \Diamond j) \rightarrow \Box(\Diamond j \rightarrow i) \tag{NG}$$

defines the following condition:

$$\forall_{xyzu}(\mathcal{R}xy \wedge \mathcal{R}xz \wedge \mathcal{R}yu \wedge \mathcal{R}zu \rightarrow y = z)$$

Fortunately, the situation slightly changes when we move to hybrid tense language.

It would be also interesting to know if we could obtain an axiomatization which is sufficient for obtaining pure completeness theorem but which is more standard in a sense that rules like (BG) or (PASTE) are derivable. As we shall see it is possible in case of stronger languages, at least partly—for instance

(BG) is derivable in $\mathbf{HKt_{H@}}$ (but with the help of (NAME) however). Full elimination of nonstandard rules is possible when we have local binder $\downarrow$ in a language, but in the case of basic language the presence of such rules is not incidental which was shown in [27].

7. Hybrid Tense Logic

7.1. The Impact of Past Operators

Hybrid tense logic shows some important differences with modal hybrid logic. It is easy to check that $\mathbf{L_{TH@}}$ (and even $\mathbf{L_{TH}}$) is strictly more expressive than $\mathbf{L_{H@}}$. In particular, three points should be stressed:

1. @ is in principle dispensable in the presence of past-operators, e.g. trichotomy may be defined by $\mathsf{P}\,i \lor i \lor \mathsf{F}\,i$. But it does not mean that @ is simply definable. [6] shows that @ is eliminable in $\mathbf{L_{TH@\downarrow}}$ from all nominal-free sentences. Different way of simulating the effect of sat-operators in $\mathbf{L_{TH}}$ is shown in Demri's sequent calculus [38] (see the section on sequent calculi in part II).

2. Some frame-conditions undefinable in $\mathbf{L_{H@}}$ by pure formulae (although definable in $\mathbf{L_M}$) are definable with the help of tense operators, e.g. Church-Rosser property (or connectedness) is defined by $\mathsf{F}\,i \land \mathsf{F}\,j \to \mathsf{F}(i \land \mathsf{F}\,\mathsf{P}\,j)$.

3. Some frame-conditions are definable that are not definable in any of $\mathbf{L_M}$, $\mathbf{L_T}$, $\mathbf{L_{H@}}$, e.g.:

- left directedness: $\forall_{xy}\exists_z(z < x \land z < y)$ is defined by $\mathsf{P}\,\mathsf{F}\,i$,
- right discreteness: $\forall_{xy}(x < y \to \exists_z(x < z \land \neg\exists_v(x < v < z)))$ is defined by $@_i(\mathsf{F}\top \to \mathsf{F}\,\mathsf{H}\,\mathsf{H}\,\neg i)$ (or $i \to (\mathsf{F}\top \to \mathsf{F}\,\mathsf{H}\,\mathsf{H}\,\neg i)$).

In fact, every Sahlqvist formula has pure formula equivalent in $\mathbf{L_{TH@}}$ (see [59]), so we have:

THEOREM 9 (Sahlqvist/pure completeness). *Let Γ be any set of pure or Sahlqvist formulae, then $\mathbf{HKt^+_{H@}} + \Gamma$ is strongly complete for the class of frames defined by Γ.*

$\mathbf{HKt^+_{H@}}$ is similar to $\mathbf{HK^+_{H@}}$—we simply replace the axioms of **HK** by the axioms of **HKt** and replace (RG) by two tense versions for G and H, respectively. But some changes are possible, namely:

1. We can use one pure axiom $@_i\,\mathsf{F}\,j \leftrightarrow @_j\,\mathsf{P}\,i$ instead of two standard interaction axioms from **HKt**: $p \to \mathsf{G}\,\mathsf{P}\,p$ and $p \to \mathsf{H}\,\mathsf{F}\,p$.

2. If we add two axioms: $@_i\,\mathsf{G}\,\mathsf{P}\,i$ and $@_i\,\mathsf{H}\,\mathsf{F}\,i$ we can derive both (for G and H) tense versions of (BG).

So in completeness theory we can avoid some strange features of $\mathbf{L_{H@}}$ but there are some disadvantages—$\mathbf{Kt_{H@}}$ is still decidable but EXPTIME-complete, whereas $\mathbf{Kt}$ is in PSPACE (as $\mathbf{K}$ and $\mathbf{K_{H@}}$). So (basic) hybrid tense logic is more complex than ordinary tense logic $\mathbf{Kt}$.

7.2. Tenses

Research on hybrid tense logic opens also a new perspective for formalization of English language tenses. Blackburn [18] noticed that standard Priorean $\mathbf{L_T}$ already has deictic nature but shows strong limitation in expressing language tenses. $\mathbf{L_{TH@}}$ yields referential perspective which makes possible to express Reichenbachian analysis of tenses in terms of three time points. The table lists details

reference	tense	example	formula
E-R-S	Pluperfect	I had seen	$\mathsf{P}(i \wedge \mathsf{P}\,p)$
E,R-S	Past	I saw	$\mathsf{P}(i \wedge p)$
R-E-S	Future-in-the-Past	I'd see	$\mathsf{P}(i \wedge \mathsf{F}\,p)$
R-S,E	Future-in-the-Past	I'd see	$\mathsf{P}(i \wedge \mathsf{F}\,p)$
R-S-E	Future-in-the-Past	I'd see	$\mathsf{P}(i \wedge \mathsf{F}\,p)$
E-S,R	Present perfect	I've seen	$\mathsf{P}\,p$
S,R,E	Present	I see	p
S,R-E	Prospective	I'm going to see	$\mathsf{F}\,p$
S-E-R	Future perfect	I'll have seen	$\mathsf{F}(i \wedge \mathsf{P}\,p)$
S,E-R	Future perfect	I'll have seen	$\mathsf{F}(i \wedge \mathsf{P}\,p)$
E-S-R	Future perfect	I'll have seen	$\mathsf{F}(i \wedge \mathsf{P}\,p)$
S-R,E	Future	I'll see	$\mathsf{P}(i \wedge p)$
S-R-E	Future-in-the-Future	(Latin: abiturus ero)	$\mathsf{F}(i \wedge \mathsf{F}\,p)$

where:
S – the point of speech
E – the point of event
R – the point of reference

[18] contains also other applications of hybrid tense languages to analysis of language temporal phenomena like indexicals, anaphora, calendar terms. Other papers of Blackburn undertake the problem of extending expressive power of hybrid language to cover interval based temporal languages, but this requires substantial changes in hybrid machinery by introducing further

sorts of atoms (see e.g. [17]). Nevertheless, for many purposes even hybrid languages with backward-looking operators are still too weak. In what follows we describe briefly the most popular extensions.

8. Language Extensions

Although the basic hybrid language offers many improvements over standard modal language it has still strong limitations which may be overcame by further strengthenings. Moreover, some of them were historically the first forms of hybrid languages. Below we consider some of the most important languages and their expressive hierarchy. We describe in particular:

- Extra modalities
  1. Global modalities
  2. Difference modalities
- Modal Binders
  1. Local binder
  2. Quantifiers

8.1. Strong Modalities

It should be stressed that early works on hybrid logics, in particular from Sofia school (like [53] or [58]), were concerned with stronger languages than those we presented so far. Studies on basic and weak hybrid language started later in the middle of 1990's. In this paragraph we briefly recall two, surprisingly strong solutions.

Global Modalities. One of the popular solutions, not necessarily connected with hybrid logic is to use the so called global modalities. Notes to the last chapter of [22] include interesting historical information on their use. They are called global because they are not characterised by accessibility relations but defined by reference to any state in a model. We use A (from Aristotle or "always") for universal (global) modality and E for its dual. Semantically they are defined as follows:

$$
\begin{array}{lll}
\mathfrak{M}, w \vDash \mathsf{A}\,\varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ for any } w' \\
\mathfrak{M}, w \vDash \mathsf{E}\,\varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ for some } w'
\end{array}
$$

Let $\mathbf{L_{H@A}}$ denote $\mathbf{L_{H@}}$ with universal modality A or (interdefinable) existential modality E. Note that $\mathbf{L_{HA}} = \mathbf{L_{H@A}}$ since @ is definable (see (Selfdual@)):

$$@_i\varphi := \mathsf{A}(i \to \varphi)$$
$$:= \mathsf{E}(i \wedge \varphi)$$

So, in the presence of global modality, the difference between $\mathbf{L_H}$ and $\mathbf{L_{H@}}$ disapears and $\mathbf{L_{HA}}$ is at least as expressive as $\mathbf{L_{H@}}$. In fact hybrid languages with A are strictly stronger which is evident if we consider computational behaviour of $\mathbf{K_{HA}}$.

$\mathbf{K_{HA}}$ is also decidable but global modalities are very strong. Even $\mathbf{K_A}$—basic logic of A in standard language (no nominals) is EXPTIME-complete [65]. But if we add nominals, the situation does not change. Even if we add A to hybrid tense logic we still have EXPTIME-completeness, so both $\mathbf{K_{HA}}$ and $\mathbf{Kt_{HA}}$ are in the same complexity class as plain $\mathbf{K_A}$. Hence, at least from the point of view of complexity of $\mathbf{Kt_H}$, we do not loose anything if we add global modalities. But if compared with $\mathbf{K_{H@}}$ (and if standard beliefs concerning relations between complexity classes are right) satisfiability problem for hybrid logic with A is harder.

Complete axiomatization of $\mathbf{K_{HA}}$ (= $\mathbf{K_{H@A}}$) may be obtained by addition of the following axioms to $\mathbf{HK_H}$:

$$\mathsf{E}\,p \leftrightarrow \neg\,\mathsf{A}\,\neg p \qquad (\mathtt{Dual}\mathsf{A})$$
$$\mathsf{A}(p \to q) \to (\mathsf{A}\,p \to \mathsf{A}\,q) \qquad (\mathtt{K}\mathsf{A})$$
$$\mathsf{A}\,p \to p \qquad (\mathtt{T}\mathsf{A})$$
$$p \to \mathsf{A}\,\mathsf{E}\,p \qquad (\mathtt{B}\mathsf{A})$$
$$\mathsf{A}\,p \to \mathsf{A}\,\mathsf{A}\,p \qquad (\mathtt{4}\mathsf{A})$$
$$\Diamond p \to \mathsf{E}\,p \qquad (\mathtt{Incl}\Diamond)$$
$$\mathsf{E}\,i \qquad (\mathtt{Incl}i)$$
$$\mathsf{E}(i \wedge p) \to \mathsf{A}(i \to p) \qquad (\mathtt{Nom}\mathsf{A})$$

Clearly to the set of rules of $\mathbf{HK_H}$ we must add:

$$\vdash \varphi \;/\; \vdash \mathsf{A}\,\varphi \qquad (\mathtt{RG}\mathsf{A})$$

If we want to strengthen $\mathbf{HK_{HA}}$ in order to get $\mathbf{HK^+_{HA}}$, a formalization suitable for general pure completeness theorem, we must add (NAME′) and:

$$\vdash \mathsf{E}(\imath \wedge \Diamond \jmath) \to \mathsf{E}(\jmath \wedge \varphi) \;/\; \vdash \mathsf{E}(\imath \wedge \Box\varphi), \quad \text{provided } \imath \neq \jmath \text{ and } \jmath \notin \varphi \qquad (\mathtt{BGE})$$

In the set of axioms, formulae (DualA)–(4A) simply reflect the fact that global modalities are normal **S5**-modalities. Interesting cases are the last three axioms. Alternatively, we could add just (Incl$i$) to $\mathbf{HK}^{+}_{\mathbf{H}}$ since the rest of the axioms is derivable.

Difference Modality. Other kind of modality very popular in early works on hybrid logic (see e.g. [53]) is difference modality. In fact, it was firstly introduced in the context of ordinary modal languages (again, see the notes in [22]). Let $\mathbf{L_{MD}}$ denote $\mathbf{L_M}$ with difference possibility D or (interdefinable) difference necessity $\bar{\mathsf{D}}$ defined as follows:

$$\begin{array}{lll} \mathfrak{M}, w \vDash \mathsf{D}\,\varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ for some } w' \neq w \\ \mathfrak{M}, w \vDash \bar{\mathsf{D}}\varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ for any } w' \neq w \end{array}$$

Note that $\mathbf{L_{MD}}$ is strictly stronger than $\mathbf{L_{MA}}$ since A is definable by $\bar{\mathsf{D}}$ but not conversely:

$$\mathsf{A}\,\varphi := \varphi \wedge \bar{\mathsf{D}}\varphi$$

In fact, difference modality is so strong, that in $\mathbf{L_{MD}}$ we can even simulate nominals: φ is true at exactly one point iff $\mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \to \neg\,\mathsf{D}\,\varphi)$ holds.[2] On the other hand, with respect to frame definability $\mathbf{L_{HA}}$ is as expressive as $\mathbf{L_{MD}}$, so addition of D to $\mathbf{L_{HA}}$ does not change its strength. The interested reader should consult [53] or [5] for details.

As a result of these language interdependencies we have the following hierarchy of expressivity:

$$\mathbf{L_{MA}} < \mathbf{L_{MD}} = \mathbf{L_{MAD}} = \mathbf{L_{HAD}} = \mathbf{L_{HA}} = \mathbf{L_{H@A}}$$

We can obtain complete formalization of hybrid logic with D very simply. It's enough to add the following pure axiom:

$$\mathsf{D}\,i \leftrightarrow \neg i$$

to $\mathbf{HK}^{+}_{\mathbf{H@}}$.[3]

In this section we consider only the expressive power of hybrid languages on the set of all frames. Interesting results concerning selected classes of frames will be mentioned later but one fact should be noticed here. One can easily check that $\mathsf{D}\,\varphi$ is definable in **Kt4.3** (basic tense logic of linear frames) by $\mathsf{P}\,\varphi \vee \mathsf{F}\,\varphi$, so on linear frames $\mathbf{L_{TH@}}$, $\mathbf{L_{THD}}$ and $\mathbf{L_T}$ have the same expressivity.

[2] We have: $\mathfrak{M}, w \vDash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \to \neg\,\mathsf{D}\,\varphi)$ iff $\exists_{w' \in W}\mathfrak{M}, w' \vDash \varphi$ and $\forall_{w', w'' \in W}$ if $\mathfrak{M}, w' \vDash \varphi$ and $\mathfrak{M}, w'' \vDash \varphi$ then $w' = w''$.

[3] We have: $\mathfrak{M}, w \vDash \mathsf{D}\,\imath$ iff $\exists_{w' \in W}(w' \neq w$ and $w' = V(\imath))$ iff $w \neq V(\imath)$ iff $\mathfrak{M}, w \vDash \neg\imath$.

8.2. Modal Binders

Applications of special modalities described in the last section do not have particularly hybrid character. They were considered independently of investigations on hybrid logic and their importance for this field is connected with nice interplay of these modalities with hybrid machinery. But hybrid languages lead to specific enrichments; if we can name states in a model we can ask why not to quantify over states? So the next step is:

- add the third sort of atoms SVAR $:= \{u, v, \dots\}$ (state variables) to the basic hybrid language.
- add some binders – quantifiers $\forall$, $\exists$ or local binder $\downarrow$.

In fact application of quantifiers is present in the earliest approach to hybrid logic due to Prior [104]. His third grade tense logic used both nominals (or rather state variables) and $\forall$. Local binder $\downarrow$ was invented much later and—in contrast to quantifiers borrowed from first-order language—is essentially hybrid concept, although some forms of it were applied outside MHL earlier (see [25] for some historical remarks and [58] for first application in hybrid languages). By the way: application of binders (in particular quantifiers) is one of the sources of the name "hybrid" meant as a combination of propositional modal language and quantification.

Addition of the third sort of atoms is strictly speaking not necessary but it is easier to have distinct state symbols: nominals and variables. The situation is in a sense analogous to that in typical Gentzen-style proof theory for first-order logic, where we distinguish bound occurrences of variables and free occurrences (parameters). But note that free state variables will be also admitted. Definitions of free and bound occurrences of (state) variables, the scope of the binder, the sentence (no free variables) and other similar concepts, are exact analogs of definitions from first-order language, so we omit details, believing in the reader's knowledge.

The definition of the frame and model is the same as for $\mathbf{L_{H@}}$ but we need also the concept of assignment a for $\mathfrak{M}$ which is a mapping $a\colon \text{SVAR} \longrightarrow W$. The satisfaction of a formula is now defined for a model and an assignment. In particular, we have the following conditions:

$$
\begin{array}{lll}
\mathfrak{M}, a, w \vDash \varphi & \text{iff} & w \in V(\varphi), \text{ for any } \varphi \in \text{VAR} \\
\mathfrak{M}, a, w \vDash v & \text{iff} & w = a(v), \text{ for any } v \in \text{SVAR} \\
\mathfrak{M}, a, w \vDash \forall_v \varphi & \text{iff} & \text{for any } w' \in W\colon \mathfrak{M}, a^v_{w'}, w \vDash \varphi \\
\mathfrak{M}, a, w \vDash \exists_v \varphi & \text{iff} & \text{for some } w' \in W\colon \mathfrak{M}, a^v_{w'}, w \vDash \varphi \\
\mathfrak{M}, a, w \vDash \downarrow_v \varphi & \text{iff} & \mathfrak{M}, a^v_w, w \vDash \varphi
\end{array}
$$

where a^{υ}_{w} is an *υ-variant of a*, i.e., for any $\upsilon' \in$ SVAR:

$$a^{\upsilon}_{w}(\upsilon') := \begin{cases} w & \text{if } \upsilon' = \upsilon \\ a(\upsilon') & \text{if } \upsilon' \neq \upsilon \end{cases}$$

We should also admit free state-variables as arguments of @, so the more general condition is:

$\mathfrak{M}, a, w \vDash @_{\imath}\varphi$ iff $\mathfrak{M}, a, V(\imath) \vDash \varphi$, for any $\imath \in$ NOM
$\mathfrak{M}, a, w \vDash @_{\upsilon}\varphi$ iff $\mathfrak{M}, a, a(\upsilon) \vDash \varphi$, for any $\upsilon \in$ SVAR

Truth clauses show that we have exact hybrid analogs of first-order quantifiers, but $\downarrow$ needs some comment. The difference between $\downarrow$ and $\forall$ is between local and global binding. $\downarrow$ enables to name current state ($\downarrow$ binds state variable to current state). Note also that $\downarrow$ is self-dual.

Let $\mathbf{L_{H\forall}}$, $\mathbf{L_{H\downarrow}}$, $\mathbf{L_{H\downarrow\forall}}$ denote weak hybrid languages with added binders and $\mathbf{L_{H@\forall}}$, $\mathbf{L_{H@\downarrow}}$, $\mathbf{L_{H@\downarrow\forall}}$ respective languages with satisfaction operators.

Fact 4. *$\mathbf{L_{H@\forall}}$ is strictly stronger than $\mathbf{L_{H@\downarrow}}$, since:*

1. *$\downarrow$ is definable in $\mathbf{L_{H@\forall}}$ by: $\downarrow_{\upsilon}\varphi := \exists_{\upsilon}(\upsilon \wedge \varphi)$, but*
2. *$\mathbf{L_{H@\downarrow}}$ is preserved under generated submodels, whereas $\mathbf{L_{H@\downarrow\forall}}$ is not.*

(*Obviously, the same applies to $\mathbf{L_{H\forall}}$ and $\mathbf{L_{H\downarrow}}$.*)

Corollary 1. *$\mathbf{L_{H@\forall}} = \mathbf{L_{H@\downarrow\forall}}$ and $\mathbf{L_{H\forall}} = \mathbf{L_{H\downarrow\forall}}$*

In contrast to weak hybrid language with A, @ is not definable in $\mathbf{L_{H\forall}}$. But if we add A to languages with binders we can obtain interesting interdefinability results stated below as:

Fact 5. *$\mathbf{L_{H@\forall}} = \mathbf{L_{H\downarrow A}} = \mathbf{L_{H\forall A}} = \mathbf{L_{H@\downarrow A}} = \mathbf{L_{H@\forall A}}$*

Some of the equations are obvious if we remember that A defines sat-operator; moreover:

1. $\forall$ is defined in $\mathbf{L_{H\downarrow A}}$: $\forall_{\upsilon}\varphi := \downarrow'_{\upsilon} \mathsf{A} \downarrow_{\upsilon} \mathsf{A}(\upsilon' \rightarrow \varphi)$, where $\upsilon' \neq \upsilon$ and $\upsilon' \notin \varphi$;
2. A is defined in $\mathbf{L_{H@\forall}}$: $\mathsf{A}\,\varphi := \forall_{\upsilon}@_{\upsilon}\varphi$, where $\upsilon \notin \varphi$.

As we shall see, the addition of binders strongly increases expressive power of hybrid languages but there are serious costs. Both basic hybrid logics with added binders $\mathbf{K_{H@\downarrow}}$ and $\mathbf{K_{H@\forall}}$ are undecidable (in fact even $\mathbf{K_{H\forall}}$ is undecidable).

Axiomatization. Let $\mathbf{HK}_{\mathbf{H}@\downarrow}$ be axiomatization of $\mathbf{K}_{\mathbf{H}@\downarrow}$, obtained from $\mathbf{HK}_{\mathbf{H}@}$ by addition of:[4]

$$@_i(\downarrow_\upsilon\varphi \leftrightarrow \varphi[\upsilon/\imath]) \tag{DA}$$

Clearly the proviso for the rule of Substitution must be changed a bit to deal with the presence of state variables. Nominals and state variables may be substituted for each other but state variables may be substituted for nominal/(free) state variable only if they are still free. One can easily prove the selfduality principle:

$$\downarrow_\upsilon\varphi \leftrightarrow \neg\downarrow_\upsilon\neg\varphi \tag{Selfdual$\downarrow$}$$

Addition of (NAME) and (BG) to $\mathbf{HK}_{\mathbf{H}@\downarrow}$ gives $\mathbf{HK}^{+}_{\mathbf{H}@\downarrow}$. Pure completeness holds for $\mathbf{HK}^{+}_{\mathbf{H}@\downarrow}$ exactly as for $\mathbf{HK}^{+}_{\mathbf{H}@}$. What's more, we can axiomatize $\mathbf{HK}^{+}_{\mathbf{H}@\downarrow}$ without (BG) and (NAME) but using more standard rules (no side conditions). Just add to $\mathbf{HK}_{\mathbf{H}@\downarrow}$ the following axioms and rules:

$$\downarrow_\upsilon(\upsilon \to \varphi) \to \varphi \qquad \text{provided } \upsilon \notin \varphi \tag{Name$\downarrow$}$$

$$@_\imath\Box\downarrow_\upsilon @_\imath\Diamond\upsilon \tag{BG$\downarrow$}$$

$$\vdash \varphi \;/\; \vdash \downarrow_\upsilon\varphi \tag{RG$\downarrow$}$$

We can axiomatize the set of all validities in the strongest hybrid language just by adding to $\mathbf{HK}^{+}_{\mathbf{H}@}$ the following axioms:

$$\forall_\upsilon(\varphi \to \psi) \to (\varphi \to \forall_\upsilon\psi) \qquad \text{if } u \notin \mathrm{VF}(\varphi) \tag{Q1}$$

$$\forall_\upsilon\varphi \to \varphi[\upsilon/\sigma] \qquad \text{if } \sigma \in \mathrm{SVAR},\ \sigma \text{ is free for } \upsilon \text{ in } \varphi \tag{Q2}$$

$$\forall_\upsilon @_\imath\varphi \leftrightarrow @_\imath\forall_\upsilon\varphi \tag{Barcan@}$$

and the following rule:

$$\vdash \varphi \;/\; \vdash \forall_\upsilon\varphi \tag{Gen}$$

However this system uses nonstandard rules (NAME) and (BG), and we already remarked that even in $\mathbf{L}_{\mathbf{H}\downarrow}$ we can avoid them completely. In $\mathbf{L}_{\mathbf{H}\forall@}$ we can eliminate these rules in favor of two additional axioms:

$$\exists_\upsilon\upsilon \tag{Name$\exists$}$$

$$\forall_\upsilon\Box\varphi \leftrightarrow \Box\forall_\upsilon\varphi \tag{Barcan$\Box$}$$

This is possible also in $\mathbf{L}_{\mathbf{H}\forall}$. Suitable axiomatic system $\mathbf{HK}_{\mathbf{H}\forall}$ consists of axioms and rules of $\mathbf{HK}_{\mathbf{H}@\forall}$ without (Barcan@), but plus for any $m, n \in \omega$:

$$\forall_\upsilon(\Diamond^m(\upsilon \wedge \varphi) \to \Box^n(\upsilon \to \varphi)) \tag{Nom}$$

All these axiomatizations are strongly complete for respective logics.

[4] For axiomatization of $\mathbf{K}_{\mathbf{H}\downarrow}$ with the use of (COV)-rules, see [25].

Expressivity. In fact, $\downarrow$, $\forall$ and $\exists$ are not the only binders considered in hybrid languages. Below we consider briefly some strong versions of quantifiers and local binder considered in [24].

$$\begin{array}{lll} \mathfrak{M}, a, w \vDash \Pi_v\varphi & \text{iff} & \text{for all } w' \in W\text{: } \mathfrak{M}, a^v_{w'}, w' \vDash \varphi \\ \mathfrak{M}, a, w \vDash \Sigma_v\varphi & \text{iff} & \text{for some } w' \in W\text{: } \mathfrak{M}, a^v_{w'}, w' \vDash \varphi \\ \mathfrak{M}, a, w \vDash \Downarrow_v\varphi & \text{iff} & \text{for some } w' \in W\text{: } \mathfrak{M}, a^v_{w}, w' \vDash \varphi \end{array}$$

where a^v_w is an *v-variant of a* (see p. 182).

If we add to $\mathbf{L_H}$ any of these binders we have the following hierarchy: $\mathbf{L_{H\downarrow}} < \mathbf{L_{H\exists}} < \mathbf{L_{H\Downarrow}}$ and $\mathbf{L_{HA}} < \mathbf{L_{H\Sigma}} < \mathbf{L_{H\Downarrow}}$. Since:

$$\begin{array}{ll} \exists_u\varphi := \Downarrow_v\Downarrow_u(v \wedge \varphi) & \text{if } v \notin \varphi \\ \mathsf{E}\,\varphi := \Sigma_u\varphi & \text{if } u \notin \varphi \\ \Sigma_u\varphi := \Downarrow_v\Downarrow_u(u \wedge \varphi) & \text{if } v \notin \varphi \\ \Downarrow_u\varphi := \downarrow_u \mathsf{E}\,\varphi & \end{array}$$

so $\mathbf{L_{H\downarrow A}} = \mathbf{L_{H\Downarrow}}$

The use of these binders may be convenient but languages containing them are not stronger than $\mathbf{L_{H@\forall}}$. In the rest of this paragraph we focus on the expressive strength of two central languages with binders: $\mathbf{L_{H@\downarrow}}$ and $\mathbf{L_{H@\forall}}$.

Let's look at the extension of the standard translation function ST from ordinary modal language to all hybrid languages discussed so far. Following [10] we may use a version, where nominals/state variables are identified with first-order constants/variables.

1. Standard Translation ST_t

$$\begin{array}{lll} \mathrm{ST}_t(s) & = & t = s \\ \mathrm{ST}_t(p) & = & P(t) \\ \mathrm{ST}_t(\neg\varphi) & = & \neg\mathrm{ST}_t(\varphi) \\ \mathrm{ST}_t(\varphi \wedge \psi) & = & \mathrm{ST}_t(\varphi) \wedge \mathrm{ST}_t(\psi) \\ \mathrm{ST}_t(\Diamond\varphi) & = & \exists_x(\mathcal{R}tx \wedge \mathrm{ST}_{\boldsymbol{x}}(\varphi)) \\ \mathrm{ST}_t(@_\sigma\varphi) & = & \mathrm{ST}_\sigma(\varphi) \\ \mathrm{ST}_t(\mathsf{E}\,\varphi) & = & \exists_{\boldsymbol{x}}\mathrm{ST}_{\boldsymbol{x}}(\varphi) \\ \mathrm{ST}_t(\downarrow_v\varphi) & = & \exists_v(v = t \wedge \mathrm{ST}_t(\varphi)) \\ \mathrm{ST}_t(\exists_v\varphi) & = & \exists_v\mathrm{ST}_t(\varphi) \\ \mathrm{ST}_t(\Sigma_v\varphi) & = & \exists_v\mathrm{ST}_u(\varphi) \\ \mathrm{ST}_t(\Downarrow_v\varphi) & = & \exists_{\boldsymbol{x}}\exists_v(v = t \wedge \mathrm{ST}_{\boldsymbol{x}}(\varphi)) \end{array}$$

where $\boldsymbol{x}$ is a variable distinct from term t and not occurring in φ.

Pure completeness of $\mathbf{HK}^{+}_{\mathbf{H@\downarrow}}$ opens the question if we have something more. There are two points worth noticing:

1. $\mathbf{L_{H@}}$ is more expressive (than $\mathbf{L_M}$) at the level of frames but even $\mathbf{L_{H\downarrow}}$ is more expressive at the level of models! For example we can distinguish between reflexive and nonreflexive states in a model ($\downarrow_u \Diamond u$ and $\downarrow_u \neg\Diamond u$).

2. Binary temporal operators U ("Until") and S ("Since") are definable in $\mathbf{L_{H@\downarrow}}$ or $\mathbf{L_{TH\downarrow}}$:

$$\mathsf{U}(\varphi,\psi) := \downarrow_u \Diamond \downarrow_v (\varphi \wedge @_u \Box(\Diamond v \to \psi))$$
$$\mathsf{U}(\varphi,\psi) := \downarrow_u \mathsf{F}(\varphi \wedge \mathsf{H}(Pu \to \psi))$$

Let's recall that U is semantically defined by the following clause:

$$\mathfrak{M}, t \vDash \mathsf{U}(\varphi,\psi) \quad \text{iff} \quad \begin{array}{l} \mathfrak{M}, t' \vDash \varphi \text{ for some } t' \text{ such that } t < t' \text{ and} \\ \mathfrak{M}, t'' \vDash \psi \text{ for any } t'' \text{ such that } t < t'' \text{ and } t'' < t' \end{array}$$

Note by the way that U or S may be locally defined also in $\mathbf{L_{TH@}}$ in the following way:

$$@_\imath(\mathsf{U}(\varphi,\psi) \leftrightarrow \mathsf{F}(\varphi \wedge \mathsf{H}(\mathsf{P}\,\imath \to \psi)))$$

In fact, we can establish exactly the expressive power of $\mathbf{L_{H@\downarrow}}$ on the level of models and frames. As for the first it holds [6]:

THEOREM 10. *A formula of first-order correspondence language φ is equivalent to standard translation of a sentence in $\mathbf{L_{H@}}$ iff φ is equivalent to strongly bounded formula.*

Recall that strongly bounded fragment of first-order language covers all formulae built up from atoms with the help of boolean constants and bounded quantification (i.e., $\exists_y(Rxy \wedge \varphi)$ and $\forall y(Rxy \to \varphi)$). This is the fragment of first-order language which is invariant under generated submodels. Concerning frame definability we have:

THEOREM 11. *An elementary class of frames is defined by pure sentences of $\mathbf{L_{H@\downarrow}}$ iff it is closed under generated subframes and reflects finitely generated subframes.*

But the class of elementary frames definable in ordinary modal language is closed under generated subframes and reflects point-generated subframes, so $\mathbf{L_{H@\downarrow}}$ covers all this class (Sahlqvist formulae in particular). For example, Church-Rosser property is definable in $\mathbf{L_{H@\downarrow}}$ by pure formula:

$$\Diamond i \wedge \Diamond j \to @_i(\Diamond \downarrow_u @_j \Diamond u)$$

On the other hand, some elementary conditions not definable in $\mathbf{L_M}$ like Predecessors ($\forall_x\exists_y\mathcal{R}yx$) are not definable by pure formulae in $\mathbf{L_{H@\downarrow}}$ either (since the class of frames with this property is not closed under generated subframes).

So far we have pointed out limitations of several hybrid languages when compared with the expressive strength of first-order language. But $\mathbf{L_{H@\forall}}$ has full first-order language expressivity. It is obvious, because now we can directly rewrite any first order formula as a formula of $\mathbf{L_{H@\forall}}$. Formally, we can define some translation function, first introduced by Prior:

Hybrid Translation HT

$$\begin{array}{lcl}
\mathrm{HT}(Rtt') & = & @_t\Diamond t' \\
\mathrm{HT}(Pt) & = & @_t p \\
\mathrm{HT}(t = t') & = & @_t t' \\
\mathrm{HT}(\neg\varphi) & = & \neg\mathrm{HT}(\varphi) \\
\mathrm{HT}(\varphi \wedge \psi) & = & \mathrm{HT}(\varphi) \wedge \mathrm{HT}(\psi) \\
\mathrm{HT}(\exists_v\varphi) & = & \exists_v\mathrm{HT}(\varphi)
\end{array}$$

Because of the full first-order expressive power, in case of $\mathbf{HK_{H@\forall}}$ we need only the basic completeness theorem.

The definition of HTmakes obvious why $\mathbf{L_{H@\forall}}$ is strictly stronger than $\mathbf{L_{H\forall}}$. It is important to note that the use of satisfaction operators are essential in the translation. Addition of quantifiers to the weak hybrid language does not yield the full first-order expressivity which at first sight may seem strange.

Note in particular, that all properties not expressible as pure formulae in $\mathbf{L_{H@}}$ (e.g. Geach axioms, Directedness) are expressible in $\mathbf{L_{H@\forall}}$ as a PUENF-formulae (pure universal existential nominal-free formulae) of the shape: $\forall_{u_1,\ldots,u_m}\exists v_1,\ldots,v_n\varphi$, where φ has no quantifiers, propositional variables, nominals (only state variables), e.g.:

- Church-Rosser property: $\forall_{u_1u_2u_3}\exists_v(@_{u_1}\Diamond u_2 \wedge @_{u_1}\Diamond u_3 \rightarrow @_{u_2}\Diamond v \wedge @_{u_3}\Diamond v)$
- Predecessors: $\forall_u\exists_v@_v\Diamond u$
- Right-directedness: $\forall_{u_1u_2}\exists_v(@_{u_1}\Diamond v \wedge @_{u_2}\Diamond v)$

THEOREM 12. *Frame condition is defined by PUENF-formula iff it is UE-closure of strongly bounded first-order formula.*

Blackburn [27] stated a conjecture that every Sahlqvist formula is expressible by PUENF-formula. PUENF-formulae are quite interesting since they allow stronger completeness result for $\mathbf{HK^+_{H@}}$ (see [28]). First note the following:

Fact 6. *Every PUENF-formula* (PF) $\forall_{u_1 \ldots u_m} \exists_{v_1 \ldots v_n} \varphi$ *corresponds to existential saturation rule of the form:*

$$\text{if} \vdash \varphi[u_1/\imath_1, \ldots, u_m/\imath_m, v_1/\jmath_1, \ldots, v_n/\jmath_n] \to \psi, \text{ then} \vdash \psi \qquad \text{(RPF)}$$

provided $\jmath_1, \ldots, \jmath_n$ *are distinct, unequal to* $\imath_1, \ldots, \imath_m$ *and do not occur in* ψ.

For example for Church-Rosser property we have the rule:

$$\text{If} \vdash (@_{\imath_1}\Diamond\imath_2 \wedge @_{\imath_1}\Diamond\imath_3 \to @_{\imath_2}\Diamond\jmath \wedge @_{\imath_3}\Diamond\jmath) \to \psi, \text{ then} \vdash \psi$$

provided $\jmath \notin \psi$ and $\jmath \neq \imath_1, \imath_2, \imath_3$.

(RPF)-rules closely resemble Gabbay's style nonstructural rules for undefinable (in standard ML) conditions. They become in the effect of skolemization of state variables with the help of nominals in PUENF-formulae. The relation between formulae and rules is clarified in the following:

Lemma 8. *If* (PF) *defines* $\mathcal{F}$, *then* (RPF) *is admissible in* $\mathcal{F}$.

As a consequence we can prove much stronger completeness result for logics in the basic language.

Theorem 13 (Extended Pure completeness). *Let* Γ *be any set of pure formulae and* R *any set of existential saturation rules, then* $\mathbf{HK}^{+}_{\mathbf{H@}} + \Gamma + R$ *is strongly complete for the class of frames defined by* Γ *and* R.

Note that application of existential saturation rules may also strengthen the scope of the pure completeness theorem for $\mathbf{HK}^{+}_{\mathbf{H@\downarrow}}$.

One should note that the concept of such nonstandard rules enriching seriously the expressive power of $\mathbf{L_{H@}}$ was first independently explored on the field of tableau methods (see [27]) and RDN—nonstandard formalization combining natural deduction with resolution (see [76]). Some details will be given in the second part.

9. Instead of a Summary

There are many developments of MHL that we did not even touch. In particular, interesting results with respect to expressivity may be obtained not only by adding new constants but also multiplying sorts of atoms. Multisorted hybrid languages and their application to analysis of linguistic phenomena of tensal discourse were investigated by Blackburn (see e.g. [18, 21]). One can find there for instance hybrid formalization of interval tense logic with two sorts of nominals, denoting instants and intervals. Below we describe briefly the extension of propositional hybrid logic to first-order modal hybrid logic. We also bring together the most basic facts concerning decidability, complexity and interpolation in MHL.

9.1. First-Order Modal Hybrid Logic QMHL

The number of papers devoted to first-order hybrid logic is rather small but the effects of such extensions obtained so far are quite promising. The use of hybrid languages makes possible to obtain interesting results concerning the formalization of nonrigid terms and expressing various conditions put on domains in models. Moreover, we will see that first-order hybrid logic is particularly good behaved with respect to interpolation property.

Below we present a logic **QMHL**, a first-order version of $\mathbf{L_{H@\downarrow}}$ from [23].

1. Vocabulary of $\mathbf{L_{H@\downarrow}}$ is enriched with:

- denumerable set of first order variables FVAR $:= \{x, y, \dots\}$,
- denumerable set of rigid constants CON $:= \{c_1, c_2, \dots\}$,
- denumerable set of nonrigid constants FUN $:= \{f_1, f_2, \dots\}$,
- denumerable set of predicate symbols of n-arity PRED $:= \{P_1, P_2, \dots\}$,
- first-order (possibilistic) quantifiers and equality predicate: $\forall$, $\exists$, $=$.

2. The set of terms contains FVAR and CON, and is closed under the rule:

- if $f \in$ FUN and $\sigma \in$ NOM $\cup$ SVAR, then $@_\sigma f$ is a term

Remark. Sat-operator is used to form both formulae and terms which is very hybrid solution indeed! In case of terms this is the way for rigidification of nonrigid terms. Let me remind you that nonrigid terms vary their denotation in different worlds, whereas rigid terms have the same denotation in all worlds. When sat-operator is attached to nonrigid term f it means: the designate of f in σ, and this compound term has constant value. This informal remark will become obvious after introduction of semantics.

3. Models are structures of the form $\mathfrak{M} = \langle \mathcal{W}, \mathcal{R}, D, V \rangle$, where D is a nonempty constant domain and V is defined as follows: $V(c) \in D$, $V(\imath) \in \mathcal{W}$, $V(P^n) \subseteq D^n \times \mathcal{W}$ and $V(f) \in D^{\mathcal{W}}$. An assignment $a = a_{\mathrm{s}} \cup a_{\mathrm{f}}$, where $a_{\mathrm{s}}\colon \mathrm{SVAR} \longrightarrow \mathcal{W}$ and $a_{\mathrm{f}}\colon \mathrm{FVAR} \longrightarrow D$. The interpretation I of the term τ

in a model and under an assignment a is defined as follows:

$$I(\tau) := \begin{cases} a(\tau) & \text{if } \tau \in \text{FVAR} \\ V(\tau) & \text{if } \tau \in \text{CON} \\ V(f).V(\imath) & \text{if } \tau = @_\imath f \text{ for } \imath \in \text{NOM and } f \in \text{FUN} \\ V(f).a(\upsilon) & \text{if } \tau = @_\upsilon f \text{ for } \upsilon \in \text{SVAR and } f \in \text{FUN} \end{cases}$$

New clauses for satisfaction are:

$$\begin{array}{lll} \mathfrak{M}, a, w \vDash P^n(\tau_1, \ldots, \tau_n) & \text{iff} & \langle I(\tau_1), \ldots, I(\tau_n), w\rangle \in V(P^n) \\ \mathfrak{M}, a, w \vDash \tau_1 = \tau_2 & \text{iff} & I(\tau_1) = I(\tau_2) \\ \mathfrak{M}, a, w \vDash \forall_x \varphi & \text{iff} & \text{for all } o \in D\text{: } \mathfrak{M}, a^x_o, w \vDash \varphi \\ \mathfrak{M}, a, w \vDash \exists_x \varphi & \text{iff} & \text{for some } o \in D\text{: } \mathfrak{M}, a^x_o, w \vDash \varphi \end{array}$$

The version of the semantics we have presented has constant domain and possibilistic quantifiers just for simplicity. But we can also add the function $d\colon \mathcal{W} \longrightarrow \mathcal{P}(D)$ and introduce actualist quantifiers—this is easy. One can do that either indirectly in the manner described in [48], by introducing existence predicate and relativization of quantifiers to this predicate, or directly by treating actualist quantifiers as primitive. The second route is taken in [28], where axiomatization of **QMHL** in $\mathbf{L_{H@}}$ is presented which satisfies general pure completeness theorem. Interestingly enough it covers not only frame conditions definable by pure axioms (and saturated rules) but also extensions obtained by considering several domain conditions, because they may be expressed by pure axioms. For example, popular conditions ordinarily defined with the help of Barcan Formula and its converse, like monotonicity or antymonotonicity are defined as follows:

$$E@_i c \to \Box E@_i c \tag{MON}$$

$$\Diamond E@_i c \to E@_i c \tag{AMON}$$

Where E is existence predicate defined in a standard way: $E\tau := \exists_x\, x = \tau$ and $x \neq \tau$. Constant domain is expressed even simpler by $@_i E @_j c \to @_k E @_j c$. Moreover, some other, not very popular, conditions may be expressed, e.g.:

Full domains:	$E@_i c$
Disjoint domains:	$@_i E@_j c \wedge @_k E@_j c \to @_i k$
Convex domains:	$E@_i c \to \Box(\Diamond E@_i c \to E@_i c)$

Note that hybrid version of **QML** allows simple form of representation of nonrigid terms which in ordinary modal language makes some troubles.

Here is an example: let c = "Caroline" (rigid term in Kripke spirit) and f = "Miss of America" (clearly nonrigid term), then the sentence "Caroline is the present Miss of America" is expressed by $\downarrow_u(c = @_u f)$. One can check that:

$$\models \downarrow_u(c = @_u f) \to \downarrow_u \mathsf{G}(c = @_u f)$$

but

$$\not\models \downarrow_u(c = @_u f) \to \mathsf{G} \downarrow_u(c = @_u f)$$

And this is in accordance with our expectations, since the first means: "If Caroline is the present Miss of America, then it always be the case, that she is the Miss of America of now", which is obviously true. On the other hand, the second means: "If Caroline is the present Miss of America, then it always be the case, that she will be the Miss of America", which is obviously false.

9.2. Decidability and Complexity

During the discussion of several hybrid languages and logics, we accidentally made some remarks concerning decidability and complexity of them. It is helpful to collect these remarks and add some more in order to get a fuller picture. We consider only the question of decidability for satisfiability problem. It's easy to note that there are three possible effects of changing ordinary modal theories into hybrid theories. We can have:

1. The same complexity class e.g. $\mathbf{K_{H@}}$
2. Worse behavior e.g. $\mathbf{Kt_{H@}}$
3. Better behavior – logics of some frame classes.

The last point is particularly interesting and we show some striking examples but first just recall the basic complexity hierarchy for easy reference:

$$\text{P} \leq \text{NP} \leq \text{PSPACE} \leq \text{EXPTIME}$$

(for our considerations we need only these classes of problems).

Some concrete results. 1. Bad impact of past operators: Even $\mathbf{Kt_H}$ with one nominal is EXPTIME-complete, whereas $\mathbf{Kt}$ is PSPACE-complete. The same applies also to monomodal hybrid logics of symmetric frames. On the other hand, addition of @ and A do not change the complexity, whereas in ordinary modal language it also jumps to EXPTIME.

2. Transitive frames: Hybrid modal logics of transitive frames are in PSPACE even with A (recall that $\mathbf{K_{HA}}$ is EXPTIME-complete). But $\mathbf{Kt4_H}$ is still EXPTIME-complete.

3. Linear frames: The best results we have for hybrid logics of linear frames. They are NP-complete even with $\downarrow$. Note that even $\mathbf{K_{H\downarrow}}$ is undecidable but on linear frames we have not only decidability but also of relatively low complexity since in this respect it is as good as **CPL**.[5]

9.3. Interpolation and Beth Definability

Hybrid logics show also remarkable advantages over standard modal logics with respect to interpolation properties. Let's recall the basic definitions in the form suitable for hybrid languages. Note that we can obtain several forms of interpolation property for MHL, if in the following definitions we change the meaning of P. If P is the set of propositional variables (VAR), it is a standard notion from ML, but we can consider also the version where P covers additionally the set of nominals (NOM), or of only nominals.

Definition. $\boldsymbol{L}$ has the *Strong Interpolation Property* iff $\models_{\boldsymbol{L}} \varphi \to \psi$ implies that $\models_{\boldsymbol{L}} \varphi \to \chi$ and $\models_{\boldsymbol{L}} \chi \to \psi$ for some χ such that $P(\chi) \subseteq P(\varphi) \cap P(\psi)$.

Definition. $\boldsymbol{L}$ has the *Weak Interpolation Property* iff $\varphi \mid\models_{\boldsymbol{L}} \psi$ implies that $\varphi \mid\models_{\boldsymbol{L}} \chi$ and $\chi \mid\models_{\boldsymbol{L}} \psi$ for some χ such that $P(\chi) \subseteq P(\varphi) \cap P(\psi)$.

The relation between the two concepts is the following:

Theorem 14. *If $\models$ is compact, then strong interpolation implies weak interpolation.*

For the two most important basic hybrid logics we have:

Theorem 15. $\mathbf{K_{H@\downarrow}}$ *has strong interpolation.* $\mathbf{K_{H@}}$ *has only weak interpolation.*

An example: there is no interpolant for $i \wedge \Diamond i \to (j \to \Diamond j)$ but if we limit P to propositional variables only, then strong interpolation holds also for $\mathbf{K_{H@}}$ and for $\mathbf{K_H}$. These results extend to Beth definability in case of $\mathbf{K_{H@}}$, since in order to derive this property we need only interpolation with

[5]Perhaps we should rather say as bad as **CPL**, if we remember that only P-complete problems are considered as practically tractable.

P limited to propositional variables. But, surprisingly enough, it does not hold for $\mathbf{K_H}$ (see [115]).

There is an interesting relation between decidability and interpolation which is expressed by the following two theorems:

Theorem 16. *Every hybrid logic formulated in extension of* $\mathbf{L_H}$ *either is decidable or has strong interpolation (over nominals).*

Theorem 17. $\mathbf{K_{H@\downarrow}}$ *is the least logic with strong interpolation; any extension axiomatizable by a set of nominal-free sentences also has this property.*

Good behavior of **QMHL** in $\mathbf{L_{H@\downarrow}}$ is particularly worth mentioning. The following theorem holds:

Theorem 18. *Strong interpolation (and Beth definability) holds for any* **QMHL** *between* **K** *and* **S5**.

This is in strong contrast to ordinary **QML** where we have the following negative result due to Fine:

Theorem 19. *Interpolation fails for any* **QML** *between* **K** *and* **S5** *with constant domains and for* **S5** *with varying domains.*

PART II

10. Introductory Remarks on Proof Methods

In this part we focus on the discussion of proof systems invented for hybrid logics. So far for hybrid logics the following proof systems, except axiomatic calculi, were devised:

- Sequent calculi (SC)
- Natural Deduction systems (ND)
- Tableau calculi
- Resolution systems

Some types of systems, like refutation systems, connexion calculi, Davis-Putnam method or goal oriented deduction systems, although applied in nonclassical logics, and in standard modal logic in particular, were not devised for hybrid logics so far. Before we describe and compare existing formalizations some remarks on the general features of proof systems are in order.

10.1. Desiderata

In the Introduction we have stressed that one of the main problems caused by the limitations of ordinary modal language is the disability to obtain good proof theory for modal logics. But what does it mean to have a "good" proof theory? This is very general question concerning very vague notion. For our purposes we may pose related but more specific question, since we are interested in practical matters only. What is a "good" proof system? Even in this form it is disputable, but here we think of the following properties:

- universal application
- uniform character
- generality
- naturalness
- simplicity
- efficiency

Except the last, all of these terms are in need of some explanation.

We say that a proof system has universal application (or simpler, that it is universal) if it may be used to perform different deductive tasks. For example, universal system allows not only to construct proofs but also to show that a formula is invalid. It makes possible to define proof search procedures, and even if the formalized logic is not decidable, it gives some ground for application in automated theorem proving. Tableau and resolution calculi, and to some extent sequent calculi, satisfy this property, whereas axiomatic systems and ND-systems, in their standard form, are not universal.

Uniform character of a system is connected with the scope of its applicability. It means that the system may serve as a general deductive framework for formalization of several nonclassical logics. The name comes from the fact that it gives us handy tool to investigate different logics in an uniform fashion. So far, axiomatic systems are unquestionable winners in this respect. But recent developments of sequent calculi, especially of nonstandard character (like display calculi or hipersequent calculi), or tableau calculi (in particular cf. [61]) offer some hope. Below we focus on a search for systems uniform at least with respect to hybrid logics.

By generality of a system we mean that it is able to simulate in a direct fashion other kinds of systems.[6] It makes possible to apply on its ground

[6]Several notions of simulation are precisely defined in e.g. [2], [91] or [108]. Because of semiformal character of these notes, we do not pursue this issue here.

deductive techniques from several sources, and use it as a tool for a comparison of different proof-search strategies and their efficiency. In our opinion, ND-systems seem to be the most general systems, but their abilities are not fully recognised so far. In the last section we present a system of hybrid character because it combines ND-system with resolution. This solution gives it rather general character, as we will try to show.

Proof system is natural if its rules are in accordance with traditional methods of inference, known from antiquity and used by humans in their common thinking, as well as in informal mathematical proofs. Once again, ND-systems seem to satisfy this requirement better than other systems, because the latter are often limited to the use of special types of rules only, regulated rather by theoretical than practical needs. It's not a surprise; both Jaśkowski and Gentzen have just this goal in mind when they have constructed the first ND-systems. Most of the later introduced variants and modifications were also generally connected with this idea. Of course, we can consider to what extent the existing ND-systems are really natural, but we should agree that they are natural at least by definition.

Naturalness seems to be in close connection with simplicity of the system, but this property is a very vague notion in general. Moreover, several possible senses are hardly subject to any objective criteria. Anyway, it is worth exploring. In the case of proof systems simplicity means, among other things:

1. simplicity of inference rules,
2. simplicity of the construction, and the number of elements of the whole system (easy to describe, to implement),
3. easy to follow proofs, readable for humans,
4. ability to construct short and direct proofs,
5. applicability of simple proof search strategies.

It is easy to observe that these features are rather independent and moreover, sometimes they even tend to be in conflict. For instance the possibility of building short and direct proofs is usually the result of the rich structure of the system. On the other hand, systems simple in the sense 1 or 2 are often unable to produce short and easy to follow (and to find) proofs. For example, axiom systems are certainly simple in the 1 and 2 sense, which is the source of their success in metalogic. Axiomatic proofs have also in a sense very simple structure, but it does not mean that they are readable or short, or easy to find! ND-systems are simple usually in the sense 1, 3 and 5, but the price for that is a complex structure of the calculus. Similar remarks may be applied to other types of proof systems which will be discussed below.

10.2. Labelled Deduction

Proof systems for hybrid logics are usually based on the solutions for ordinary modal logics and, as we shall see, they often improve them in many respects. The first non-axiomatic proof systems for the most important modal logics were invented quite early; it is enough to mention Ohnishi and Matsumoto sequent calculi [92, 93], Kripke tableau systems [86], or Fitch's ND-systems [43]. But for many years the work on their extension to other logics in a uniform way was rather limited. The problem was noticed for example by Bull and Segerberg [32]. On the other hand, one can say that the surveys like that of Goré [61](or even much earlier [121]) show that a lot of logics can be formalized with the help of tableaux or sequent calculi. It's true, but it is easily seen that many of the systems are based on ad hoc solutions that are incompatible with the most natural requirements connected with natural and practically useful tableau or sequent calculi. These problems for modal sequent calculi are discussed extensively in [119].

Restricted application of standard proof methods to standard modal logics generated two strategies: either construct nonstandard proof methods better suited to formalization of standard modal logics, or change the language into something more sensitive to the application of standard methods. The first approach, usually based on the use of richer metalogical apparatus, appeared very fruitful and, in particular on the ground on the methodology of sequent calculi, has led to the invention of many interesting general frameworks suitable not only for modal logics. We can mention here for example the method of hypersequent calculi due to Avron, or Belnap's general theory of display calculi (in particular for modal logics, for presentation of both see again [119]). The second choice has led to invention of hybrid logics[7] that—as we shall see—represent far better behavior when formalized with standard tools.

So in the case of hybrid logics rather the second strategy is applied, but not only. Calculi for hybrid logics are closely related to one of the nonstandard approaches in constructing proof systems, based on the use of

[7] But not only; we can mention also description logics in this context despite its different origin.

labels (prefixes), introduced by Fitting ([44] in fact refers to earlier note of Fitch [42] as a source of inspiration) and extensively studied by Gabbay [50]. Although, historically (and also statistically), labelling is associated with tableau methods, it must be stressed that this technique is independent of the kind of proof system we use.

As we have pointed out in the Introduction, labelling is the external form of representation of states in a model, whereas hybrid languages present internal form. In fact, if we use the term labelled deduction in its most general form, almost all existing proof systems for hybrid logics belong to this category. So, before the presentation of concrete proof systems for hybrid logics we focus on the question of applicability of labels in proof theory. This technique is not only connected with modal logics. Dov Gabbay in his general theory of LDS's (labelled deductive systems) considered several applications on different fields. Labels may be used to represent e.g.:

- fuzzy reliability value n $(0 \leqslant n \leqslant 1)$ used mainly in expert systems,
- the situation where the infon holds in situation semantics,
- the set of assumptions for a formula (e.g. Anderson/Belnap [4] ND-systems for relevant logics),
- truth values or the sets of truth values for a formula (e.g. Carnielli [34] or Hahnle [62] tableau systems for many-valued logics),
- possible world (point of time) satisfying a formula in modal (temporal) logics.

Of course for our aims the last item is the most important. In general, we will call labelled deduction system, every proof system where labels (in a wide sense) are used. Blackburn [19] distinguishes three kinds of labelled deduction systems:

1. external – labels as an additional technical apparatus,
2. internalized – labels as a part of a language (in particular nominals in hybrid languages),
3. mixed – both nominals (in a language) and labels (metalinguistic devices) present.

In external approach we can additionaly distinguish a variety of solutions:

1. Weak labelling – labels as a very limited device supporting proof construction, e.g. tableau systems of Marx, Mikulas, Reynolds [87] for linear tense

logics based on the use of three labels, multisequent calculi of Indrzejczak [74] for tense logics.

2. Strong labelling – system of labels as an exact representation of an attempted falsifying model, fusion of 2 systems: object language calculus + calculus for the algebra of labels, e.g. Gabbay's theory of labelled systems, Russo [107] ND-systems for modal logics, Basin, Mathews, Vigano [13, 14] ND-systems for nonclassical logics.

3. Medium labelling – with no special calculus for labels but still sufficient for construction of falsifying model e.g. Fitting's [45] tableau calculi for modal logics, explored by Massaci [88, 89] and Goré [61] under the name explicit systems.

The application of labels in weakly labelled systems is very restrictive. They usually work as an additional mechanism supporting the proof construction but not sufficient for extraction of falsifying models for nonvalid formulae. Since they were not applied in hybrid logics we are not going to discuss them here.

The situation with strong labelling is similar since hybrid languages in itself are strong enough to express everything usually represented by algebra of labels in such systems. What is of some importance for hybrid logics is the approach of Fitting, called here medium labelling, because it was also applied by Tzakova [117]. It is one of the most popular solution for modal deduction—simple, and natural. In this approach labels connected to formulae are finite sequences of natural numbers encoding both the name of a state where this formula is evaluated and the place of this state in a (falsifying) model we search for.

In fact, the restriction of our attention to systems for hybrid logics leave us with the internalized and mixed systems. In particular, internalized systems are the most popular for hybrid logics. But even in this group, where essentially we have the situation of application of standard proof systems to hybrid logics it is handy to distinguish additional group. These are systems which do not use external labels but where the rules are defined only on sat-formulae or data structures (like sequents, clauses e.t.c.) built up only from sat-formulae. Although such systems are naturally limited to logics in languages with sat-operators, within this group of languages they have sufficient generality. It follows from the admissibility of (NAME). So if we want to prove a thesis φ which is not sat-formula we must try to prove $@_\imath\varphi$ with $\imath \notin \varphi$. The reason to distinguish sat-calculi as a group of its own lies in the fact that they form the most numerous group of proof systems for hybrid logics. So finally we have the folowing groups of systems:

1) ordinary calculi: Seligman's sequent calculus, ND-system of Indrzejczak;
2) sat-calculi (rules defined on sat-formulae): Blackburn's tableau system, Demri's sequent calculus[8], Braüner's ND-system, Areces' HyLoRes resolution system, RND-system (resolution-based ND) of Indrzejczak;
3) mixed calculi (with external labels): sequent system of Seligman and tableau system of Tzakova.

In what follows we focus first on sequent calculi: ordinary, due to Seligman, sat-calculus of Blackburn, and nonstandard calculus of Demri. Then we go to natural deduction systems—ordinary (due to Indrzejczak) and sat-calculus of Braüner. In the section on tableau systems we present sat-calculus of Blackburn and mixed calculus of Tzakova. Finally, we present two sat-calculi defined on clauses and based on resolution, namely HyLoRes due to Areces, and RND (resolution based ND) due to Indrzejczak. In each case we will present the basic system and its main features. In particular, we will focus on the problem of uniform extension of the basic system to stronger languages and logics.

11. Sequent Calculi

We start our presentation of proof systems with sequent calculi (SC) since they seem to be the most important proof systems applied in proof theory. Moreover, the first non-axiomatic systems constructed for hybrid logics were of this type. There were several versions od SC constructed by Seligman in the early 1990's, for situation theory (see [109, 110]). These systems deal with languages without modalities, so we do not describe them in detail focusing rather on the system from [112] which extends earlier results and contains formalization of strong modal hybrid logic. But this is not the only SC for logics considered in this survey.

In what follows, we will describe three calculi for MHL. Except Seligman's ordinary sequent calculus, we present two nonstandard ones. One of them, due to Blackburn [19], is an example of a sat-calculus, constructed rather indirectly, by transformation of suitable tableau sat-system. Although

[8]In fact this system is not a sat-calculus in the strict sense. There are no even sat-operators in the language! But rules are defined only on formulae of the shape $\imath \to \varphi$, which naturally put this calculus in this group.

this SC uses exclusively sat-formulae, it is rather an extension than modification of standard Gentzen approach. The second proposal, due to Demri [38], shows more serious departure from ordinary SC since it is based on sequent version of KE-system.

From the variety of other nonstandard sequent calculi applied for modal logics, like hipersequent calculus, only display calculus was used by Demri and Goré [39], to formalize tense hybrid logic. We do not present this SC because it would require prior presentation of principles of display calculi, and there is no space for that.

11.1. Seligman's SC

We start with the sequent calculus of Seligman complete for the basic hybrid logic in $\mathbf{L_{H@}}$ and some stronger languages. It this case we have simply an extension of ordinary SC with additional rules. So the first group of rules is standard:

General rules

(AX) $\varphi, \Gamma \Rightarrow \Delta, \varphi$

(S)[1] $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}$

($\neg\Rightarrow$) $\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$

($\Rightarrow\neg$) $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

($\wedge\Rightarrow$) $\dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$

($\Rightarrow\wedge$) $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$

($\vee\Rightarrow$) $\dfrac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$

($\Rightarrow\vee$) $\dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$

($\rightarrow\Rightarrow$) $\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}$

($\Rightarrow\rightarrow$) $\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$

Side conditions: 1. where Γ, Γ' and Δ, Δ' contain the same sets of formulae.

Rules are defined on ordinary Gentzen sequents $\Gamma \Rightarrow \Delta$, where both Γ, Δ are finite lists of formulae. But the actual set of rules is slightly different, in particular there is no ordinary Gentzen structural rules of contraction, permutation and weakening. Note however, that (S) captures the effect of contraction and permutation; weakening is admissible since (AX) has generalised form (with side formulae). To the original set of Seligman's rules we've added rules for $\wedge$ and $\rightarrow$; Seligman treats these constants as definable and suitable rules as derivable.

The second group of rules is defined to deal with nominals.

Nominal rules

$$(\mathtt{@I}\Rightarrow)\ \frac{\imath, \varphi, \Gamma \Rightarrow \Delta}{\imath, @_\imath\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\mathtt{@I})\ \frac{\imath, \Gamma \Rightarrow \Delta, \varphi}{\imath, \Gamma \Rightarrow \Delta, @_\imath\varphi}$$

$$(\mathtt{@E}\Rightarrow)\ \frac{\imath,\ @_\imath\varphi, \Gamma \Rightarrow \Delta}{\imath, \varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow @E)\ \frac{\imath, \Gamma \Rightarrow \Delta, @_\imath\varphi}{\imath, \Gamma \Rightarrow \Delta, \varphi}$$

$$(\mathtt{N}_1)^1\ \frac{\imath, \jmath, \Gamma[\imath] \Rightarrow \Delta[\imath]}{\imath, \jmath, \Gamma[\jmath] \Rightarrow \Delta[\jmath]} \qquad (\mathtt{TERM})^2\ \frac{\imath, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\mathtt{NAME})^3\ \frac{\imath, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Side conditions:

1. where $\Gamma[\imath]$ means that $\imath$ occur in Γ and $\Gamma[\jmath]$ is the result of replacement of $\jmath$ for $\imath$ in Γ,
2. where all elements of Γ and Δ are sat-formulae,
3. where $\imath$ does not occur in Γ and Δ.

$(\mathtt{N}_1)$ is a kind of substitution rule; we substitute $\jmath$ for $\imath$ at once through all the sequent.

The last group of rules is defined for modal constants.

Modal rules

$$(\Diamond\Rightarrow)^1\ \frac{\Diamond\imath, @_\imath\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Diamond)\ \frac{\Gamma \Rightarrow \Delta, \Diamond\imath \qquad \Gamma \Rightarrow \Delta, @_\imath\varphi}{\Gamma \Rightarrow \Delta, \Diamond\varphi}$$

$$(\Box\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \Diamond\imath \qquad @_\imath\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Box)^1\ \frac{\Diamond\imath, \Gamma \Rightarrow \Delta, @_\imath\varphi}{\Gamma \Rightarrow \Delta, \Box\varphi}$$

Side condition: 1. where $\imath$ does not occur in Γ, Δ and φ.

This time we've added rules for $\Box$ that are not present in [112], but are easy to obtain.

Proofs in the system are defined in standard way, as trees of sequents, constructed with the help of rules, with axioms as leaves and deduced sequents as roots. Clearly, proof search is performed in an upside-down manner; we start with the root-sequent and systematically add above sequents-premises of suitable rules. Below we display an example of a proof (applications of (S) ignored).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Diamond j, i \Rightarrow @_j p, \Diamond j \quad @_j p, \Diamond j, i \Rightarrow @_j p}{\Box p, \Diamond j, i \Rightarrow @_j p}\,(\Box\Rightarrow)}{i, @_i\Box p, \Diamond j \Rightarrow @_j p}\,(@\mathtt{I}\Rightarrow)}{i, @_i\Box p, @_i\Diamond j \Rightarrow @_j p}\,(@\mathtt{I}\Rightarrow)}{@_i\Box p, @_i\Diamond j \Rightarrow @_j p}\,(\mathtt{TERM})}{@_i\Box p \wedge @_i\Diamond j \Rightarrow @_j p}\,(\wedge\Rightarrow)}{\Rightarrow @_i\Box p \wedge @_i\Diamond j \to @_j p}\,(\Rightarrow\to)
$$

The notions of derivable and admissible rules are easily redefined for SC, if we introduce the relation of deducibility between sequents, and in the definitions taken from axiom system we put sequents in place of formulae. Formally:

Definition. Let S_i denote a sequent, then:

1. $S_1, \ldots, S_k \vdash_{\mathtt{SC}} S_{k+1}$ iff there is a proof in SC of S_{k+1}, where leaves are not only axioms but also sequents $S_i, i \leqslant k$ (if $k = 0$ we have ordinary SC-proof).
2. $S_1, \ldots, S_k \,/\, S_{k+1}$ is SC-derivable iff $S_1, \ldots, S_k \vdash_{\mathtt{SC}} S_{k+1}$
3. $S_1, \ldots, S_k \,/\, S_{k+1}$ is SC-admissible iff $\vdash_{\mathtt{SC}} S_{k+1}$, if $\vdash_{\mathtt{SC}} S_1, \ldots, \vdash_{\mathtt{SC}} S_k$.

In particular, proofs of admissibility of several forms of cut in the context of SC are usually called cut-elimination proofs.

Concepts of satisfiability and validity of a sequent $\Gamma \Rightarrow \Delta$ may be reduced to satisfiability of the corresponding implication $\wedge\Gamma \to \vee\Delta$. Seligman's SC is sound and complete with respect to $\mathbf{K_{H@}}$.

Properties. Let us consider some important features of Seligman's SC, namely:

- The lack of restrictions on formulae in sequents
- The construction and generality of the hybrid rules
- The presence of elimination rules for nominals and sat-operators
- Admissibility of the cut of the form:

$$
(\mathtt{Cut}) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}
$$

By the first, we simply mean that it is the extension of ordinary SC to hybrid language. All formulae are permitted as elements of sequents which is the most fundamental difference with sat-calculus of Blackburn which will be discussed next.[9] Due to this feature, Seligman is able to obtain SC for hybrid logic just by addition of new rules to standard SC; no modification of classical basis is needed.

Such a modular approach makes easier the comparison of this calculus with axiomatic system. We can easily prove the following:

Lemma 9. *If* $\varphi \in \mathrm{Th}(\mathbf{HK}^{+}_{\mathbf{H@}})$, *then* $\Rightarrow \varphi$ *is derivable in Seligman's* SC.

We omit the proof; it is sufficient to prove all the axioms and show (with the help of cut) derivability of all the rules of $\mathbf{HK}^{+}_{\mathbf{H@}}$, which is routine.

Nevertheless the proof of the above lemma may be instructive; one can find that application of $(\mathtt{N}_1)$ is not necessary for the proofs of axioms (although it can shorten them). Also the rule (NAME) is needed only for the proof of derivability of (NAME) from axiom system, so if we need SC equivalent to $\mathbf{HK_{H@}}$ it is also dispensable. In [109] we have direct evidence for this redundancy, because completeness proof is provided for SC formalization of non-modal hybrid language with sat-operators and nominals. In this case, to ordinary SC with cut only three rules are added, a version of (TERM) and two @-introduction rules of the form:

$$(@\Rightarrow')\ \frac{\Gamma \Rightarrow \Delta, \imath \qquad \varphi, \Gamma \Rightarrow \Delta}{@_\imath\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow@')\ \frac{\Gamma \Rightarrow \Delta, \imath \qquad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, @_\imath\varphi}$$

Although these rules are different from rules described above, it may be shown that they are equivalent in the sense of mutual derivability. The lack of rules for @-elimination shows that they are also redundant if we search for complete system with cut, but they are required for obtaining cut-free version. In fact, for stronger languages we need them also in cut-free proofs, as well as (NAME).

Seligman's rules are also very natural. In [112] the rules are obtained by the series of syntactical transformations, but [110] and [111] contains a justification of them by reference to intuitively plausible patterns of reasoning. We just comment on the sense of (TERM) and (NAME). The former means that if some sat-formulae follow from other sat-formulae locally (in some state $\imath$), then this entailment holds generally, independent of the state of evaluation.

[9] In fact Seligman considers in [112] also sat-calculus and mixed calculus as stages in the series of transformations from SC for first-order logic.

The latter is justified similarly: if in arbitrary state (new $\imath$) we have local entailment, then it holds generally.

But still this calculus may seem strange for researchers familiar with Gentzen approach, since for nominals and sat-operators we have both introduction and elimination rules. It is not a standard solution in SC, where constants are usually characterised by introduction rules only. Also some other of the Seligman's rules lack several properties required from "good" sequent calculi (see e.g. properties discussed by Avron or Wansing [119], like the lack of symmetry for nominal rules). Despite these drawbacks, Seligman's system presents quite good behaviour. In particular, cut-elimination theorem holds for this calculus. The proof of this fact is not performed directly for this form of SC, but for the form of SC adequate for first-order logic. For this SC admissibility of cut is proved in standard way by induction on the rank and the degree of cut applications. Seligman then obtains his calculus for hybrid logic by the series of transformations from this origin-SC. These transformations preserve many important properties, among them cut admissibility. Direct proofs of cut-elimination for SC without modals may be found in [109] and [111].

From the point of view of practical utility, as a tool for proof-search, Seligman's system has some drawbacks however, connected with the lack of analyticity. Of course cut is admissible, but cut-elimination is not in itself the sufficient condition for obtaining practically useful system for proof-search. One can easily note that in Seligman's SC cut is not the only nondeterministic rule. Because of (TERM), (NAME) and two @-elimination rules, cut-free Seligman's system does not satisfy subformula-property. It is interesting to note, that the version of SC for nonmodal logic from [111] satisfies some generalised form of subformula-property, namely:

Every formula occurring in the derivation of $\Gamma \Rightarrow \Delta$ is a quasi-subformula of elements in $\Sigma = \Gamma \cup \Delta \cup N$, where N is a finite set of nominals and φ is a quasi-subformula of Σ iff either φ is ordinary subformula of some $\psi \in \Sigma$ or $\varphi := @_i\psi$ and both i and ψ are subformulae of some formulae in Σ. But in this form of SC both rules for @-elimination additionally satisfy side conditions to the effect, that φ in eliminated $@_i\varphi$, is not itself sat-formula.

Such a property makes possible to define proof-search procedure and to redefine SC system for Hintikka-style tableau calculus by simply turning upside down all the rules and change all sequents $\Gamma \Rightarrow \Delta$ on sets $\Gamma, \neg\Delta$ like in ordinary modal logic (in fact some modifications of rules ($\Rightarrow\Diamond$) and ($\Box\Rightarrow$) are also necessary; cf. next section). But it is not clear if similar form of subformula-property may be obtained for considered SC.

Other properties of Seligman's SC are responsible for that we cannot define on the basis of this system any sort of tableau system operating on formulae (like Smullyan's system for classical logic). It is impossible, or at least very difficult, because of the global character of rules for sat-operators and nominal rules described in respective side-conditions. For example, ($\mathtt{N}_1$) is a rule difficult to simulate in a proof system where proof consists of single formulae as basic items, because application of such a rule requires performing a global transformation on actual proof. In systems like ND or Smullyan's tableau, more natural solution is to use some kind of rewrite rules that operate locally. We will return to this question later. Simulation of rules like (TERM), (NAME) and rules for sat-operators in tableau system would demand a presence of some nominal as a context for whole branch, which is possible but rather artificial solution in such systems.

Also the transfer of these rules into the context of ND systems may be difficult in some respect. For example, one can show that in ordinary modal logic, the rules of Ohnishi/Matsumoto SC have natural ND-counterparts in Fitch's style system. One can ask if such a transfer is possible with respect to Seligman's rules. In the next section, devoted to ND-systems, we will focus on this problem.

Extensions. In fact, Seligman's system is stronger than the reduct we've discussed. This is a consequence of the mode of its construction, namely by the transformation of SC for first-order logic. Final calculus contains also rules for $\downarrow$, E and $\exists$.

$$(\downarrow\Rightarrow)\ \frac{\imath, \varphi[\upsilon/\imath], \Gamma \Rightarrow \Delta}{\imath, \downarrow_\upsilon\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\downarrow)\ \frac{\imath, \Gamma \Rightarrow \Delta, \varphi[\upsilon/\imath]}{\imath, \Gamma \Rightarrow \Delta, \downarrow_\upsilon\varphi}$$

$$(\mathsf{E}\Rightarrow)^1\ \frac{@_\imath\varphi, \Gamma \Rightarrow \Delta}{\mathsf{E}\,\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\mathsf{E})\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi}{\Gamma \Rightarrow \Delta, \mathsf{E}\,\varphi}$$

$$(\exists\Rightarrow)^1\ \frac{\varphi[\upsilon/\imath], \Gamma \Rightarrow \Delta}{\exists_\upsilon\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi[\upsilon/\imath]}{\Gamma \Rightarrow \Delta, \exists_\upsilon\varphi}$$

Side condition: 1. where i does not occur in Γ, Δ and φ.

Since the system is modular, by combining these rules over basic system we can obtain adequate formalizations of basic hybrid logics in these languages. Seligman does not consider any extension of his system to stronger modal logics than **K**. The fact that it is not sat-calculus opens the problem if this system may be modified for logics in languages without sat-operators.

But it is not so obvious how to obtain a system complete (and cut-free) for $\mathbf{K_H}$. It is not enough to get rid of 4 rules for sat-operators, since @ is present also in the rules for modals (they do not satisfy properties of separation and explicitness, see [119])—different rules are necessary. Tzakova offered a tableau system for sat-operators free logics which may be transformed into cut-free SC, but it applies also external labels (see the section on tableau calculi).

11.2. Sequent Sat-calculus of Blackburn

As we mentioned earlier Blackburn's system is defined on sat-formulae only, so we have in a sense nonstandard form of SC. On the other hand, the form of rules is quite close to standard Gentzen format as we shall see. All the definitions concerning proof, derivable and admissible rules e.t.c. are the same as for Seligman's SC.

General rules

(AX) $\Gamma \Rightarrow \Delta$, where $\Gamma \cap \Delta \neq \varnothing$

$$(\mathtt{C}\Rightarrow)\ \frac{@_\imath\varphi, @_\imath\varphi, \Gamma \Rightarrow \Delta}{@_\imath\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\mathtt{C})\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi, @_\imath\varphi}{\Gamma \Rightarrow \Delta, @_\imath\varphi}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi}{@_\imath\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{@_\imath\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, @_\imath\neg\varphi}$$

$$(\wedge\Rightarrow)\ \frac{@_\imath\varphi, @_\imath\psi, \Gamma \Rightarrow \Delta}{@_\imath(\varphi\wedge\psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi \qquad \Gamma \Rightarrow \Delta, @_\imath\psi}{\Gamma \Rightarrow \Delta, @_\imath(\varphi\wedge\psi)}$$

$$(\vee\Rightarrow)\ \frac{@_\imath\varphi, \Gamma \Rightarrow \Delta \qquad @_\imath\psi, \Gamma \Rightarrow \Delta}{@_\imath(\varphi\vee\psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi, @_\imath\psi}{\Gamma \Rightarrow \Delta, @_\imath(\varphi\vee\psi)}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi \qquad @_\imath\psi, \Gamma \Rightarrow \Delta}{@_\imath(\varphi\rightarrow\psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{@_\imath\varphi, \Gamma \Rightarrow \Delta, @_\imath\psi}{\Gamma \Rightarrow \Delta, @_\imath(\varphi\rightarrow\psi)}$$

Rules are defined on sequents $\Gamma \Rightarrow \Delta$, where both Γ and Δ are finite multisets of sat-formulae. This is why Blackburn needs structural rules of contraction ((C⇒) and (⇒C)). In fact, he uses axioms of the form: $@_\imath\varphi \Rightarrow @_\imath\varphi$ and in consequence he needs also rules of weakening.

Modal rules

$$(@\Rightarrow)\ \frac{@_\imath\varphi, \Gamma \Rightarrow \Delta}{@_\jmath@_\imath\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow@)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi}{\Gamma \Rightarrow \Delta, @_\jmath@_\imath\varphi}$$

$$(\Diamond\Rightarrow)^1 \ \frac{@_i\Diamond j, @_j\varphi, \Gamma \Rightarrow \Delta}{@_i\Diamond\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Diamond) \ \frac{\Gamma \Rightarrow \Delta, @_j\varphi}{@_i\Diamond j, \Gamma \Rightarrow \Delta, @_i\Diamond\varphi}$$

$$(\Box\Rightarrow) \ \frac{@_j\varphi, \Gamma \Rightarrow \Delta}{@_i\Box\varphi, @_i\Diamond j, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Box)^1 \ \frac{@_i\Diamond j, \Gamma \Rightarrow \Delta, @_j\varphi}{\Gamma \Rightarrow \Delta, @_i\Box\varphi}$$

Side condition: 1. where j does not occur in $\Gamma \cup \Delta \cup \{\varphi\}$.

Special rules

$$(\mathtt{Ref}) \ \frac{@_i i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (\mathtt{Sym}) \ \frac{@_i j, \Gamma \Rightarrow \Delta}{@_j i, \Gamma \Rightarrow \Delta}$$

$$(\mathtt{Nom}) \ \frac{@_i\varphi, \Gamma \Rightarrow \Delta}{@_i j, @_j\varphi, \Gamma \Rightarrow \Delta} \qquad (\mathtt{Bridge}) \ \frac{@_i\Diamond j, \Gamma \Rightarrow \Delta}{@_i\Diamond\kappa, @_\kappa j, \Gamma \Rightarrow \Delta}$$

In fact both ($\mathtt{Sym}$) and ($\mathtt{Bridge}$) are derivable with the help of cut. Below we display a proof of the derivability of ($\mathtt{Sym}$):

$$\dfrac{\dfrac{@_i j, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, @_i\neg j}\,(\Rightarrow\neg) \qquad \dfrac{\dfrac{\dfrac{\dfrac{@_j j \Rightarrow @_j j}{@_j j, @_j\neg j \Rightarrow}\,(\neg\Rightarrow)}{@_j\neg j \Rightarrow}\,(\mathtt{Ref})}{@_j i, @_i\neg j \Rightarrow}\,(\mathtt{Nom})}{@_j i, \Gamma \Rightarrow \Delta}\,(\mathtt{Cut})$$

Blackburn sat-SC is also cut-free but this fact is not proved constructively but rather shown indirectly. The calculus is obtained from cut-free tableau system (see section on tableau calculi) for which Hintikka style constructive completeness proof is provided (by constructing suitably defined downward-saturated sets from open branches). It is instructive to compare this calculus with previously presented SC of Seligman. One can easily notice that rules of Blackburn differ from those of Seligman not only with respect to the kind of formulae they use. In fact Seligman obtains also sat-calculus as one of the stages in the process of transformations leading to SC described in the previous paragraph. So our comparison will be more direct if we refer to this sat-calculus of Seligman, instead of the final form of his SC.

The fact that sequents in Blackburn SC are defined on multisets is not essential, we can define this SC also on sequents made of lists of sat-formulae and just add rules of permutation. More serious differences concern some rules:

- for ($\Rightarrow\Diamond$) and ($\Box\Rightarrow$),
- forms of ($\mathtt{Ref}$).
- nominal rules.

Seligman's rules for modalities look as follows:

$$(\Rightarrow\Diamond)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\jmath \quad \Gamma \Rightarrow \Delta, @_\jmath\varphi}{\Gamma \Rightarrow \Delta, @_\imath\Diamond\varphi} \qquad (\Box\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, @_\imath\Diamond\jmath \quad @_\jmath\varphi, \Gamma \Rightarrow \Delta}{@_\imath\Box\varphi, \Gamma \Rightarrow \Delta}$$

They are interderivable (i.e., mutually derivable) with Blackburn's rules which may be shown by referring to the more general result [80]:

LEMMA 10. *If one of the following rules (or a sequent) belongs to* SC *with ordinary structural rules (including cut), then the rest is derivable:*

1. $\varphi, \psi \Rightarrow \chi$
2. $\chi, \Gamma \Rightarrow \Delta \ / \ \varphi, \psi, \Gamma \Rightarrow \Delta$
3. $\Gamma \Rightarrow \Delta, \varphi \ / \ \psi, \Gamma \Rightarrow \Delta, \chi$
4. $\Gamma \Rightarrow \Delta, \psi \ / \ \varphi, \Gamma \Rightarrow \Delta, \chi$
5. $\Gamma \Rightarrow \Delta, \varphi$ *and* $\Gamma' \Rightarrow \Delta', \psi \ / \ \Gamma, \Gamma' \Rightarrow \Delta, \Delta', \chi$
6. $\Gamma \Rightarrow \Delta, \varphi$ *and* $\chi, \Gamma' \Rightarrow \Delta' \ / \ \psi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$
7. $\Gamma \Rightarrow \Delta, \psi$ *and* $\chi, \Gamma' \Rightarrow \Delta' \ / \ \varphi, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$
8. $\Gamma \Rightarrow \Delta, \varphi$ *and* $\Gamma' \Rightarrow \Delta', \psi$ *and* $\chi, \Pi \Rightarrow \Sigma \ / \ \Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Sigma$

One can easily notice that $(\Box\Rightarrow)$ of Blackburn and Seligman are instances of 2 and 7, whereas $(\Rightarrow\Diamond)$ are instances of 4 and 5 respectively. Seligman's variants are better from proof-theoretical perspective, but Blackburn's rules are better if we need a calculus for actual proof-search. In such case it is handy to redefine the calculus in terms of sequents built up from sets of formulae, but two respective rules must be changed in order to keep the effect of contraction:

$$(\Rightarrow\Diamond')\ \frac{@_\jmath\Diamond\imath, \Gamma \Rightarrow \Delta, @_\jmath\Diamond\varphi, @_\imath\varphi}{@_\jmath\Diamond\imath, \Gamma \Rightarrow \Delta, @_\jmath\Diamond\varphi} \qquad (\Box\Rightarrow')\ \frac{@_\imath\varphi, @_\jmath\Box\varphi, @_\jmath\Diamond\imath, \Gamma \Rightarrow \Delta}{@_\jmath\Box\varphi, @_\jmath\Diamond\imath, \Gamma \Rightarrow \Delta}$$

Seligman's sat calculus makes use of one more axiom ($\mathtt{R@}$) of the form: $\Gamma \Rightarrow \Delta, @_\imath\imath$ instead of Blackburn's rule ($\mathtt{Ref}$). They are also interderivable.

Finally note that Blackburn's SC in addition to ($\mathtt{Ref}$) has three additional special rules, whereas Seligman's system uses only one pair of rules:

$$(@\mathrm{L}_1)\ \frac{@_\imath\jmath, \Gamma[\imath] \Rightarrow \Delta[\imath]}{@_\imath\jmath, \Gamma[\jmath] \Rightarrow \Delta[\jmath]} \qquad (@\mathrm{L}_2)\ \frac{@_\imath\jmath, \Gamma[\jmath] \Rightarrow \Delta[\jmath]}{@_\imath\jmath, \Gamma[\imath] \Rightarrow \Delta[\imath]}$$

where $\Gamma[\imath]$ means that $\imath$ occur in Γ and $\Gamma[\jmath]$ is the result of replacement of $\jmath$ for $\imath$ in Γ.

This pair of rules is contracted to one—($\mathtt{N}_1$) in the final calculus, since $@_\imath\jmath$ is replaced by a pair of nominals $\imath$, $\jmath$. One can check that three rules of Blackburn are derivable in Seligman's version and that his substitution rules are admissible in Blackburn's SC (which needs more complicated proof). If we use Seligman's substitution rules we can often obtain shorter proofs than in Blackburn's version. It is because of the global character of their application. On the other hand, local rewrite rules, like these in Blackburn's version, may be naturally applied in ND systems or tableau systems defined on formulae. In fact—as we remarked above—Blackburn obtains his SC from such tableau system which explains why he prefers local rules as primitives.

Neither Seligman nor Blackburn do consider extensions to stronger logics. We return to this question in the next section by the way of presenting ND-system of Braüner. On the basis of his ND-system Braüner presents yet another variant of sat SC-calculus of a uniform character.

11.3. Nonstandard Sequent Calculi

There is a lot of nonstandard sequent calculi for ordinary modal logic, substantially enriching and modifying original Gentzen ideas (see [119, 120] for an overview). In contrast, the number of nonstandard sequent calculi for hybrid modal logic is poor. Sat-calculus of Blackburn, although nonstandard, represents rather small departure from original Gentzen approach. It is perhaps due to the fact that hybrid languages are more expressive, and all this metalogical apparatus applied in nonstandard calculi to deal with the limitations of ordinary languages, is indeed of no use.

Anyway, one should note that two really nonstandard calculi were devised. One of them, due to Goré and Demri [39], belongs to the familly of display calculi, so we are not going to describe it here, because such a presentation would require introduction of too much technical details. So we only point out that [39] contains display calculus for hybrid tense logic with difference modality. One can dind a good exposition of display calculi for modal logics in [119].

The second system, due to Demri [38], is the calculus for $\mathbf{Kt_H}$ and a huge class of its extensions. Because in this case the departures from standard SC are not so great we describe briefly its main distinctive features.

1. The calculus is based on the idea of using "implicit prefixes" applied by Konikowska [84]. This role is played by nominals. Since sat-operators are not present in the language, all rules are defined on the formulae of

the shape $\imath \to \varphi$. It means that even rules for boolean constants must be suitably transformed. For instance ($\Rightarrow\to$) has a form:

$$(\Rightarrow\to)\quad \frac{\imath \to \varphi, \Gamma \Rightarrow \Delta, \imath \to \psi}{\Gamma \Rightarrow \Delta, \imath \to (\varphi \to \psi)}$$

Similarly for other rules. So $\imath \to$ plays the role of $@_\imath$ in sat-calculi, and that's why we put Demri's system in the class of sat-calculi in our taxonomy of proof systems. Demri's solution shows how to dispense with sat-operators in the presence of backward-looking modalities. Transmission between such formulae and ordinary hybrid formulae we want to prove, is realized by the rule:

$$(\mathtt{Start})\quad \frac{\Rightarrow \imath \to \varphi}{\Rightarrow \varphi} \qquad \text{where } \imath \notin \varphi$$

This is clearly a sequent version of ($\mathtt{NAME}'$) discussed on the ground of axiomatic formulations. This rule is applied only once in a proof—its place is just at the root of the proof-tree.

2. This form of calculus is based on tableau-like system KE of D'Agostino and Mondadori [2] which does not use branching rules, except cut (called there (BP) – bivalence principle). It means that instead of the usual tableau branching β-rules[10] like:

$$(\neg\wedge)\quad \neg(\varphi \wedge \psi)\ /\ \neg\varphi\ \ \|\ \ \neg\psi,$$

where $\|$ denotes branching

We use more natural deduction-like elimination rules of the form:

$$(\neg\wedge\mathtt{E})\quad \neg(\varphi \wedge \psi),\ \varphi\ /\ \neg\psi \quad \text{and} \quad \neg(\varphi \wedge \psi),\ \psi\ /\ \neg\varphi$$

In Demri's sequent calculus these rules are realised in the following form:

$$(\Rightarrow\wedge_1)\quad \frac{\Gamma, \imath \to \varphi \Rightarrow \Delta, \imath \to \psi}{\Gamma, \imath \to \varphi \Rightarrow \Delta, \imath \to \varphi \wedge \psi} \qquad (\Rightarrow\wedge_2)\quad \frac{\Gamma, \imath \to \psi \Rightarrow \Delta, \imath \to \varphi}{\Gamma, \imath \to \psi \Rightarrow \Delta, \imath \to \varphi \wedge \psi}$$

Note that $\imath \to \varphi(\psi)$ from the antecedent of conclusion-sequent must be still present in the premise-sequent, otherwise the calculus would be incomplete (it may be used more than once in the course of proof-search). Similar pairs of one-premise rules for $\to$ and $\vee$ must be introduced instead of ordinary two-premise rules ($\vee\Rightarrow$) and ($\to\Rightarrow$).

3. KE and its sequential version is incomplete without cut, but since cut without restriction on its applicability makes the calculus practically useless, it raises the question of convenient delimitation of its applications. In

[10] See the next section for explanation of β-, α-notation

ordinary KE for **CPL** it is sufficient to use cut only for introducing lacking minor premises (and their negations) for nonbranching β-rules. Such a calculus is analytic (has a form of subformula property) and in fact provides often essentially shorter proofs than ordinary tableau calculus, due to reduction of branches. Such an improvement was the main reason for introducing KE, following the result of Boolos [29]. Demri's system also satisfies some restrictions on the applicability of cut, namely cut-formula $\imath \to \psi$ introduced in the proof of φ must obey the following:

- $\imath$ is the nominal which was introduced as "new" by the application of rules that have suitable side-condition,
- ψ is either subformula of φ, or $\jmath$ introduced as "new", or has a form $\mathsf{G}\,\neg\jmath$ with $\jmath$ introduced as "new".

The shape of all rules and some more restrictions put on their applications make this calculus satisfy three conditions which give it almost analytic character. However Demri himself is sceptical with respect to usefulness of this calculus in the field of automated deduction, he is rather concerned with defining uniform complete framework.

4. The number of rules for nominals and temporal constants is rather numerous (and redundant), so we display below only four central rules for G and H:

$$(\mathsf{G}\Rightarrow)\ \frac{\Gamma, \imath \to \mathsf{G}\,\varphi, \jmath \to \varphi \Rightarrow \Delta, \imath \to \mathsf{G}\,\neg\jmath}{\Gamma, \imath \to \mathsf{G}\,\varphi \Rightarrow \Delta, \imath \to \mathsf{G}\,\neg\jmath} \qquad (\Rightarrow\mathsf{G})\ \frac{\Gamma \Rightarrow \Delta, \jmath \to \varphi, \imath \to \mathsf{G}\,\neg\jmath}{\Gamma \Rightarrow \Delta, \imath \to \mathsf{G}\,\varphi}$$

$$(\mathsf{H}\Rightarrow)\ \frac{\Gamma, \jmath \to \mathsf{H}\,\varphi, \imath \to \varphi \Rightarrow \Delta, \imath \to G\neg\jmath}{\Gamma, \jmath \to \mathsf{H}\,\varphi \Rightarrow \Delta, \imath \to G\neg\jmath} \qquad (\Rightarrow\mathsf{H})\ \frac{\Gamma \Rightarrow \Delta, \jmath \to \varphi, \jmath \to \mathsf{G}\,\neg\imath}{\Gamma \Rightarrow \Delta, \imath \to \mathsf{H}\,\varphi}$$

Clearly, in $(\Rightarrow\mathsf{G})$ and $(\Rightarrow\mathsf{H})$, $\jmath$ does not occur in the conclusion sequent (this is the proviso concerning "new" nominals when rules are read off from bottom to top). One can easily note that in these rules, formulae of the form $\imath \to \mathsf{G}\,\neg\jmath$ in the succedent of a sequent just inform that $\jmath$ is after $\imath$ in the flow of time. Similarly in many other rules, formulae of the form $\imath \to \jmath$ in the antecedent identify points $\imath$ and $\jmath$, whereas in succedent they serve as an information on their inequality. Interested reader should consult [38] for the exposition of the whole calculus.

5. Last thing of a great importance is the definition of the schema of one more (multi-) branching rule which covers a huge class of first-order frame defining conditions. We omit the details because of their complexity and the lack of space, but in the next section we briefly comment on the scope of this extension.

12. Natural Deduction Systems

Strangely enough, there is no precise definition of ND-systems which is generally accepted. Informally we characterize ND-system as any proof systems in which we have:

- some means for entering assumptions into a proof and also for discharging them,
- rules for introduction and elimination of logical constants,
- possibility of applying several proof constructions.

This is very broad characteristics and it allows a lot of freedom in concrete realizations. The main point is that real ND-system should be open for different proof constructions. The user is free in constructing direct, indirect or conditional proofs. He may build more complex formulae or decompose them, as respective introduction/elimination rules allow. Instead of using axioms or already proved theses, he is rather encouraged to introduce assumptions and derive consequences from them (although the presence of axioms is permitted). This flexibility of proof construction in ND is in striking contrast to strict form of admissible rules and proof formats in ordinary SC (cumulative proofs and only introduction rules) or tableau systems (indirect proofs and elimination rules only).

Many existing systems satisfy this loose characteristics but differ in many other respects. The most important differences are two: the format of the proof, and the kind of basic items from which the proof is constructed. Proofs in ND-systems are setting down generally as trees (tree- or Gentzen-format, see [54]) or sequences (linear- or Jaśkowski format, see [81]). Basic items of these proofs (nodes of proof-tree) may be formulae, sequents or other structured data (e.g. formulae with labels). In what follows, we will be concerned only with ND-systems using formulae, since sequent based ND-systems were not applied in hybrid logics (and in ordinary modal logics too).[11]

The distinction between ND-systems using tree- or linear-format is important because in the latter we may use the same formula many times. So we must have some devices for canceling the part of a proof which is in

[11] Note that sequent ND-systems, although also introduced by Gentzen [55], should not be identified with sequent calculi of the sort described in the previous section. The former use both introduction and elimination rules but usually only in the succedent; antecedent simply displays active assumptions.

the scope of an assumption already discharged. Otherwise we could "prove" everything. This is not possible in tree-proofs because we are operating not on formulae but on their single occurrences; premises of a rule must always be displayed directly over the conclusion, so we cannot use something which depends on discharged assumptions. Tree format then requires less complicated machinery and is very good in representing ready proofs—that's why it is very popular in theoretical works on ND-systems. On the other hand, linear format, despite the above mentioned inconvenience, is much more useful for actual proof-search—that's why it is present in many ND-variants in logic textbooks.

In ND-systems we have two types of rules: *rules of inference* and *proof construction rules*. Rules of inference of the discussed system have the form $\Gamma \ / \ \varphi$; we read them as follows: if we have all formulae from Γ in the derivation we can add φ to this derivation. By derivation we mean an attempted proof, i.e., unfinished tree or sequence.

In ND-system we need also some *proof construction rules* that allow us to build a proof, enter additional assumptions which open nested subderivations, and show under what conditions we may discharge these assumptions and close respective subderivations. For systems we will consider they have general form:

$$\text{If } \Gamma \vdash \varphi, \text{ then } \Delta \vdash \psi.$$

In this schema the antecedent refers to the subderivation which, if completed (φ is inferred from Γ), gives a justification for ψ (on the basis of Δ). Typical proof construction rules formalize old and well known proof techniques like conditional proof, indirect proof, proof by cases e.t.c.

These rules may be realized in different ways, depending on the variant of ND-system. In what follows we will apply as a formal basis, a variant of Jaśkowski linear format ND-system [81] due to Kalish and Montague [83]. The detailed presentation is beyond the limits of this paper, below we point out only things necessary to understand what follows.

Applicability of ND-systems to modal logics is rather limited; the number of different approaches is even smaller than the number of sequent calculi. Moreover, most of them are formalizations of only small number of modal logics (c.f. [73]). Among a few approaches to modal logics in ND, only the Fitch's system [43] has more extensive range of application. It was extended in Fitting [45] to many monomodal normal and regular logics. [70] contains the extension to many tense logics, it is also popular in other nonclassical logics e.g. relevant logics (see [4]) and conditional logics [116]. The main idea of Fitch's approach is the use of so called *strict subderivations*. In ordinary ND-system if we start a new subderivation by entering additional assumption into a proof, we can use any formulae above as the premises of inferences inside this subderivation, provided they are not dependant on already discharged assumptions (i.e., they do not belong to closed subderivations). Any strict subderivation restricts the use of formulae only to some modal formulae. Technically it is executed by reiteration rule that specifies what kind of formulae are admissible for transfer. So, in Fitch's approach it is necessary to distinguish two types of a derivation and to block the unrestricted transfer of the formulae to the strict ones.

ND-systems for ordinary modal logic of Basin, Mathews, Vigano [13] and of Russo [107] represent different approach. They represent labelled systems in the strong sense, following the tradition of Gabbay's LDS's. The first investigation on ND-systems for hybrid logics was in fact undertaken by Seligman [109], although it is in a slightly different context of so called logic of correct description. Much of the work on the sat-ND was done by Braüner, in particular in [30]. Below we present two systems: the first due to Indrzejczak, and the second due to Braüner.

The first system is an extension of Fitch-style ND-system for ordinary modal logic to hybrid logic by addition of new rules. It was constructed with practical applications in mind, ease of use with paper and pencil, above all. In consequence it is highly redundant. There are in fact two versions of this system, one closer to axiomatic system, and the second based on Seligman's SC.

The second one is a sat-calculus due to Braüner. It is more elegant and concise, because it was constructed mainly for theoretical purposes. Its set of rules is defined with particular goal in mind—the proof of normalization theorem, being ND counterpart of cut-elimination theorem from sequent calculi.

12.1. Ordinary ND-system for $\mathbf{K_H}@$

In order to present our system in a simple way we recall well known convention from [114], concerning the type of a formula:

α	α_1	α_2	β	β_1	β_2
$\varphi \wedge \psi$	φ	ψ	$\neg(\varphi \wedge \psi)$	$\neg\varphi$	$\neg\psi$
$\neg(\varphi \vee \psi)$	$\neg\varphi$	$\neg\psi$	$\varphi \vee \psi$	φ	ψ
$\neg(\varphi \to \psi)$	φ	$\neg\psi$	$\varphi \to \psi$	$\neg\varphi$	ψ

One additional convention: $-\varphi$ is a conjugate of φ; it denotes the negation of φ, if it is unnegated formula, otherwise it refers to the formula where negation is deleted. Moreover, $-\Gamma$ is the result of turning all elements of Γ into their conjugates.

The first set of rules comprises just a system for **CPL** divided into inference rules and proof construction rules.

1. Standard ND-system for CPL.

Inference rules

(αE)	$\alpha \;/\; \alpha_i$, where $i \in \{1,2\}$
(αI)	$\alpha_1 \;,\; \alpha_2 \;/\; \alpha$
(βE)	$\beta \;,\; -\beta_i \;/\; \beta_j$, where $i \neq j \in \{1,2\}$
(βI)	$\beta_i \;/\; \beta$, where $i \in \{1,2\}$
($\bot$)	$\varphi, \; -\varphi \;/\; \bot$
($\neg\neg$)	$\neg\neg\varphi \;/\; \varphi$

Proof Construction rules

[COND]	If $\Gamma, -\beta_i \vdash \beta_j$, then $\Gamma \vdash \beta$
[RED]	If $\Gamma, \; -\varphi \vdash \bot$, then $\Gamma \vdash \varphi$

This set of rules is obviously highly redundant but it is practically simpler to use many rules. We follow in this respect the way of presentation due to Fitting [45]. But of course, for theoretical purposes, it's better to show this redundancy and limit the set of rules to the collection of pairs of intro- and elim-rules for each constant.

From the point of view of practical applications, one of the most important things is to establish some form of setting a proof in ND-system. Theoretical formulation of rules presented above is independent of the form of a proof e.g. is it linear or tree-like?. We prefer to use linear format based on the idea of nested subproofs due to Jaśkowski [81] (but commonly called Fitch-style) and particularly useful for modal logics. This type of ND-systems requires of course some machinery for making clear, which part of a proof is active and which is not (completed subproofs based on discharged assumptions). Without going into details we just state that concrete realization of the system we prefer, is based on Kalish/Montague form of setting out proofs [83], where not only completed subproofs are put into boxes, but also every subproof (and the main proof as well) is introduced via so called Show-line. Actually, application of Show-lines is a distinctive feature of Kalish/Montague system giving it a dynamic character. Such line displays the formula which is current (sub)goal of this (sub)proof at a given stage. Technically, Show-line is a formula preceded with the prefix SHOW and it is introduced into every proof at least once. We put "SHOW:φ" always at the beginning of a proof of φ, but we enter "SHOW:ψ" also if we realise that ψ is what we need to complete the (current stage of the) proof. One must remember that formula in show-line is in a sense not a part of a proof; it simply displays the immediate goal. We cannot use such a formula as a premise of any inference rule but it can be turned into ordinary formula if this goal is reached through some subsidiary derivation.

After completion of a subproof initialized by Show-line, this subproof is put in the box and the prefix is canceled (which looks like SHØW). From now on, the formula from this (canceled) show-line is ordinary element of the proof or—to put it other terms—it is not show-line but ordinary line of a proof. On the other hand, no formula from the box is further available in the proof. Shortly—formulae in show-lines and in boxes are inactive but formula with canceled SHOW is active. Obviously no subproof can be completed in which we have non-canceled show-lines. It means that inside open subderivation we can start new subderivations but all such nested subderivations must be completed first.

Completed proof of a formula consists of this formula preceded with canceled SHOW and a box containing the proof itself. In this format the application of both proof construction rules may be displayed pictorially by the following diagrams:

$$
\begin{array}{ll}
 & \mathit{\Gamma} \\
i & \text{SHØW: } \beta \\
\begin{array}{l} i+1 \\ \\ k \end{array} & \qquad\qquad \boxed{\begin{array}{l} -\beta_i \\ \vdots \\ \beta_j \end{array}}
\end{array}
\qquad\qquad
\begin{array}{ll}
 & \mathit{\Gamma} \\
i & \text{SHØW: } \varphi \\
\begin{array}{l} i+1 \\ \\ k \end{array} & \qquad\qquad \boxed{\begin{array}{l} -\varphi \\ \vdots \\ \bot \end{array}}
\end{array}
$$

where there is no show-lines in a box.

2. Fitch-style ND-rules for K

The second group covers two proof construction rules which are complete for **K**. Once again, for simplicity, we use convention concerning type of a formula, this time due to Fitting [45].

π	ν	$\pi' = \nu'$
$\Diamond\varphi$	$\Box\varphi$	φ
$\neg\Box\varphi$	$\neg\Diamond\varphi$	$\neg\varphi$

[NEC] If $\Gamma \vdash \nu'$, then $\Box\Gamma \vdash \nu$

[POS] If $\Gamma,\ \pi'_1 \vdash \pi'_2$, then $\Box\Gamma,\ \pi_1 \vdash \pi_2$
where $\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\} \cup \{\neg\Diamond\varphi : \neg\varphi \in \Gamma\}$

This set of rules is also redundant since we can prove the admissibility of [NEC] in the presence of [POS] and vice versa. In fact, Fitting [45] presents two types of ND-systems for modal logics, one based on [NEC] (so called A-systems) and one based on [POS] (I-systems), but, because of practical reasons, it is better to have them both. In Kalish/Montague style of setting proofs the application of these rules looks like this.

$$
\begin{array}{lll}
 & \Box\Gamma & \\
i & \text{SHØW: } \nu & \\
 & & \boxed{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \\ \nu' \end{array}} \\
k & &
\end{array}
\qquad\qquad
\begin{array}{lll}
 & \Box\Gamma \cup \{\pi_1\} & \\
i & \text{SHØW: } \pi_2 & \\
i+1 & & \boxed{\begin{array}{c} \pi'_1 \\ \Gamma \\ \vdots \\ \pi'_2 \end{array}} \\
k & &
\end{array}
$$

Formulae in Γ are said to be transported into the subproof with the help of the reiteration rule from the outside. Subproofs of this kind are called strict or modal.

3. Inference Hybrid rules

Now, there are two ways of extending the modal ND-system to hybrid logic. The first version depends rather on axiomatic formulation. To obtain ND-system for the basic hybrid logic $\mathbf{K_{H@}}$ we need two additional sets of rules:

(S-D) $\neg @_\imath\varphi \ // \ @_\imath\neg\varphi$

$(\bot @)$ $@_\imath\bot \ // \ \bot$

(@I) $\imath,\ \varphi \ / \ @_\imath\varphi$

(@E) $\imath,\ @_\imath\varphi \ / \ \varphi$

(I@E) $@_\jmath @_\imath\varphi \ // \ @_\imath\varphi$

(Ref) $\varnothing \ / \ @_\imath\imath$

$(\Diamond\mathrm{E})$ $\Diamond @_\imath\varphi \ / \ @_\imath\varphi$

$(\Box\mathrm{I})$	$@_\imath\varphi \;/\; \Box @_\imath\varphi$
$(\vdash @\mathrm{E})$	$@_\imath\varphi \;/\; \varphi$, provided $\vdash @_\imath\varphi$

where // means that the rule is doubly sound, i.e., the premise may be inferred from the conclusion. Needless to say that this set of rules is also redundant e.g. (@I) is interderivable with (@E) and ($\Box$I) with ($\Diamond$E).

4. Hybrid proof construction rules

[@I]	If $\Gamma \vdash \varphi$, then $@_\imath\Gamma \vdash @_\imath\varphi$, where $@_\imath\Gamma = \{@_\imath\varphi : \varphi \in \Gamma\}$
[@$\Box$]	If Γ, $@_\imath\Diamond\jmath \vdash @_\jmath\varphi$, then $\Gamma \vdash @_\imath\Box\varphi$, where $\jmath$ is not in φ or in any undischarged assumption in Γ

In Kalish/Montague format both rules are represented as follows:

$$
\begin{array}{ll}
 & @_\imath\Gamma \\
i & \text{SHØW: } @_\imath\varphi \\
 & \boxed{\begin{array}{l} \Gamma \\ \vdots \\ \vdots \\ \varphi \end{array}} \\
k &
\end{array}
\qquad\qquad
\begin{array}{ll}
 & \Gamma \\
i & \text{SHØW: } @_\imath\Box\varphi \\
i+1 & \boxed{\begin{array}{l} @_\imath\Diamond\jmath \\ \Gamma \\ \vdots \\ @_\jmath\varphi \end{array}} \\
k &
\end{array}
$$

One should note that [@*I*] creates a strict subproof since we must use reiteration on sat-formulae changing them into ordinary (perhaps different sat-) formulae, whereas [@$\Box$] makes ordinary subproof (all formulae from the above are permitted in the subproof). Both rules are necessary if we want to have ND-equivalent of $\mathbf{HK}^{+}_{\mathbf{H@}}$. But note that ($\vdash$@E) is a counterpart of axiomatic (`NAME`) and [@$\Box$] is just a ND-realisation of (`BG`), so both may be omitted if we need only complete ND-formalization of $\mathbf{K}_{\mathbf{H@}}$.

Although the set of rules is redundant, it does not include the rules corresponding to symmetry of @, or to some forms of Leibniz rule (like (`Nom`) in Blackburn's SC); rules like (@E) and (@I) are sufficiently strong to make them derivable (cf. with axiomatic formulation of $\mathbf{K}_{\mathbf{H@}}$ in Part I).

Below we present two proofs as an illustration of how this system works. The first is one of the form of (`Nom`), the second should be compared with ($\Box$E) in Braüner's system.

1	SHØW: $@_i j \wedge @_j p \to @_i p$	$[6, \mathrm{COND}]$
2	$@_i j \wedge @_j p$	ass.
3	$@_i j$	$(2, \alpha\mathrm{E})$
4	$@_j p$	$(2, \alpha\mathrm{E})$
5	$@_i @_j p$	$(4, \mathrm{I@E})$
6	SHØW: $@_i p$	$[9, @\mathrm{I}]$
7	$@_j p$	$(5, \mathrm{Reit.})$
8	j	$(3, \mathrm{Reit.})$
9	p	$(7, 8, @\mathrm{E})$

1	SHØW: $@_i \Box p \wedge @_i \Diamond j \to @_j p$	$[13, \mathrm{COND}]$
2	$@_i \Box p \wedge @_i \Diamond j$	ass.
3	$@_i \Box p$	$(2, \alpha\mathrm{E})$
4	$@_i \Diamond j$	$(2, \alpha\mathrm{E})$
5	SHØW: $@_i @_j p$	$[12, @\mathrm{I}]$
6	$\Box p$	$(3, \mathrm{Reit.})$
7	$\Diamond j$	$(4, \mathrm{Reit.})$
8	SHØW: $\Diamond @_j p$	$[11, \mathrm{POS}]$
9	j	mod. ass.
10	p	$(6, \mathrm{Reit.})$
11	$@_j p$	$(9, 10, @\mathrm{I})$
12	$@_j p$	$(8, \Diamond\mathrm{E})$
13	$@_j p$	$(5, \mathrm{I@E})$

The second choice in defining ND for hybrid logic is to follow SC of Seligman. Now, first we display proof construction rules:

[NAME] If $\Gamma, \imath \vdash \varphi$, then $\Gamma \vdash \varphi$, where Γ and φ are sat-formulae or $\imath$ is not in φ and in Γ

[$\Box$] If $\Gamma,\ \Diamond\jmath \vdash @_\jmath\varphi$, then $\Gamma \vdash \Box\varphi$, where $\jmath$ is not in φ or in any undischarged assumption in Γ

In Kalish/Montague format they are represented as follows:

	Γ			Γ
i	SHØW: φ		i	SHØW: $\Box\varphi$
$i+1$	$\imath$		$i+1$	$\Diamond\jmath$
	Γ			Γ
	$\vdots$			$\vdots$
k	φ		k	$@_\jmath\varphi$

One can easily note that [NAME] covers both (TERM) and(NAME) from Seligman's SC, whereas [$\Box$] corresponds to his ($\Rightarrow\Box$) (and ($\Diamond\Rightarrow$) by interdefinability). The first of these rules is deductively so strong that the set of inference rules may be limited to: ($\bot@$), ($@$I), ($@$E), ($\Diamond$E), ($\Box$I). All the rest is derivable, including two inelegant (from the point of view of ND-systems) rules (Ref) and ($\vdash@$E). If we want to have rules more strictly following Seligman's rules we can also replace ($\Diamond$E), ($\Box$I) by ND-counterparts of: ($\Box\Rightarrow$) and ($\Rightarrow\Diamond$):

($\Diamond$E) $\quad \neg\Diamond\varphi,\ \Diamond\jmath\ /\ @_\jmath\neg\varphi$

($\Box$E) $\quad \Box\varphi,\ \Diamond\jmath\ /\ @_\jmath\varphi$

Such global rule like Seligman's (N_1) is not easy to obtain for ND-system like this. But we do not need this rule because the system is already complete for $\mathbf{K}^+_{\mathbf{H}@}$. One may doubt about this only when comparing [$\Box$] with [$\Box@$] which was exact counterpart of (BG). But the latter rule is easily proved admissible in the current system. Namely, every application of [$\Box@$] in a proof may be substituted by the subproof using [NAME] and [$\Box$], as the following schema shows:

1	SHØW: $@_\imath\Box\varphi$	[8, NAME]
2	$\imath$	ass.
3	SHØW: $\Box\varphi$	[7, $\Box$]
4	$\Diamond\jmath$	ass.
5	$@_\imath\Diamond\jmath$	(2, 4, $@$I)
6	$\vdots$	
7	$@_\jmath\varphi$	(5 by assumption)
8	$@_\imath\Box\varphi$	(2, 3, $@$I)

The question of extension of this system for stronger languages does not generate any problems. We can add suitable rules from Braüner's system. The extension to many stronger modal logics may be done as in standard modal logics, by modifying reiteration rule (see [45] or [70] for tense logics). But in the light of our pure completeness results we may obtain much more uniform system by addition of new inference rules modeled on pure axioms. Interesting and very general solution of this kind will be considered in the next paragraph.

12.2. Braüner's ND-system

As we already remarked, ND-system of Braüner is an example of sat-calculus and shows many similarities to Blackburn's SC and tableau calculus. It is a great value of this system that it represents a uniform formalization of a wide class of logics. Before we specify what kind of strengthenings is dealt with, we present a basic system for $\mathbf{K_{H@}}$.

1. Inference rules

($\wedge$E)	$@_\imath(\varphi \wedge \psi)\ /\ @_\imath\varphi,\ @_\imath\psi$
($\wedge$I)	$@_\imath\varphi,\ @_\imath\psi\ /\ @_\imath(\varphi \wedge \psi)$
($\rightarrow$E)	$@_\imath(\varphi \rightarrow \psi),\ @_\imath\varphi\ /\ @_\imath\psi$
($\bot$I)	$@_\imath\bot\ /\ @_\jmath\bot$
(@I)	$@_\imath\varphi\ /\ @_\jmath@_\imath\varphi$
(@E)	$@_\jmath@_\imath\varphi\ /\ @_\imath\varphi$
(Ref)	$\varnothing\ /\ @_\imath i$
(Nom_1)	$@_\imath\jmath,\ @_\imath\varphi\ /\ @_\jmath\varphi$, where $\varphi \in AT$
(Nom_2)	$@_\imath\jmath,\ @_\imath\Diamond k\ /\ @_\jmath\Diamond k$
($\Box$E)	$@_\imath\Box\varphi,\ @_\imath\Diamond\jmath\ /\ @_\jmath\varphi$

2. Proof construction rules

[COND]	If $\Gamma,\ @_\imath\varphi \vdash @_\imath\psi$, then $\Gamma \vdash @_\imath(\varphi \rightarrow \psi)$
[RAA]	If $\Gamma,\ @_\imath\neg\varphi \vdash @_\imath\bot$, then $\Gamma \vdash @_\imath\varphi$, where $\varphi \in AT$
[$\Box$]	If $\Gamma,\ @_\imath\Diamond\jmath \vdash @_\jmath\varphi$, then $\Gamma \vdash @_\imath\Box\varphi$, where $\jmath$ is not in φ or in any undischarged assumption in Γ

Since rules of Braüner differ remarkably both from sat SC-calculus of Blackburn and from axiomatic formulation, we make some comments on them. It is obvious that for booleans we have ordinary ND-rules but with $@_\imath$ as added context; it applies to inference rules, and to proof construction rules as well. Since the proof of normalization theorem is the main goal of Braüner, the calculus is $\neg$-free, with $\bot$ instead; $\neg$ and $\Diamond$ are used in proof-schemata as obvious definitional shorthands. Moreover, note that in this system $\bot$ is treated locally (with $@_\imath$ added), that's why Braüner needs ($\bot$I) as a kind of (inconsistency) propagation rule, necessary to perform [RAA].

As for hybrid rules, [□] is the same as [@□] and—as we remarked earlier—it is just ND-form of (BG).(Ref), (@I) and (@E) are direct counterparts of axioms, but note that (Nom_2) is not the same as thesis (Nom2) and it is also different from (Bridge). So, despite appearances, it is not that (Nom_1) corresponds to Blackburn's (Nom) and (Nom_2) to (Bridge). In fact, both rules (Nom_1) and (Nom_2) may be covered by one general rule:

(Nom′) $\quad @_\imath \jmath,\ @_\imath \varphi \ /\ @_\jmath \varphi$

which is interderivable with:

(Nom) $\quad @_\imath \jmath,\ @_\jmath \varphi \ /\ @_\imath \varphi$

which, by Lemma 10, is equivalent to Blackburn's rule.

On the other hand, to derive (Bridge) we need a rule (□E) which is modeled rather on (□⇒) from SC, but in fact plays a role of axiom (Back).

Regarding extensions, one should note first that Braüner provided normalization theorem for ND-system adequate not only for special extensions over $\mathbf{K_{H@}}$ but also over $\mathbf{K_{H@\downarrow}}$ and $\mathbf{K_{H@\forall}}$. The rules for $\downarrow$ and $\forall$ are the following:

(↓E) $\quad @_\imath {\downarrow_\upsilon} \varphi,\ @_\imath \jmath \ /\ @_\jmath \varphi[\upsilon/\jmath]$

[↓I] $\quad$ if $\Gamma,\ @_\imath \jmath \vdash @_\jmath \varphi[\upsilon/\jmath]$, then $\Gamma \vdash @_\imath {\downarrow_\upsilon} \varphi$, where $\jmath$ is not in φ or in any undischarged assumption in Γ

(∀E) $\quad @_\imath \forall_\upsilon \varphi \ /\ @_\imath \varphi[\upsilon/\jmath]$

(∀I) $\quad @_\imath \varphi[\upsilon/\jmath] \ /\ @_\imath \forall_\upsilon \varphi$, where $\jmath$ is not in $@_\imath \forall_\upsilon \varphi$ or any undischarged assumption

In fact, the formulation of rules for binders in Braüner's system is a bit different since he uses only state variables, so ordinary conditions concerning proper substitution of variables must be satisfied and side conditions forbid only free occurences of substituted variable which in our formulation is just nominal $\jmath$.

Now we consider what kind of extensions is considered by Braüner. He has proven general completeness theorem (and general normalization theorem) for all logics (in one of the hybrid languages $\mathbf{L_{H@}}$, $\mathbf{L_{H@\downarrow}}$, $\mathbf{L_{H@\forall}}$) whose classes of frames (i.e., accessibility conditions) are expressed by so called geometric theories (see [118]).

DEFINITION. A first-order formula is geometric, if it is built up from atoms of the form $\mathcal{R}xy$ and $x = y$ with the help of $\bot$, $\wedge$, $\vee$ and $\exists$ only.

Geometric theory is a finite set of first-order sentences of the form:

$$\forall_{x_1 \dots x_k}(\varphi \to \psi),$$

where φ and ψ are geometric formulae.

Simpson [113] has proved that each geometric theory is equivalent to basic geometric theory, where each formula has the form:

(bgf) $\forall_{x_1 \dots x_k}(\varphi_1 \wedge \dots \wedge \varphi_n \to \exists_{y_1,\dots,y_l}(\psi_1 \vee \dots \vee \psi_m)),$

where $k \geqslant 1$ and $l, n, m \geqslant 0$, each φ_i is an atom and each ψ_i is an atom or finite conjunction of atoms.

Every formula of basic geometric theory corresponds in hybrid language via HT translation function to ND-rule of the following form:

[BGR] If $\Gamma_1, \Psi_1 \vdash \chi, \dots, \Gamma_m, \Psi_m \vdash \chi$, then $\Delta, \Gamma_1, \dots, \Gamma_m, \varphi'_1, \dots, \varphi'_n \vdash \chi$

where $n, m \geqslant 0$, each $\varphi'_i = \mathrm{HT}(\varphi_i)$, each Ψ_i is a set of HT-translations of atoms that form conjunction ψ_i and no nominal that corresponds to y_i occurs in χ, $\Gamma_1 - \Gamma_m$, Δ, $\varphi'_1 - \varphi'_n$.

This rather complicated general characteristics may become clearer if we take a look at some examples. For instance, known conditions of symmetry, asymmetry, antisymmetry, transitivity, irreflexivity belong to this category. In case of irreflexivity and asymmetry because of the lack of negation we have in mind the following formulae:

$$\forall_x(\mathcal{R}xx \to \bot)$$
$$\forall_{xy}(\mathcal{R}xy \wedge \mathcal{R}yx \to \bot)$$

This is not the whole story—we note the following cases of bgf-s:

- every instance of Geach axiom, in particular Church-Rosser property,
- every Horn clause.

As for Horn clauses, just note that it is bgf with $l = 0, m = 1$ and ψ_1 being an atom. In case of Horn clauses the schema of the corresponding rule may be simplified:

(HR) $\varphi_1, \dots, \varphi_n \;/\; \psi_1$

Note that this result is a generalization of that obtained by Basin, Mathews and Vigano [13], since their labelled ND-system covers only logics axiomatized by Horn clauses. One should also note that not every bgf is expressible by pure formula unless $\forall$ is present. So in weaker hybrid languages Braüner's completeness theorem covers some logics not captured by pure completeness theorem.

The result of Braüner is similar to that of Demri [38] mentioned in the previous section. His general rule corresponds to the class of restricted Π^0_2-formulae of the form:

$$\forall_{x_1 \dots x_k} \exists_{y_1 \dots y_l} (\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow (\psi_1 \vee \dots \vee \psi_m)),$$

where $k \geqslant 1$, and $l, n, m \geqslant 0$, each φ_i is a literal (atom in the sense defined above or its negation) with variables only from $\{x_1, \dots, x_k\}$ and each ψ_i is a literal or finite conjunction of literals.

This class of formulae also includes Horn formulae and is equivalent to the class of all first-order formulae which are primitive in the sense of Kracht [85].

We finish this discussion with two examples of rules corresponding to concrete bgf-s: antisymmetry and Church-Rosser:

[ANTISYM] If $\Gamma, @_{\imath}\jmath \vdash \chi$, then $\Delta, \Gamma, @_{\imath}\Diamond\jmath, @_{\jmath}\Diamond\imath \vdash \chi$

[C-R] If $\Gamma, @_{\jmath}\Diamond\lambda, @_{\kappa}\Diamond\lambda \vdash \chi$, then $\Delta, \Gamma, @_{\imath}\Diamond\jmath, @_{\imath}\Diamond\kappa \vdash \chi$
where λ is a nominal not in χ, Γ, Δ

The first of them may be simplified to inference rule, since it is Horn clause:

(Antisym) $@_{\imath}\Diamond\jmath, @_{\jmath}\Diamond\imath \;/\; @_{\imath}\jmath$

Braüner uses tree-format of proof-representation, since it is well behaved with respect to proving normalization theorem. But of course we can display proofs in his system as Kalish-Montague style proofs. Here is an example:

1	SHØW: $@_i(\Diamond j \wedge @_j p \rightarrow \Diamond p)$	$[6, \mathrm{COND}]$
2	$@_i(\Diamond j \wedge @_j p)$	*ass.*
3	$@_i\Diamond j$	$(2, \wedge\mathrm{E})$
4	$@_i @_j p$	$(2, \wedge\mathrm{E})$
5	$@_j p$	$(4, @E)$
6	SHØW: $@_i\Diamond p$	$[10, \mathrm{RAA}]$
7	$@_i\Box(p \rightarrow \bot)$	ass.
8	$@_j(p \rightarrow \bot)$	$(3, 7, \Box\mathrm{E})$
9	$@_j\bot$	$(5, 8, \rightarrow \mathrm{E})$
10	$@_i\bot$	$(9, \bot I)$

Results of Braüner may be transferred to SC; in fact Braüner himself did it. On the basis of his ND-system he defines cut-free sat SC similar to that of Blackburn. The differences are the following:

- One more axiom of the form: $@_\imath\bot, \Gamma \Rightarrow \Delta$,
- $(\Box \Rightarrow)$ like in Seligman's calculus with 2 premises,
- different special rules.

As for the last point: (`Ref`) is the same, but instead of (`Sym`), (`Nom`) and (`Bridge`) there are two counterparts of ($\mathtt{Nom}_1$) and ($\mathtt{Nom}_2$):

$$(\mathtt{Nom}_1)\ \frac{\Gamma \Rightarrow \Delta, @_\imath \jmath \qquad \Gamma \Rightarrow \Delta, @_\imath\varphi}{\Gamma \Rightarrow \Delta, @_\jmath\varphi} \qquad \text{where } \varphi \in \mathrm{AT}$$

$$(\mathtt{Nom}_2)\ \frac{\Gamma \Rightarrow \Delta, @_\imath \jmath \qquad \Gamma \Rightarrow \Delta, @_\imath\Diamond\kappa \qquad @_\jmath\Diamond\kappa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

For $\downarrow$ and $\forall$ the rules are as follows:

$$(\downarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, @_\imath \jmath \qquad @_\jmath\varphi[\upsilon/\jmath], \Gamma \Rightarrow \Delta}{@_\imath{\downarrow_\upsilon}\varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\downarrow)^1\ \frac{@_\imath\jmath, \Gamma \Rightarrow \Delta, @_\jmath\varphi[\upsilon/\jmath]}{\Gamma \Rightarrow \Delta, @_\imath{\downarrow_\upsilon}\varphi}$$

$$(\forall\Rightarrow)\ \frac{@_\imath\varphi[\upsilon/\jmath], \Gamma \Rightarrow \Delta}{@_\imath\forall_\upsilon\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\forall)^1\ \frac{\Gamma \Rightarrow \Delta, @_\imath\varphi[\upsilon/\jmath]}{\Gamma \Rightarrow \Delta, @_\imath\forall_\upsilon\varphi}$$

where: 1. $\jmath$ does not occur in the conclusion.

SC schema of rules for (bgf) is of the form:

$$(\mathtt{BGR})\ \frac{\Gamma \Rightarrow \Delta, \varphi_1' \quad \ldots \quad \Gamma \Rightarrow \Delta, \varphi_n' \quad \Psi_1, \Gamma \Rightarrow \Delta \quad \ldots \quad \Psi_m, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where $n, m \geqslant 0$, each $\varphi_i' = \mathrm{HT}(\varphi_i)$, each Ψ_i is a set of HT-translations of atoms that form conjunction ψ_i and no nominal that corresponds to y_i occurs in $\Gamma_1 - \Gamma_m$, Δ, $\varphi_1' - \varphi_n'$.

This calculus satisfies the following quasi-subformula property:

Every formula occurring in the derivation of $\Gamma \Rightarrow \Delta$ is a quasi-subformula of elements in $\Gamma \cup \Delta$ or a quasi-subformula of some $@_\imath\jmath$ or $@_\imath\Diamond\jmath$, where $@_\imath\varphi$ is a quasi-subformula of $@_\jmath\psi$ iff φ is ordinary subformula of ψ.

Clearly his ND-systems also satisfies this property, but the formulation in this context is more complicated, so we address the reader to [30] for details.

Finally, we should note that the possibility of defining such a uniform calculus is essentially dependent on the capability of hybrid languages to express such things like state identity, state succession and internalization of satisfaction statements. For ordinary modal languages it is possible only if we use strong labelling like in [13] or [107]. This may suggest that application of external labels is of no use in hybrid logics. Next section will show that it is not the whole truth, at least if we are concerned with hybrid logics without sat-operators.

13. Tableau Systems

There are two different tableau systems for MHL. The first one is due to Tzakova [117] and has mixed character, i.e., except nominals it applies extra metalinguistic labels called (after Fitting) prefixes. The second one is a sat-calculus due to Blacburn [19], close to sat-SC presented earlier. In fact, tableau calculus was presented as primary system in [19], then SC was extracted from them. Both systems are of Smullyan's type, which means that nodes of a tableau are single formulae not sets of formulae like in Hintikka style tableau calculi.

The definition of a tableau for φ in both systems is standard; it is a tree of formulae with φ (with a prefix in Tzakova system) as a root (on the top) which is expanded by expansion rules, typically decomposing formulae on its parts. Some of the rules are branching, i.e., they lead to independent outputs. A branch is closed if it contains $\perp$, otherwise it is open. A tableau is closed if all its branches are closed. $\vdash_T \varphi$ iff there is a closed tableau for $\neg\varphi$ (again with prefix in Tzakova system; see below).

In order to state compactly expansion rules we will apply Smullyan's α-, β-notation. In the schemata $\|$ is used to represent branching. Hence nonbranching expansion rules have the general form: Γ / Δ, which reads: if all elements of Γ are on the branch, then extend the branch adding all elements of Δ. Branching expansion rules have the form: Γ / $\Delta \parallel \Pi$, which reads: if all elements of Γ are on the branch, then divide this branch into two subbranches and extend them adding all elements of Δ on the first, and all elements of Π on the second. We do not consider rules extending to more than two subbranches.

13.1. Mixed Calculus of Tzakova

As we mentioned above, Tzakova system is a labelled system of mixed character with extra prefixes added to formulae. We will use σ and τ to denote prefixes, and $\sigma : \varphi$ to denote prefixed formula. The application of metalinguistic prefixes is not the only characteristic feature of this system. Tzakova uses two types of formulae:

- *prefixed sentences* of the form $\sigma : \varphi$, where φ is hybrid formula and σ is metalinguistic prefix,
- *accessibility sentences* of the form $\sigma < \tau$, where both σ and τ are prefixes.

Prefixes are defined like in [45] as finite sequences of natural numbers with root prefix 1. For example 1.1.1.2, 1.4.2.6 e.t.c. The prefix is not only a name of a state in a model but additionaly, its structure encodes the place of this state in a falsifying model we are trying to build. So we can define also the relation of accessibility between prefixes. We say that τ is accessible from σ if either $\tau = \sigma.i$ or $\sigma < \tau$ is on the branch. Hence the proof of φ in Tzakova system is a closed tableau for $1 : \neg\varphi$ built up with the help of the following rules:

The rules for the weakest logic $\mathbf{K_H}$:

$(\bot_1)$	$\sigma : \varphi,\ \sigma : \neg\varphi\ /\ \bot$
$(\bot_2)$	$\sigma : \imath,\ \tau : \imath,\ \sigma : \varphi,\ \tau : \neg\varphi\ /\ \bot$
$(\neg)$	$\sigma : \neg\neg\varphi\ /\ \sigma : \varphi$
(α)	$\sigma : \alpha\ /\ \sigma : \alpha_1,\ \sigma : \alpha_2$
(β)	$\sigma : \beta\ /\ \sigma : \beta_1 \parallel \sigma : \beta_2$
$(\Box E)$	$\sigma : \Box\varphi\ /\ \tau : \varphi$, for any τ accessible from σ
$(\neg\Box E)$	$\sigma : \neg\Box\varphi\ /\ \tau : \neg\varphi$, where τ is a new label accessible from σ
$(\Diamond E)$	$\sigma : \Diamond\varphi\ /\ \tau : \varphi$, where τ is a new label accessible from σ
$(\neg\Diamond E)$	$\sigma : \neg\Diamond\varphi\ /\ \tau : \varphi$, for any τ accessible from σ
(Lab)	$\sigma : \varphi\ /\ \sigma : \imath$, where $\imath$ is a new nominal
(S-Id)	$\sigma : \imath,\ \tau : \imath\ /\ \sigma < \sigma'$, provided σ' is accessible from τ
(L-Id)	$\sigma : \imath,\ \tau : \imath,\ \sigma' : \jmath,\ \tau : \jmath\ /\ \sigma : \jmath,\ \sigma' : \imath$

Note that there are two rules for closing branches. One is standard, whereas the second reflects identity of prefixes by two additional premises ($\sigma : \imath$ and $\tau : \imath$). Similar situation is present in case of (S-Is) and (L-Is).

Tzakova provided also rules for @, $\downarrow$ and $\forall$:

(@)	$\sigma : @_\imath\varphi\ /\ \tau : \imath,\ \tau : \varphi$
$(\neg@)$	$\sigma : \neg@_\imath\varphi\ /\ \tau : \imath,\ \tau : \neg\varphi$

($\downarrow$) $\quad \sigma : \downarrow_\upsilon \varphi,\ \sigma : \imath \ /\ \sigma : \varphi[\upsilon/\imath]$

($\neg\downarrow$) $\quad \sigma : \neg\downarrow_\upsilon \varphi,\ \sigma : \imath \ /\ \sigma : \neg\varphi[\upsilon/\imath]$

($\forall$) $\quad \sigma : \forall_\upsilon \varphi \ /\ \sigma : \varphi[\upsilon/\imath]$

($\neg\forall$) $\quad \sigma : \neg\forall_\upsilon \varphi \ /\ \sigma : \neg\varphi[\upsilon/\imath]$, where $\imath$ is a new nominal

In both rules for @ if $\tau : \imath$ is already present on the branch we just add the second conclusion of a rule, otherwise τ is a new prefix on the branch.

The reproduced proof of $\Diamond(i \wedge \Box(j \to p)) \to \neg\Diamond(i \wedge \Diamond(j \wedge \neg p))$ shows well the application of rules, including (S-Id):

1	$1 : \neg(\Diamond(i \wedge \Box(j \to p)) \to \neg\Diamond(i \wedge \Diamond(j \wedge \neg p)))$	
2	$1 : \Diamond(i \wedge \Box(j \to p))$	$(1, \alpha)$
3	$1 : \neg\neg\Diamond(i \wedge \Diamond(j \wedge \neg p))$	$(1, \alpha)$
4	$1.1 : i \wedge \Box(j \to p)$	(2, $\Diamond$E)
5	$1.1 : i$	$(4, \alpha)$
6	$1.1 : \Box(j \to p)$	$(4, \alpha)$
7	$1 : \Diamond(i \wedge \Diamond(j \wedge \neg p))$	$(3, \neg)$
8	$1.2 : i \wedge \Diamond(j \wedge \neg p)$	(7, $\Diamond$E)
9	$1.2 : i$	$(8, \alpha)$
10	$1.2 : \Diamond(j \wedge \neg p)$	$(8, \alpha)$
11	$1.2.1 : j \wedge \neg p$	(10, $\Diamond$E)
12	$1.2.1 : j$	$(11, \alpha)$
13	$1.2.1 : \neg p$	$(11, \alpha)$
14	$1.1 < 1.2.1$	(5, 9, S-Id)
15	$1.2.1 : j \to p$	(6, 14, $\Box$E)
16	$1.2.1 : \neg j \quad \Vert \quad 1.2.1 : p$	(15, β)
17	$\bot\ (16, 12, \bot_1) \quad \Vert \quad \bot$	$(16, 13, \bot_1)$

System of Tzakova is weakly complete for $\mathbf{K_H}$, $\mathbf{K_{H@}}$, $\mathbf{K_{H\downarrow}}$, $\mathbf{K_{H@\downarrow}}$, $\mathbf{K_{H\forall}}$ and $\mathbf{K_{H@\forall}}$. There are no rules for stronger logics but they can be obtained by adding e.g. Fitting's rules. The proof of completeness is via construction of suitable downward saturated sets but it does not yield decision procedure for $\mathbf{K_H}$ and $\mathbf{K_{H@}}$ which are decidable logics. Problems with loop generation may be avoided if we change (S-Id) into the following rules:

(S-Id$'$) $\quad \sigma : \imath,\ \tau : \imath,\ \sigma : \Box\varphi \ /\ \sigma' : \varphi$, provided σ' is accessible from τ

(S-Id$''$) $\quad \sigma : \imath,\ \tau : \imath,\ \sigma : \neg\Diamond\varphi \ /\ \sigma' : \neg\varphi$, provided σ' is accessible from τ

Since Tzakova uses Fitting's prefixes it is natural to consider if accessibility sentences are really needed. In standard Fitting's tableau system for modal logics, construction of labels gives all the required information necessary for extraction of falsifying model from open branch. But in hybrid logic there is an interplay between prefixes and nominals. The latter give additional information about links between states in attempted falsifying model. Note that (S-Id) is the only rule that enters accessible sentences as nodes of a tableau. In all other rules having provisos referring to accessibility between prefixes, the presence of such sentences on a branch is not necessary for application since all the required information is implicit in the shape of prefixes. When (S-Id) is applied, such accessibility sentences appear in a tableau and may be used as actual second premise for application of ($\Box$E), ($\neg\Diamond$E) or (S-Id). Because the presence of accessibility sentences is necessary for hybrid logics, it would be in fact simpler to formulate this system with the help of strong labelling (cf. [50] or [13]) using just natural numbers (or any other symbols) as labels and accessibility sentences as the only explicit source of information about structure of attempted falsifying model. Obviously in such a variant we must change a formulation of some rules:

($\Box$E) $\quad \sigma < \tau,\ \sigma : \Box\varphi \ /\ \tau : \varphi$

($\neg\Box$E) $\quad \sigma : \neg\Box\varphi \ /\ \sigma < \tau,\ \tau : \neg\varphi$ where τ is a new label

($\Diamond$E) $\quad \sigma : \Diamond\varphi \ /\ \sigma < \tau,\ \tau : \varphi$ where τ is a new label

($\neg\Diamond$E) $\quad \sigma < \tau,\ \sigma : \neg\Diamond\varphi \ /\ \tau : \varphi$

(S-Id) $\quad \sigma : \imath,\ \tau : \imath,\ \tau < \sigma' \ /\ \sigma < \sigma'$

It is important to note that Tzakova system is the only non-axiomatic formalization of hybrid logics without sat-operators. In fact, it is exactly the lack of internalized sat-operators in a language, which makes sense to consider external labels in addition to nominals. One should also note that not all rules are typical expansion rules of tableau calculi. (S-Id), (L-Id) and both rules for $\downarrow$ have more than one premise, so their application requires scanning of all the branch above to find the additional premises. The same remark applies to some rules in Blackburn's system.

13.2. Blacburn's Sat-calculus

We still use α- and β-notation but in a slightly modified form represented in the table. Such reformulation is needed for uniform representation of rules, because tableau calculus of Blackburn is defined on all sat-formulae including negated forms, in contrast to his SC or Braüner's ND-system where there is no negation. The proof of φ is a closed tableau for $\neg @_\imath \varphi$, where $\imath$ is not in φ.

α	α_1	α_2	β	β_1	β_2
$@_\imath(\varphi \wedge \psi)$	$@_\imath \varphi$	$@_\imath \psi$	$\neg @_\imath(\varphi \wedge \psi)$	$\neg @_\imath \varphi$	$\neg @_\imath \psi$
$\neg @_\imath(\varphi \vee \psi)$	$\neg @_\imath \varphi$	$\neg @_\imath \psi$	$@_\imath(\varphi \vee \psi)$	$@_\imath \varphi$	$@_\imath \psi$
$\neg @_\imath(\varphi \rightarrow \psi)$	$@_\imath \varphi$	$\neg @_\imath \psi$	$@_\imath(\varphi \rightarrow \psi)$	$\neg @_\imath \varphi$	$@_\imath \psi$

Rules for $\mathbf{K_{H@}}$

1. Sat-versions of classical expansion rules:

(¬) $@_\imath \neg\varphi \ / \ \neg @_\imath \varphi$

(¬¬) $\neg @_\imath \neg\varphi \ / \ @_\imath \varphi$

(⊥) $@_\imath \varphi,\ \neg @_\imath \varphi \ / \ \bot$

(α) $\alpha \ / \ \alpha_1,\ \alpha_2$

(β) $\beta \ / \ \beta_1 \parallel \beta_2$

2. Modal and nominal expansion rules:

(@E) $@_\jmath @_\imath \varphi \ / \ @_\imath \varphi$

(¬@E) $\neg @_\jmath @_\imath \varphi \ / \ \neg @_\imath \varphi$

(Ref) $\varnothing \ / \ @_\imath \imath$ provided $\imath$ is on the branch

(Sym) $@_\imath \jmath \ / \ @_\jmath \imath$

(Nom) $@_\imath \jmath,\ @_\jmath \varphi \ / \ @_\imath \varphi$

(Bridge) $@_\imath \jmath,\ @_\kappa \Diamond \imath \ / \ @_\kappa \Diamond \jmath$

(□E) $@_\imath \Box \varphi,\ @_\imath \Diamond \jmath \ / \ @_\jmath \varphi$

(¬□E) $\neg @_\imath \Box \varphi \ / \ @_\imath \Diamond \jmath,\ \neg @_\jmath \varphi$ $\jmath$ new on the branch

(◇E) $@_\imath \Diamond \varphi \ / \ @_\imath \Diamond \jmath,\ @_\jmath \varphi$ $\jmath$ new on the branch

(¬◇E) $\neg @_\imath \Diamond \varphi,\ @_\imath \Diamond \jmath \ / \ \neg @_\jmath \varphi$

Blackburn provides also rules for ↓:

(↓E) $@_\imath \downarrow_\upsilon \varphi \ / \ @_\imath \varphi[\upsilon/\imath]$

(¬↓E) $\neg @_\imath \downarrow_\upsilon \varphi \ / \ \neg @_\imath \varphi[\upsilon/\imath]$

It is easy to observe that the rules like (Sym), (Nom), (Bridge) are not of a kind characteristic for tableau systems. In fact, we have a set of rewrite rules added to standard expansion rules. These rules are necessary to handle theory of equality of nominals ((Sym) is redundant). Also two rules for modal operators: ($\Box$E) and ($\neg\Diamond$E) are of different kind, like in ND-system.

One can easily check that this calculus is in exact one-to-one correspondence with earlier described SC. Blackburn considers also its variant for basic tense hybrid logic $Kt_{H@}$. We replace four rules for modalities by 8 rules for F and P instead of $\Diamond$, and G, H instead of $\Box$; also (Bridge) must be doubled for F and P. Additionally we need rules:

(Transpose-P) $\quad @_\imath \mathsf{P}\jmath \ / \ @_\jmath \mathsf{F}\imath$

(Transpose-F) $\quad @_\imath \mathsf{F}\jmath \ / \ @_\jmath \mathsf{P}\imath$

In [19], Blackburn considered only extensions to stronger logics obtained by the addition of pure axioms. For such calculi he proved strong completeness theorem by Hintikka method, using downward saturated sets. In [27] there is a considerable extension provided by the use of so called node creating rules. These rules were already mentioned in Part I in connection with PUENF-formulae; they are tableau counterparts of existential saturated rules. Fact 6 (p. 187) concerning existential saturated rules defined for axiomatic systems may be restated:

Fact 7. *Every PUENF-formula* ($\mathtt{PF}$) $\forall u_1 \ldots u_m \exists_{v_1 \ldots, v_n} \varphi$ *corresponds to node creating rule of the form:*

$$\varnothing \ / \ \varphi[u_1/\imath_1, \ldots, u_m/\imath_m, v_1/\jmath_1, \ldots, v_n/\jmath_n] \qquad \text{(NCR)}$$

provided $\imath_1, \ldots, \imath_m$ *occur on the branch,* $\jmath_1, \ldots, \jmath_n$ *are distinct, unequal to* $\imath_1, \ldots, \imath_m$ *and do not occur on the branch.*

Strong completeness theorem for tableau system with (NCR) rules holds with respect to every class of frames defined by respective PUENF-formulae. The drawback of such solution lies in the shape of these rules. Instead of expansion rules we must use in fact special instances of suitable axioms. In many respects these may be replaced by tableau-like rules. For example, to every property defined by Geach Axiom (see ($\star$) on p. 175) there corresponds the rule covered by the following schema:

$$@_\imath\Diamond\imath_1, @_{\imath_1}\Diamond\imath_2, \ldots, @_{\imath_{m-1}}\Diamond\imath_m, @_\imath\Diamond\jmath_1, @_{\jmath_1}\Diamond\jmath_2, \ldots, @_{\jmath_{s-1}}\Diamond\jmath_s \ /$$
$$@_{\imath_m}\Diamond\kappa_1, @_{\kappa_1}\Diamond\kappa_2, \ldots, @_{\kappa_{n-1}}\Diamond\mu, @_{\jmath_s}\Diamond\lambda_1, @_{\lambda_1}\Diamond\lambda_2, \ldots, @_{\lambda_{t-1}}\Diamond\mu$$

where, $\kappa_1, \ldots, \kappa_{n-1}, \lambda_1, \ldots, \lambda_{t-1}, \mu$ are new nominals.

In order to understand the sense of this rather complicated schema one should recognize, that $\imath$ is the denotation of x, $\imath_m$ of y, $\jmath_s$ of z and μ of v in ($\star$), p. 175. For example, Church-Rosser property is defined by the rule:

(CR) $\quad @_{\imath_1}\Diamond\imath_2, @_{\imath_1}\Diamond\imath_3 \ / \ @_{\imath_2}\Diamond\jmath, @_{\imath_3}\Diamond\jmath$ $\quad$ where $\jmath$ is new

In [23] there is a tableau formalization of **QMHL** presented in Section 9. To the set of rules for $\mathbf{K_{H@\downarrow}}$ one should add:

(∀E) $\quad @_\sigma\forall_{\boldsymbol{x}}\varphi \ / \ @_\sigma\varphi[\boldsymbol{x}/t]$ $\quad$ where t is any grounded term

(¬∀E) $\quad \neg@_\sigma\forall_{\boldsymbol{x}}\varphi \ / \ \neg@_\sigma\varphi[\boldsymbol{x}/c]$ $\quad$ where c is a new parameter

(∃E) $\quad @_\sigma\exists_{\boldsymbol{x}}\varphi \ / \ @_\sigma\varphi[\boldsymbol{x}/c]$ $\quad$ where c is a new parameter

(¬∃E) $\quad \neg@_\sigma\exists_{\boldsymbol{x}}\varphi \ / \ \neg@_\sigma\varphi[\boldsymbol{x}/t]$ $\quad$ where t is any grounded term

(Ref) $\quad \varnothing \ / \ t = t$

(RR) $\quad t = u, \varphi(t) \ / \ \varphi(t/u)$ $\quad$ where (t/u) denotes the replacement of at least one occurrence of t by u

(DD) $\quad @_{\sigma_1}\sigma_2 \ / \ @_{\sigma_1}f = @_{\sigma_2}f$

(@=) $\quad @_\sigma(t_1 = t_2) \ / \ t_1 = t_2$

(¬@=) $\quad \neg@_\sigma(t_1 = t_2) \ / \ \neg(t_1 = t_2)$

where σ is nominal or state variable, t and u are terms, and a term is grounded if it is a constant (rigid), a parameter or rigidified term (i.e., $@_\imath f$). Rules for quantifiers are classical since these are possibilistic quantifiers. The last two rules state that equality is rigid and make possible to keep standard rules (RR) and (Ref) (not the sat-versions!) since they delete sat-operators. The system is proved complete by translation of every tableau into the tableau calculus of the corresponding first order logic.

Although the tableau system of Blackburn was designed rather for practical paper and pencil application, not for automated deduction, one can find an implementation of it called Hydra, accessible on hybrid web page.

14. Resolution

Since Robinson has introduced the first form of resolution in 1965, it became almost industrial standard in the field of automated theorem proving. This popularity is a consequence of its striking simplicity. In the simplest form it is just the application of special form of cut to the set of clauses (sets of atoms) until we get an empty clause (⊥) or irreducible set of atoms. Enriching resolution with skolemization and unification gave the most popular method of automated theorem proving for first-order logic, despite undecidability of this logic.

But despite recent advances, the application of resolution to modal logic seem to be rather limited.[12] The basic problem is connected with the lack of simple normal forms for modal languages. In standard resolution for classical logic, if we search for a proof of φ, preliminary step consists in transforming $\neg\varphi$ to conjunctive normal form where each disjunction represents a clause, then we can perform resolution on this set of clauses. In modal logic usually normal forms are complex and resolution must be performed inside the scope of modal operators and even in propositional logic it requires some form of skolemization (cf. e.g. the system of Enjalbert and Fariñas del Cerro in [40]).

Problems with modal normal forms were responsible for great popularity of indirect resolution systems for modal logics. They are based on some form of translation of modal language into first-order language or some other richer language (e.g. relation calculus, see [95]). Indirect approach offers many advantages since we can apply ready to use provers and plenty of effective strategies tested during last 40 years of work with optimization of classical resolution. On the other hand, indirect resolution has some disadvantages due to fact that decidable modal logics are translated into undecidable first-order (or second-order) logic. This operation requires introduction of additional specific strategies for termination of the fragment of this rich language that corresponds to suitable modal language and usually leads to implementations that shows worse performance than tableau based provers. But it should be noted that recent investigations on indirect methods for some rich modal logics ([91, 108] open new perspectives in showing that by smart translation we can not only profit from first-order resolution strategies but even develop tableau calculi for these logics. Extending these techniques to hybrid logics seems promising.

But in here, we are not concerned with indirect methods since they were not investigated for MHL yet. Proposed resolution calculi for hybrid logic belong to the group of direct resolution methods (no translation), sometimes called non-clausal because they do not need to convert the input into some normal form. We prefer to say that they use generalized clauses (see below). Resolution systems operating on generalized clauses are in general more appropriate for nonclassical logics since, as we remarked, such normal forms

[12] One can find a useful survey in [40] and [7].

are usually quite complicated. In fact such non-clausal forms of resolution seem to be more effective even in classical logics, since they reduce complexities connected with the first phase of translation (it may lead to exponential blow up, but see e.g. [91, 108] for methods of structure-preserving reduction working in polynomial time). In modal logic, systems of this sort were offered by e.g. Fitting [46] and Abadi [1]. But there are only a few systems of this sort, and they are adequate for only a few particular logics, so the general resolution approach for ordinary modal logic is still to come.

Hybrid languages seem to offer far reaching simplification due to machinery of nominals and sat-operators. But the presence of @ is essential; without sat-operators we are unable to take a formula out of the scope of modal operator. Because of that, the two resolution systems for hybrid logics presented below belong to the group of sat-calculi, i.e., both are defined on clauses containing only sat-formulae. On the other hand, clauses are in generalized form; they contain not only literals prefixed with $@_\imath$ but any sat-formulae.

First of these calculi was constructed by Areces, de Nivelle, de Rijke and Heguiabehere [7] and later implemented [8] under the name HyLoRes. The main motivation was to find an efficient reasoning system useful for automated theorem proving. Recent form of this system applies many optimization techniques discovered for first-order resolution, like ordering and selection functions (see [9]), and is still improved.

The second system, due to Indrzejczak [76] is of different character. It was not constructed with automated deduction in mind but rather for obtaining a general, user's friendly framework, open for free use of several proof techniques. The basic idea was to combine natural deduction interface with resolution calculus in order to mix natural (from definition) and effective.

We apply the following notational conventions in this section:

- Γ, Δ, Σ and Θ stand for generalised clauses; empty clause is understood as $\bot$, a single formula is a unit clause.
- X, Y refer to any set of clauses, including empty set.
- In schemata of the rules ';' is used to separate clauses (it works like metalinguistic $\wedge$) and ',' is used to separate elements in clauses (it works like $\vee$).
- Rules of inference have the form X/Y or $X//Y$; we read them as follows: if we have all clauses from X in the derivation we can add all clauses from Y to this derivation. $//$ means that the rule may be applied also from Y to X.

14.1. HyLoRes

Resolution system constructed by Areces and others [7] is an effective system defined for automated deduction. Its implementation is called HyLoRes and may be download from hybrid logic web page. For convenience we will apply the name of a prover to the system as well. The version presented below is from [7], where it is embedded in a more general setting of labelled resolution containing system for ordinary modal logics (**K** and some of its extensions), description logic **ALCR** and $\mathbf{K_{H\downarrow@}}$.

As we already remarked the system does not operate on ordinary clauses obtained as a result of previous transformation to normal form. But the formulae in clauses are assumed in negation normal form by the application of the following rewriting procedure nf:

$$
\begin{aligned}
\mathrm{nf}(\neg\neg\varphi) &= \varphi \\
\mathrm{nf}(\Diamond\varphi) &= \neg\Box\neg\varphi \\
\mathrm{nf}(\varphi \vee \psi) &= \neg(\neg\varphi \wedge \neg\psi)
\end{aligned}
$$

As a result special rules for negation are dispensable. The rules of the basic version for $\mathbf{K_{H@}}$ are the following:

(Res)	$\Gamma, @_\imath\varphi \ ; \ \Delta, @_\imath\neg\varphi \ / \ \Gamma, \Delta$
$(\wedge)$	$\Gamma,\ @_\imath(\varphi \wedge \psi) \ / \ \Gamma,\ @_\imath\varphi \ \ ; \ \ \Gamma,\ @_\imath\psi$
$(\neg\wedge)$	$\Gamma,\ @_\imath\neg(\varphi \wedge \psi) \ / \ \Gamma,\ @_\imath nf(\neg\varphi),\ @_\imath nf(\neg\psi)$
$(\neg\Box)$	$\Gamma, @_\imath\neg\Box\varphi \ / \ \Gamma, @_\imath\neg\Box\neg\jmath \ ; \ \Gamma, @_\jmath nf(\neg\varphi)$ where $\jmath$ is a new nominal
$(\Box)$	$\Gamma, @_\imath\neg\Box\neg\jmath \ ; \ \Delta, @_\imath\Box\varphi \ / \ \Gamma, \Delta, @_\jmath\varphi$
$(@)$	$\Gamma, @_\imath @_\jmath\varphi \ / \ \Gamma, @_\jmath\varphi$
(Ref)	$\Gamma, @_\imath\neg\imath \ / \ \Gamma$
(Sym)	$\Gamma, @_\imath\jmath \ / \ \Gamma, @_\jmath i$
(Param)	$\Gamma, @_\imath\jmath \ ; \ \Delta, \varphi(\imath) \ / \ \Gamma, \Delta, \varphi(\imath/\jmath)$

Note the similarity of these rules to the rules of Blackburn's tableau calculus. In fact, most of the tableau rules are just special forms of these resolution rules with $\Gamma = \varnothing = \Delta$ and with obvious differences of β-rules, since here we have no branching but just transforming of a clause. The only important differences concern (Ref) and (Param) (from paramodulation).

The latter covers Blackbourn's rules (Nom) and (Bridge) but in more general form. (Ref) obviously in both systems covers reflexivity of the identity relation between nominals, but note that present form has genuinely resolution character (deletion, not addition of a suitable formula). Needless to say that (Sym) is derivable, as in other calculi with the similar set of rules for nominals.

Essential similarities of rules in non-clausal forms of resolution systems to tableau expansion rules are rather unavoidable since resolution steps are interleaved with simplification steps. It is more natural and simpler to use resolution on any formulae, not only on literals, especially in the context of modal logics.

In fact essential resolution steps are connected not only with application of (Res) and (Ref). Closer analysis of ($\Box$) also shows that it is a kind of resolution rule. If we apply standard translation we can see that it is ordinary resolution on $\mathcal{R}\imath\jmath$ and $\neg\mathcal{R}\imath x$ with unification on x. One can ask if it is possible to define more rules that are resolution-like rather than tableau-like. For example instead od (Sym) we can use:

(Sym′) $\quad \Gamma, @_\imath\jmath, \Delta, \neg @_\jmath\imath \;/\; \Gamma, \Delta$

We return to this question in a more detailed way after presentation of the second system. One should also observe that ($\neg\Box$) is a kind of skolemization but limited to introduction of constants only.

Deduction of a clause Γ from a set of clauses X ($X \vdash \Gamma$) in HyLoRes is defined as a finite sequence of sets of clauses $X_1, \ldots, X_n$, where $X_1 = X$, $\Gamma \in X_n$ and each X_i, $i > 1$, consists of set of clauses obtained by application of one of the rules to X_{i-1}. If $X_n = \bot$ then we have a refutation of X. Obviously the proof of φ is a refutation of $\{@_\imath \mathrm{nf}(\neg\varphi)\}$, where $\imath \notin \varphi$ exactly as in other sat-calculi.

Since HyLoRes is universal proof system we can use it also for constructing falsifying models (model extraction from non-successful refutations). [7] contains constructive completeness proof of resolution system for description logic which applies with small modifications to the system above and from which suitable terminating procedure may be obtained.

HyLoRes may be extended also to undecidable $\mathbf{K_{H@\downarrow}}$ just by adding one rule:

($\downarrow$) $\quad \Gamma, @_\imath{\downarrow_\upsilon}\varphi \;/\; \Gamma, @_\imath\varphi[\upsilon/\imath]$

Below we reproduce an example of a proof. Let our $\varphi := {\downarrow_u}\Diamond(u \wedge p) \to p$ which is a thesis of $\mathbf{K_{H@\downarrow}}$. Then $@_i\mathrm{nf}(\neg\varphi) := @_i(({\downarrow_u}\Diamond(u \wedge p)) \wedge \neg p)$. The proof in HyLoRes looks like that:

1	$@_i((\downarrow u \neg\Box\neg(u \wedge p)) \wedge \neg p)$	
2	$@_i(\downarrow u \neg\Box\neg(u \wedge p))$; $@_i\neg p$	$(\wedge)$
3	$@_i(\neg\Box\neg(i \wedge p))$; $@_i\neg p$	$(\downarrow)$
4	$@_i\neg\Box\neg j$; $@_j(i \wedge p)$; $@_i\neg p$	$(\neg\Box)$
5	$@_j i$; $@_j p$; $@_i\neg p$	$(\wedge)$
6	$@_i p$; $@_i\neg p$	(Param)
7	$\bot$	(Res)

No extension to other logics over $\mathbf{K_{H@}}$ or $\mathbf{K_{H@\downarrow}}$ is considered but three such rules are presented for labelled resolution system for ordinary modal logic that may be applied also in hybrid setting so we display suitable transformations below:

(T) $\Gamma, @_\imath\Box\varphi \ / \ \Gamma, @_\imath\varphi$

(D) $\Gamma, @_\imath\Box\varphi \ / \ \Gamma, @_\imath\neg\Box\mathrm{nf}(\neg\varphi)$

(4) $\Gamma, @_\imath\Box\varphi \ ; \ \Delta, @_\imath\neg\Box\neg\jmath \ / \ \Gamma, \Delta, @_\jmath\Box\varphi$

Note that these rules do not correspond to pure axioms. Moreover (4) introduces the risk of a loop, so procedure from completeness proof must be modified accordingly in order to save termination. We will turn to the problem of extension to stronger logics in the next subsection.

Note that in the actual form of HyLoRes several constraints on the applicability of rules are involved that serve to increase its efficiency. Since discussion of optimization techniques is beyond the scope of this text, in our simplified presentation conditions on selection functions etc. are omitted.

14.2. RND – Resolution Based ND-system

RND means Resolution based Natural Deduction so it is the system which is hybrid in itself by mixing ND-system and resolution method. The main goal of this combination was to obtain a general deductive framework allowing simulation of several proof-search strategies and construction of short and readable proofs. It seems that in order to get such a formalization it is good to combine some features of natural deduction and resolution.

The latter—as we remarked—is still the most popular technique in automated theorem proving. Formal simplicity and effectiveness were the source of the success of resolution in automated deduction because they lead to direct implementations. Current versions of resolution are in fact not so simple, but this is a price for increasing efficiency. During almost 40 decades of experience, a lot of efficient proof-search strategies and optimization techniques was tested in this area.

But standard resolution is rather not a system that is good for humans. Neither actual searching for a proof, nor even reading the result of machine performance, is easy. It is not a drawback if we are just interested in quick response: is it provable or not? But if we are interested not only in the result obtained for given input but we want to see the actual derivation we have a problem. It is of particular importance if we are interested in the construction of a proof itself, or in finding the most simple and direct deduction. It is also important for different kinds of checkers and other interactive programs needed for teaching of logic. In fact, some efforts were undertaken (see e.g. [68, 69]) recently to make resolution proofs more readable.

Natural deduction is the other extreme—a natural and flexible tool of deduction for humans but very rarely used for automated proof-search. Both making derivations and reading proofs is rather easy from practical point of view. But rich machinery of rules and the structure of proofs (nested subderivations e.t.c.) lead to some troubles for automatization. Moreover, standard ND-systems are not universal since they allow only proofs, in contrast to e.g. tableau systems that allow easy falsifying-model extraction from open branches.

Is it possible to mix both approaches in order to get a system that is at the same time simple, efficient and user's friendly? In our opinion the richness of ND apparatus makes possible simulation of other systems like tableau, even on the ground of quite standard basis, since [RED] may replace applications of branching rules. Difficulties with the lack of universality may be easily overcome by admitting also open derivations, and by defining suitable saturation procedures (see [75]). In what follows, we will show this in a more detail. In order to simulate resolution we need only one simple generalization; admission of generalized clauses, instead of single formulae, as basic items in derivations. Resulting system, called RND, is one of the possible solutions to the posed question.

Basically, the system presented below is Jaśkowski style ND-system connected with the treatment of resolution due to Fitting [47]. Several versions of RDN were already presented: in [78] the system adequate for first-order classical and free logic is examined, [76] contains calculi for ordinary modal logics in standard and labelled form. Here we present the sat-calculus for basic hybrid logic. In particular, we pay an attention to the problem of simulation of several proof methods in RND framework, and to the problem of extensions to stronger logics.

RND sat-calculus defined on generalised clauses consists of:

1. Sat-versions of classical inference rules

(¬) $\Gamma, \neg @_\imath \varphi \;//\; \Gamma, @_\imath \neg\varphi$

(Res) $\Gamma, @_\imath \varphi \;;\; \Gamma, @_\imath - \varphi \;/\; \Gamma$

(NN) $\Gamma, @_\imath \neg\neg\varphi \;//\; \Gamma, @_\imath \varphi$

(α) $\Gamma, @_\imath \alpha \;//\; \Gamma, @_\imath \alpha_1 \;;\; \Gamma, @_\imath \alpha_2$

(β) $\Gamma, @_\imath \beta \;//\; \Gamma, @_\imath \beta_1, @_\imath \beta_2$

2. Modal inference rules

(π) $\Gamma, @_\imath \pi \;/\; \Gamma, @_\imath \Diamond \jmath \;;\; \Gamma, @_\jmath \pi'$
where $\jmath$ is a new nominal in derivation

(ν) $\Gamma, @_\imath \nu \;;\; \Delta, @_\imath \Diamond \jmath \;/\; \Gamma, \Delta, @_\jmath \nu'$

(@) $\Gamma, @_\imath @_\jmath \varphi \;//\; \Gamma, @_\jmath \varphi$

(Ref) $\Gamma, @_\imath \neg\imath \;/\; \Gamma$

(Sym) $\Gamma, @_\imath \jmath \;/\; \Gamma, @_\jmath \imath$

(Nom) $\Gamma, @_\imath \jmath \;;\; \Delta, @_\jmath \varphi \;/\; \Gamma, \Delta, @_\imath \varphi$

(Bridge) $\Gamma, @_\imath \jmath \;;\; \Delta, @_\kappa \Diamond \imath \;/\; \Gamma, \Delta, @_\kappa \Diamond \jmath$

The set of rules is similar to that of HyLoRes, but note that we admit also building up rules (// in some rules instead of /). It is because we want to have a system of more general character than typical analytical systems like tableau calculi. This generality is needed for a system which may be easily tailored in order to simulate different proof systems.

3. One proof-construction rule:

Recall that in ND-system we need also some *proof construction rules* that allow us to build up a proof, enter additional assumptions which open nested subderivations and show under what conditions we may discharge these assumptions and close respective subderivations. In RND we need only one such a rule called *Subsumption*:

[SUB] if $X; -\varphi_1; \ldots; -\varphi_i \;/\; \Delta$, then $X \;/\; \Gamma$
where: Γ is nonempty, $\Delta \subseteq \Gamma$, $\{\varphi_1, \ldots, \varphi_i\} \subseteq \Gamma$, $i \geqslant 0$.

Remark. Every φ in the schema is a sat-formula and X is a set of generalised clauses built up from sat-formulae only.

Schematically its application in Kalish/Montague format looks like this:

$$
\begin{array}{lll}
 & X & \\
k & \text{SHØW:} & \Gamma \\
k+1 & & \multicolumn{1}{|c|}{-\varphi_1} \\
. & & \multicolumn{1}{|c|}{.} \\
. & & \multicolumn{1}{|c|}{.} \\
k+i & & \multicolumn{1}{|c|}{-\varphi_i} \\
. & & \multicolumn{1}{|c|}{.} \\
. & & \multicolumn{1}{|c|}{.} \\
n & & \multicolumn{1}{|c|}{\Delta}
\end{array}
$$

It is rather easy to check that we have for $\mathbf{K_{H@}}$:

THEOREM 20 (Soundness). *RND* $\vdash -\Gamma, \varphi$ *implies* $\Gamma \models \varphi$.

PROOF. Soundness of the inference rules is easy to provide if we interpret clauses as n-ary disjunctions. For [SUB] assume that $\vee\Delta$ follows from X and, possibly empty, set of assumptions: $-\varphi_1, \ldots, -\varphi_k$, then clearly $\vee\Gamma$ follows either, since $\Delta \subseteq \Gamma$. Hence, if $k = 0$, we are done, otherwise X implies $-\varphi_1 \rightarrow (-\varphi_2 \rightarrow \ldots(-\varphi_k \rightarrow \vee\Gamma)\ldots)$, which is equivalent to $\varphi_1 \vee (\varphi_2 \vee \ldots(\varphi_k \vee (\vee\Gamma)\ldots)$ which is equivalent to $\vee\Gamma$ since each $\varphi_i \in \Gamma$. Therefore, $\vee\Gamma$ follows from X. $\square$

The form of the soundness theorem shows how to construct derivations justifying arguments. We just start a proof of a clause which consists of a conclusion and conjugates of all premises.

Completeness of RND will be proved in the next subsection as a byproduct of more general considerations.

Simplicity, universality and generality of RND. Now we consider the behavior of RND with respect to properties of proof systems we found desirable. In particular, we sketch briefly how RND can simulate in a step-wise manner (i.e., each inference in simulated system is duplicated by n inference steps in RND, see more accurate exposition in [108]) not only proofs but all kinds of deductive tasks, including model-extraction, which are realizable in several known systems. Our strategy is based on the enrichment of the basic RND with additional derivable or admissible rules that enable close following of derivations in simulated systems.

Let us consider the following rules:

(Res′) $\quad \Gamma, @_\imath\varphi \; ; \; \Delta, @_\imath\neg\varphi \; / \; \Gamma, \Delta$

(α') $\quad \Gamma, @_\imath\alpha_1 \ ; \ \Delta, @_\imath\alpha_2 \ / \ \Gamma, \Delta, @_\imath\alpha$

$(\beta\mathrm{E})$ $\quad \Gamma, @_\imath\beta \ ; \ \Delta, @_\imath - \beta_\imath \ / \ \Gamma, \Delta, @_\imath\beta_\jmath$ $\quad$ where $i \neq j \in \{1, 2\}$

$(\beta\mathrm{I})$ $\quad \Gamma, @_\imath\beta_n \ / \ \Gamma, @_\imath\beta$ $\quad$ where $n \in \{1, 2\}$

[COND] if $X \ ; \ @_\imath - \beta_m \ / \ @_\imath\beta_n$, then $X \ / \ @_\imath\beta$
where $m \neq n \in \{1, 2\}$

[TAB] if $X \ ; \ @_\imath\beta \ ; \ @_\imath\beta_m \ / \ \bot$, then $X \ ; \ @_\imath\beta \ / \ @_\imath\beta_n$
where $m \neq n \in \{1, 2\}$

[SEP] if $X \ ; \ \Gamma, \Delta \ ; \ \Gamma \ / \ \bot$, then $X \ ; \ \Gamma, \Delta \ / \ \Delta$

FACT 8. *The rules* [COND], [TAB], [SEP] *are admissible, remaining ones are derivable in RND.*

PROOF. For the sake of illustration we prove that [COND] which is sat-generalization of ordinary ND rule of this name, is admissible as the following schema shows:

$$
\begin{array}{llll}
 & X & & \\
k & \text{SHØW: } @_\imath\beta & & [n+1, \text{SUB}] \\
k+1 & & \text{SHØW: } @_\imath\beta_m, @_\imath\beta_n & [n, \text{SUB}] \\
k+2 & & \quad \neg @_\imath\beta_m & \text{ass.} \\
k+3 & & \quad @_\imath - \beta_m & (k+2, \neg) \\
 & & \quad \vdots & \\
n & & \quad @_\imath\beta_n & (k+3, \ldots) \\
n+1 & & @_\imath\beta & (k+1, \beta\mathrm{D}')
\end{array}
$$

□

Recall that in ordinary ND-systems we need many proof construction rules formalizing conditional proof, reductio ad absurdum, proof by cases e.t.c. They give us sufficient flexibility in constructing proofs but make the system-description, metalogical proofs of system features, definitions of proof-search procedures, and many other things, rather complicated. Here we need only one proof construction rule since $[SUB]$ is sufficiently general to cover all usual ND proof construction rules. In fact all proof resources of standard ND-system are present in RND which is easy to show. For the sake of concreteness, and because it is also sat-calculus, let's consider Braüner's system.

LEMMA 11. *Inference rules of Braüner's ND-system are derivable.*

PROOF. ($\wedge$E), ($\wedge$I), ($\rightarrow$E), (@I), (@E), ($\Box$E) are just special cases of (α), (βE) (see the Fact stated above), (@) and (ν), where $\Gamma = \emptyset = \Delta$). Braüner's (Ref) is derived easily by our version of the rule. Both (Nom_1) and (Nom_2) have simple and similar proofs. Below we display one of them:

$$
\begin{array}{lll}
1 & \text{SHØW: } \neg @_{\imath}\jmath,\ \neg @_{\imath}\Diamond\kappa,\ @_{\jmath}\Diamond\kappa & [5, \text{SUB}] \\
2 & \quad @_{\imath}\jmath & \text{ass} \\
3 & \quad @_{\imath}\Diamond\kappa & \text{ass} \\
4 & \quad @_{\jmath}\imath & (2, \text{Sym}) \\
5 & \quad @_{\jmath}\Diamond\kappa & (3, 4, \text{Nom})
\end{array}
$$

($\perp$I) is just a special feature of Braüner system, where inconsistency is local (with @) hence we need some way of its propagation into others states. In RND inconsistency is global so this rule is not required. $\Box$

The same applies to proof construction rules.

LEMMA 12. *Proof construction rules of Braüner system are admissible in RND.*

PROOF. [RAA] is a special form of [SUB] with Γ being unit clause, $k = 1$ and $\Delta = \perp$. [COND] is admissible as we have shown. Every application of [$\Box$] is eliminable in favor of the following subderivation:

$$
\begin{array}{lll}
 & X & \\
k & \text{SHØW: } @_{\imath}\Box\varphi & [l+1, \text{SUB}] \\
k+1 & \quad \neg @_{\imath}\Box\varphi & \text{ass} \\
k+2 & \quad @_{\imath}\neg\Box\varphi & (k+1, \neg) \\
k+3 & \quad @_{\imath}\Diamond\jmath & (k+2, \pi) \\
k+4 & \quad @_{\jmath}\neg\varphi & (k+2, \pi) \\
 & \quad \vdots & \\
l & \quad @_{\jmath}\varphi & (X, k+3, \text{by assumption}) \\
l+1 & \quad \perp & (k+4, l, \text{Res})
\end{array}
$$

where X is the set of elements of Γ from the formulation of [$\Box$] but treated as unit clauses. $\Box$

By these two lemmata and by completeness theorem for Braüner system we got:

THEOREM 21. *RND-system is strongly complete with respect to* $\mathbf{K_{H@}}$.

Hence RND is a ND-system of a very simple structure with only one proof construction rule that covers all standard rules as special instances.

Moreover, proofs in this system are simple, certainly simpler than in any standard ND. In particular, we are not forced to procure as many subderivations as in ordinary ND-systems, due to β-rule. In fact, in case we make an indirect derivation (we write down all possible assumptions) for a thesis we do not need more show-lines, than (the starting) one. This is a consequence of the fact that in RND we can directly simulate a resolution system (see below). What's more, very often, even if we do not enter all possible assumptions, we can avoid entering subderivations. Here is an example from classical propositional logic:

1	SHØW: $@_i(p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r))$	$[8, \text{COND}]$
2	$@_i(p \vee (q \wedge r))$	ass
3	$@_ip, @_\imath(q \wedge r)$	$(2, \beta)$
4	$@_ip, @_\imath q$	$(3, \alpha)$
5	$@_ip, @_\imath r$	$(3, \alpha)$
6	$@_i(p \vee q)$	$(4, \beta)$
7	$@_i(p \vee r)$	$(5, \beta)$
8	$@_i((p \vee q) \wedge (p \vee r))$	$(6, 7, \alpha)$

One can easily recognize well known classical propositional thesis but in sat-formulae form. If we just delete all occurences of $@_i$ we obtain a proof in ordinary RDN-system for **CPL**. One should compare this proof with ordinary ND-proof to see the difference in length and complexity, measured by the number of subderivations. Moreover, we can just turn upside down all the proof and obtain the proof of the converse implication. For simplicity we have used [COND] which is admissible as we have shown. Very often we can obtain direct proofs, and with no subderivations, of theses that are normally proved in ND only by indirect proofs. Try, for example, to prove Peirce law in RND.

RND is general enough to simulate proof techniques from many known systems which easily yields decision procedures. It is not evident by inspection of the system; some rules are certainly not analytical in any sense, so the upper bound on the number of possible choices in proof-search is not limited in advance. But we can easily obtain several decision procedures by imposing some restrictions on full RND. We briefly comment on this question.

First of all, we can simulate resolution system HyLoRes by stipulating that we always write down all possible assumptions (hence we attempt indirect proof), then we apply only elimination rules (only one direction of

($\neg$), (α) and (β)). Applications of (Param) are easily simulated by (Nom) and (Bridge). The difference is only in the form of setting out a proof: each line of derivation in HyLoRes corresponds to a stage of construction of a derivation in RND, where each clause is put in sole line. So by vertical displaying of horizontally oriented elements (with omitting of clauses that occur in more than one line) we can simulate every proof and disproof from HyLoRes in RND. This way we do not need to apply building-up rules at all, and our derivations obey subformula property. We can apply known strategies used for resolution and tested on HyLoRes and, in our opinion, thus obtained derivations have more readable character.

Making a derivation for a nonderivable clause opens a lot of interesting options concerning termination and building falsifying models, not necessarily by using strategies taken from resolution. We can, for instance, use the strategy of Davis/Putnam procedure which so far was not applied for modal logics ([37], see especially [47] for clear presentation). Basically it is performed by marking as used all clauses that cannot help to derive $\perp$. Recall that they are clauses containing tautologies, containing pure literals (no occurence of conjugate literal in other clauses) and being supersets of other clauses (in [37] these are called Affirmative-negative rule and Subsumption rule). The Splitting rule is simulated by introducing unit clause with suitable literal $@_\imath\varphi_1$ as show-line and its conjugate as indirect assumption. First, we perform (Res$'$) using our assumption as one of the premises and clauses contained above the last Show-line as the material for the second premise. All the time we are marking as used, all superset-clauses. If this subderivation is completed, we put it in a box, cancel SHOW before our chosen literal and repeat this procedure now using $@_\imath\varphi_1$ as one of the premises for (Res$'$). Otherwise we will start the next Show-line with the next literal $@_\jmath\varphi_2$ and its conjugate as an assumption and repeat the procedure. One-literal rule is simply a special form of (Res$'$) with one premise being unit atomic clause.

Another suitable system for **CPL**, being an improvement of ordinary tableau system, is KE-system based on the application of (analytic) cut. Although this system was used to formalize many nonclassical logics it was not used for hybrid logics so far, except the influence on SC of Demri (see section on sequent calculi). Since the analytic cut is just Splitting rule of Davis/Putnam (but not limited to literals!), RND can simulate KE by applying [SUB] sufficiently often to produce downward saturated sets in open derivations.

In both cases (DP-procedure and KE), the ease of reproducing derivations in RND is the consequence of the fact that cut is simulated by indirect proof in ND and such proof, as we have explained, is a special case of [SUB] in RND. Whenever in KE we introduce two branches, one with $@_\imath\varphi$ and the other with $\neg@_\imath\varphi$, in the corresponding RND-derivation we simply enter $@_\imath\varphi$ as a Show-line and its conjugate as an assumption. The nested subderivation starting with $\neg@_\imath\varphi$ corresponds to the one branch, and if it closes, the outer derivation (now containing $@_\imath\varphi$ as ordinary clause ready to use) corresponds to the other branch. This way, due to the machinery of the nested subderivations, we can display any binary tree in a linear fashion. The only difference is in performance; any proof-search procedure in RND must proceed in a depth-first manner, whereas in KE or tableau we are free in designing how our procedure traverses the tree (we may apply breath-first manner techniques as well).

But RND is by no means devoted only to follow strategies from systems based on application of cut. We mentioned already that we can easily simulate also cut-free tableau systems, in particular Blackburn's system. Consider [TAB]; it is a proof construction rule that displays in a linear fashion the effect of branching due to ordinary tableau β-rules. The rule [SEP] is just a generalization of [TAB] to any clause. The possibility of simulating tableau calculi is particularly important in the case of modal logics, because it allows to make use in RND of the optimization techniques for proof-search, developed recently in the area of tableau methods (see e.g. [66, 67]).

The above remarks show informally that RND may be treated as a simple frame suitable for direct simulation of several systems and strategies, despite their apparent differences. We can ask what properties of RND are responsible for its flexibility. Why RND can simulate and combine proof search procedures from resolution and tableau based systems like KE? One important reason is that RND applies cut in both directions. We should make some comments on these two manners of cut applications involved in RND.

First note that resolution is a special case of forward-application of cut.

$$\text{(Res)}\quad \frac{\varGamma, \varphi \qquad -\varphi, \varDelta}{\varGamma, \varDelta}$$

On the other hand in tableau systems we can have backward-application of cut e.g. in Hintikka-style systems it has a form:

$$\text{(B-Cut)}\quad \frac{\varGamma}{\varGamma, \varphi \quad \| \quad \varGamma, -\varphi}$$

Clearly (B-Cut) in general, is rather destructive for proof-search since it introduces the indeterminacy. But elimination of cut is, from the point of view of effectiveness, too strong as was already noticed by Boloos [29]. If we put sensible constraints on the use of cut leading to its analyticity, we can obtain better proof-search procedures and much shorter proofs (see e.g. [2]). After all, resolution is analytic form of cut and it leads to the most efficient automated deduction. It seems that using two forms of cut can give even better results. In fact, there are some systems, like Davis-Putnam procedure or KE, that involve both forms of Cut but in a very limited, special way. Still for classical propositional logic Davis-Putnam procedure is considered as better than resolution. In RND we have both forms in full generality since we have resolution as direct form of (F-Cut) and [SUB] can simulate every application of (B-Cut).

Extensions. Extending RND to stronger hybrid languages does not make serious problems, e.g. we can use HyLoRes rules for $\downarrow$. Defining suitable rules for quantifiers is also easy. We can extend RND also to first-order logic. One way is to use clausal versions of Blackburn's rules put in the section on tableau calculi. The other consists in generalizing the rule [SUB] and it was presented in [78] in two versions: for classical and free logic quantifiers. In what follows we rather focus on defining rules for several modal logics over $\mathbf{K_{H@}}$. RDN may be extended to stronger modal logics in different ways. In [76] three forms of rules are considered:

- with 1-parameter-formula φ

  (1R-A) $\quad \Gamma, \varphi \ / \ \Gamma$
- with 2-parameter-formulae φ and ψ

  (2Exp-A) $\quad \Gamma, \varphi \ / \ \Gamma, \psi \qquad$ or

  (2R-A) $\quad \Gamma, \varphi \ ; \ \Delta, -\psi \ / \ \Gamma, \Delta$
- with 3-parameter-formulae φ, ψ and χ

  (3Exp-A) $\quad \Gamma, \varphi \ / \ \Gamma, \psi, \chi \qquad$ or

  (3RExp-A) $\quad \Gamma, \varphi \ ; \ \Delta, -\psi \ / \ \Gamma, \Delta, \chi \qquad$ or

  (3R-A) $\quad \Gamma, \varphi \ ; \ \Delta, -\psi \ ; \ \Sigma, -\chi \ / \ \Gamma, \Delta, \Sigma$

Theorem 22. *Rules of the type* (2Exp-A), (2R-A) *and* (3RExp-A), (3Exp-A), (3R-A) *are interderivable in RND.*

Proof. Assume Γ, φ in RND+(2R-A). We write down Γ, ψ as a Show-line and $-\psi$ as the only assumption of this subderivation. From Γ, φ and $-\psi$

we obtain, by (2R-A), Γ which is sufficient to close this subderivation by [SUB] and makes Γ, ψ a clause inferred only from our first assumption.

If we assume both Γ, φ and Δ, $-\psi$, then from the first one we deduce, by (2Exp-A), Γ, ψ and this clause together with the second assumption gives us, by (Res′), a clause Γ, Δ.

Proofs of interderivability of (3RExp-A), (3Exp-A) and (3R-A) are similar. □

Clearly we may also introduce the contrapositives of these rules obtained by interchanging conclusion-clause with one of the premise-clause and changing parameters with theirs conjugates. For example both:

(3RExp-A′) $\quad \Gamma, \varphi \ ; \ \Delta, -\chi \ / \ \Gamma, \Delta, \psi \qquad$ and

(3RExp-A″) $\quad \Gamma, -\chi \ ; \ \Delta, -\psi \ / \ \Gamma, \Delta, -\varphi$

are contrapositives of (3RExp-A), obviously every schema of rules is interderivable with its contrapositives either.

In fact some of these types of rules were already present in the basic set. The rule (Ref) represents particular case of a schema (1R-A). It may seem as an expansion rule but it is rather a kind of one-premise resolution-rule. There is no reason to look for some equivalent.

Many rules represent the schema (2Exp-A). In general it is tableau-like expansion rule which may be replaced by interderivable resolution rule (2R-A).

Finally, one may observe that the rules (ν), (Nom) and (Bridge) represent a schema (3RExp-A). It may seem to have essentially resolution-character. This is not the whole truth however, since some additional formula appears in the conclusion which makes it partly expansion rule as well. This kind of a rule may be also interchanged with an equivalent pure expansion rule of the form (3Exp-A) or with a more involved but pure kind of a resolution rule of the form (3R-A).

Basing on this variety of forms one can extend RND to stronger logics in different ways depending on the proof strategy which is under consideration. The type of a rule is encoded in its name where the number says how many parameters must be specified, ‘R’ – means resolution, ‘Exp’ – means expansion, and ‘A’ is the variable in the name substituted by the name of the suitable axiom when parameters in the rule-schema are specified. In particular, all the rules of type Exp are tableau-like, whereas rules of type R are forms of resolution modulo substitution of parameters. The scheme (3RExp-A) denotes rules of mixed character—something is cut out in premises and something new is added in the conclusion.

The table specifies what substitutions for parameters we must perform in order to obtain rules equivalent to suitable pure axioms in a modular way. If the places under the heading ψ and χ are blank it means that we have only unique rule of the form ($1R$-A); if only the place under χ is blank we can introduce either the rule of the form ($2Exp$-A) or ($2R$-A), otherwise we have three possible characterizations.

axiom	φ	ψ	χ
(ConvD)	$@_\imath\Diamond\jmath$	$\neg@_\imath\Diamond\kappa$	$@_\jmath\kappa$
(T′)	$\neg@_\imath\Diamond\imath$	–	–
(Irr)	$@_\imath\Diamond\imath$	–	–
(4′)	$@_\imath\Diamond\jmath$	$\neg@_\jmath\Diamond\kappa$	$@_\imath\Diamond\kappa$
(5′)	$@_\imath\Diamond\jmath$	$\neg@_\imath\Diamond\kappa$	$@_\jmath\Diamond\kappa$
(B′)	$@_\imath\Diamond\jmath$	$@_\jmath\Diamond\imath$	–
(As)	$@_\imath\Diamond\jmath$	$\neg@_\jmath\Diamond\imath$	–
(Ant)	$@_\imath\Diamond\jmath$	$\neg@_\jmath\Diamond\imath$	$@_\imath\jmath$
(Dich)	$\neg@_\imath\Diamond\jmath$	$@_\jmath\Diamond\imath$	–
(Tri)	$\neg@_\imath\Diamond\jmath$	$@_\jmath\Diamond\imath$	$@_\imath\jmath$
(3′)	$@_\imath\Diamond\jmath$	$\neg@_\imath\Diamond\kappa$	$@_\jmath\Diamond\kappa$, $@_\kappa\Diamond\jmath$
(L′)	$@_\imath\Diamond\jmath$	$\neg@_\imath\Diamond\kappa$	$@_\jmath\Diamond\kappa$, $@_\kappa\Diamond\jmath$, $@_\jmath\kappa$

Note that for (3′) and (L′) we have in fact more general schemata since χ does not refer to the single formula but to the clause. So we have χ_1, χ_2 and χ_1, χ_2, χ_3 respectively instead of a single χ. For instance (3RExp-L′) has a form Γ, φ ; $\Delta, -\psi$ / $\Gamma, \Delta, \chi_1, \chi_2, \chi_3$. Due to this multiplication of the third parameter we should rather generalize the schemata of rules for more than 3 parameters. For example in case of (3′) such forms (and their contrapositives) are possible as:

(4RExp-A) $\quad \Gamma, \varphi \;;\; \Delta, -\psi \;;\; \Sigma, -\chi_1 \;/\; \Gamma, \Delta, \Sigma, \chi_2$

(4R-A) $\quad \Gamma, \varphi \;;\; \Delta, -\psi \;;\; \Sigma, -\chi_1 \;;\; \Pi, -\chi_2 \;/\; \Gamma, \Delta, \Sigma, \Pi$

In Table 1 is an example of a proof.

Nominal existence rules. Some of the important conditions need rules of different form similar to node-creating rules of Blackburn (see section on tableau systems) defined for instances of Geach Axiom. Below we give examples of rules for density and for Church-Rosser property:

1	SHØW: $@_i(j \to \Box(\Diamond j \to j))$	$[12, \mathrm{SUB}]$
2	$\neg @_i(j \to \Box(\Diamond j \to j))$	ass
3	$@_i j$	$(2, \alpha)$
4	$@_i \neg \Box(\Diamond j \to j)$	$(2, \alpha)$
5	$@_i \Diamond k$	$(4, \pi)$
6	$@_k \neg(\Diamond j \to j)$	$(4, \pi)$
7	$@_k \Diamond j$	$(6, \alpha)$
8	$@_k \neg j$	$(6, \alpha)$
9	$\neg @_k j$	$(8, \mathrm{NN})$
10	$@_j i$	$(3, \mathrm{Sym})$
11	$@_j \Diamond k$	$(10, 5, \mathrm{Nom})$
12	$\bot$	$(7, 11, 9, \mathrm{3R\text{-}Ant})$

Table 1.

(RDN-Conv4) $\Gamma, @_\imath \Diamond \jmath \ / \ \Gamma, @_\imath \Diamond \kappa \ ; \ \Gamma, @_\kappa \Diamond \jmath$
where κ is a new nominal

(RDN-CR) $\Gamma, @_\imath \Diamond \jmath \ ; \ \Delta, @_\imath \Diamond \kappa \ / \ \Gamma, \Delta, @_\jmath \Diamond \lambda \ ; \ \Gamma, \Delta, @_\kappa \Diamond \lambda$
where λ is a new nominal

Here is an example of a proof:

1	SHØW: $@_i(\Diamond\Box j \to \Box\Diamond j)$	$[13, \mathrm{SUB}]$
2	$\neg @_i(\Diamond\Box j \to \Box\Diamond j)$	ass
3	$@_i \Diamond\Box j$	$(2, \alpha)$
4	$@_i \neg\Box\Diamond j$	$(2, \alpha)$
5	$@_i \Diamond k$	$(3, \pi)$
6	$@_k \Box j$	$(3, \pi)$
7	$@_i \Diamond l$	$(4, \pi)$
8	$@_l \neg\Diamond j$	$(4, \pi)$
9	$@_k \Diamond m$	$(5, 7, \mathrm{RDN\text{-}CR})$
10	$@_l \Diamond m$	$(5, 7, \mathrm{RDN\text{-}CR})$
11	$@_m j$	$(6, 9, \nu)$
12	$@_m \neg j$	$(8, 10, \nu)$
13	$\bot$	$(11, 12, \mathrm{Res})$

Such rules differ from those considered above not only by the presence of side condition but also because they admit more than one conclusion-clause. We can easily define general schema for rules corresponding to Geach axiom by generalizing the schema from Blackburn's tableau calculus. It takes the form:

$$\Gamma_1, \varphi_1 \; ; \ldots ; \Gamma_{m+s}, \varphi_{m+s} \; / \; \Gamma_1, \ldots, \Gamma_{m+s}, \psi_1 \; ; \ldots ; \Gamma_1, \ldots, \Gamma_{m+s} \psi_{n+t},$$

where formulae φ_i $(i \leqslant m+s)$ are sat-formulae displayed as premises, and ψ_i $(i \leqslant n+t)$ are sat-formulae displayed as consequences of this tableau rule-scheme.

RND enables also simulation of other generalizations obtained so far on the ground of several proof methods. To close the discussion we recall Braüner's general result for geometric theories. We can define RND counterpart of Braüner's rule from his SC version. It is enough to change every sequent $\Gamma \Rightarrow \Delta$ into corresponding clause $-\Gamma, \Delta$ and rewrite the rule-schema (BGR) accordingly.

Presented proof systems show that passing to hybrid languages may help to overcome many limitations of proof theory for standard modal logics. They also open new perspectives for development of reasoning methods and inventing new techniques of proof theory in general. We have closed the presentation of proof methods for hybrid logics with this rather lengthy discussion of RND not because it is better than other systems, but because we believe that its hybrid character fits pretty well with the spirit of MHL. In particular, easiness of simulation of other systems, shows that it may be used as a convenient framework for uniform treatment of a great number of modal logics based on solutions from different fields. Perhaps RND may be also used for experimentation with different strategies of proof-search in order to measure their efficiency. But this claim requires further investigation.

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