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NATURAL DEDUCTION SYSTEMS FOR SOME NON-COMMUTATIVE LOGICS

Abstract. Varieties of natural deduction systems are introduced for Wansing's paraconsistent non-commutative substructural logic, called a constructive sequential propositional logic (COSPL), and its fragments. Normalization, strong normalization and Church-Rosser theorems are proved for these systems. These results include some new results on full Lambek logic (FL) and its fragments, because FL is a fragment of COSPL.

Keywords: Constructive sequential propositional logic (COSPL), full Lambek logic (FL), natural deduction, (strong) normalization.

1. Introduction

Wansing's paraconsistent non-commutative substructural logic, called a *constructive sequential propositional logic* (COSPL) in [15], is a conservative extension of full Lambek logic (FL) by adding the strong negation connective $\sim$, and is a structural rule-free variant (i.e., without any structural rules) of Nelson's paraconsistent logic N4. In this paper, various Gentzen-type natural deduction systems are introduced for COSPL and its fragments. Normalization, strong normalization and Church-Rosser theorems are shown for some such proposed systems. The results of this paper include some new results on FL and its fragments.

Natural deduction systems (and typed λ-calculi) for some fragments of FL have been proposed by several researchers [5, 12, 14, 15, 16]. A detailed

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presentation of Curry-Howard correspondences for some fragments of various sybsystems of intuitionistic logic (including some fragments of FL) was studied by Wansing [14, 15] in order to consider the relationship between cut-elimination and normalization. It was also shown in [14, 15] that the strong normalization theorem holds for a two-directional typed λ-calculus for the $\{/, \backslash, *\}$-fragment of FL, where $/$ and $\backslash$ denote the two kinds of implication connectives, and $*$ denotes the multiplicative conjunction (or fusion) connective. Some natural deduction systems for the $\{/, \backslash\}$-fragments of (the original) Lambek calculus and FL were studied by van Benthem [1] and Tiede [12] to deal with some applications to formal grammars. In [1, 12], sequence-type assumptions were adopted for the underlying natural deduction systems. Some natural deduction systems for FL were studied by Watari et al. [16] to investigate normalization. Varieties of natural deduction systems for substructural logics including the $\{/, *, \wedge\}$-fragment of FL were introduced systematically by Mouri [5] based on the notion of labeled assumptions. Until now, a natural deduction system for COSPL has not been proposed.

The contents of the present paper are then summarized as follows.

In Section 2, a sequent calculus COSPL is introduced, and some basic properties for COSPL are reviewed. Three sequent calculi FL, C and L are also introduced as the $\sim$-free, $\{/, \backslash, *, \sim\}$- and $\{/, \backslash, *\}$-fragments of COSPL, respectively. An illustrative example for medical reasoning based on COSPL is shown by the virtue of non-commutativity and paraconsistency.

In Section 3, three natural deduction systems N_L, N_C and N^2_C are introduced for L, C and also C, respectively. In these systems, the construction by Mouri [5] using labelled assumptions is adopted. The equivalences between N_L and L, between N_C and C, and between N^2_C and N_C are proved.

In Section 4, in order to prove the strong normalization and Church-Rosser theorems for N_L, N_C and N^2_C, the corresponding typed λ-calculi λ_L, λ_C and λ^2_C are introduced based on the Curry-Howard correspondences. The definition of the two-directional λ-term using two kinds of abstraction operators by Wansing [14, 15] is adopted to these calculi. The strong normalization theorems for λ_L, λ_C and λ^2_C are proved, and hence the same theorems for N_L, N_C and N^2_C are shown. The Church-Rosser theorems for λ_L, λ_C, λ^2_C, N_L, N_C and N^2_C are shown as a corollary. In addition, two alternative typed calculi λ^3_C and λ^4_C, which are nearly equal to λ_C and λ^2_C, respectively, are introduced. The strong normalization and Church-Rosser theorems are proved for these calculi. The results in Sections 3 and 4 are considered to be difficult to extend to the full system COSPL, and hence other frameworks are needed to give some natural deduction systems for COSPL and also for FL.

In Section 5, firstly, a general natural deduction system N-COSPL (with general elimination rules) for COSPL is introduced based on the framework by Negri [6] for intuitionistic linear logic. Secondly, a uniform natural deduction system U-COSPL (with general elimination and introduction rules) are introduced following the framework by Negri [7]. The normalization theorems for N-COSPL and U-COSPL are proved by using the relationships between cut-free COSPL, N-COSPL and U-COSPL. It is known that the frameworks by Negri [6, 7] have a simple definition of normalization, and can obtain a natural correspondence between normal proofs and cut-free proofs. The discussion of this section obviously include the same results for FL, and hence such a discussion on FL is omitted.

In Section 6, some remarks on the proposed systems are given.

In Section 7, the conclusion of this paper is addressed.

2. Sequent calculus and illustrative example

2.1. Sequent calculus

Formulas are constructed from propositional variables, propositional constants: $\mathbf{1}$ (multiplicative truth), $\top$ (additive truth) and $\bot$ (additive falsity), two kinds of implications: $/$ and $\backslash$, $*$ (fusion), $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (strong negation). Lower-case letters $p, q, \ldots$ are used to denote propositional variables, Greek lower-case letters $\alpha, \beta, \ldots$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ are used to represent finite (possibly empty) sequences of formulas. Parentheses for $*$ are sometimes omitted because $*$ is associative. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$. The symbol $\equiv$ is used to denote equality of sequences of symbols. Since all logics discussed in this paper are formulated as sequent calculi, a sequent calculus will occasionally be identified with the logic determined by it.

DEFINITION 2.1 (COSPL, FL, C and L). The initial sequents of COSPL are of the form:

$$\alpha \Rightarrow \alpha \qquad \Rightarrow \mathbf{1} \qquad \Gamma \Rightarrow \top \qquad \Gamma, \bot, \Delta \Rightarrow \gamma$$

$$\Gamma, {\sim}\mathbf{1}, \Delta \Rightarrow \gamma \qquad \Gamma, {\sim}\top, \Delta \Rightarrow \gamma \qquad \Gamma \Rightarrow {\sim}\bot.$$

The inference rules of COSPL are of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta, \Gamma, \Sigma \Rightarrow \gamma}\ (\text{cut}) \qquad \frac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, \mathbf{1}, \Delta \Rightarrow \gamma}\ (\mathbf{1}\text{we})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta, \beta/\alpha, \Gamma, \Sigma \Rightarrow \gamma}\ (/\text{left}) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (/\text{right})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \gamma}\ (\backslash\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\backslash\text{right})$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha * \beta, \Delta \Rightarrow \gamma}\ (*\text{left}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha * \beta}\ (*\text{right})$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge\text{left1}) \quad \frac{\Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \gamma}\ (\wedge\text{left2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge\text{right}) \quad \frac{\Gamma, \alpha, \Delta \Rightarrow \gamma \quad \Gamma, \beta, \Delta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \gamma}\ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee\text{right2})$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \gamma}{\Gamma, {\sim}{\sim}\alpha, \Delta \Rightarrow \gamma}\ ({\sim}\text{left}) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow {\sim}{\sim}\alpha}\ ({\sim}\text{right})$$

$$\frac{\Gamma, \alpha, {\sim}\beta, \Delta \Rightarrow \gamma}{\Gamma, {\sim}(\beta/\alpha), \Delta \Rightarrow \gamma}\ ({\sim}/\text{left}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow {\sim}\beta}{\Gamma, \Delta \Rightarrow {\sim}(\beta/\alpha)}\ ({\sim}/\text{right})$$

$$\frac{\Gamma, {\sim}\beta, \alpha, \Delta \Rightarrow \gamma}{\Gamma, {\sim}(\alpha\backslash\beta), \Delta \Rightarrow \gamma}\ ({\sim}\backslash\text{left}) \quad \frac{\Gamma \Rightarrow {\sim}\beta \quad \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow {\sim}(\alpha\backslash\beta)}\ ({\sim}\backslash\text{right})$$

$$\frac{\Gamma, {\sim}\alpha, {\sim}\beta, \Delta \Rightarrow \gamma}{\Gamma, {\sim}(\alpha * \beta), \Delta \Rightarrow \gamma}\ ({\sim} * \text{left}) \quad \frac{\Gamma \Rightarrow {\sim}\alpha \quad \Delta \Rightarrow {\sim}\beta}{\Gamma, \Delta \Rightarrow {\sim}(\alpha * \beta)}\ ({\sim} * \text{right})$$

$$\frac{\Gamma \Rightarrow {\sim}\alpha}{\Gamma \Rightarrow {\sim}(\alpha \wedge \beta)}\ ({\sim} \wedge \text{right1}) \quad \frac{\Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha \wedge \beta)}\ ({\sim} \wedge \text{right2})$$

$$\frac{\Gamma, {\sim}\alpha, \Delta \Rightarrow \gamma \quad \Gamma, {\sim}\beta, \Delta \Rightarrow \gamma}{\Gamma, {\sim}(\alpha \wedge \beta), \Delta \Rightarrow \gamma}\ ({\sim} \wedge \text{left}) \quad \frac{\Gamma \Rightarrow {\sim}\alpha \quad \Gamma \Rightarrow {\sim}\beta}{\Gamma \Rightarrow {\sim}(\alpha \vee \beta)}\ ({\sim} \vee \text{right})$$

$$\frac{\Gamma, {\sim}\alpha, \Delta \Rightarrow \gamma}{\Gamma, {\sim}(\alpha \vee \beta), \Delta \Rightarrow \gamma}\ ({\sim} \vee \text{left1}) \quad \frac{\Gamma, {\sim}\beta, \Delta \Rightarrow \gamma}{\Gamma, {\sim}(\alpha \vee \beta), \Delta \Rightarrow \gamma}\ ({\sim} \vee \text{left2}).$$

The $\sim$-free fragment of COSPL is called FL (full Lambek logic). The $\{/, \backslash, *\}$-fragment of FL and the $\{/, \backslash, *, \sim\}$-fragment of COSPL are called here L and C, respectively.

It can be observed that Nelson's logic N4 is obtained from the $\{/, \wedge, \vee, \sim\}$-fragment of COSPL by adding the exchange, contraction and weakening rules respectively of the form:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}\ (\text{ex}) \quad \frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}\ (\text{co}) \quad \frac{\Gamma, \Delta \Rightarrow \gamma}{\Gamma, \alpha, \Delta \Rightarrow \gamma}\ (\text{we}).$$

The (non-modal propositional) intuitionistic linear logic is obtained from FL by adding (ex).

The following theorem is known [15].

Theorem 2.2 (Cut-Elimination for COSPL). *The rule* (cut) *is admissible in cut-free* COSPL.

The same theorem holds for FL, L and C.

Using Theorem 2.2, we can derive the following property [15].

Corollary 2.3 (Constructible Falsity). *If* $\Rightarrow\ {\sim}(\alpha \wedge \beta)$ *is provable in* COSPL, *then either* $\Rightarrow {\sim}\alpha$ *or* $\Rightarrow {\sim}\beta$ *is provable in* COSPL.

Definition 2.4. A logic L is called *explosive* if for any formulas α and β, the sequent $\alpha, {\sim}\alpha \Rightarrow \beta$ or ${\sim}\alpha, \alpha \Rightarrow \beta$ is provable in L. A logic L is called *paraconsistent* if L is not explosive[1].

Corollary 2.5 (Paraconsistency). COSPL *is paraconsistent.*

2.2. Illustrative example

In the following, it is shown that COSPL can be used in medical reasoning by the virtue of non-commutativity and paraconsistency.

Paraconsistency. It is known that logics with paraconsistency can deal with inconsistency-tolerant reasoning more appropriately. An example using paraconsistency is briefly explained below. Assume a large medical knowledge-base MKB of symptoms and diseases, such as an expert system based on COSPL. It can also be assumed that MKB is inconsistent in the sense that there is a symptom predicate $s(x)$ such that ${\sim}s(x), s(x) \in MKB$, where ${\sim}s(x)$ means "a person x does not have a symptom s." This assumption is very realistic, because symptom is a vague concept, which is difficult to determine by any diagnosis. Then, MKB does not derive arbitrary disease $d(x)$, which means "a person x suffers form a disease d", since paraconsistency ensures the fact that for some formulas α and β, both the sequents ${\sim}\alpha, \alpha \Rightarrow \beta$ and $\alpha, {\sim}\alpha \Rightarrow \beta$ are not provable. The paraconsistent COSPL-based MKB is thus inconsistency-tolerant. In the classical and intuitionistic logics, the sequent ${\sim}s(x), s(x) \Rightarrow d(x)$ is provable for any

[1] Paraconsistency is usually defined with respect to consequence relations instead of sequents [11], and the definition presented here is for a non-commutative version. Thus, this definition is not a standard one.

disease d, and hence the non-paraconsistent formulation based on the logics are regarded as inappropriate to the application of medical knowledge base.

Constructible falsity. It is known that the property of *constructible falsity* guarantees the constructiveness of the underlying negation connective [9]. The disjunction connective $\vee$ of the intuitionistic logic is known to be constructive, since it has the disjunction property: if $\Rightarrow \alpha \vee \beta$ is provable, then either $\Rightarrow \alpha$ or $\Rightarrow \beta$ is provable. The property of constructible falsity, which does not hold for the intuitionistic logic, is regarded as the dual notion of the disjunction property. It is also known that logics with this property can allow to express *inexact predicates*. An inexact predicate is an incomplete predicate in an empirical domain. An example of an inexact predicate is a disease or symptom predicate such as $melancholia(x)$, which means "a person x suffers from the first-stage melancholia." This predicate is incomplete in the sense that we can not determine exactly that the formula ${\sim}melancholia(x) \vee melancholia(x)$ is true. For more detailed discussions and examples, see e.g. [13].

Resource-sensitivity. It is known that logics without the contraction rule (co) can elegantly represent the concept of "resource consumption". For example, we consider a sequent: $coin, coin \Rightarrow coffee$, which means "if we consume two coins, then we can take a cup of coffee." Then, if assuming the classical or intuitionistic logic, this sequent is logically equivalent to the sequent: $coin \Rightarrow coffee$, because of the presence of the contraction rule. On the other hand, we desire to distinguish such two sequents in the sense of the "resource-sensitivity", i.e., one coin and two coins have the different effect as resources. It is noted that COSPL is one of such resource-sensitive logics, since it has no contraction rule.

An appropriate resource consumption example is medicine consumption in medical reasoning. Consider a medicine m as a resource. An expression $m(x) \Rightarrow recover(x)$ means "if a person x uses a medicine m to recover from a disease, then x makes a recovery from the disease with the medicine." In this case, $m(x), m(x) \Rightarrow recover(x)$ and $m(x) \Rightarrow recover(x)$ have the completely different meaning in the real world, because two medicines and one medicine have the different effect in general.

Priority. In the case of medicine consumption discussed above, it may not be sufficient to consider the effects of medicines. For example, if we consider two distinct medicines m_1 and m_2, then the meanings of the following two expressions are regarded as different: $m_1(x), m_2(x) \Rightarrow recover(x)$ and $m_2(x), m_1(x) \Rightarrow recover(x)$, because the order of using medicines change the effect of the medicines. In other words, the *time priority* of using medicines is more important in general. A more detailed example is expressed as follows. An expression $meal(x)$ means “a person x have a meal." Then, $m(x), meal(x) \Rightarrow recover(x)$ and $meal(x), m(x) \Rightarrow recover(x)$ have the different meaning, i.e., the effect of the medicine m is different whether the medicine is used after or before the meal.

To express such fine-grained medical reasoning, we have to use a non-commutative logic, such as COSPL, because, for example, logics with the exchange rule (ex) can not express the priority of the use of medicines. It can be known that in a sequent expression $\gamma_1, \gamma_2, ..., \gamma_n \Rightarrow \beta$ in COSPL, the antecedent $(\gamma_1, \gamma_2, ..., \gamma_n)$ can express the time priority of consuming the resources $\gamma_1, \gamma_2, ..., \gamma_n$, in fact, $(\gamma_1, \gamma_2, ..., \gamma_n)$ is a sequence of formulas in COSPL, since COSPL has no exchange rule. It is remarked that two sequents $\gamma_1, \gamma_2, ..., \gamma_n \Rightarrow \beta$ and $\gamma_1 * \gamma_2 * \cdots * \gamma_n \Rightarrow \beta$ are logically equivalent in COSPL, and hence an expression $\gamma_1 * \gamma_2$ means “first γ_1 is consumed, next so is γ_2.” It is also noted that in two expressions β/α and $\alpha\backslash\beta$, the implications / and \ represent resource consumption with priority, e.g. / means the consumption of (subscription) ascending order priority, and \ means the consumption of descending order priority.

In order to give an intuitive and natural formulation for the prioritized (or ordered) human reasoning as discussed above, some natural deduction systems using labelled assumptions will be introduced in the next section.

3. Natural deduction systems for L and C

3.1. $\mathbf{N}_L$, $\mathbf{N}_C$ and equivalences

In order to formulate natural deduction systems for L and C, the notion of labeled assumptions is introduced.

Definition 3.1. If α is a formula and n is a natural number, then α^n is called an assumption (with respect to the underlying natural deduction system).

Let α, β be formulas and n, m be natural numbers. Let $<$ and $\leq$ be strict partial order and partial order, respectively, on the set of natural numbers. Then, the strict partial and partial orders on the set of assumptions are defined as follows:

1. $\alpha^n = \beta^m$ iff $\alpha \equiv \beta$ and $n = m$,
2. $\alpha^n < \beta^m$ iff $n < m$,
3. $\alpha^n \leq \beta^m$ iff $n < m$ or $\alpha^n = \beta^m$.

Let Γ, Δ be sets of assumptions. Then, the strict partial and partial orders on the powerset of the set of assumptions are defined as follows:

1. $\Gamma < \Delta$ iff $\forall \alpha^n \in \Gamma, \forall \beta^m \in \Delta\ [\alpha^n < \beta^m]$,
2. $\Gamma \leq \Delta$ iff $\forall \alpha^n \in \Gamma, \forall \beta^m \in \Delta\ [\alpha^n \leq \beta^m]$.

It is remarked that if α and β are different as symbols, i.e., not$(\alpha \equiv \beta)$, then α^n and β^n are incomparable.

DEFINITION 3.2 (N_L). Let Γ, Δ be sets of assumptions. The inference rules of N_L (a natural deduction system for L) are of the form:

$$\dfrac{\begin{array}{c}\Gamma - \{\alpha^n\} \\ \vdots \\ \beta\end{array}}{\beta/\alpha}\ (/I)^n \quad \textit{where } \alpha^n \in \Gamma \textit{ and } \Gamma \leq \{\alpha^n\},$$

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \beta/\alpha\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \alpha\end{array}}{\beta}\ (/E) \quad \textit{where } \Gamma \cap \Delta = \emptyset \textit{ and } \Gamma \leq \Delta,$$

$$\dfrac{\begin{array}{c}\Gamma - \{\alpha^n\} \\ \vdots \\ \beta\end{array}}{\alpha\backslash\beta}\ (\backslash I)^n \quad \textit{where } \alpha^n \in \Gamma \textit{ and } \{\alpha^n\} \leq \Gamma,$$

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \alpha\backslash\beta\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \alpha\end{array}}{\beta}\ (\backslash E) \quad \textit{where } \Gamma \cap \Delta = \emptyset \textit{ and } \Delta \leq \Gamma,$$

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \alpha\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \beta\end{array}}{\alpha * \beta}\ (*I) \quad \textit{where } \Gamma \cap \Delta = \emptyset \textit{ and } \Gamma \leq \Delta,$$

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ (\gamma/\beta)/\alpha\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \alpha * \beta\end{array}}{\gamma}\ (*E1) \quad \textit{where } \Gamma \cap \Delta = \emptyset \textit{ and } \Gamma \leq \Delta,$$

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \alpha\backslash(\beta\backslash\gamma)\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \beta * \alpha\end{array}}{\gamma}\ (*E2) \quad \textit{where } \Gamma \cap \Delta = \emptyset \textit{ and } \Delta \leq \Gamma.$$

It is remarked that although inference rules in N_L may be applied to assumptions, yet labels are not inherited by conclusions of these rules applications.

DEFINITION 3.3 (N_C). N_C (a natural deduction system for C) is obtained from N_L by adding the inference rules of the form:

$$\frac{\alpha}{\sim\sim\alpha}\ (\sim I) \qquad \frac{\sim\sim\alpha}{\alpha}\ (\sim E) \qquad \frac{\alpha * \sim\beta}{\sim(\beta/\alpha)}\ (\sim/I) \qquad \frac{\sim(\beta/\alpha)}{\alpha * \sim\beta}\ (\sim/E)$$

$$\frac{\sim\beta * \alpha}{\sim(\alpha\backslash\beta)}\ (\sim\backslash I) \qquad \frac{\sim(\alpha\backslash\beta)}{\sim\beta * \alpha}\ (\sim\backslash E) \qquad \frac{\sim\alpha * \sim\beta}{\sim(\alpha * \beta)}\ (\sim * I) \qquad \frac{\sim(\alpha * \beta)}{\sim\alpha * \sim\beta}\ (\sim * E).$$

The inference rules $(/I)$, $(\backslash I)$, $(*I)$, $(\sim I)$, $(\sim/I)$, $(\sim\backslash I)$ and $(\sim * I)$ are called *introduction rules*, and the inference rules $(/E)$, $(\backslash E)$, $(*E1)$, $(*E2)$, $(\sim E)$, $(\sim/E)$, $(\sim\backslash E)$ and $(\sim * E)$ are called *elimination rules*. The usual terminologies of *major* or *minor premise* of some inference rules are used in the following. In particular, the right premises of the rules $(*E1)$ and $(*E2)$ are called the major premises of the rules. The notion of *proof*, *assumptions of proof*, and *end-formula of proof* are defined as usual. It is remarked that an assumption α^n is itself a proof. A formula α is said to be *provable* in a natural deduction system if there is a proof in the system with no open assumption whose end-formula is α.

In the definitions of N_L and N_C, the condition $\alpha^n \in \Gamma$ of $(/I)$ and $(\backslash I)$ corresponds to the fact that the underlying logics have no weakening rule (we). The condition $\Gamma \cap \Delta = \emptyset$ of $(/E)$, $(\backslash E)$, $(*I)$, $(*E1)$ and $(*E2)$ corresponds to the fact that the underlying logics have no contraction rule (co). The conditions $\Gamma \leq \{\alpha^n\}$ and $\{\alpha^n\} \leq \Gamma$ of $(/I)$ and $(\backslash I)$, respectively, and the conditions $\Gamma \leq \Delta$ and $\Delta \leq \Gamma$ of $\{(/E), (*E1)\}$ and $\{(\backslash E), (*E2)\}$, respectively, correspond to the fact that the underlying logics have no exchange rule (ex). We call the conditions concerning (we), (co) and (ex), the *weakening, contraction and exchange conditions*, respectively. By deleting any of these conditions, we can obtain the corresponding natural deduction system, systematically. For example, a natural deduction system for L+(ex), i.e., the $\{/, *\}$-fragment of intuitionistic linear logic, is obtained from N_L by deleting the exchange conditions.

DEFINITION 3.4. Let P be a proof in a natural deduction system. An expression $\mathrm{oa}(P)$ denotes the set of open assumptions of P, and an expression $\mathrm{end}(P)$ denotes the end-formula of P.

To prove the equivalence between N_C and C (and also between N_L and L), we need the following lemma.

LEMMA 3.5 (Label shift). *Let* Γ *be a (possibly empty) set of assumptions. If* P *is a proof in* N_C *such that* $\mathrm{oa}(P) = \Gamma \cup \{\alpha_1^{n_1}, ..., \alpha_k^{n_k}\}$ $(\Gamma < \{\alpha_1^{n_1}\} < \cdots < \{\alpha_k^{n_k}\})$ *and* $\mathrm{end}(P) = \beta$, *then for any natural number* $m \geq 0$, *there is a proof* P' *in* N_C *such that* $\mathrm{oa}(P') = \Gamma \cup \{\alpha_1^{n_1+m}, ..., \alpha_k^{n_k+m}\}$ *and* $\mathrm{end}(P') = \beta$.

PROOF. By induction on P. □

Using Lemma 3.5, we will prove Lemma 3.7. In the proofs of Lemmas 3.6 and 3.7, we use the notation Γ' for a corresponding labelled assumption set for a sequence Γ of formulas. We also use an expression Γ'^{+m} which means that Γ'^{+m} is obtained from a set Γ' of assumptions by m shifting to the labels of all the assumptions in Γ'.

LEMMA 3.6. *If* P *is a proof in* N_C *such that* $\mathrm{oa}(P) = \{\alpha_1^{n_1}, \cdots, \alpha_k^{n_k}\}$ $(\alpha_1^{n_1} < \cdots < \alpha_k^{n_k})$ $(0 \leq k)$[2] *and* $\mathrm{end}(P) = \beta$, *then the sequent* $\alpha_1, ..., \alpha_k \Rightarrow \beta$ *is provable in* C.

PROOF. We prove this lemma by induction on P. We distinguish the cases according to the last inference in P. We only show the following case.

Case $(*E1)$: P is of the form:

$$\dfrac{\begin{matrix}\Gamma' \\ \vdots \\ (\gamma/\beta)/\alpha\end{matrix} \qquad \begin{matrix}\Delta' \\ \vdots \\ \alpha * \beta\end{matrix}}{\gamma}\ (*E1)$$

where $\Gamma' \cap \Delta' = \emptyset$ and $\Gamma' \leq \Delta'$. By the hypothesis of induction, we have that the sequents $\Gamma \Rightarrow (\gamma/\beta)/\alpha$ and $\Delta \Rightarrow \alpha * \beta$ are provable in C. Then, we obtain:

$$\dfrac{\Delta \Rightarrow \alpha * \beta \qquad \dfrac{\dfrac{\Gamma \Rightarrow (\gamma/\beta)/\alpha \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow \beta \quad \gamma \Rightarrow \gamma}{\gamma/\beta, \beta \Rightarrow \gamma}\,(/\mathrm{left})}{(\gamma/\beta)/\alpha, \alpha, \beta \Rightarrow \gamma}\,(/\mathrm{left})}{\Gamma, \alpha, \beta \Rightarrow \gamma}\,(\mathrm{cut})}{\Gamma, \alpha * \beta \Rightarrow \gamma}\,(*\mathrm{left})}{\Gamma, \Delta \Rightarrow \gamma}\,(\mathrm{cut}).$$

□

LEMMA 3.7. *If a sequent* $\alpha_1, ..., \alpha_k \Rightarrow \beta$ $(0 \leq k)$ *is provable in* C, *then there is a proof* Q *in* N_C *such that* $\mathrm{oa}(Q) = \{\alpha_1^{n_1}, ..., \alpha_k^{n_k}\}$ $(\alpha_1^{n_1} < \cdots < \alpha_k^{n_k})$ *and* $\mathrm{end}(Q) = \beta$.

[2] The case for $k = 0$ means $\mathrm{oa}(P) = \{\beta^n\} = \mathrm{end}(P) = P$.

Proof. We prove this lemma by induction on a proof P of $\alpha_1, ..., \alpha_k \Rightarrow \beta$ in C. We distinguish the cases according to the last inference in P. We show some cases.

Case (\left): P is of the form:

$$\dfrac{\Gamma \Rightarrow \gamma \quad \Delta, \delta, \Sigma \Rightarrow \beta}{\Delta, \Gamma, \gamma\backslash\delta, \Sigma \Rightarrow \beta} \ (\backslash\text{left}).$$

By the hypothesis of induction, there are proofs in N_C such that

$$\begin{array}{c} \Gamma' \\ \vdots \\ \gamma, \end{array} \qquad \begin{array}{c} \Delta' \cup \{\delta^n\} \cup \Sigma' \\ \vdots \\ \beta \end{array}$$

where $\Delta' < \{\delta^n\} < \Sigma'$. Let Γ be $\{\gamma_1^{n_1}, ..., \gamma_m^{n_m}\}$ $(0 \leq m)$. By Lemma 3.5, we have proofs in N_C such that

$$\begin{array}{c} \Gamma'^{+r} \\ \vdots \\ \gamma, \end{array} \qquad \begin{array}{c} \Delta' \cup \{\delta^n\} \cup \Sigma'^{+z} \\ \vdots \\ \beta \end{array}$$

where $n < r$ and $n_m + r < z$, and hence $\Delta' < \Gamma'^{+r} < \Sigma'^{+z}$. Then, we have a required proof in N_C:

$$\Delta' \quad \begin{array}{c} \dfrac{(\gamma\backslash\delta)^{n_m+r} \qquad \begin{array}{c}\Gamma'^{+r} \\ \vdots \\ \gamma\end{array}}{\delta} \ (\backslash E) \\ \vdots \\ \beta \end{array} \quad \Sigma'^{+z}$$

where $\Gamma'^{+r} < \{(\gamma\backslash\delta)^{n_m+r}\}$. It is remarked that the required condition $\Delta' < \Gamma'^{+r} < \{(\gamma\backslash\delta)^{n_m+r}\} < \Sigma'^{+z}$ holds for this proof.

Case ($\sim$/left): P is of the form:

$$\dfrac{\Gamma, \gamma, \sim\delta, \Delta \Rightarrow \beta}{\Gamma, \sim(\delta/\gamma), \Delta \Rightarrow \beta} \ (\sim/\text{left}).$$

By the hypothesis of induction, we have that there is a proof in N_C of the form:

$$\begin{array}{c} \Gamma' \cup \{\gamma^m\} \cup \{{\sim}\delta^n\} \cup \Delta' \\ \vdots \\ \beta \end{array}$$

where $\Gamma' < \{\gamma^m\} < \{{\sim}\delta^n\} < \Delta'$. In the following, β/Δ denotes $(\cdots((\beta/\delta_n)/\delta_{n-1})/\cdots/\delta_1)$ if Δ is a sequence $(\delta_1, ..., \delta_n)$ where $1 \leq n$, and also denotes β if $\Delta = \emptyset$. Then, we obtain a required proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma'[\gamma^m][{\sim}\delta^n][\Delta'] \\ \vdots \\ \beta \\ \vdots\ (/I) \\ \beta/\Delta\end{array}}{(\beta/\Delta)/{\sim}\delta}\,(/I)^n}{((\beta/\Delta)/{\sim}\delta)/\gamma}\,(/I)^m \quad \dfrac{{\sim}(\delta/\gamma)^m}{\gamma * {\sim}\delta}\,({\sim}/E)}{\beta/\Delta}\,(*E1) \qquad \Delta'}{\beta}\ \vdots\ (/E)$$

where $\Gamma' < \{{\sim}(\delta/\gamma)^m\} < \Delta'$. □

Lemmas 3.6 and 3.7 imply the following theorem.

THEOREM 3.8 (Equivalence between N_C and C). *A formula α is provable in N_C if and only if the sequent $\Rightarrow \alpha$ is provable in* C.

We also have the following theorem as the subproof of Theorem 3.8.

THEOREM 3.9 (Equivalence between N_L and L). *A formula α is provable in N_L if and only if the sequent $\Rightarrow \alpha$ is provable in* L.

In order to define a reduction relation $\rhd$ on the set of proofs in N_L (and N_C), we assume the usual definition of substitution for proofs, i.e., an assumption α^n occurring in a proof P is replaced by a proof D with end(D) $= \alpha$. This substitution can be defined exactly, and can also be observed that the set of proofs in N_L (and N_C) is closed under the substitution[3].

[3] Strictly speaking, this fact is presented as follows. Let D be a proof such that oa(D) $= \Gamma \cup \{\alpha^n\}$, end$(D) = \beta$, $\alpha^n \notin \Gamma$ and $\Gamma < \{\alpha^n\}$. Let E be a proof such that oa$(E) =$ Δ and end$(E) = \alpha$. If $\Gamma < \Delta$, i.e., $\Gamma \cap \Delta = \emptyset$ and $\Gamma \leq \Delta$, then the figure F which is obtained from D by substituting E for α^n in D is a proof such that oa$(F) = \Gamma \cup \Delta$ and end(F) $= \beta$.

Definition 3.10. Let α be a formula occurring in a proof P. Then, α is called a maximum formula in P if α satisfies the following conditions: (1) α is a conclusion of an introduction rule and (2) α is the major premise of an elimination rule. A proof is said to be *normal* if it contains no maximum formula.

Definition 3.11 (Reduction for N_L). Let δ be a maximum formula in a proof which is the conclusion of an inference rule R and is the major premise of an elimination rule R'. The *reduction* relation (for N_L) $\rhd$ at δ is defined as follows.

1. R is $(/I)^n$ and δ is β/α:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma - \{\alpha^n\}\\ \vdots\, D\\ \beta\end{array}}{\beta/\alpha}\,R \qquad \begin{array}{c}\vdots\, E\\ \alpha\end{array}}{\beta} \quad \rhd \quad \begin{array}{c}\vdots\, E\\ \Gamma', \alpha\\ \vdots\, D\\ \beta\end{array}$$

where $\Gamma' \cup \{\alpha^n\} = \Gamma$.

2. R is $(\backslash I)^n$ and δ is $\alpha\backslash\beta$:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma - \{\alpha^n\}\\ \vdots\, D\\ \beta\end{array}}{\alpha\backslash\beta}\,R \qquad \begin{array}{c}\vdots\, E\\ \alpha\end{array}}{\beta} \quad \rhd \quad \begin{array}{c}\vdots\, E\\ \alpha, \Gamma'\\ \vdots\, D\\ \beta\end{array}$$

where $\Gamma' \cup \{\alpha^n\} = \Gamma$.

3. R is $(*I)$, R' is $(*E1)$, and δ is $\alpha * \beta$:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ (\gamma/\beta)/\alpha\end{array} \qquad \dfrac{\begin{array}{c}\Delta\\ \vdots\\ \alpha\end{array} \quad \begin{array}{c}\Sigma\\ \vdots\\ \beta\end{array}}{\alpha * \beta}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ (\gamma/\beta)/\alpha\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ \alpha\end{array}}{\gamma/\beta}\,(/E) \qquad \begin{array}{c}\Sigma\\ \vdots\\ \beta\end{array}}{\gamma}\,(/E).$$

4. R is $(*I)$, R' is $(*E2)$, and δ is $\beta * \alpha$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots \\ \alpha\backslash(\beta\backslash\gamma)\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ \beta\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ \alpha\end{matrix}}{\beta * \alpha}\, R}{\gamma}\, R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \alpha\backslash(\beta\backslash\gamma)\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ \alpha\end{matrix}}{\beta\backslash\gamma}\,(\backslash E) \quad \begin{matrix}\Delta\\ \vdots\\ \beta\end{matrix}}{\gamma}\,(\backslash E).$$

5. Let D, D', E be proofs. If $D \rhd D'$, then

$$\dfrac{D}{\alpha}\,(I) \quad \rhd \quad \dfrac{D'}{\alpha}\,(I),$$

$$\dfrac{D \quad E}{\alpha}\,(R) \quad \rhd \quad \dfrac{D' \quad E}{\alpha}\,(R), \qquad \dfrac{E \quad D}{\alpha}\,(R) \quad \rhd \quad \dfrac{E \quad D'}{\alpha}\,(R)$$

where $I \in \{/I, \backslash I\}$ and $R \in \{/E, \backslash E, *I, *E1, *E2\}$.

It is remarked that in this definition, the conditions of proofs are preserved with respect to $\rhd$, e.g., for the case 4, the conditions $\Gamma \cap \Delta \cap \Pi = \emptyset$ and $\Delta \leq \Sigma \leq \Gamma$ of the left hand side of $\rhd$ are preserved in the right hand side of $\rhd$. Thus, the set of proofs in N_L is closed under $\rhd$[4].

DEFINITION 3.12 (Reduction for N_C). Let γ be a maximum formula in a proof which is the conclusion of an inference rule R. The reduction relation $\rhd$ of N_C at γ is obtained from those of N_L by adding the following conditions.

6. R is $(\sim I)$, and γ is $\sim\sim\alpha$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\, D\\ \alpha\end{matrix}}{\sim\sim\alpha}\, R}{\alpha} \quad \rhd \quad \begin{matrix}\vdots\, D\\ \alpha.\end{matrix}$$

7. R is $(\sim/I)$, and γ is $\sim(\beta/\alpha)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\, D\\ \alpha * \sim\beta\end{matrix}}{\sim(\beta/\alpha)}\, R}{\alpha * \sim\beta} \quad \rhd \quad \begin{matrix}\vdots\, D\\ \alpha * \sim\beta.\end{matrix}$$

[4] Strictly speaking, this fact is presented as follows. Let D be a proof in N_L such that $\mathrm{oa}(D) = \Gamma$ and $\mathrm{end}(D) = \alpha$. If $D \rhd E$, then E is also a proof in N_L such that $\mathrm{oa}(E) = \Gamma$ and $\mathrm{end}(E) = \alpha$.

8. R is $(\sim\backslash I)$, and γ is $\sim(\alpha\backslash\beta)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, D\\ \sim\beta * \alpha\end{array}}{\sim(\alpha\backslash\beta)}\, R}{\sim\beta * \alpha} \quad \rhd \quad \begin{array}{c}\vdots\, D\\ \sim\beta * \alpha.\end{array}$$

9. R is $(\sim * I)$, and γ is $\sim(\alpha * \beta)$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, D\\ \sim\alpha * \sim\beta\end{array}}{\sim(\alpha * \beta)}\, R}{\sim\alpha * \sim\beta.} \quad \rhd \quad \begin{array}{c}\vdots\, D\\ \sim\alpha * \sim\beta.\end{array}$$

10. Let D, D', E be proofs. If $D \rhd D'$, then

$$\frac{D}{\alpha}\,(I) \quad \rhd \quad \frac{D'}{\alpha}\,(I)$$

where $I \in \{\sim I, \sim E, \sim/I, \sim/E, \sim\backslash I, \sim\backslash E, \sim * I, \sim * E\}$.

It is remarked that the set of proofs in N_C is closed under $\rhd$.

Definition 3.13. A sequence $D_0, D_1, \ldots$ of proofs is called a *reduction sequence* if it satisfies the following conditions (1) $D_i \rhd D_{i+1}$ for all $0 \leq i$ and (2) the last proof in the sequence is normal if the sequence is finite. A proof D is called *strongly normalizable* if each reduction sequence starting from D terminates.

3.2. N_C^2 and equivalence

Definition 3.14 (N_C^2). N_C^2 is obtained from N_C by replacing the inference rules $(\sim/I)$, $(\sim/E)$, $(\sim\backslash I)$, $(\sim\backslash E)$, $(\sim * I)$ and $(\sim * E)$ by the inference rules of the form: for $\Gamma \cap \Delta = \emptyset$ and $\Gamma \leq \Delta$,

$$\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ \alpha & \sim\beta\end{array}}{\sim(\beta/\alpha)}\,(\sim/I^*) \qquad \dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ (\gamma/\sim\beta)/\alpha & \sim(\beta/\alpha)\end{array}}{\gamma}\,(\sim/E1^*) \qquad \dfrac{\begin{array}{cc}\Delta & \Gamma\\ \vdots & \vdots\\ \sim\beta\backslash(\alpha\backslash\gamma) & \sim(\beta/\alpha)\end{array}}{\gamma}\,(\sim/E2^*)$$

$$\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ \sim\beta & \alpha\end{array}}{\sim(\alpha\backslash\beta)}\,(\sim\backslash I^*) \qquad \dfrac{\begin{array}{cc}\Delta & \Gamma\\ \vdots & \vdots\\ \alpha\backslash(\sim\beta\backslash\gamma) & \sim(\alpha\backslash\beta)\end{array}}{\gamma}\,(\sim\backslash E1^*) \qquad \dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ (\gamma/\alpha)/\sim\beta & \sim(\alpha\backslash\beta)\end{array}}{\gamma}\,(\sim\backslash E2^*)$$

$$\dfrac{\begin{matrix}\Gamma & \Delta\\ \vdots & \vdots\\ {\sim}\alpha & {\sim}\beta\end{matrix}}{{\sim}(\alpha * \beta)}\,({\sim} * I^*) \quad \dfrac{\begin{matrix}\Gamma & \Delta\\ \vdots & \vdots\\ (\gamma/{\sim}\beta)/{\sim}\alpha & {\sim}\alpha * {\sim}\beta\end{matrix}}{\gamma}\,({\sim} * E1^*) \quad \dfrac{\begin{matrix}\Delta & \Gamma\\ \vdots & \vdots\\ {\sim}\beta\backslash({\sim}\alpha\backslash\gamma) & {\sim}\alpha * {\sim}\beta\end{matrix}}{\gamma}\,({\sim} * E2^*).$$

Theorem 3.15 (Equivalence between N_C^2 and N_C). *A formula γ is provable in N_C^2 if and only if γ is provable in N_C.*

Proof. ($\Longleftarrow$): We prove the statement by induction on a proof P of γ in N_C. We distinguish the cases according to the last inference in P. We show some cases.

Case $({\sim}/I)$: P is of the form:

$$\dfrac{\begin{matrix}\vdots\, Q\\ \alpha * {\sim}\beta\end{matrix}}{{\sim}(\beta/\alpha)}\,({\sim}/I).$$

By the hypothesis of induction, we get a proof of $\alpha * {\sim}\beta$ in N_C^2 such that

$$\begin{matrix}\vdots\, Q'\\ \alpha * {\sim}\beta.\end{matrix}$$

Then, we obtain a required proof in N_C^2:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha^1] \quad [{\sim}\beta^2]}{{\sim}(\beta/\alpha)}\,({\sim}/I^*)}{{\sim}(\beta/\alpha)/{\sim}\beta}\,(/I)^2}{({\sim}(\beta/\alpha)/{\sim}\beta)/\alpha}\,(/I)^1 \quad \begin{matrix}\vdots\, Q'\\ \alpha * {\sim}\beta\end{matrix}}{{\sim}(\beta/\alpha)}\,(*E1).$$

Case $({\sim}/E)$: P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q\\ {\sim}(\beta/\alpha)\end{matrix}}{\alpha * {\sim}\beta}\,({\sim}/E).$$

By the hypothesis of induction, we get a proof of ${\sim}(\beta/\alpha)$ in N_C^2 such that

$$\begin{matrix}\vdots\, Q'\\ {\sim}(\beta/\alpha).\end{matrix}$$

Then, we obtain a required proof in N_C^2:

$$\cfrac{\cfrac{\cfrac{\cfrac{[\alpha^1] \quad [{\sim}\beta^2]}{\alpha * {\sim}\beta}\,(*I)}{(\alpha * {\sim}\beta)/{\sim}\beta}\,(/I)^2}{((\alpha * {\sim}\beta)/{\sim}\beta)/\alpha}\,(/I)^1 \qquad \begin{array}{c}\vdots\, Q' \\ {\sim}(\beta/\alpha)\end{array}}{\alpha * {\sim}\beta}\,({\sim}/E1^*).$$

($\Longrightarrow$): We prove the statement by induction on a proof P of γ in N_C^2. We distinguish the cases according to the last inference in P. We only show the following case.

Case $({\sim}/E1^*)$: P is of the form:

$$\cfrac{\begin{array}{c}\vdots\, Q_1 \\ (\gamma/{\sim}\beta)/\alpha\end{array} \quad \begin{array}{c}\vdots\, Q_2 \\ {\sim}(\beta/\alpha)\end{array}}{\gamma}\,({\sim}/E1^*).$$

By the hypothsis of induction, we obtain the proofs in N_C of the form:

$$\begin{array}{c}\vdots\, Q_1' \\ (\gamma/{\sim}\beta)/\alpha\end{array} \qquad\qquad \begin{array}{c}\vdots\, Q_2' \\ {\sim}(\beta/\alpha)\end{array}$$

and hence a required proof is

$$\cfrac{\begin{array}{c}\vdots\, Q_1' \\ (\gamma/{\sim}\beta)/\alpha\end{array} \quad \cfrac{\begin{array}{c}\vdots\, Q_2' \\ {\sim}(\beta/\alpha)\end{array}}{\alpha * {\sim}\beta}\,({\sim}/E)}{\gamma}\,(*E1).$$

□

Definition 3.16 (Reduction for N_C^2). Let δ be a maximum formula in a proof which is the conclusion of an inference rule R. The reduction relation $\rhd$ of N_C^2 at δ is obtained from that of N_L by adding the following conditions.

6. R is $({\sim}I)$, and δ is ${\sim}{\sim}\alpha$:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\, D \\ \alpha\end{array}}{{\sim}{\sim}\alpha}\,R}{\alpha} \quad \rhd \quad \begin{array}{c}\vdots\, D \\ \alpha.\end{array}$$

7. R is $({\sim}/I^*)$, R' is $({\sim}/E1^*)$, and δ is ${\sim}(\beta/\alpha)$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ (\gamma/{\sim}\beta)/\alpha\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ \alpha\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\beta\end{matrix}}{{\sim}(\beta/\alpha)}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ (\gamma/{\sim}\beta)/\alpha\end{matrix} \quad \begin{matrix}\Delta\\ \vdots\\ \alpha\end{matrix}}{\gamma/{\sim}\beta}\,(/E) \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\beta\end{matrix}}{\gamma}\,(/E).$$

8. R is $({\sim}/I^*)$, R' is $({\sim}/E2^*)$, and δ is ${\sim}(\beta/\alpha)$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ {\sim}\beta\backslash(\alpha\backslash\gamma)\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ \alpha\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\beta\end{matrix}}{{\sim}(\beta/\alpha)}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ {\sim}\beta\backslash(\alpha\backslash\gamma)\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\beta\end{matrix}}{\alpha\backslash\gamma}\,(\backslash E) \quad \begin{matrix}\Delta\\ \vdots\\ \alpha\end{matrix}}{\gamma}\,(\backslash E).$$

9. R is $({\sim}\backslash I^*)$, R' is $({\sim}\backslash E1^*)$, and δ is ${\sim}(\alpha\backslash\beta)$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \alpha\backslash({\sim}\beta\backslash\gamma)\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ {\sim}\beta\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ \alpha\end{matrix}}{{\sim}(\alpha\backslash\beta)}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \alpha\backslash({\sim}\beta\backslash\gamma)\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ \alpha\end{matrix}}{{\sim}\beta\backslash\gamma}\,(\backslash E) \quad \begin{matrix}\Delta\\ \vdots\\ {\sim}\beta\end{matrix}}{\gamma}\,(\backslash E).$$

10. R is $({\sim}\backslash I^*)$, R' is $({\sim}\backslash E2^*)$, and δ is ${\sim}(\alpha\backslash\beta)$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ (\gamma/\alpha)/{\sim}\beta\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ {\sim}\beta\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ \alpha\end{matrix}}{{\sim}(\alpha\backslash\beta)}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ (\gamma/\alpha)/{\sim}\beta\end{matrix} \quad \begin{matrix}\Delta\\ \vdots\\ {\sim}\beta\end{matrix}}{\gamma/\alpha}\,(/E) \quad \begin{matrix}\Sigma\\ \vdots\\ \alpha\end{matrix}}{\gamma}\,(/E).$$

11. R is $({\sim}*I^*)$, R' is $({\sim}*E1^*)$, and δ is ${\sim}(\alpha*\beta)$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ (\gamma/{\sim}\beta)/{\sim}\alpha\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ {\sim}\alpha\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\beta\end{matrix}}{{\sim}(\alpha*\beta)}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ (\gamma/{\sim}\beta)/{\sim}\alpha\end{matrix} \quad \begin{matrix}\Delta\\ \vdots\\ {\sim}\alpha\end{matrix}}{\gamma/{\sim}\beta}\,(/E) \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\beta\end{matrix}}{\gamma}\,(/E).$$

12. R is $({\sim}*I^*)$, R' is $({\sim}*E2^*)$, and δ is ${\sim}(\beta*\alpha)$:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ {\sim}\alpha\backslash({\sim}\beta\backslash\gamma)\end{matrix} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ {\sim}\beta\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\alpha\end{matrix}}{{\sim}(\beta*\alpha)}\,R}{\gamma}\,R' \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ {\sim}\alpha\backslash({\sim}\beta\backslash\gamma)\end{matrix} \quad \begin{matrix}\Sigma\\ \vdots\\ {\sim}\alpha\end{matrix}}{{\sim}\beta\backslash\gamma}\,(\backslash E) \quad \begin{matrix}\Delta\\ \vdots\\ {\sim}\beta\end{matrix}}{\gamma}\,(\backslash E).$$

13. Let D, D', E be proofs. If $D \rhd D'$, then

$$\frac{D}{\alpha}\ (I) \quad \rhd \quad \frac{D'}{\alpha}\ (I),$$

$$\frac{D \quad E}{\alpha}\ (R) \quad \rhd \quad \frac{D' \quad E}{\alpha}\ (R), \qquad \frac{E \quad D}{\alpha}\ (R) \quad \rhd \quad \frac{E \quad D'}{\alpha}\ (R)$$

where $I \in \{\sim I, \sim E\}$ and $R \in \{\sim/I^*, \sim/E1^*, \sim/E2^*, \sim\backslash I^*, \sim\backslash E1^*, \sim\backslash E2^*, \sim * I^*, \sim * E1^*, \sim * E2^*\}$.

The set of proofs in N_C^2 is closed under $\rhd$.

The notions of reduction sequence and strong normalizability are the same as in the Definition 3.13.

The strong normalizability for N_L, N_C and N_C^2 will be proved in the next section, by using the corresponding typed λ-calculi based on the Curry-Howard correspondences.

4. Typed λ-calculus and strong normalization

4.1. λ_L and strong normalization

Terms are constructed from variables, two kinds of λ-abstractions λ^r, λ^l concerning the two directional implication connectives $/, \backslash$ in L, usual (left and right) applications, an application operator $\circ$ concerning the fusion connective $*$, and a pairing function $[\ ,\ ]$ concerning $*$. *Types* are constructed from atomic types, $/, \backslash$, and $*$. Variables are denoted as $x, x_n, y, ...$, untyped terms are denoted as $M, M_n, N, ...$, types are denoted as $\alpha, \beta, \gamma, ...$, and typed terms are denoted as $M^\alpha, N^\beta, L^\gamma, ...$ Typed terms are sometimes denoted as $M, N, L, ...$ by omitting the types. It is assumed that in a λ-term, the same variables do not occur simultaneously as both free and bound variables. It is also assumed that in a λ-term, there are no iterated occurrences of the same bound variable x, such as $\cdots \lambda^r x^\alpha.(\cdots \lambda^r x^\alpha.(\cdots)\cdots)\cdots$. An expression $FV(M^\alpha)$ means the set of all (typed) free variables in the typed term M^α. An expression $[N^\alpha/x^\alpha]M^\beta$ means, in a usual sense, the substitution of N^α to a free variable x^α in M^β. To avoid the clash of bound variables by substitutions, α-conversions are occasionally assumed.

DEFINITION 4.1. Assume that the set of variables is countable. Then, an expression x^α denotes a typed variable if x is a variable and α is a type.

Let x, y be variables, and α, β be types. Suppose that the strict partial and partial orders $<, \leq$, respectively, on the set of variables are defined.

Then, the strict partial and partial orders on the set of typed variables are defined as follows:

1. $x^\alpha = y^\beta$ iff $x = y$ and $\alpha \equiv \beta$,
2. $x^\alpha < y^\beta$ iff $x < y$,
3. $x^\alpha \leq y^\beta$ iff $x < y$ or $x^\alpha = y^\beta$.

Let Γ, Δ be sets of typed variables. Then, the strict partial and partial orders on the powerset of the set of typed variables are defined as follows:

1. $\Gamma < \Delta$ iff $\forall x^\alpha \in \Gamma, \forall y^\beta \in \Delta\ [x^\alpha < y^\beta]$,
2. $\Gamma \leq \Delta$ iff $\forall x^\alpha \in \Gamma, \forall y^\beta \in \Delta\ [x^\alpha \leq y^\beta]$.

DEFINITION 4.2 (Typed λ-term for λ_L). Suppose that types and terms are defined as usual. Then, typed λ-terms (for λ_L) are inductively defined as follows.

1. if x^α is a typed variable, then it is a typed λ-term.
2. if x^α and M^β are typed λ-terms, $FV(M^\beta) \leq \{x^\alpha\}$, and $x^\alpha \in FV(M^\beta)$, then $(\lambda^r x^\alpha.M^\beta)^{\beta/\alpha}$ is a typed λ-term.
3. if x^α and M^β are typed λ-terms, $\{x^\alpha\} \leq FV(M^\beta)$, and $x^\alpha \in FV(M^\beta)$, then $(\lambda^l x^\alpha.M^\beta)^{\alpha\backslash\beta}$ is a typed λ-term.
4. if $M^{\beta/\alpha}$ and N^α are typed λ-terms, $FV(M^{\beta/\alpha}) \leq FV(N^\alpha)$ and $FV(M^{\beta/\alpha}) \cap FV(N^\alpha) = \emptyset$, then $(M^{\beta/\alpha}N^\alpha)^\beta$ is a typed λ-term.
5. if $M^{\alpha\backslash\beta}$ and N^α are typed λ-terms, $FV(N^\alpha) \leq FV(M^{\alpha\backslash\beta})$ and $FV(M^{\alpha\backslash\beta}) \cap FV(N^\alpha) = \emptyset$, then $(N^\alpha M^{\alpha\backslash\beta})^\beta$ is a typed λ-term.
6. if M^α and N^β are typed λ-terms, $FV(M^\alpha) \leq FV(N^\beta)$ and $FV(M^\alpha) \cap FV(N^\beta) = \emptyset$, then $[M^\alpha, N^\beta]^{\alpha*\beta}$ is a typed λ-term.
7. if $M^{(\gamma/\beta)/\alpha}$ and $N^{\alpha*\beta}$ are typed λ-terms, $FV(M^{(\gamma/\beta)/\alpha}) \leq FV(N^{\alpha*\beta})$ and $FV(M^{(\gamma/\beta)/\alpha}) \cap FV(N^{\alpha*\beta}) = \emptyset$, then $(M^{(\gamma/\beta)/\alpha} \circ N^{\alpha*\beta})^\gamma$ is a typed λ-term.

8. if $M^{\alpha\backslash(\beta\backslash\gamma)}$ and $N^{\beta*\alpha}$ are typed λ-terms, $FV(N^{\beta*\alpha}) \leq FV(M^{\alpha\backslash(\beta\backslash\gamma)})$ and $FV(M^{\alpha\backslash(\beta\backslash\gamma)}) \cap FV(N^{\beta*\alpha}) = \emptyset$, then $(N^{\beta*\alpha} \circ M^{\alpha\backslash(\beta\backslash\gamma)})^{\gamma}$ is a typed λ-term.

It is observed that, in Definition 4.2, the items 2—8 correspond to the inference rules $(/I)$, $(\backslash I)$, $(/E)$, $(\backslash E)$, $(*I)$, $(*E1)$ and $(*E2)$, respectively. Hence, in Definition 4.2, the assumptions $FV(M^{\beta}) \leq \{x^{\alpha}\}$, $\{x^{\alpha}\} \leq FV(M^{\beta})$, $FV(M^{\beta/\alpha}) \leq FV(N^{\alpha})$, $FV(N^{\alpha}) \leq FV(M^{\alpha\backslash\beta})$, $FV(M^{\alpha}) \leq FV(N^{\beta})$, $FV(M^{(\gamma/\beta)/\alpha}) \leq FV(N^{\alpha*\beta})$ and $FV(N^{\beta*\alpha}) \leq FV(M^{\alpha\backslash(\beta\backslash\gamma)})$ in the conditions 2—8 correspond to the fact that the underlying logic has no exchange rule. The assumption $x^{\alpha} \in FV(M^{\beta})$ in the conditions 2 and 3 corresponds to the fact that the underlying logic has no weakening rule. The empty-variable assumptions such as $FV(M^{\beta/\alpha}) \cap FV(N^{\alpha}) = \emptyset$ in the conditions 4—8 correspond to the fact that the underlying logic has no contraction rule. These empty-variable assumptions concerning the contraction rule have a crucial role in the simplicity of the strong normalization proof. The definition discussed is based on the definition of linear λ-term by Hindley [3], and on the definition of two directional λ-term using two kinds of abstractions by Wansing [14, 15]. Hence the Curry-Howard correspondence can naturally be obtained.

Definition 4.3 (λ_L). In the following, a typed λ-calculus for L, called λ_L, is defined by the reductions for the typed λ-terms defined in Definition 4.2. The transformation process from the left hand side of $\rhd$ to the right hand side of $\rhd$ is called a *reduction*, and the term of the left hand side of $\rhd$ is called a *redex*.

1. $((\lambda^r x^{\alpha}.M^{\beta})^{\beta/\alpha} N^{\alpha})^{\beta} \rhd [N^{\alpha}/x^{\alpha}]M^{\beta}$.

2. $(N^{\alpha}(\lambda^l x^{\alpha}.M^{\beta})^{\alpha\backslash\beta})^{\beta} \rhd [N^{\alpha}/x^{\alpha}]M^{\beta}$.

3. $(M^{(\gamma/\beta)/\alpha} \circ [N^{\alpha}, L^{\beta}]^{\alpha*\beta})^{\gamma} \rhd ((M^{(\gamma/\beta)/\alpha} N^{\alpha})^{\gamma/\beta} L^{\beta})^{\gamma}$.

4. $([L^{\beta}, N^{\alpha}]^{\beta*\alpha} \circ M^{\alpha\backslash(\beta\backslash\gamma)})^{\gamma} \rhd (L^{\beta}(N^{\alpha} M^{\alpha\backslash(\beta\backslash\gamma)})^{\beta\backslash\gamma})^{\gamma}$.

5.[5] if $M \rhd N$, then $\lambda^r x.M \rhd \lambda^r x.N$, $\lambda^l x.M \rhd \lambda^l x.N$, $ML \rhd NL$, $LM \rhd LN$, $[M, L] \rhd [N, L]$, $[L, M] \rhd [L, N]$, $M \circ L \rhd N \circ L$ and $L \circ M \rhd L \circ N$.

[5]For the sake of simplicity of the expressions, types are omitted in 5, but the corresponding types can be attached appropriately.

It can be observed that the set of typed λ-terms for λ_L is closed under $\triangleright$.

Definition 4.4. A typed λ-term is said to be *normal* if it contains no redex. A sequence $M_0^{\alpha_0}, M_1^{\alpha_1}, \cdots$ of typed λ-terms is called a *reduction sequence* if it satisfies the following conditions (1) $M_i^{\alpha_i} \triangleright M_{i+1}^{\alpha_{i+1}}$ for all $0 \leq i$ and (2) the last typed λ-term in the sequence is normal if the sequence is finite. A typed λ-term M^α is called *strongly normalizable* if each reduction sequence starting from M^α terminates.

It is known that the strong normalization theorem for the linear λ-terms for the implicational fragment can be proved using the function l from the set of terms to the set of natural numbers defined inductively by $l(x) = 0$, $l(\lambda x.M) = l(M)+1$ and $l(MN) = l(M)+l(N)$. Since the linear λ-terms have the restriction related to the absence of the contraction rule, the fact "$M \triangleright N$ implies $l(M) > l(N)$" can be obtained, and using this fact, the strong normalization theorem can be proved. To prove the strong normalization for λ_L, the function l is extended with the addition of the cases for $\circ$ and $[\ ,\ ]$.

Definition 4.5. A function f from the set of all typed λ-terms to the set of natural numbers is inductively defined by

1. $f(x) = 0$,
2. $f(MN) = f([M, N]) = f(M) + f(N)$,
3. $f(\lambda^r x.M) = f(\lambda^l x.M) = f(M) + 1$,
4. $f(M \circ N) = f(M) + f(N) + 1$.

Lemma 4.6. *Let $(\lambda^r x.M)N$, $N(\lambda^l x.M)$ be typed λ-terms for λ_L.*

(1) $f((\lambda^r x.M)N) > f([N/x]M)$.

(2) $f(N(\lambda^l x.M)) > f([N/x]M)$.

Proof. We only prove (1) of this lemma by induction on M. (2) can be proved similarly. We show some cases. Let $n = f(N)$, $p = f(P)$ and $q = f(Q)$.

(Case $M \equiv x$): The left hand side of the inequality is $f((\lambda^r x.x)N) = f(\lambda^r x.x) + f(N) = f(x) + 1 + f(N) = n + 1$. The right hand side is $f([N/x]x) = f(N) = n$. Therefore we have the required fact.

(Case $M \equiv [P, Q]$): The left hand side is $L = f((\lambda^r x.[P, Q])N) = f(\lambda^r x.[P, Q]) + f(N) = f(P) + f(Q) + 1 + f(N) = p + q + n + 1$. The right hand side is $R = f([N/x][P, Q]) = f([[N/x]P, Q])$ or $f([P, [N/x]Q])$[6]. We only consider the former case: $f([[N/x]P, Q])$. The latter case can be treated similarly. By the hypothesis of induction, we have $R = f([N/x]P) + f(Q) < f((\lambda^r x.P)N) + f(Q) = f(\lambda^r x.P) + f(N) + f(Q) = p + q + n + 1$. Therefore $L > R$.

(Case $M \equiv \lambda^r y.P$ where $y \neq x$ and $y \notin FV(N)$): The left hand side is $L = f((\lambda^r x.(\lambda^r y.P))N) = f(\lambda^r x.(\lambda^r y.P)) + f(N) = f(\lambda^r y.P) + 1 + f(N) = f(P) + 1 + 1 + f(N) = p + n + 2$. The right hand side is $R = f([N/x](\lambda^r y.P)) = f(\lambda^r y.[N/x]P) = f([N/x]P) + 1 < f((\lambda^r x.P)N) + 1 = f(\lambda^r x.P) + f(N) + 1 = f(P) + 1 + f(N) + 1 = p + n + 2$ by the hypothesis of induction[7]. Therefore $L > R$.

(Case $M \equiv P \circ Q$): The left hand side is $L = f((\lambda^r x.(P \circ Q))N) = f(P \circ Q) + 1 + f(N) = f(P) + f(Q) + 1 + 1 + f(N) = p + q + n + 2$. The right hand side is $R = f([N/x](P \circ Q)) = f([N/x]P \circ Q)$ or $f(P \circ [N/x]Q)$. We only consider the former case. By the hypothesis of induction, we have $f([N/x]P \circ Q) = f([N/x]P) + f(Q) + 1 < f((\lambda^r x.P)N) + f(Q) + 1 = f(P) + 1 + f(N) + f(Q) + 1 = p + q + n + 2$. Therefore $L > R$. □

Using this lemma, we can prove the following lemma.

LEMMA 4.7. *Let M be a typed λ-term for λ_L. Then, $M \rhd N$ implies $f(M) > f(N)$.*

PROOF. We prove this lemma by induction on M. We show some cases.

(Case $M \equiv ((\lambda^r x.P)Q)$, $N \equiv [Q/x]P$): By Lemma 4.6, we obtain $f(M) > f(N)$.

(Case $M \equiv (P \circ [Q, R])$, $N \equiv ((PQ)R)$): We have $f(M) = f(P \circ [Q, R]) = f(P) + f([Q, R]) + 1 = f(P) + f(Q) + f(R) + 1$, and $f(N) = f((PQ)R) = f(P) + f(Q) + f(R)$. Therefore $f(M) > f(N)$. □

Using this lemma, we obtain the following theorem.

THEOREM 4.8 (Strong normalization for λ_L). *All typed λ-terms for λ_L are strongly normalizable.*

[6] This is because of the restriction $FV(P) \cap FV(Q) = \emptyset$ concerning the absence of the contraction rule.

[7] We assume the restriction $x \in FV(\lambda^r y.P)$ concerning the absence of the weakening rule.

PROOF. Suppose that there is an infinite reduction sequence starting form M_0 such that $M_0 \rhd M_1 \rhd \cdots$. Then, by Lemma 4.7, we have the fact that $f(M_0) > f(M_1) > \cdots$ is infinite. However this is the contradiction for the fact that $f(M_0)$ is a natural number, and hence the reductions terminate. □

Using this theorem, we obtain the following theorem.

THEOREM 4.9 (Strong normalization for N_L). *All proofs for N_L are strongly normalizable.*

PROOF. We only give a sketch of the proof. First, we consider a usual type-assignment system TAN_L[8] which has appropriate typed inference rules such as

$$\dfrac{\begin{matrix}\vdots \\ M^{(\gamma/\beta)/\alpha} : (\gamma/\beta)/\alpha\end{matrix} \quad \begin{matrix}\vdots \\ N^{\alpha*\beta} : \alpha * \beta\end{matrix}}{(M \circ N)^{\gamma} : \gamma}$$

where $FV(M^{(\gamma/\beta)/\alpha}) \cap FV(N^{\alpha*\beta}) = \emptyset$ and $FV(M^{(\gamma/\beta)/\alpha}) \leq FV(N^{\alpha*\beta})$. By introducing such a system, we can clarify the Curry-Howard correspondence as a result. It is noted that a figure which is obtained from a proof of $(M^{\alpha} : \alpha)$ in TAN_L by deleting all typed λ-terms is a proof of α in N_L, and conversely, for a proof of α in N_L, there is an appropriate proof of $(M^{\alpha} : \alpha)$ in TAN_L. If D is a proof of $(M^{\alpha} : \alpha)$ in TAN_L, and D' is the proof in N_L obtained from D by deleting all typed λ-terms, then M^{α} in λ_L is said to be *assignable* to D'. We then have the following fact, because the reductions of N_L just correspond to the reductions of λ_L.

(∗) Let D be a proof of a formula α in N_L, and M^{α} be assignable to D. If $D \rhd D'$, then there is an assignable typed λ-term N^{β} in λ_L to D' such that $M^{\alpha} \rhd N^{\beta}$.

Let D_0 be a proof of N_L. Suppose that there is an infinite reduction sequence starting from D_0 with assignable $M_0^{\alpha_0}$ such that $D_0 \rhd D_1 \rhd D_2 \rhd \cdots$. Then, by the statement (∗), there are typed λ-terms $M_1^{\alpha_1}, M_2^{\alpha_2}, ...$ assignable respectively to $D_1, D_2, ...$ such that $M_0^{\alpha_0} \rhd M_1^{\alpha_1} \rhd M_2^{\alpha_2} \cdots$ is an infinite reduction sequence. This contradicts Theorem 4.8. □

THEOREM 4.10 (Church-Rosser property). *Let $\rhd$ be the reduction relation on the set of terms (or proofs) in λ_L (or N_L, respectively). Let $\rhd_\beta$ be the*

[8]The precise definition of TAN_L can be given by combining the way of defining N_L and λ_L, but such a way is just a repetition of the same thing.

reflexive transitive closure of $\rhd$*. For any terms (or proofs)* M, M_1 *and* M_2*, if* $M \rhd_\beta M_1$ *and* $M \rhd_\beta M_2$*, then there is a term (or proof)* N *such that* $M_1 \rhd_\beta N$ *and* $M_2 \rhd_\beta N$*.*

Proof. We only consider the case for N_L. By Newman's Lemma, it suffices to show the following weak Church-Rosser property, since we have already established the strong normalization theorem (Theorem 4.9).

> (Weak Church-Rosser property): For any proofs M, M_1 and M_2, if $M \rhd M_1$ and $M \rhd M_2$, then there is a proof N such that $M_1 \rhd_\beta N$ and $M_2 \rhd_\beta N$.

This can straightforwardly be proved. □

4.2. λ_C and strong normalization

First, a typed λ-calculus with strong negation type, denoted λ_C, is introduced, and next the strong normalization theorem for λ_C is proved. Terms for λ_C are obtained from that for λ_L by adding eight new functions $\iota_1, \iota_2, \iota_3, \iota_4$, $\iota_1^{-1}, \iota_2^{-1}, \iota_3^{-1}$ and ι_4^{-1}. The functions ι_j and ι_j^{-1} are analogues of the pairing function $\langle , \rangle$ and projection functions π_1, π_2 for the conjunction type, respectively, which appear in a usual typed λ-calculus with the conjunction type. For example, the intended meaning of ι_1 and ι_1^{-1} can be presented as the equations: $(\iota_1^{-1}(\iota_1 M^\alpha)^{\sim\sim\alpha})^\alpha = M^\alpha$ and $(\iota_1(\iota_1^{-1} M^{\sim\sim\alpha})^\alpha)^{\sim\sim\alpha} = M^{\sim\sim\alpha}$, which are analogues of the equations: $(\pi_1\langle M^\alpha, N^\beta\rangle^{\alpha\wedge\beta})^\alpha = M^\alpha$ and $\langle(\pi_1 M^{\alpha\wedge\beta})^\alpha, (\pi_2 M^{\alpha\wedge\beta})^\beta\rangle^{\alpha\wedge\beta} = M^{\alpha\wedge\beta}$. Types for λ_C are obtained from that for λ_L by adding the strong negation symbol $\sim$. The notion of substitution, denoted as $[N^\alpha/x^\alpha]M^\beta$, is extended for λ_C. For example, we have $[N^\alpha/x^\alpha](\iota_1 M^\beta)^{\sim\sim\beta} = (\iota_1[N^\alpha/x^\alpha]M^\beta)^{\sim\sim\beta}$.

Definition 4.11 (Typed λ-term for λ_C). Let a pair $\langle\delta, \delta'\rangle$ of types be $\langle\alpha, {\sim\sim}\alpha\rangle$, $\langle{\sim}(\beta/\alpha), \alpha * {\sim}\beta\rangle$, $\langle{\sim}(\alpha\backslash\beta), {\sim}\beta * \alpha\rangle$ or $\langle{\sim}(\alpha * \beta), {\sim}\alpha * {\sim}\beta\rangle$. Let j be 1, 2, 3 or 4. Typed λ-terms for λ_C are obtained from Definition 4.2 by adding the following conditions.

9. if M^δ is a typed λ-term, then $(\iota_j M^\delta)^{\delta'}$ is a typed λ-term.
10. if $M^{\delta'}$ is a typed λ-term, then $(\iota_j^{-1} M^{\delta'})^\delta$ is a typed λ-term.

Definition 4.12 (λ_C). Let a pair $\langle \delta, \delta' \rangle$ of types be $\langle \alpha, {\sim}{\sim}\alpha \rangle$, $\langle {\sim}(\beta/\alpha), \alpha * {\sim}\beta \rangle$, $\langle {\sim}(\alpha \backslash \beta), {\sim}\beta * \alpha \rangle$ or $\langle {\sim}(\alpha * \beta), {\sim}\alpha * {\sim}\beta \rangle$. The reduction relation $\rhd$ with respect to λ_C is obtained from Definition 4.3 by adding the following conditions.

6. $(\iota_j^{-1}(\iota_j M^{\delta})^{\delta'})^{\delta} \rhd M^{\delta}$ for any $j \in \{1, 2, 3, 4\}$.

7. if $M \rhd N$, then $\iota_j M \rhd \iota_j N$ and $\iota_j^{-1} M \rhd \iota_j^{-1} N$ for any $j \in \{1, 2, 3, 4\}$.

It can be observed that the set of typed λ-terms for λ_C is closed under $\rhd$.

It is remarked that the Curry-Howard correspondence with respect to λ_C and N_C can naturally be obtained. We can thus introduce a type-assignment system TAN_C with respect to λ_C and N_C in an appropriate way. For example, the following rules are used for TAN_C:

$$\frac{M^{\alpha} : \alpha}{(\iota_1 M^{\alpha})^{{\sim}{\sim}\alpha} : {\sim}{\sim}\alpha} \qquad \frac{M^{{\sim}{\sim}\alpha} : {\sim}{\sim}\alpha}{(\iota_1^{-1} M^{{\sim}{\sim}\alpha})^{\alpha} : \alpha}$$

and the following reduction condition can also be presented:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, D \\ M^{\alpha} : \alpha\end{array}}{(\iota_1 M^{\alpha})^{{\sim}{\sim}\alpha} : {\sim}{\sim}\alpha}}{(\iota_1^{-1}(\iota_1 M^{\alpha})^{{\sim}{\sim}\alpha})^{\alpha} : \alpha} \quad \rhd \quad \begin{array}{c}\vdots\, D \\ M^{\alpha} : \alpha.\end{array}$$

In order to prove the strong normalization theorem for λ_C, the function f defined in Definition 4.5 is modified as follows.

Definition 4.13. A function k from the set of all untyped λ-terms to the set of natural numbers is obtained from the same conditions as 1–4 for f in Definition 4.5 by adding the following condition.

5. $k(\iota_j M) = k(\iota_j^{-1} M) = k(M) + 1$ for any $j \in \{1, 2, 3, 4\}$.

The following lemma, which is the same as Lemma 4.6, holds for k.

Lemma 4.14. *Let* $(\lambda^r x.M)N$, $N(\lambda^l x.M)$ *be typed* λ*-terms for* λ_C.

(1) $k((\lambda^r x.M)N) > k([N/x]M)$.

(2) $k(N(\lambda^l x.M)) > k([N/x]M)$.

PROOF. We only prove (1) of this lemma by induction on M. (2) can be proved similarly. The proof of the subpart which is just related to λ_L is the same as those in Lemma 4.6. Thus, it is enough to prove the cases for the ι_j, ι_j^{-1}-functions. We only show the case for the ι_j-functions. The case for ι_j^{-1}-functions can be proved similarly. Let $n = k(N)$ and $p = k(P)$.

(Case $M \equiv \iota_j P$ for $j \in \{1, 2, 3, 4\}$): The left hand side of the inequality is $L = k((\lambda^r x.\iota_j P)N) = k(\lambda^r x.\iota_j P) + k(N) = k(\iota_j P) + 1 + k(N) = k(P) + 1 + 1 + k(N) = p + n + 2$. The right hand side of the inequality is $R = k([N/x]\iota_j P)$ $= k(\iota_j [N/x]P) = k([N/x]P) + 1$. By the hypothesis of induction, we have $k([N/x]P) < k((\lambda^r x.P)N) = k(P) + 1 + f(N) = p + n + 1$. Therefore $L > R$. □

LEMMA 4.15. *Let M be a typed λ-term for λ_C. Then, $M \rhd N$ implies $k(M)$ $> k(N)$.*

PROOF. By induction on M. By using Lemma 4.14, the λ_L-cases are proved by the same way as in Lemma 4.7. Thus, it is enough to show $k(\iota_j^{-1}\iota_j M) >$ $k(M)$. This is obvious by the definition of k. □

Using this lemma, we obtain the following theorem.

THEOREM 4.16 (Strong normalization for λ_C). *All typed λ-terms for λ_C are strongly normalizable.*

We also obtain the following theorems.

THEOREM 4.17 (Strong normalization for N_C). *All proofs for N_C are strongly normalizable.*

THEOREM 4.18 (Church-Rosser property). *Let $\rhd$ be the reduction relation on the set of terms (or proofs) in λ_C (or N_C, respectively). Let $\rhd_\beta$ be the reflexive transitive closure of $\rhd$. For any terms (or proofs) M, M_1 and M_2, if $M \rhd_\beta M_1$ and $M \rhd_\beta M_2$, then there is a term (or proof) N such that $M_1 \rhd_\beta N$ and $M_2 \rhd_\beta N$.*

4.3. λ_C^2 and strong normalization

Next, we introduce a typed λ-calculus, denoted λ_C^2. In λ_C^2, the functions ι_1 and ι_1^{-1} are also used, but the other functions ι_j, ι_j^{-1} for $j \in \{2, 3, 4\}$ are needless. The Curry-Howard correspondence with respect to λ_C^2 and N_C^2 can also be obtained. Since the proof of the strong normalization theorems for λ_C^2 and N_C^2 is similar to that of λ_C and N_C, the proof will be omitted.

DEFINITION 4.19 (Typed λ-term for λ^2_C). Typed λ-termes for λ^2_C are obtained from Definition 4.2 by adding the following conditions.

9. if M^{α} is a typed λ-term, then $(\iota_1 M^{\alpha})^{\sim\sim\alpha}$ is a typed λ-term.
10. if $M^{\sim\sim\alpha}$ is a typed λ-term, then $(\iota_1^{-1} M^{\sim\sim\alpha})^{\alpha}$ is a typed λ-term.
11. if M^{α} and $N^{\sim\beta}$ are typed λ-terms, $FV(M^{\alpha}) \leq FV(N^{\sim\beta})$ and $FV(M^{\alpha}) \cap FV(N^{\sim\beta}) = \emptyset$, then $[M^{\alpha}, N^{\sim\beta}]^{\sim(\beta/\alpha)}$ is a typed λ-term.
12. if $M^{(\gamma/\sim\beta)/\alpha}$ and $N^{\sim(\beta/\alpha)}$ are typed λ-terms, $FV(M^{(\gamma/\sim\beta)/\alpha}) \leq FV(N^{\sim(\beta/\alpha)})$ and $FV(M^{(\gamma/\sim\beta)/\alpha}) \cap FV(N^{\sim(\beta/\alpha)}) = \emptyset$, then $(M^{(\gamma/\sim\beta)/\alpha} \circ N^{\sim(\beta/\alpha)})^{\gamma}$ is a typed λ-term.
13. if $M^{\sim(\beta/\alpha)}$ and $N^{\sim\beta\backslash(\alpha\backslash\gamma)}$ are typed λ-terms, $FV(M^{\sim(\beta/\alpha)}) \leq FV(N^{\sim\beta\backslash(\alpha\backslash\gamma)})$ and $FV(M^{\sim(\beta/\alpha)}) \cap FV(N^{\sim\beta\backslash(\alpha\backslash\gamma)}) = \emptyset$, then $(M^{\sim(\beta/\alpha)} \circ N^{\sim\beta\backslash(\alpha\backslash\gamma)})^{\gamma}$ is a typed λ-term.
14. if $M^{\sim\beta}$ and N^{α} are typed λ-terms, $FV(M^{\sim\beta}) \leq FV(N^{\alpha})$ and $FV(M^{\sim\beta}) \cap FV(N^{\alpha}) = \emptyset$, then $[M^{\sim\beta}, N^{\alpha}]^{\sim(\alpha\backslash\beta)}$ is a typed λ-term.
15. if $M^{(\gamma/\alpha)/\sim\beta}$ and $N^{\sim(\alpha\backslash\beta)}$ are typed λ-terms, $FV(M^{(\gamma/\alpha)/\sim\beta}) \leq FV(N^{\sim(\alpha\backslash\beta)})$ and $FV(M^{(\gamma/\alpha)/\sim\beta}) \cap FV(N^{\sim(\alpha\backslash\beta)}) = \emptyset$, then $(M^{(\gamma/\alpha)/\sim\beta} \circ N^{\sim(\alpha\backslash\beta)})^{\gamma}$ is a typed λ-term.
16. if $M^{\alpha\backslash(\sim\beta\backslash\gamma)}$ and $N^{\sim(\alpha\backslash\beta)}$ are typed λ-terms, $FV(N^{\sim(\alpha\backslash\beta)}) \leq FV(M^{\alpha\backslash(\sim\beta\backslash\gamma)})$ and $FV(M^{\alpha\backslash(\sim\beta\backslash\gamma)}) \cap FV(N^{\sim(\alpha\backslash\beta)}) = \emptyset$, then $(N^{\sim(\alpha\backslash\beta)} \circ M^{\alpha\backslash(\sim\beta\backslash\gamma)})^{\gamma}$ is a typed λ-term.
17. if $M^{\sim\alpha}$ and $N^{\sim\beta}$ are typed λ-terms, $FV(M^{\sim\alpha}) \leq FV(N^{\sim\beta})$ and $FV(M^{\sim\alpha}) \cap FV(N^{\sim\beta}) = \emptyset$, then $[M^{\sim\alpha}, N^{\sim\beta}]^{\sim(\alpha*\beta)}$ is a typed λ-term.
18. if $M^{(\gamma/\sim\beta)/\sim\alpha}$ and $N^{\sim(\alpha*\beta)}$ are typed λ-terms, $FV(M^{(\gamma/\sim\beta)/\sim\alpha}) \leq FV(N^{\sim(\alpha*\beta)})$ and $FV(M^{(\gamma/\sim\beta)/\sim\alpha}) \cap FV(N^{\sim(\alpha*\beta)}) = \emptyset$, then $(M^{(\gamma/\sim\beta)/\sim\alpha} \circ N^{\sim(\alpha*\beta)})^{\gamma}$ is a typed λ-term.

19. if $M^{\sim\beta\backslash(\sim\alpha\backslash\gamma)}$ and $N^{\sim(\alpha*\beta)}$ are typed λ-terms, $FV(N^{\sim(\alpha*\beta)}) \leq FV(M^{\sim\beta\backslash(\sim\alpha\backslash\gamma)})$ and $FV(M^{\sim\beta\backslash(\sim\alpha\backslash\gamma)}) \cap FV(N^{\sim(\alpha*\beta)}) = \emptyset$, then $(N^{\sim(\alpha*\beta)} \circ M^{\sim\beta\backslash(\sim\alpha\backslash\gamma)})^{\gamma}$ is a typed λ-term.

Definition 4.20 (λ^2_C). The reduction relation $\triangleright$ with respect to λ^2_C is obtained from Definition 4.3 by adding the following conditions.

6. $(\iota_1^{-1}(\iota_1 M^{\alpha})^{\sim\sim\alpha})^{\alpha} \triangleright M^{\alpha}$.
7. $(M^{(\gamma/\sim\beta)/\alpha} \circ [N^{\alpha}, L^{\sim\beta}]^{\sim(\beta/\alpha)})^{\gamma} \triangleright ((M^{(\gamma/\sim\beta)/\alpha} N^{\alpha})^{\gamma/\sim\beta} L^{\sim\beta})^{\gamma}$.
8. $([N^{\alpha}, L^{\sim\beta}]^{\sim(\beta/\alpha)} \circ M^{\sim\beta\backslash(\alpha\backslash\gamma)})^{\gamma} \triangleright (N^{\alpha}(L^{\sim\beta} M^{\sim\beta\backslash(\alpha\backslash\gamma)})^{\alpha\backslash\gamma})^{\gamma}$.
9. $([L^{\sim\beta}, N^{\alpha}]^{\sim(\alpha\backslash\beta)} \circ M^{\alpha\backslash(\sim\beta\backslash\gamma)})^{\gamma} \triangleright (L^{\sim\beta}(N^{\alpha} M^{\alpha\backslash(\sim\beta\backslash\gamma)})^{\sim\beta\backslash\gamma})^{\gamma}$.
10. $(M^{(\gamma/\alpha)/\sim\beta} \circ [L^{\sim\beta}, N^{\alpha}]^{\sim(\alpha\backslash\beta)})^{\gamma} \triangleright ((M^{(\gamma/\alpha)/\sim\beta} L^{\sim\beta})^{\gamma/\alpha} N^{\alpha})^{\gamma}$.
11. $(M^{(\gamma/\sim\beta)/\sim\alpha} \circ [N^{\sim\alpha}, L^{\sim\beta}]^{\sim(\alpha*\beta)})^{\gamma}$
$$\triangleright ((M^{(\gamma/\sim\beta)/\sim\alpha} N^{\sim\alpha})^{\gamma/\sim\beta} L^{\sim\beta})^{\gamma}.$$
12. $([L^{\sim\beta}, N^{\sim\alpha}]^{\sim(\beta*\alpha)} \circ M^{\sim\alpha\backslash(\sim\beta\backslash\gamma)})^{\gamma}$
$$\triangleright (L^{\sim\beta}(N^{\sim\alpha} M^{\sim\alpha\backslash(\sim\beta\backslash\gamma)})^{\sim\beta\backslash\gamma})^{\gamma}.$$
13. if $M \triangleright N$, then $\iota_1 M \triangleright \iota_1 N$ and $\iota_1^{-1} M \triangleright \iota_1^{-1} N$.

Theorem 4.21 (Strong normalization for λ^2_C and N^2_C). *All typed λ-terms for λ^2_C and all proofs for N^2_C are strongly normalizable.*

Theorem 4.22 (Church-Rosser property). *Let $\triangleright$ be the reduction relation on the set of terms (or proofs) in λ^2_C (or N^2_C, respectively). Let $\triangleright_\beta$ be the reflexive transitive closure of $\triangleright$. For any terms (or proofs) M, M_1 and M_2, if $M \triangleright_\beta M_1$ and $M \triangleright_\beta M_2$, then there is a term (or proof) N such that $M_1 \triangleright_\beta N$ and $M_2 \triangleright_\beta N$.*

4.4. Alternative calculi and strong normalization

In the previous subsections, we considered two typed λ-calculi with the ι_j, ι_j^{-1}-functions. These systems, for example, have the following type assignment rules:

$$\frac{M : \alpha}{\iota_1 M : \sim\sim\alpha} \qquad \frac{M : \sim\sim\alpha}{\iota_1^{-1} M : \alpha}.$$

On the other hand, we can consider another forms of the simple type assignment rules:

$$\frac{M : \alpha}{M : \sim\sim\alpha} \qquad \frac{M : \sim\sim\alpha}{M : \alpha}.$$

The latter rules are regarded as analogues of the intersection type rules of certain intersection type assignment systems. A usual intersection type assignment system has the following simple rules for the intersection type $\cap$:

$$\frac{M : \alpha \quad M : \beta}{M : \alpha \cap \beta} \qquad \frac{M : \alpha \cap \beta}{M : \alpha} \qquad \frac{M : \alpha \cap \beta}{M : \beta}.$$

A type assignment system which corresponds to a logic has the following rules for the conjunction type $\wedge$:

$$\frac{M : \alpha \quad N : \beta}{\langle M, N\rangle : \alpha \wedge \beta} \qquad \frac{M : \alpha \wedge \beta}{\pi_1 M : \alpha} \qquad \frac{M : \alpha \wedge \beta}{\pi_2 M : \beta}$$

where $\langle M, N\rangle$ denotes the pairing function and π_i denotes the projection functions.

In the following, we introduce two calculi λ^3_C and λ^4_C which are analogues of the intersection type assignment system. The calculi λ^3_C and λ^4_C are nearly equal to λ_C and λ^2_C, respectively, but the equivalences have not yet been clarified. Types for λ^3_C and λ^4_C are obtained from those of λ_L by adding the strong negation symbol $\sim$.

DEFINITION 4.23 (Typed λ-term for λ^3_C). Typed λ-termes for λ^3_C are obtained from Definition 4.2 by adding the following conditions.

9′. $M^{\sim\sim\alpha}$ is a typed λ-term iff M^{α} is a typed λ-term.

10′. $M^{\sim(\beta/\alpha)}$ is a typed λ-term iff $M^{\alpha*\sim\beta}$ is a typed λ-term.

11′. $M^{\sim(\alpha\backslash\beta)}$ is a typed λ-term iff $M^{\sim\beta*\alpha}$ is a typed λ-term.

12′. $M^{\sim(\alpha*\beta)}$ is a typed λ-term iff $M^{\sim\alpha*\sim\beta}$ is a typed λ-term.

The definition with respect to $\sim$ is from the definition of the λ^c-terms (w.r.t. Nelson's logic N4) posed by Wansing [15].

DEFINITION 4.24 (λ^3_C). Let a pair $\langle\delta, \delta'\rangle$ of types be $\langle\alpha, \sim\sim\alpha\rangle$, $\langle\sim(\beta/\alpha), \alpha * \sim\beta\rangle$, $\langle\sim(\alpha\backslash\beta), \sim\beta * \alpha\rangle$ or $\langle\sim(\alpha * \beta), \sim\alpha * \sim\beta\rangle$. The reduction relation $\rhd$ with respect to λ^3_C is obtained from Definition 4.3 by adding the following conditions.

7′. $M^{\delta} \rhd M^{\delta'}$.

8′. If $M^{\delta} \rhd M^{\delta'}$, then $(\lambda^r x^{\gamma}.M^{\delta})^{\delta/\gamma} \rhd (\lambda^r x^{\gamma}.M^{\delta'})^{\delta'/\gamma}$, $(\lambda^l x^{\gamma}.M^{\delta})^{\gamma\backslash\delta} \rhd (\lambda^l x^{\gamma}.M^{\delta'})^{\gamma\backslash\delta'}$, $[M^{\delta}, L^{\gamma}]^{\delta*\gamma} \rhd [M^{\delta'}, L^{\gamma}]^{\delta'*\gamma}$, $[L^{\gamma}, M^{\delta}]^{\gamma*\delta} \rhd [L^{\gamma}, M^{\delta'}]^{\gamma*\delta'}$.

It can be observed that the set of typed λ-terms for λ^3_C is closed under $\rhd$.

In order to prove the strong normalization theorem for λ^3_C, the function f defined in Definition 4.5 is modified with the addition of the cases for $\sim$. For example, for the case of $M^{\sim(\alpha*\beta)} \rhd M^{\sim\alpha*\sim\beta}$, we have to show $h(M^{\sim(\alpha*\beta)}) > h(M^{\sim\alpha*\sim\beta})$ for an appropriate function h.

Definition 4.25. Let f be the function from the set of all *un*typed λ-terms to the set of natural numbers which is defined in Definition 4.5. A function g from the set of all types to the set of natural numbers is inductively defined by

1. if p is atomic type, then $g(p) = g(\sim p) = 1$,
2. $g(\beta/\alpha) = g(\alpha\backslash\beta) = g(\alpha * \beta) = g(\alpha) + g(\beta)$,
3. $g(\sim\sim\alpha) = g(\sim\alpha) + 1$,
4. $g(\sim(\beta/\alpha)) = g(\sim(\alpha\backslash\beta)) = g(\alpha) + g(\sim\beta) + 1$,
5. $g(\sim(\alpha * \beta)) = g(\sim\alpha) + g(\sim\beta) + 1$.

A function from the set of all typed λ-terms to the set of natural numbers is defined by $h(M^{\alpha}) = f(M) + g(\alpha)$.

The lemma similar to Lemma 4.6 holds for f, and using this lemma, we can prove the following lemma.

Lemma 4.26. *Let M^{α} be a typed λ-term for λ^3_C. Then, $M^{\alpha} \rhd N^{\beta}$ implies $h(M^{\alpha}) > h(N^{\beta})$.*

Proof. By induction on M^{α}. We show some cases.

(Case $M^{\alpha} \equiv ((\lambda x^{\gamma}.P^{\alpha})^{\alpha/\gamma} Q^{\gamma})^{\alpha}$, $N^{\beta} \equiv N^{\alpha} \equiv [Q^{\gamma}/x^{\gamma}]P^{\alpha}$): By the lemma similar to Lemma 4.6, we obtain ($*$): $f(M) > f(N)$. Then, we can obtain $h(M^{\alpha}) = f(M) + g(\alpha) > f(N) + g(\alpha) = h(N^{\alpha})$ by ($*$).

(Case $M^{\alpha} \equiv M^{\sim\sim\gamma}$ and $N^{\beta} \equiv M^{\gamma}$): We have $h(M^{\sim\sim\gamma}) = f(M) + g(\sim\sim\gamma) = f(M) + g(\sim\gamma) + 1$, and $h(M^{\gamma}) = f(M) + g(\gamma)$. Here we have

$g({\sim}\gamma) \geq g(\gamma)$, because for the case that γ is atomic, $g({\sim}\gamma) = g(\gamma) = 1$, otherwise $g({\sim}\gamma) \geq g(\gamma) + 1$. Therefore $h(M^{\sim\sim\gamma}) > h(M^{\gamma})$.

(Case $M^{\alpha} \equiv M^{\sim(\delta/\gamma)}$ and $N^{\beta} \equiv M^{\gamma * \sim\delta}$): We have $h(M^{\sim(\delta/\gamma)}) = f(M) + g({\sim}(\delta/\gamma)) = f(M) + g(\gamma) + g({\sim}\delta) + 1$, and $h(M^{\gamma * \sim\delta}) = f(M) + g(\gamma) + g({\sim}\delta)$. Therefore $h(M^{\sim(\delta/\gamma)}) > h(M^{\gamma * \sim\delta})$.

(Case $M^{\alpha} \equiv (\lambda^{r} x^{\gamma}.P^{\sim\sim\delta})^{\sim\sim\delta/\gamma}$ and $N^{\beta} \equiv (\lambda^{r} x^{\gamma}.P^{\delta})^{\delta/\gamma}$): We have $h((\lambda^{r} x^{\gamma}.\, P^{\sim\sim\delta})^{\sim\sim\delta/\gamma}) = f(\lambda^{r} x.P) + g({\sim\sim}\delta/\gamma) = f(\lambda^{r} x.P) + g({\sim\sim}\delta) + g(\gamma) = f(\lambda^{r} x.P) + g({\sim}\delta) + 1 + g(\gamma)$, and $h((\lambda^{r} x^{\gamma}.P^{\delta})^{\delta/\gamma}) = f(\lambda^{r} x.P) + g(\delta/\gamma) = f(\lambda^{r} x.P) + g(\delta) + g(\gamma)$. Since $g({\sim}\delta) \geq g(\delta)$, we have the required fact. $\square$

Using this lemma, we obtain the following theorem.

THEOREM 4.27 (Strong normalization for λ^{3}_{C}). *All typed λ-terms for λ^{3}_{C} are strongly normalizable.*

THEOREM 4.28 (Church-Rosser property for λ^{3}_{C}). *Let $\rhd$ be the reduction relation on the set of terms for λ^{3}_{C}. Let $\rhd_{\beta}$ be the reflexive transitive closure of $\rhd$. For any terms M, M_1 and M_2, if $M \rhd_{\beta} M_1$ and $M \rhd_{\beta} M_2$, then there is a term N such that $M_1 \rhd_{\beta} N$ and $M_2 \rhd_{\beta} N$.*

Next, we introduce the typed λ-calculus λ^{4}_{C}. Since the proof of the strong normalization theorem for λ^{4}_{C} is similar to that for λ^{3}_{C}, the proof is omitted.

DEFINITION 4.29 (Typed λ-term for λ^{4}_{C}). Typed λ-terms for λ^{4}_{C} are obtained from Definition 4.19 by replacing the conditions 9 and 10 by the following condition.

9″. $M^{\sim\sim\alpha}$ is a typed λ-term iff M^{α} is a typed λ-term.

DEFINITION 4.30 (λ^{4}_{C}). The reduction relation $\rhd$ with respect to λ^{4}_{C} is obtained from Definition 4.20 by replacing the conditions 6 and 13 by the following condition 6″, and moreover add the condition 7′ in Definition 4.24 for λ^{3}_{C}.

6″. $M^{\sim\sim\alpha} \rhd M^{\alpha}$.

THEOREM 4.31 (Strong normalization for λ^{4}_{C}). *All typed λ-terms for λ^{4}_{C} are strongly normalizable.*

THEOREM 4.32 (Church-Rosser property for λ^{4}_{C}). *Let $\rhd$ be the reduction relation on the set of terms in λ^{4}_{C}. Let $\rhd_{\beta}$ be the reflexive transitive closure of $\rhd$. For any terms M, M_1 and M_2, if $M \rhd_{\beta} M_1$ and $M \rhd_{\beta} M_2$, then there is a term N such that $M_1 \rhd_{\beta} N$ and $M_2 \rhd_{\beta} N$.*

5. Natural deduction systems for COSPL

5.1. N-COSPL and normalization

We introduce a natural deduction system N-COSPL in sequent calculus style, following the framework by Negri [6] for intuitionistic linear logic. The notations used in Section 2 are also adopted in this section. In N-COSPL, a derivability relation $\vdash$ is used instead of the vertical expression in a usual natural deduction system. Although the forms of inference rules are different from those of N_C, the same names such as $(/E)$ and $(*E)$ are used in N-COSPL.

Definition 5.1 (N-COSPL). The axioms of N-COSPL are of the form:

$$\alpha \vdash \alpha \qquad \vdash \mathbf{1} \qquad \Gamma \vdash \top \qquad \Gamma \vdash {\sim}\bot.$$

The inference rules of N-COSPL are of the form:

$$\frac{\Pi \vdash \mathbf{1} \quad \Gamma, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ (\mathbf{1}E) \qquad \frac{\Pi \vdash \bot}{\Gamma, \Pi, \Delta \vdash \gamma}\ (\bot E)$$

$$\frac{\Pi \vdash {\sim}\mathbf{1}}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}\mathbf{1}E) \qquad \frac{\Pi \vdash {\sim}\top}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}\top E)$$

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \beta/\alpha}\ (/I) \qquad \frac{\Pi \vdash \beta/\alpha \quad \Gamma \vdash \alpha \quad \Delta, \beta, \Sigma \vdash \gamma}{\Delta, \Pi, \Gamma, \Sigma \vdash \gamma}\ (/E)$$

$$\frac{\alpha, \Gamma \vdash \beta}{\Gamma \vdash \alpha\backslash\beta}\ (\backslash I) \qquad \frac{\Pi \vdash \alpha\backslash\beta \quad \Gamma \vdash \alpha \quad \Delta, \beta, \Sigma \vdash \gamma}{\Delta, \Gamma, \Pi, \Sigma \vdash \gamma}\ (\backslash E)$$

$$\frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha * \beta}\ (*I) \qquad \frac{\Pi \vdash \alpha * \beta \quad \Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ (*E)$$

$$\frac{\Pi \vdash \alpha \wedge \beta \quad \Gamma, \alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ (\wedge E1) \qquad \frac{\Pi \vdash \alpha \wedge \beta \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ (\wedge E2)$$

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}\ (\wedge I) \qquad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta}\ (\vee I1) \qquad \frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta}\ (\vee I2)$$

$$\frac{\Pi \vdash \alpha \vee \beta \quad \Gamma, \alpha, \Delta \vdash \gamma \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ (\vee E)$$

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash {\sim}{\sim}\alpha}\ ({\sim}I) \qquad \frac{\Pi \vdash {\sim}{\sim}\alpha \quad \Gamma, \alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}E)$$

$$\frac{\Gamma \vdash \alpha \quad \Delta \vdash {\sim}\beta}{\Gamma, \Delta \vdash {\sim}(\beta/\alpha)}\ ({\sim}/I) \qquad \frac{\Pi \vdash {\sim}(\beta/\alpha) \quad \Gamma, \alpha, {\sim}\beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}/E)$$

$$\frac{\Gamma \vdash {\sim}\beta \quad \Delta \vdash \alpha}{\Gamma, \Delta \vdash {\sim}(\alpha\backslash\beta)}\ ({\sim}\backslash I) \qquad \frac{\Pi \vdash {\sim}(\alpha\backslash\beta) \quad \Gamma, {\sim}\beta, \alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}\backslash E)$$

$$\frac{\Gamma \vdash {\sim}\alpha \quad \Delta \vdash {\sim}\beta}{\Gamma, \Delta \vdash {\sim}(\alpha * \beta)} \ ({\sim} * I) \qquad \frac{\Pi \vdash {\sim}(\alpha * \beta) \quad \Gamma, {\sim}\alpha, {\sim}\beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma} \ ({\sim} * E)$$

$$\frac{\Pi \vdash {\sim}(\alpha \wedge \beta) \quad \Gamma, {\sim}\alpha, \Delta \vdash \gamma \quad \Gamma, {\sim}\beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma} \ ({\sim} \wedge E)$$

$$\frac{\Gamma \vdash {\sim}\alpha}{\Gamma \vdash {\sim}(\alpha \wedge \beta)} \ ({\sim} \wedge I1) \qquad \frac{\Gamma \vdash {\sim}\beta}{\Gamma \vdash {\sim}(\alpha \wedge \beta)} \ ({\sim} \wedge I2) \qquad \frac{\Gamma \vdash {\sim}\alpha \quad \Gamma \vdash {\sim}\beta}{\Gamma \vdash {\sim}(\alpha \vee \beta)} \ ({\sim} \vee I)$$

$$\frac{\Pi \vdash {\sim}(\alpha \vee \beta) \quad \Gamma, {\sim}\alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma} \ ({\sim} \vee E1) \qquad \frac{\Pi \vdash {\sim}(\alpha \vee \beta) \quad \Gamma, {\sim}\beta, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma} \ ({\sim} \vee E2).$$

The inference rules $(\mathbf{1}E)$, $(\bot E)$, $({\sim}\mathbf{1}E)$, $({\sim}\top E)$, $(/E)$, $(\backslash E)$, $(*E)$, $(\wedge E1)$, $(\wedge E2)$, $(\vee E)$, $({\sim}E)$, $({\sim}/E)$, $({\sim}\backslash E)$, $({\sim} * E)$, $({\sim} \wedge E)$, $({\sim} \vee E1)$ and $({\sim} \vee E2)$ are called *elimination rules*, and the other inference rules are called *introduction rules*. These elimination rules presented here are called in [6] *general elimination rules*. In elimination rules, the premise containing the logical connective or constant is called *major premise*. The other premises are called *minor premises*. It is remarked that the rule of the form:

$$\frac{\Gamma \vdash \alpha \quad \Delta, \alpha, \Sigma \vdash \gamma}{\Delta, \Gamma, \Sigma \vdash \gamma} \ (\text{subst})$$

is admissible in N-COSPL.

DEFINITION 5.2. A proof in N-COSPL is in *general normal form* if all major premises of the elimination rules in the proof are assumptions.

In order to distinguish it from the usual notion of (weak) normalization with respect to reduction, the term "general normal form" in this definition is used.

THEOREM 5.3 (Equivalence between N-COSPL and COSPL). (1) *If* $\Gamma \Rightarrow \gamma$ *is provable in cut-free* COSPL*, then there is a general normal proof of* $\Gamma \vdash \gamma$ *in* N-COSPL*.* (2) *If there is a proof of* $\Gamma \vdash \gamma$ *in* N-COSPL*, then* $\Gamma \Rightarrow \gamma$ *is provable in* COSPL*.*

PROOF. We prove (1) by induction on the cut-free proof P of $\Gamma \Rightarrow \gamma$ in COSPL. We distinguish the cases according to the last inference of P. We show some cases.

Case (/left): P is of the form:

$$\frac{\begin{matrix} \vdots \ P_1 \\ \Gamma \Rightarrow \alpha \end{matrix} \quad \begin{matrix} \vdots \ P_2 \\ \Delta, \beta, \Sigma \Rightarrow \gamma \end{matrix}}{\Delta, \beta/\alpha, \Gamma, \Sigma \Rightarrow \gamma} \ (/\text{left}).$$

By the hypothesis of induction, there are general normal proofs P_1' and P_2' of $\Gamma \vdash \alpha$ and $\Delta, \beta, \Sigma \vdash \gamma$, respectively, in N-COSPL, and hence a required proof is

$$\dfrac{\beta/\alpha \vdash \beta/\alpha \quad \begin{array}{c}\vdots\ P_1' \\ \Gamma \vdash \alpha\end{array} \quad \begin{array}{c}\vdots\ P_2' \\ \Delta, \beta, \Sigma \vdash \gamma\end{array}}{\Delta, \beta/\alpha, \Gamma, \Sigma \vdash \gamma}\ (/E).$$

Case ($\sim$/left): P is of the form:

$$\dfrac{\begin{array}{c}\vdots\ P_1 \\ \Gamma, \alpha, {\sim}\beta, \Delta \Rightarrow \gamma\end{array}}{\Gamma, {\sim}(\beta/\alpha), \Delta \Rightarrow \gamma}\ ({\sim}/\text{left}).$$

By the hypothesis of induction, there is a general normal proof P_1' of $\Gamma, \alpha, {\sim}\beta, \Delta \vdash \gamma$ in N-COSPL, and hence a required proof is

$$\dfrac{{\sim}(\beta/\alpha) \vdash {\sim}(\beta/\alpha) \quad \begin{array}{c}\vdots\ P_1' \\ \Gamma, \alpha, {\sim}\beta, \Delta \vdash \gamma\end{array}}{\Gamma, {\sim}(\beta/\alpha), \Delta \vdash \gamma}\ ({\sim}/E).$$

Next, we prove (2) by induction on a proof Q of $\Gamma \vdash \gamma$ in N-COSPL. We distinguish the cases according to the last inference of Q. We show some cases.

Case ($/E$): Q is of the form:

$$\dfrac{\begin{array}{c}\vdots\ Q_1 \\ \Pi \vdash \beta/\alpha\end{array} \quad \begin{array}{c}\vdots\ Q_2 \\ \Gamma \vdash \alpha\end{array} \quad \begin{array}{c}\vdots\ Q_3 \\ \Delta, \beta, \Sigma \vdash \gamma\end{array}}{\Delta, \Pi, \Gamma, \Sigma \vdash \gamma}\ (/E).$$

By the hypothesis of induction, the sequents $(\Pi \Rightarrow \beta/\alpha)$, $(\Gamma \Rightarrow \alpha)$ and $(\Delta, \beta, \Sigma \Rightarrow \gamma)$ are provable in COSPL, and hence a required proof is

$$\dfrac{\begin{array}{c}\vdots\ Q_1' \\ \Pi \Rightarrow \beta/\alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\ Q_2' \\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\ Q_3' \\ \Delta, \beta, \Sigma \Rightarrow \gamma\end{array}}{\Delta, \beta/\alpha, \Gamma, \Sigma \Rightarrow \gamma}\ (/\text{left})}{\Delta, \Pi, \Gamma, \Sigma \Rightarrow \gamma}\ (\text{cut}).$$

Case (${\sim}/E$): Q is of the form:

$$\dfrac{\begin{array}{c}\vdots\ Q_1 \\ \Pi \vdash {\sim}(\beta/\alpha)\end{array} \quad \begin{array}{c}\vdots\ Q_2 \\ \Gamma, \alpha, {\sim}\beta, \Delta \vdash \gamma\end{array}}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}/E).$$

By the hypothesis of induction, the sequents $\Pi \Rightarrow {\sim}(\beta/\alpha)$ and $\Gamma, \alpha, {\sim}\beta, \Delta \Rightarrow \gamma$ are provable in COSPL, and hence a required proof is

$$\dfrac{\dfrac{\vdots\, Q'_1}{\Pi \Rightarrow {\sim}(\beta/\alpha)} \quad \dfrac{\dfrac{\vdots\, Q'_2}{\Gamma, \alpha, {\sim}\beta, \Delta \Rightarrow \gamma}}{\Gamma, {\sim}(\beta/\alpha), \Delta \Rightarrow \gamma}\,({\sim}/\text{left})}{\Gamma, \Pi, \Delta \Rightarrow \gamma}\,(\text{cut}).$$

□

THEOREM 5.4 (Normalization for N-COSPL). *Every proof P of $\Gamma \vdash \gamma$ in* N-COSPL *can be transformed into a general normal proof P' of $\Gamma \vdash \gamma$ in* N-COSPL.

PROOF. Let P be a proof of $\Gamma \vdash \gamma$ in N-COSPL. Then, the sequent $\Gamma \Rightarrow \gamma$ is provable in COSPL by Theorem 5.3 (2), and hence $\Gamma \Rightarrow \gamma$ is provable in cut-free COSPL by Theorem 2.2. By Theorem 5.3 (1), there is a general normal proof P' of $\Gamma \vdash \gamma$ in N-COSPL. □

5.2. U-COSPL and normalization

We introduce a *uniform calculus* U-COSPL for COSPL, following the framework by Negri [7]. U-COSPL has not only general elimination rules, but also *general introduction rules*.

DEFINITION 5.5 (U-COSPL). U-COSPL is obtained from N-COSPL by replacing the introduction rules and axioms ($\vdash \mathbf{1}$), ($\Gamma \vdash \top$), ($\Gamma \vdash {\sim}\bot$) by the general introduction rules of the form:

$$\frac{\Gamma, \mathbf{1}, \Delta \vdash \gamma}{\Gamma, \Delta \vdash \gamma}\,(\mathbf{1}I) \qquad \frac{\Gamma, \top, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\,(\top I) \qquad \frac{\Gamma, {\sim}\bot, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\,({\sim}\bot I)$$

$$\frac{\Gamma, \beta/\alpha, \Delta \vdash \gamma \quad \Pi, \alpha \vdash \beta}{\Gamma, \Pi, \Delta \vdash \gamma}\,(/I^u) \qquad \frac{\Gamma, \alpha\backslash\beta, \Delta \vdash \gamma \quad \alpha, \Pi \vdash \beta}{\Gamma, \Pi, \Delta \vdash \gamma}\,(\backslash I^u)$$

$$\frac{\Gamma, \alpha * \beta, \Delta \vdash \gamma \quad \Pi \vdash \alpha \quad \Sigma \vdash \beta}{\Gamma, \Pi, \Sigma, \Delta \vdash \gamma}\,(*I^u) \qquad \frac{\Gamma, \alpha \wedge \beta, \Delta \vdash \gamma \quad \Pi \vdash \alpha \quad \Pi \vdash \beta}{\Gamma, \Pi, \Delta \vdash \gamma}\,(\wedge I^u)$$

$$\frac{\Gamma, \alpha \vee \beta, \Delta \vdash \gamma \quad \Pi \vdash \alpha}{\Gamma, \Pi, \Delta \vdash \gamma}\,(\vee I1^u) \qquad \frac{\Gamma, \alpha \vee \beta, \Delta \vdash \gamma \quad \Pi \vdash \beta}{\Gamma, \Pi, \Delta \vdash \gamma}\,(\vee I2^u)$$

$$\frac{\Gamma, {\sim}{\sim}\alpha, \Delta \vdash \gamma \quad \Pi \vdash \alpha}{\Gamma, \Pi, \Delta \vdash \gamma}\,({\sim}I^u)$$

$$\frac{\Gamma, {\sim}(\beta/\alpha), \Delta \vdash \gamma \quad \Pi \vdash \alpha \quad \Sigma \vdash {\sim}\beta}{\Gamma, \Pi, \Sigma, \Delta \vdash \gamma}\,({\sim}/I^u)$$

$$\frac{\Gamma, {\sim}(\alpha \backslash \beta), \Delta \vdash \gamma \quad \Pi \vdash {\sim}\beta \quad \Sigma \vdash \alpha}{\Gamma, \Pi, \Sigma, \Delta \vdash \gamma}\ ({\sim}\backslash I^u)$$

$$\frac{\Gamma, {\sim}(\alpha * \beta), \Delta \vdash \gamma \quad \Pi \vdash {\sim}\alpha \quad \Sigma \vdash {\sim}\beta}{\Gamma, \Pi, \Sigma, \Delta \vdash \gamma}\ ({\sim} * I^u)$$

$$\frac{\Gamma, {\sim}(\alpha \wedge \beta), \Delta \vdash \gamma \quad \Pi \vdash {\sim}\alpha}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}\wedge I1^u) \quad \frac{\Gamma, {\sim}(\alpha \wedge \beta), \Delta \vdash \gamma \quad \Pi \vdash {\sim}\beta}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}\wedge I2^u)$$

$$\frac{\Gamma, {\sim}(\alpha \vee \beta), \Delta \vdash \gamma \quad \Pi \vdash {\sim}\alpha \quad \Pi \vdash {\sim}\beta}{\Gamma, \Pi, \Delta \vdash \gamma}\ ({\sim}\vee I^u).$$

It is remarked that the rule (subst) is also admissible in U-COSPL. In the inference rules of U-COSPL, the premise containing the logical connective or constant is called *major premise*.

We must modify the notion of general normal form as follows.

Definition 5.6. A proof in U-COSPL is in *general normal form* if all major premises of all the inference rules in the proof are assumptions.

Theorem 5.7 (Equivalence between U-COSPL and COSPL). (1) *If* $\Gamma \Rightarrow \gamma$ *is provable in cut-free* COSPL*, then there is a general normal proof of* $\Gamma \vdash \gamma$ *in* U-COSPL. (2) *If there is a proof of* $\Gamma \vdash \gamma$ *in* U-COSPL*, then* $\Gamma \Rightarrow \gamma$ *is provable in* COSPL.

Theorem 5.8 (Normalization for U-COSPL). *Every proof* P *of* $\Gamma \vdash \gamma$ *in* U-COSPL *can be transformed into a general normal proof* P' *of* $\Gamma \vdash \gamma$ *in* U-COSPL.

6. Remarks

6.1. Adding structural rules

As mentioned in Section 3, in the framework of N_L, N_C and N_C^2, we can define the natural deduction systems for any logics over L and C by adding any combinations of the structural rules (ex), (we) and (co). We can also show, in a similar way as in Section 4, the strong normalization theorems for the natural deduction systems for L+(ex), L+(we), L+(ex)+(we), C+(ex), C+(we) and C+(ex)+(we). On the other hand, we cannot prove the strong normalization theorems for the systems for L+(co), L+(we)+(co), C+(co) and C+(we)+(co), because the set of proofs in these systems is not closed under substitution, i.e., the set of proofs is not closed under $\rhd$. Also the strong normalization theorems for the natural deduction systems for the other logics with (co) cannot be proved in the same way as discussed in this paper. But, by using the method of Mouri [5], we can prove the strong normalization theorems for the corresponding natural deduction systems for L+(ex)+(co) and L+(ex)+(co)+(we). In fact, a systematic proof for the $\{/, *, \wedge\}$-fragments of intuitionistic substructural logics was proposed in [5]. The strong normalization theorems for the corresponding natural deduction systems for C+(ex)+(co) and C+(ex)+(co)+(we) have not been proved yet.

In the framework related to N-COSPL and U-COSPL, we can give some normalizing natural deduction systems for other substructural logics. Indeed, the natural deduction systems for intuitionistic and classical full linear logics were introduced by Negri [6, 7]. In the same framework, natural deduction systems for intuitionistic and classical logics were studied by Negri and von Plato [8]. The framework by Negri and von Plato [8] was extended to Nelson's logic N4 by Kamide [4].

6.2. Adding $\wedge$ and $\vee$ to N_C

The $\{/, \backslash, *, \wedge, \vee, \sim\}$-fragment of COSPL is denoted as COSPL$^-$.

A set Γ of assumptions is called *total* if $\forall \alpha^n, \beta^m \in \Gamma$ $[\alpha^n < \beta^m$ or $\beta^m < \alpha^n$ or $\alpha^n = \beta^m]$. Let Γ and Δ be sets of assumptions. A natural deduction system N_C^- for COSPL$^-$ is obtained from N_C by adding the inference rules of the form:

$$\cfrac{\begin{matrix}\Gamma \\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}\Gamma \\ \vdots \\ \beta\end{matrix}}{\alpha \wedge \beta}$$

where Γ is total,

$$\frac{\alpha_1 \wedge \alpha_2}{\alpha_1}\,, \qquad \frac{\alpha_1 \wedge \alpha_2}{\alpha_2}\,, \qquad \frac{\alpha_1}{\alpha_1 \vee \alpha_2}\,, \qquad \frac{\alpha_2}{\alpha_1 \vee \alpha_2}\,,$$

$$\cfrac{\begin{matrix}\Sigma \\ \vdots \\ \alpha_1 \vee \alpha_2\end{matrix} \quad \begin{matrix}\Gamma - \{\alpha_1^n\} \\ \vdots \\ \beta\end{matrix} \quad \begin{matrix}\Delta - \{\alpha_2^n\} \\ \vdots \\ \beta\end{matrix}}{\beta}$$

with the conditions that $\Gamma = \Gamma_1 \cup \{\alpha_1^n\} \cup \Gamma_2$, $\Gamma_1 < \{\alpha_1^n\} < \Gamma_2$, $\Delta = \Gamma_1 \cup \{\alpha_2^n\} \cup \Gamma_2$, $\Gamma_1 < \{\alpha_2^n\} < \Gamma_2$, $\Gamma_1 < \Sigma < \Gamma_2$, and $\Gamma_1 \cup \Sigma \cup \Gamma_2$ is total,

$$\frac{\sim\alpha \vee \sim\beta}{\sim(\alpha \wedge \beta)}\,, \qquad \frac{\sim(\alpha \wedge \beta)}{\sim\alpha \vee \sim\beta}\,, \qquad \frac{\sim\alpha \wedge \sim\beta}{\sim(\alpha \vee \beta)}\,, \qquad \frac{\sim(\alpha \vee \beta)}{\sim\alpha \wedge \sim\beta}\,.$$

We then have the following facts. (1) If P is a proof in N_C^- such that $oa(P) = \{\alpha_1^{n_1}, \cdots, \alpha_k^{n_k}\}$ $(\alpha_1^{n_1} < \cdots < \alpha_k^{n_k})$ $(0 \leq k)$ and $end(P) = \beta$, then the sequent $\alpha_1, ..., \alpha_k \Rightarrow \beta$ is provable in COSPL$^-$. (2) If a sequent $\alpha_1, ..., \alpha_k \Rightarrow \beta$ $(0 \leq k)$ is provable in COSPL$^-$, then there is a proof Q in N_C^- such that $oa(Q) = \{\alpha_1^{n_1}, ..., \alpha_k^{n_k}\}$ $(\alpha_1^{n_1} < \cdots < \alpha_k^{n_k})$ and $end(Q) = \beta$. (3) A formula α is provable in N_C^- if and only if the sequent $\Rightarrow \alpha$ is provable in COSPL$^-$.

We have not yet obtained a (strong) normalization theorem for N_C^- because of the following reasons. In general, it is difficult to obtain a direct proof of (strong) normalization theorem with $\vee$. For example, it is known that the permutation conversion with respect to $\vee$ makes difficult the proof. Moreover, the conditions for labelled assumptions in N_C^- are very complex to define a reduction relation. If we try to prove a weak normalization theorem with respect to N_C^- indirectly using a correspondence between N_C^- and cut-free COSPL$^-$, then we have to modify the words "COSPL$^-$" and "proof Q" in the item (2) in the facts discussed above to "cut-free COSPL$^-$" and "normal proof Q", respectively. But, this cannot be proved easily, because the corresponding "label shift lemma" is not proved for such a formulation.

6.3. Related works

Introducing the exchange conditions in N_L is essentially the same as introducing the sequence-type assumptions instead of the set-type labeled assumptions. If we adopt such sequence-type assumptions, then we have to modify the formulation of the inference rules. For example, $(\backslash E)$ is modified as

$$\dfrac{\begin{matrix}\Delta \\ \vdots \\ \alpha\end{matrix} \qquad \begin{matrix}\Gamma \\ \vdots \\ \alpha\backslash\beta\end{matrix}}{\beta}$$

where (Δ, Γ) is a sequence of assumptions. Also, $(\backslash I)$ is modified as

$$\dfrac{\begin{matrix}[\alpha]\Gamma \\ \vdots \\ \beta\end{matrix}}{\alpha\backslash\beta}$$

where the cancelled assumption $[\alpha]$ was the leftmost uncancelled assumption before applying this rule. Such a formulation using the sequence-type assumptions has already been studied by van Benthem [1] and Tiede [12].

The following inference rules for $*$ were also considered [1, 12]:

$$\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ \alpha & \beta\end{array}}{\alpha * \beta} \qquad\qquad \dfrac{\begin{array}{cc}\Gamma & [\alpha][\beta]\Delta\\ \vdots & \vdots\\ \alpha * \beta & \gamma\end{array}}{\gamma}\ (*\mathrm{Eg})$$

where α and β in $(*\mathrm{Eg})$ have to be the two left-most uncancelled assumptions of the proof of γ.

A natural deduction system for intuitionistic non-commutative linear logic was studied by Polakow and Pfenning [10]. Some systematic treatments of the Curry-Howard correspondences for various substructural logics were established by Gabbay and de Queiroz [2] and by Wansing [14].

7. Conclusion

In this paper, firstly, the natural deduction systems N_L, N_C and N_C^2 for the $\{/, \backslash, *\}$-fragment L of FL (full Lambek logic), the $\{/, \backslash, *, \sim\}$-fragment C of COSPL (constructive sequential propositional logic) and also C, respectively, were introduced. The strong normalization and Church-Rosser theorems for these systems were proved using the corresponding typed λ-calculi λ_L, λ_C and λ_C^2 via Curry-Howard correspondences. Two alternative calculi λ_C^3 and λ_C^4 were also discussed. Secondly, the natural deduction systems N-COSPL and U-COSPL for COSPL were introduced, and the normalization theorems for these systems were proved. The framework for COSPL also works for FL.

The merits of the framework of N_L, N_C and N_C^2 are summarized as follows. (1) The corresponding typed λ-calculi λ_L, λ_C and λ_C^2 can be obtained via the Curry-Howard correspondences. (2) A simple proof of the strong normalization theorems can be given. (3) A systematic treatment of other substructural logics is available by deleting the labelled assumption conditions concerning the structural rules. (4) A natural and intuitive formulation for prioritized (or ordered) human reasoning as discussed in Section 2 can be given. The demerits of the framework of N_L, N_C and N_C^2 are as follows. (1) The subformula property does not hold. (2) Extending the framework both with $\wedge$ and $\vee$ is not easy. To improve such demerits, the systems N-COSPL and U-COSPL are introduced, i.e., the merits of the framework of N-COSPL and U-COSPL are as follows. (1) The full set of connectives can be treated. Indeed, the quantifiers $\forall$ and $\exists$ can also be handled, although such extensions are not discussed in this paper. (2) The subformula property holds for the corresponding systems for FL. On the other hand, the subformula property does not hold for N-COSPL and U-COSPL, e.g. $\sim\alpha$ or $\sim\beta$ is not a subformula of $\sim(\alpha \wedge \beta)$, but such a non-subformula can appear in a normal proof.

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References

[1] van Benthem, J., *Language in action — categories, lambdas and dynamic logic*, Elsevier Science Publishers, 1991.

[2] Gabbay D. M., and Ruy. J. G. B. de Queiroz, "Extending the Curry-Howard interpretation to linear, relevant and other resource logics", *Journal of Symbolic Logic* 57 (4), 1319–1365, 1992.

[3] Hindley, J. R., "BCK-combinators and linear λ-term have types", *Theoretical Computer Science* 64, 97–105, 1986.

[4] Kamide, N., "Natural deduction systems for Nelson's paraconsistent logic and its neighbors", *Journal of Applied Non-Classical Logics* 15 (4), 405–435, 2005.

[5] Mouri, M., *A proof-theoretic study of non-classical logics — natural deduction systems for intuitionistic substructural logics and implemetation of proof assistant system. (in Japanese).* Doctoral dissertation, Japan Advanced Institute of Science and Technology, 2002.

[6] Negri, S., "A normalizing system of natural deduction for intuitionistic linear logic", *Archive for Mathematical Logic* 41, 789–810, 2002.

[7] Negri, S., "Varieties of linear calculi", *Journal of Philosophical Logic* 31, 569–590, 2002.

[8] Negri, S., and J. von Plato, "Sequent calculus in natural deduction style", *Journal of Symbolic Logic* 66 (4), 1803–1816, 2001.

[9] Nelson, D., Constructible falsity. *Journal of Symbolic Logic* 14, 16–26, 1949.

[10] Polakow, J., and F. Pfenning, "Natural deduction for intuitionistic non-commutative linear logic", *Lecture Notes in Computer Science* 1581, 295–309, 1999.

[11] Priest, G., and R. Routley, "Introduction: paraconsistent logics", *Studia Logica* 43, 3–16, 1982.

[12] Tiede, H.-J., "Proof theory and formal grammars — applications of normalization", in T. Raesch et al. (eds.), Foundations of the Formal Sciences II: *Applications of Mathematical Logic in Philosophy and Linguistics*, Kluwer Academic Publishers, 2003.

[13] Wagner, G., "Logic programming with strong negation and inexact predicates", *Journal of Logic and Computation* 1 (6), 835–859, 1991.

[14] Wansing, H., "Formulas-as-types for a hierarchy of sublogics of intuitionistic propositional logic", *Lecture Notes in Artificial Intelligence* 619, 125–145, 1990.

[15] Wansing, H., *The logic of information structures*, Lecture Notes in Artificial Intelligence 681, 1993, 163 pages.

[16] Watari, O., K. Nakatogawa, and T. Ueno, "Normalization theorems for substructural logics in Gentzen-style natural deduction (abstract)", *Bulletin of Symbolic Logic* 6 (3), 390–391, 2000.

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