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CONSTRUCTION OF TABLEAUX FOR CLASSICAL LOGIC: Tableaux as Combinations of Branches, Branches as Chains of Sets

Abstract. The paper is devoted to an approach to analytic tableaux for propositional logic, but can be successfully extended to other logics. The distinguishing features of the presented approach are:

(i) a precise set-theoretical description of tableau method;
(ii) a notion of tableau consequence relation is defined without help of a notion of tableau, in our universe of discourse the basic notion is a branch;
(iii) we define a tableau as a finite set of some chosen branches which is enough to check; hence, in our approach a tableau is only a way of choosing a minimal set of closed branches;
(iv) a choice of tableau can be arbitrary, it means that if one tableau starting with some set of premisses is closed in the defined sense, then every branch in the power set of the set of formulas, that starts with the same set, is closed.

Keywords: Analytic tableaux, propositional logic, set-theoretical approach to a description of tableaux, branches as chains of sets of formulas, tableaux consequence relation, choice of branches, tableau combined with branches.

1. Introduction

Tableau methods are very convenient way of proving theorems and checking correctness of inferences. They are a kind of analytic procedure of proof.

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But their main property is that they are refutation methods. Using tableau methods we begin by negating a conclusion of an inference under research and analyze consequences by researching a tree structure of possibilities. If every possibility provides a contradiction of some kind, we conclude that the initial inference is correct.

The above idea seems to be very intuitive and, as it is easy to observe, it has many beneficial features. The basic one is that thanks to tableau procedures we can usually find out a model or a counter-model for an inference we are examining.

Maybe this advantage is a reason for which in many books the presentation begins and ends only at the intuitive level, without developing formal and precise notions (see for example [4]).

However, tableau methods can be treated as a syntactical counterpart of axiomatic approach or natural deduction and then, as a kind of a formal tool, they need to be presented in a formal and clear way. It means that their use does have to depend neither on intuitions of any sort nor additional assumptions concerning the way we understand how they work in practice. Tableau methods understood as a kind of syntactic operations should be only rules of transformations of expressions of a language we work with.

Although there are many approaches to tableau methods, in this presentation we shall concentrate only on the approach and metatheorems proof style presented in the book of Graham Priest [5], but this book is a kind of inspiration. The main aim is not only to define the precise set-theoretical description of basic notions of this approach and to present how they work for propositional logic. Such descriptions—less or more precise and formal—are available almost since the beginning of tableau approach [6]. We will make some effective and basic modifications.

Here we are working in the domain of power set of a set of all formulas. The start point are precise definitions of rules of inference. Rules are understood as ways of enriching initial sets of formulas. Thanks to them we can give a very precise definition of a branch. Our construction of branches excludes trivial extensions (applying still the same rule to decompose the same formula) and has an internal mechanism of stopping an extension of contradictory sets. Next, we define a notion of logical consequence relation. It is based only on a notion of branch and in fact at this level we do not need a notion of tableau as a kind of tree structure. The defined relation is equal to classical consequence. The proofs are in Henkin's manner. The interesting property of the presented strategy is that its key point lemmas are very easy lemmas 1, 3. Both of them describe connections between rules and semantics. If this framework is generalized to rules for non-classical logics (where we meet an additional problem, the problem of infinite branches), to prove metatheorems (soundness and completeness) it is enough to check those lemmas. The extended framework we present in the paper [1].

The notion of a tableau is required only in a practical perspective. We show that if one wants to check whether some inference is correct, he can choose a set of a relatively small number of branches, here called a tableau. (Hence, a tableau is a combination of branches that are independent objects of considerations. We construct a tableau of branches. Usually, branches are treated as parts of a tree structure [2], [3].) Next, we prove that it is enough to build only one tableau to decide correctness of inference under research. One of the differences is also that we are extending a tableau inference to infinite sets of premisses.

The distinguishing feature of the presented approach is that we shall treat a tableau as a special kind of family of sets ordered by inclusion relation $\subset$. Intuitively, as a start point we take a certain set of formulas and extend it by use of rules of some kind, obtaining a set of chains under inclusion. If any maximal subset contains a contradiction, then the inference under examination is correct; if not, it is incorrect.

2. Bases

Language and grammar. The alphabet of propositional logic (in short: PL) can be in an usual way described as the union of separate sets: logical connectives: $\mathsf{Con} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$, sentential letters: $\mathsf{Sl} = \{p_1, q_1, r_1, p_2, q_2, r_2, \dots\}$, moreover, we need some auxiliary signs: $\{), (\}$. Obviously, we assume also that the set Sl is infinite, but countable. Furthermore, we define a notion of a PL formula:

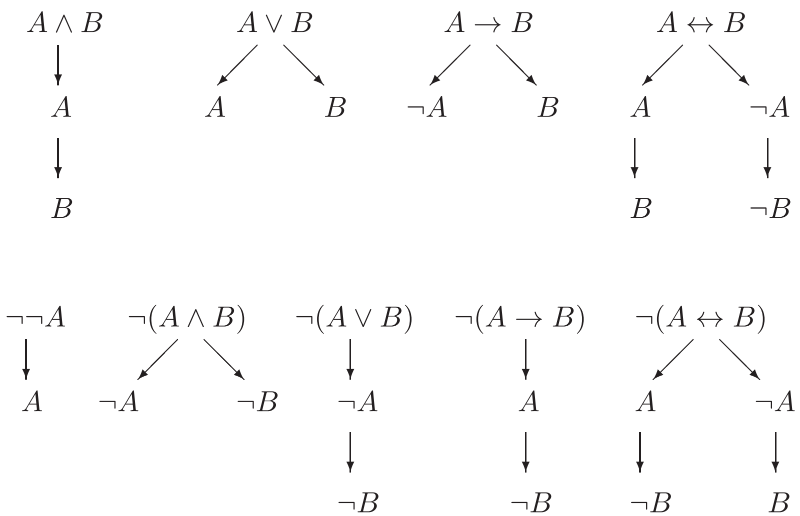
Definition 1. To the set of formulas For belong all and only such expressions that satisfy one of the following conditions:

1. are members of Sl,
2. have one of the forms: $\ulcorner\neg(A)\urcorner$, $\ulcorner(A) \wedge (B)\urcorner$, $\ulcorner(A) \vee (B)\urcorner$, $\ulcorner(A) \rightarrow (B)\urcorner$, $\ulcorner(A) \leftrightarrow (B)\urcorner$, where $A, B \in \mathsf{For}$.

Members of For are called formulas.

Semantics. An interpretation of the sentential letters is any function $V\colon \mathsf{Sl} \longrightarrow \{0, 1\}$. Any V can be extended in a classical way to a Boolean interpretation or valuation of the language $V'\colon \mathsf{For} \longrightarrow \{0, 1\}$. With help of this notion we define in an usual way the notions of classical consequence relation $\models$ and tautology.

Tableau approach to deductive tools. A standard tableau approach to the relation of classical inference is usually presented graphically, by diagrams of the following kinds:



A formula is a tableau consequence of a set of formulas, if we can built a tableau which starts with this set and a negation of the formula, whose all branches end with contradictions. This is the basic and very intuitive idea of tableau inference. The problem is that as a kind of deduction it should depend only on shapes of expressions, not on the intuitions. Simultaneously, in the literature often we do not find any precise definitions of objects like a tableau, a branch, a leaf, a complete branch etc. Consequently, omitting our intuition, we can still ask about them, which provides another questions like: for how long should we apply tableau rules?, when should we stop?, etc. These questions grow and are of great importance, when we are trying to prove metatheorems and are made to apply the tableau notions.

Basic notions. We need some very basic notions. They will be presented in turn.

Definition 2. A set $X \subseteq \mathsf{For}$ is called *contradictory* iff $\exists_{A \in \mathsf{For}}\, A, \neg(A) \in X$. If a set is not contradictory, we call it *non-contradictory*.

Definition 3. A set $X \subseteq \mathsf{For}$ is called *negative* iff $\exists_{A \in \mathsf{For}}\, \neg(A) \in X$.

By a rule we mean any set of ordered pairs or triples defined on the set 2^{For}. The first member of it is called *an initial set*, while the second and third ones are called *result sets*.

Definition 4. Let the initial sets be non-contradictory and negative. A *rule of extending* R_i ($1 \leqslant i \leqslant 9$) has one of the following forms:

$$R_1\colon \frac{X\cup\{(A)\wedge(B)\}}{X\cup\{(A)\wedge(B), A, B\}} \qquad R_2\colon \frac{X\cup\{(A)\vee(B)\}}{X\cup\{(A)\vee(B), A\}, X\cup\{(A)\vee(B), B\}}$$

$$R_3\colon \frac{X\cup\{(A)\to(B)\}}{X\cup\{(A)\to(B), \neg(A)\}, X\cup\{(A)\to(B), B\}}$$

$$R_4\colon \frac{X\cup\{(A)\leftrightarrow(B)\}}{X\cup\{(A)\leftrightarrow(B), A, B\}, X\cup\{(A)\leftrightarrow(B), \neg(A), \neg(B)\}}$$

$$R_5\colon \frac{X\cup\{\neg(\neg(A))\}}{X\cup\{\neg(\neg(A)), A\}}$$

$$R_6\colon \frac{X\cup\{\neg((A)\wedge(B))\}}{X\cup\{\neg((A)\wedge(B)), \neg(A)\}, X\cup\{\neg((A)\wedge(B)), \neg(B)\}}$$

$$R_7\colon \frac{X\cup\{\neg((A)\vee(B))\}}{X\cup\{\neg((A)\vee(B)), \neg(A), \neg(B)\}} \qquad R_8\colon \frac{X\cup\{\neg((A)\to(B))\}}{X\cup\{\neg((A)\to(B)), A, \neg(B)\}}$$

$$R_9\colon \frac{X\cup\{\neg((A)\leftrightarrow(B))\}}{X\cup\{\neg((A)\leftrightarrow(B)), \neg(A), B\}, X\cup\{\neg((A)\leftrightarrow(B)), A, \neg(B)\}}$$

Definition 5. Let X, $Y \subseteq \mathsf{For}$. We say that Y is *an extension of* X iff there is $1 \leqslant i \leqslant 9$ and a set of formulas Z, such that one of the conditions is satisfied:

1. $\langle X, Y\rangle \in R_i$,
2. $\langle X, Y, Z\rangle \in R_i$,
3. $\langle X, Z, Y\rangle \in R_i$.

In the cases 2. and 3. we say that Y, Z are *indirect extensions of* X by the rule R_i ($1 \leqslant i \leqslant 9$).

Definition 6. Let $K \subseteq \mathbb{N}$ be such a subset of natural numbers that: $1 \in K$ and for any $i, j \in K$, if $i < j$, then $i + 1 \in K$. Let X be a negative set. Any function $f\colon K \longrightarrow 2^{\mathsf{For}}$ which satisfies the conditions:

1. $f(1) = X$,
2. for every $i < j \in K$: if $\neg\exists_{k\in K}\ i < k < j$, then $f(j)$ is an extension of $f(i)$.

we call *a sequence of extensions of* X.

We see that any sequence of extensions f is a monotonic function, i.e., for any $i < j$ in a domain of f: $f(i) \subseteq f(j)$. If a context is clear we write for sequences f or $\langle X_1, \ldots, X_n\rangle_{n\in\mathbb{N}}$, where $\{1, \ldots, n\}$ is a domain K of f and $\{X_1, \ldots, X_n\}$ is an image $\{1, \ldots, n\}$ under f, sometimes omitting brackets.

Definition 7 (Maximal branch complexity of formulas). *Function of complexity* we call the function $*\colon \mathsf{For} \longrightarrow \mathbb{N}$, defined for any $x \in \mathsf{SI}$ and for any A, $B \in \mathsf{For}$ by the following conditions:

1. $*(x) = 1$,
2. $*(\neg(x)) = 1$,
3. $*((A) \wedge (B)) = *(A) + *(B)$,
4. $*((A) \vee (B)) = \max\{*(A), *(B)\} + 1$,
5. $*((A) \to (B)) = \max\{*(\neg(A)), *(B)\} + 1$,
6. $*((A) \leftrightarrow (B)) = \max\{*(A) + *(B), *(\neg(A)) + *(\neg(B))\}$,
7. $*(\neg(\neg(A))) = *(A) + 1$,
8. $*(\neg((A) \wedge (B))) = \max\{*(\neg(A)), *(\neg(B))\} + 1$,
9. $*(\neg((A) \vee (B))) = *(\neg(A)) + *(\neg(B))$,
10. $*(\neg((A) \to (B))) = *(A) + *(\neg(B))$,
11. $*(\neg((A) \leftrightarrow (B))) = \max\{*(\neg(A)) + *(B), *(A) + *(\neg(B))\}$.

Fact 1. *Let* $\langle X \cup \{A_1^1\},\ X \cup \{A_1^1, A_2^1, A_2^{k_2}\},\ \ldots,\ X \cup \{A_1^1, A_2^1, A_2^{k_2},\ \ldots,$ $A_n^1, A_n^{k_n}\}\rangle_{n\in\mathbb{N},\,1\leqslant k_2,\ldots,k_n\leqslant 2}$ *be an injective sequence of extensions* f *(i.e., such that* $\neg\exists_{\,i\neq j\,\in K}\ \ f(i) = f(j)$*). Then* $n \leqslant *(A_1^1)$.

Proof. Let a sequence $\langle X\cup\{A_1^1\}, X\cup\{A_1^1, A_2^1, A_2^{k_2}\}, \ldots, X\cup\{A_1^1, A_2^1, A_2^{k_2},$ $\ldots, A_n^1,\ A_n^{k_n}\}\rangle_{n\in\mathbb{N},\,1<k_2,\ldots,k_n\leqslant 2}$ be an injective sequence of extensions f. We see, that since it is an injective sequence, so for any natural numbers $i < j$ $\leqslant n$ and $1 \leqslant k_i, \ldots, k_j \leqslant 2$:

1. $X \cup \{A_1^1,\ \ldots,\ A_i^1, A_i^{k_i}\} \subset X \cup \{A_1,\ \ldots,\ A_i^1, A_i^{k_i}, \ldots, A_j^1, A_j^{k_j}\}$,
2. there is a rule of extending R_m ($1 \leqslant m \leqslant 9$) and a set of formulas Z, such that one of the following conditions is satisfied:

(a) $\langle X \cup \{A_1^1, \ldots, A_i^1, A_i^{k_i}\}, X \cup \{A_1^1, \ldots, A_i^1, A_i^{k_i}, A_{i+1}^1, A_{i+1}^{k_{i+1}}\}\rangle \in R_m$,

(b) $\langle X \cup \{A_1^1, \ldots, A_i^1, A_i^{k_i}\}, X \cup \{A_1^1, \ldots, A_i^1, A_i^{k_i}, A_{i+1}^1, A_{i+1}^{k_{i+1}}\}, Z\rangle \in R_m$,

(c) $\langle X \cup \{A_1^1, \ldots, A_i^1, A_i^{k_i}\}, Z, X \cup \{A_1^1, \ldots, A_i^1, A_i^{k_i}, A_{i+1}^1, A_{i+1}^{k_{i+1}}\}\rangle \in R_m$.

The proof is by induction based on complexity of formulas.

Initial step. *i)* Let $A_1^1 = x$, for some $x \in \mathsf{SI}$. Then there is no rule to be applied and $n = 1$. Since $*(x) = 1$, hence $n \leqslant *(x)$. When $A_1^1 = \neg(x)$, for some $x \in \mathsf{SI}$, this part of the proof is almost identical.

Inductive step. We assume that the hypothesis holds for any such a formula B, that $*(A_1^1) > *(B)$. Hence, by induction hypothesis, if $*(A_1^1) > *(B)$, then for any injective sequence of extensions: $\langle X \cup \{B_1^1\}$, $X \cup \{B_1^1, B_2^1, B_2^{k_2}\}, \ldots, X \cup \{B_1^1, B_2^1, B_2^{k_2} \ldots, B_m^1, B_m^{k_m}\}\rangle_{m\in\mathbb{N}, 1\leqslant k_2,\ldots,k_m\leqslant 2}$, where $B_1^1 = B$: $m \leqslant *(B)$. Such a sequence we will call *B-sequence*. To simplify notations in our proof, m will be called *a length of B-sequence*.

i) Let $A_1^1 = \neg(\neg(B))$. Therefore, by definition of function $*$, $*(A_1^1) > *(B)$. By induction hypothesis, $*(B) \geqslant m$, where m is a length of any B-sequence. Since $n = m_1 + 1$, for some m_1, so $*(A_1^1) = *(B) + 1 \geqslant m_1 + 1 = n$.

ii) Let $A_1^1 = (B) \wedge (C)$. Hence, by definition of function $*$, $*(A_1^1) > *(B), *(C)$. By induction hypothesis, $*(B) \geqslant m$, where m is a length of any B-sequence and $*(C) \geqslant k$, where k is a length of any C-sequence. As a consequence, since $n = m_1 + k_1$, for some m_1, k_1, so $*(A_1^1) = *((B) \wedge (C)) = *(B) + *(C) \geqslant m_1 + k_1 = n$.

iii) Let $A_1^1 = (B) \vee (C)$. Hence, by definition of function $*$, $*(A_1^1) > *(B), *(C)$. By induction hypothesis, $*(B) \geqslant m$, where m is a length of any B-sequence and $*(C) \geqslant k$, where k is a length of any C-sequence. As a consequence, since $n = m_1 + 1$ or $n = k_1 + 1$, for some m_1, k_1, so: if $\max\{*(B), *(C)\} = *(B)$, then $*(A_1^1) = *((B) \vee (C)) = *(B) + 1 \geqslant m_1 + 1 = n$, if $\max\{*(B), *(C)\} = *(C)$, then $*(A_1^1) = *((B) \vee (C)) = *(C) + 1 \geqslant k_1 + 1 = n$.

iv) Let $A_1^1 = (B) \rightarrow (C)$. Consequently, by definition of function $*$, $*(A_1^1) > *(\neg(B)), *(C)$. By induction hypothesis, $*(\neg(B)) \geqslant m$, where m is a length of any $\neg(B)$-sequence and $*(C) \geqslant k$, where k is a length of any C-sequence. As a consequence, since $n = m_1 + 1$ or $n = k_1 + 1$, for some m_1, k_1, so: if $\max\{*(\neg(B)), *(C)\} = *(\neg(B))$, then $*(A_1^1) = *((B) \rightarrow (C)) = *(\neg(B)) + 1 \geqslant m_1 + 1 = n$, if $\max\{*(\neg(B)), *(C)\} = *(C)$, then $*(A_1^1) = *((B) \rightarrow (C)) = *(C) + 1 \geqslant k_1 + 1 = n$.

v) Let $A_1^1 = (B) \leftrightarrow (C)$. Hence, by definition of function $*$, $*(A_1^1) > *(B), *(C)$, $*(\neg(B)), *(\neg(C))$. By induction hypothesis, $*(B) \geqslant m$, where

m is a length of any B-sequence, $*(C) \geqslant k$, where k is a length of any C-sequence, $*(\neg(B)) \geqslant l$, where l is a length of any $\neg(B)$-sequence, $*(\neg(C)) \geqslant o$, where o is a length of any $\neg(C)$-sequence. As a consequence, since $n = m_1 + k_1$ or $n = l_1 + o_1$, for some m_1, k_1, l_1, o_1, so: if $\max\{*(B) + *(C), *(\neg(B)) + *(\neg(C))\} = *(B) + *(C)$, then $*(A_1^1) = *((B) \leftrightarrow (C)) = *(B) + *(C) \geqslant m_1 + k_1 = n$, if $\max\{*(B) + *(C), *(\neg(B)) + *(\neg(C))\} = *(\neg(B)) + *(\neg(C))$, then $*(A_1^1) = *((B) \leftrightarrow (C)) = *(\neg(B)) + *(\neg(C)) \geqslant l_1 + o_1 = n$.

vi) Let $A_1^1 = \neg((B) \wedge (C))$. So, by definition of function $*$, $*(A_1^1) > *(\neg(B))$, $*(\neg(C))$. By induction hypothesis, $*(\neg(B)) \geqslant m$, where m is a length of any $\neg(B)$-sequence and $*(\neg(C)) \geqslant k$, where k is a length of any $\neg(C)$-sequence. As a consequence, since $n = m_1 + 1$ or $n = k_1 + 1$, for some m_1, k_1, so: if $\max\{*(\neg(B)), *(\neg(C))\} = *(\neg(B))$, then $*(A_1^1) = *(\neg((B) \wedge (C))) = *(\neg(B)) + 1 \geqslant m_1 + 1 = n$, if $\max\{*(\neg(B)), *(\neg(C))\} = *(\neg(C))$, then $*(A_1^1) = *(\neg((B) \wedge (C))) = *(\neg(C)) + 1 \geqslant k_1 + 1 = n$.

vii) Let $A_1^1 = \neg((B) \vee (C))$. Hence, by definition of function $*$, $*(A_1^1) > *(\neg(B))$, $*(\neg(C))$. By induction hypothesis, $*(\neg(B)) \geqslant m$, where m is a length of any $\neg(B)$-sequence and $*(\neg(C)) \geqslant k$, where k is a length of any $\neg(C)$-sequence. As a consequence, since $n = m_1 + k_1$, for some m_1, k_1, so $*(A_1^1) = *(\neg((B) \vee (C))) = *(\neg(B)) + *(\neg(C)) \geqslant m_1 + k_1 = n$.

viii) Let $A_1^1 = \neg((B) \to (C))$. Hence, by definition of function $*$, $*(A_1^1) > *(B), *(\neg(C))$. By induction hypothesis, $*(B) \geqslant m$, where m is a length of any B-sequence and $*(\neg(C)) \geqslant k$, where k is a length of any $\neg(C)$-sequence. As a consequence, since $n = m_1 + k_1$, for some m_1, k_1, so $*(A_1^1) = *((B) \to (C)) = *(B) + *(\neg(C)) \geqslant m_1 + k_1 = n$.

ix) Let $A_1^1 = \neg((B) \leftrightarrow (C))$. Since, by definition of function $*$, $*(A_1^1) > *(B), *(C)$, $*(\neg(B)), *(\neg(C))$, so by induction hypothesis, $*(B) \geqslant m$, where m is a length of any B-sequence, $*(C) \geqslant k$, where k is a length of any C-sequence, $*(\neg(B)) \geqslant l$, where l is a length of any $\neg(B)$-sequence, $*(\neg(C)) \geqslant o$, where o is a length of any $\neg(C)$-sequence. As a consequence, since $n = l_1 + k_1$ or $n = m_1 + o_1$, for some m_1, k_1, l_1, o_1, so: if $\max\{*(\neg(B)) + *(C), *(B) + *(\neg(C))\} = *(\neg(B)) + *(C)$, then $*(A_1^1) = *(\neg((B) \leftrightarrow (C))) = *(\neg(B)) + *(C) \geqslant l_1 + k_1 = n$, if $\max\{*(\neg(B)) + *(C), *(B) + *(\neg(C))\} = *(B) + *(\neg(C))$, then $*(A_1^1) = *(\neg((B) \leftrightarrow (C))) = *(B) + *(\neg(C)) \geqslant m_1 + o_1 = n$. □

Remark 1. If we decompose any formula A, step by step using rules of extending to its components, then any injective sequence of results is at most $*(A)$ long.

Example 1. Let us consider a set $Y \cup \{\neg((p) \leftrightarrow (\neg(q)))\}$. We decompose the formula $\neg((p) \leftrightarrow (\neg(q)))$ applying to the set $Y \cup \{\neg((p) \leftrightarrow (\neg(q)))\}$ the rule R_9 and obtain two sequences of extensions: $X_1' = Y \cup \{\neg((p) \leftrightarrow (\neg(q)))\}, X_2' = Y \cup \{\neg((p) \leftrightarrow (\neg(q))), \neg(p), \neg(q)\}$ and $X_1'' = Y \cup \{\neg((p) \leftrightarrow (\neg(q)))\}, X_2'' = Y \cup \{\neg((p) \leftrightarrow (\neg(q))), p, \neg(\neg(q))\}$. In the first case we cannot already decompose the components of the initial formula, but in the case of X_2'' it is still possible to apply the rule R_5, obtaining $X_3'' = Y \cup \{\neg((p) \leftrightarrow (\neg(q))), p, \neg(\neg(q)), q\}$. The last sequence is the longest one among those based on decomposing of the initial formula. It is at most $*(\neg((p) \leftrightarrow (\neg(q))))$ long. We count: $*(p) = *(\neg(q)) = *(\neg(p)) = 1$, $*(\neg(\neg(q)) = 1 + *(q) = 2$. Hence $\max\{*(\neg(p)) + *(\neg(q)), *(p) + *(\neg(\neg(q))\} = \max\{2, 3\}$. Consequently, $*(\neg((p) \leftrightarrow (\neg(q)))) = 3$.

Fact 2. *If X is a finite, negative set, then every injective sequence $f\colon K \longrightarrow 2^{\mathsf{For}}$ of extensions of X is finite.*

Proof. By double induction on the number of members of X, their complexity, and by use of Fact 1. Let X be any negative, finite set and $f\colon K \longrightarrow 2^{\mathsf{For}}$ be any injective sequence of extensions of X

Initial step. We assume $|X| = 1$, so some formula $A \in X$. *i)* Let $*(A) = 1$. Hence, $A = \neg(x)$, for some $x \in \mathsf{SI}$, since X is negative. Then, by Fact 1, $|K| \leqslant *(A) = 1$, and f is finite. *ii)* Let $*(A) \neq 1$. By Fact 1 $|K| \leqslant *(A)$ and f is finite.

Inductive step. Let $A \in X$ and $|X \setminus \{A\}| = n \neq 0$. We assume that for a negative set $X \setminus \{A\}$ any injective sequence $f'\colon K' \longrightarrow 2^{\mathsf{For}}$ of extensions of $X \setminus \{A\}$ is finite, and hence $|K'| \leqslant k$, for some $k \in \mathbb{N}$. Obviously, $|X| = n+1$.

i) Let $*(A) = 1$. If $\{A\} \cup f'(i)$ is contradictory, for some $i \in K'$, then $|K| \leqslant j$, where j is the smallest number in K', such that $\{A\} \cup f'(j)$ is contradictory, $|K| \leqslant k$ and f is finite. If $\{A\} \cup f'(i)$ is non-contradictory, for any $i \in K'$, then, by 1, $|K| \leqslant k$, since $*(A) = 1$, so f is finite.

ii) Let $*(A) \neq 1$. If $\{A\} \cup f'(i)$ is contradictory, for some $i \in K'$, then $|K| \leqslant j$, where j is the smallest number in K', such that $\{A\} \cup f'(j)$ is contradictory, and $|K| \leqslant k$. If $\{A\} \cup f'(i)$ is non-contradictory, for any $i \in K'$, then, by 1, $|K| \leqslant k + *(A)$, so f is finite. □

Definition 8. A sequence of extensions $f\colon K \longrightarrow 2^{\mathsf{For}}$ is called *a branch* iff

1. f is injection,
2. for any $j > 1$ and any $Y \subseteq \mathsf{For}$, if $j \in K$ and Y are indirect extensions of $f(j-1)$ by some rule of extending R_i ($i \in \{2, 3, 4, 6, 7, 9\}$), then $Y \neq f(j-1)$.

Example 2. Consider a negative set $\{\neg p, \neg p \vee q\} = X_1$. Applying the rule R_2 we obtain two sequences: X_1, $X_2' = \{\neg p, \neg p \vee q\}$ and X_1, $X_2'' = \{\neg p, \neg p \vee q, q\}$, but neither branch, since $X_1 = X_2'$.

Henceforth, speaking about branches, for convenience we will use: 1) sequences: $X_1, \ldots, X_n$, where $n \geqslant 1$, 2) abbreviations for functions: f_M (where M is a domain of f, i.e., $f \colon M \longrightarrow 2^{\mathsf{For}}$), or, in order to designate them, 3) small Greek letters: ϕ, ψ, etc.

FACT 3. *Let X_1 be a finite negative set. Then there is no branch of the form $X_1, \ldots, X_n, \ldots$ (no infinite branch).*

PROOF. By the previous fact and the definition of a branch. □

DEFINITION 9. A branch $X_1, \ldots, X_n$, where $n \geqslant 1$, is called *complete* iff there is no branch of the form $X_1, \ldots, X_n, X_{n+1}$.

FACT 4. *Let X_1 be a finite negative set. Then there is a branch of the form $X_1, \ldots, X_n$, where $n \geqslant 1$, which is complete.*

PROOF. By the definition of being complete and Fact 3. □

FACT 5. *If X_1 is a finite, negative set, then for every branch of the form $X_1, \ldots, X_n$, where $n \geqslant 1$, there is a complete branch of the form: $X_1, \ldots, X_n, \ldots, X_{n+m}$, where $m \geqslant 0$.*

PROOF. Let X_1 be any finite, negative set. Let ϕ be any branch of the form $X_1, \ldots, X_n$ with $n \geqslant 1$. Hence, X_n is a finite extension. Because by Fact 4 there is a complete branch $X_n^1, \ldots, X_{n+m}^{1+m}$, so there is a complete branch of the form: $X_1, \ldots, X_n, \ldots, X_{n+m}$, where $m \geqslant 0$. □

DEFINITION 10. A branch $X_1, \ldots, X_n$, where $n \geqslant 1$, is called *closed* iff X_n is contradictory. A branch is called *open* iff it is not closed.

At the end of this paragraph we point at the obvious fact, which follows from Definition 4 and the definition of a branch and a complete branch.

FACT 6. *For any branch ϕ: if ϕ is closed, then it is complete.*

DEFINITION 11. Let Φ be a set of branches and a branch $X_1, \ldots, X_m$, where $m \geqslant 1$, belong to Φ. We call it *maximal in* Φ iff there is no branch $Y_1, \ldots, Y_k$ in Φ, where $k > m \geqslant 1$. Simultaneously, by $\mathrm{B}(X_1)$ we mean the set of all branches of the form $X_1, \ldots, X_n$, where $n \geqslant 1$, and by $\mathrm{MB}(X_1)$ the set of all maximal branches of the form $X_1, \ldots, X_n$, where $n \geqslant 1$.

Fact 7. *Let X_1 be a finite, negative set and* $\mathrm{B}(X_1)$ *be a set of all branches of the form* $X_1, \dots, X_n$*, where* $n \geqslant 1$*. Then in* $\mathrm{B}(X_1)$ *there is a non-empty subset of the maximal branches* $\mathrm{MB}(X_1)$*.*

Proof. Let X_1 be an arbitrary, finite and negative set of formulas. For any $A \in X_1$, $*(A) \in \mathbb{N}$. Since X_1 is negative, so also non-empty, and $\mathrm{B}(X_1)$ is non-empty, too.

Initial step. Let $|X_1| = 1$. Then, there is a formula $\neg(A) \in X$. Consequently, by Fact 1, for any branch $X_1, \dots, X_m$ with $1 \leqslant m$, $m \leqslant *(\neg(A))$. Hence, there is at least one branch $X_1, \dots, X_k$, for some $1 \leqslant k \leqslant *(\neg(A))$, which is maximal one and the set $\mathrm{MB}(X_1)$ is non-empty.

Inductive step. Let for any negative set Y with $|Y| \leqslant |X_1|$ our hypothesis be satisfied. So we assume that $\mathrm{MB}(Y)$ is non-empty. Let the branches in $\mathrm{MB}(Y)$ be k-long and A be any formula such that $X_1 = Y \cup \{A\}$. We need to consider two cases.

i) Let $*(A) = 1$. Then for all branches of the form $X_1 = Y \cup \{A\}, \dots, X_m = Y_m \cup \{A\}$, where $1 \leqslant m \leqslant k$. If $f(i) \cup \{A\}$ is non-contradictory for some $f \in \mathrm{MB}(Y)$ and any $i \leqslant k$, then the branches in $\mathrm{MB}(X_1)$ are those which again are k-long. If not, then there is a branch f in $\mathrm{B}(Y)$, such that for some $i < k$, $f(i) \cup \{A\}$ is contradictory and there is no branch g in $\mathrm{B}(Y)$, such that $g(i+1) \cup \{A\}$ is non-contradictory. Hence, the set $\mathrm{MB}(X_1)$ is non-empty and branches which contains are i-long.

ii) Let $*(A) \neq 1$. Then for all branches of the form $X_1 = Y \cup \{A\}, \dots, X_m = Y_m \cup \{A\}$, where $m \geqslant 1$, $m \leqslant k + *(A)$. If $f(i) \cup \{A\}$ is non-contradictory for some $f \in \mathrm{MB}(Y)$ and any $i \leqslant k$, then the branches in $\mathrm{MB}(X_1)$ are among those which are at most $k + *(A)$-long. Precisely speaking, there is at least one branch f, which is l-long, for some $k < l \leqslant k + *(A)$. It is either $k + *(A)$-long or for some $k < i < k + *(A)$, $f(i) \cup \{A\}$ is a contradictory set. If $f(i) \cup \{A\}$ is contradictory for any $f \in \mathrm{MB}(Y)$ for some $i \leqslant k$, then branches in $\mathrm{MB}(X_1)$ are among those which are at most k, and in $\mathrm{MB}(X_1)$ is at least one branch i-long with $1 \leqslant i \leqslant k$. □

Tableau inference. Now, we are coming to the most important notion: tableau inference.

Definition 12. We say that a formula A is a tableau consequence of a set $X \subseteq \mathsf{For}$ (in short: $X \rhd A$) iff there is a finite $Y \subseteq X$ and every complete branch of the form: $X_1 = Y \cup \{\neg(A)\}, X_2, \dots, X_n$, where $n \geqslant 1$, is closed.

Example 3. Here is a simple example of tableau inference: $\{p\} \rhd (p) \vee (q)$ is a correct tableau inference, because every complete branch of the form

$X_1 = \{p, \neg((p) \vee (q))\}, \ldots, X_n$ is closed. In fact, there is only one complete branch of the form: $\{p, \neg((p) \vee (q))\} \subset \{p, \neg((p) \vee (q)), \neg(p), \neg(q)\}$, and it is obviously closed.

Soundness Theorem. Now, having the formal definitions, we can prove the basic metatheorems. We start with soundness theorem, but at the beginning we need some auxiliary lemmas.

Lemma 1. *Let V' be any Boolean valuation and X be any negative set. Let $\mathrm{E}(X)$ be a set of all extensions of X different of it. If $\mathrm{E}(X)$ is not empty and $V'(X) = 1$, then there is at least one extension $Y \in \mathrm{E}(X)$, such that $V'(Y) = 1$.*

Proof. By inspection of the rules of extending. For example, if $(A) \vee (B) \in X$, then $V'((A) \vee (B)) = 1$, for some assumed V'. Let $X \cup \{A\}$, $X \cup \{B\} \in \mathrm{E}(X)$ (by application of the rule R_2). Hence, $V'(X \cup \{A\}) = 1$ or $V'(X \cup \{B\}) = 1$. □

Lemma 2. *Let V' be any Boolean valuation and X_1 be a finite negative set. If $V'(X_1) = 1$, then there is at least one complete and open branch $X_1, \ldots, X_n$, where $n \geqslant 1$.*

Proof. We take some finite negative set X_1 and a Boolean valuation $V'(X_1) = 1$. From Fact 5 we know that for every branch $X_1, \ldots, X_n$, where X_1 is finite and $n \geqslant 1$, there is at least one complete branch: $X_1, \ldots, X_n, \ldots, X_{n+m}$, where $m \geqslant 0$. By Fact 3 we know also that there are not infinite branches, so we take only finite and complete branches $X_1, \ldots, X_j$, where $1 \leqslant j$. By Fact 7, some of them belong to the set of the maximal branches $\mathrm{MB}(X_1)$, having the same length k. We assume they all are closed. Hence, for every contradictory extension X_k, $V'(X_k) = 0$. Let us take any $n \geqslant 2$ and assume that for any branch f_M under consideration, if $n \in M$, then $V'(X_n) = 0$. Any X_{n-1} on any branch is either a contradictory extension, and then $V'(X_{n-1}) = 0$, or every its extension $V'(X_n) = 0$, and then by the previous lemma $V'(X_{n-1}) = 0$. Therefore, for any X_n, where $1 \leqslant n \leqslant k$, $V'(X_n) = 0$. But it contradicts the assumption that $V'(X_1) = 1$, when $n = 1$. In consequence, there is at least one complete, open branch $X_1, \ldots, X_n$, where $n \geqslant 1$. □

Theorem 1 (Soundness). *For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, if $X \rhd A$, then $X \models A$.*

Proof. We take some X, A and assume that $X \rhd A$, but $X \not\models A$, i.e., there is a Boolean valuation $V'(X \cup \{\neg(A)\}) = 1$. On the other hand there is a finite set $Y \subseteq X$, such that every complete branch of the form $X_1 = Y \cup \{\neg(A)\}$, X_2, ..., X_n, where $n \geqslant 1$, is closed. But because we know that $V'(Y \cup \{\neg(A)\}) = 1$, there must be, by Lemma 2, at least one complete, open branch $X_1 = Y \cup \{\neg(A)\}$, X_2, ..., X_n, where $n \geqslant 1$, which provides a contradiction. Hence, $X \models A$. □

Completeness Theorem. At first we remaind the compactness property of a classical consequence.

FACT 8. *For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, $X \models A$ iff there is a finite $Y \subseteq X$, such that $Y \models A$.*

Moreover, we still need one important lemma.

LEMMA 3. *Let $X_1, \ldots, X_n$, where $n \geqslant 1$, be any complete and open branch. Let $\mathrm{L}(X_n)$ be a set defined as follows: $\mathrm{L}(X_n) = \{x \in X_n \colon x = y$ or $x = \neg(y)$, where $y \in \mathsf{SL}\}$. Let V' be any Boolean valuation. If $V'(\mathrm{L}(X_n)) = 1$, then $V'(X_n) = 1$.*

PROOF. By inspection of the rules of extending. □

LEMMA 4. *Let X_1 be an extension and $X_1, \ldots, X_n$, where $n \geqslant 1$, be a complete and open branch. Then there is a Boolean valuation V', such that $V'(X_1) = 1$.*

PROOF. Since $X_1, \ldots, X_n$ is a complete and open branch, so for every $i \leqslant n$, X_i is non-contradictory. Furthermore, there is no rule to apply to X_n. We define the set $\mathrm{L}(X_n)$ to which belong all and only these formulas of the form x or $\neg(x)$ that belong to X_n, where $x \in \mathsf{Sl}$. The $\mathrm{L}(X_n)$ is a non-contradictory and non-empty set, otherwise the branch would be not complete or closed. Using $\mathrm{L}(X_n)$ we define a valuation $V \colon \mathsf{Sl} \longrightarrow \{1, 0\}$, such that for every x: $V(x) = 1$, if $x \in \mathrm{L}(X_n)$ and $V(x) = 0$, if $\neg x \in \mathrm{L}(X_n)$ and extend it to a Boolean valuation V'. By Lemma 3 we obtain that $V'(X_n) = 1$, hence $V'(X_1) = 1$. □

THEOREM 2 (Completeness). *For any $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, if $X \models A$, then $X \rhd A$.*

PROOF. We take some X and A and assume that $X \models A$, but not $X \rhd A$. By Fact 8 we know that there is a finite $Y \subseteq X$ and $Y \models A$. From the second hypothesis we have that for neither Z, finite subset of X, all complete branches of the form: $X_1 = Z \cup \{\neg(A)\}$, ..., X_n, where $n \geqslant 1$, are closed. It means that for any Z there is a complete and open branch $X_1 = Z \cup \{\neg(A)\}, \ldots, X_n$. In particular for Y there is a complete and open branch $X_1 = Y \cup \{\neg(A)\}, \ldots, X_n$. By Lemma 4 we have that there is a Boolean valuation $V'(Y \cup \{\neg(A)\}) = 1$. In consequence, it contradicts the crucial hypothesis, $Y \not\models A$, so, by Fact 8, $X \not\models A$. Hence, $X \rhd A$. □

3. How these notions work

The described notions are very inconvenient in practice. We cannot usually check, if all complete branches, even starting with a finite, negative set, are closed. It provides a question how to choose a quite small set of privileged branches that is enough to check. To define this shortened way of establishing the correctness of an inference we shall use a notion of a tableau. All the following auxiliary notions will be introduced in turn. We begin with the most basic ones.

Definition 13. Let X be a negative extension and $Y \subseteq X$. Let R_i, with $1 \leqslant i \leqslant 9$, be a rule of extending. By $R_i(Y)$ we shall mean a set of all extensions of X by use of a rule of extending R_i applied to all elements of Y, to which it is possible to apply it. More formally, $Z \in R_i(Y)$ iff there is a formula $A \in Y$ for which exactly one of the following conditions is satisfied:

1. $\langle X, Z \rangle \in R_i$ and $\langle X \setminus \{A\}, Z \setminus \{A\} \rangle \notin R_i$,
2. $\langle X, Z, U \rangle \in R_i$ and $\langle X \setminus \{A\}, Z \setminus \{A\}, U \setminus \{A\} \rangle \notin R_i$, for some $U \subseteq \mathsf{For}$,
3. $\langle X, U, Z \rangle \in R_i$ and $\langle X \setminus \{A\}, U \setminus \{A\}, Z \setminus \{A\} \rangle \notin R_i$, for some $U \subseteq \mathsf{For}$.

Example 4. Let $X \supseteq Y$ be a negative set of formulas. Let $Y = \{(A) \vee (B)\}$. If $i = 1$, then $R_i(Y) = \emptyset$. But, if $i = 2$, then $R_i(Y) = \{X \cup \{A\}, X \cup \{B\}\}$.

We can introduce the next notion. Further considerations require, among others, a notion of a relation of being subbranch $\sqsubseteq$.

Definition 14. Let f_M, f'_K be branches. $f_M \sqsubseteq f'_K$ iff $M \subseteq K$ and $\forall_{i \in M}\, f(i) = f'(i)$.

Now we may define a general notion of a tableau for classical propositional logic.

Definition 15. Let $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$. Let Φ be a set of branches. Φ is called *a tableau for* $\langle X, A \rangle$ (in short: $\langle X,\ A, \Phi \rangle$) iff:

1. $\forall_{f \in \Phi}\, f(1) = X \cup \{\neg(A)\}$,

2. $\forall_{i\in\mathbb{N}}\forall_{f'_N,\, f''_O,\, f'''_P\in\Phi}\,(i, i+1 \in N, O, P \;\&\; f'(i) = f''(i) = f'''(i) \Rightarrow$
$f(i+1) = f'(i+1)$ or $f(i+1) = f''(i+1)$ or $f'(i+1) = f''(i+1))$,
3. $\forall_{i\in\mathbb{N}}\forall_{f'_N, f''_O\in\Phi}\,(i,\, i+1 \in N, O \,\&\, f'(i) = f''(i) \,\&\, f'(i+1) \neq f''(i+1) \Rightarrow$
$\exists_{B\in f'(i)}\exists_{R_{k(k\in\{2,3,4,6,7,9\})}}\exists_{X\neq Y\subseteq \mathsf{For}}(X,\, Y \in R_k(\{B\}) \,\&$
$X \cup f'(i) = f'(i+1) \,\&\, Y \cup f''(i) = f''(i+1)))$,
4. $\forall_{f_M\in\Phi}\forall_{j\in M}\forall_{X_{j+1}\in f(M)}\forall_{R_{k(k\in\{2,3,4,6,7,9\})}}\forall_{B\in X_j}(X_{j+1} \in R_k(X_j) \Rightarrow$
$\forall_{Y\in R_k(X_j)}\forall_{f'_K}(f_{\{x\in\mathbb{N}:x\leqslant j\}} \sqsubseteq f'_K \,\&\, Y = f'(j+1) \,\&$
$\Phi \cup \{f'_K\}$ satisfies conditions 1-3 $\Rightarrow f'_K \in \Phi))$.

DEFINITION 16. Let $\langle X, A, \Phi\rangle$ be a tableau and ϕ be a branch. We say that ϕ is $\langle X, A, \Phi\rangle$ *complete* iff ϕ is complete and $\phi \in \Phi$.

DEFINITION 17. Let $\langle X, A, \Phi\rangle$ be any tableau. We say that $\langle X, A, \Phi\rangle$ is *complete* iff for every branch $\phi \in \Phi$ there is a $\langle X, A, \Phi\rangle$ complete branch ψ, such that $\phi \sqsubseteq \psi$.

DEFINITION 18. Let $\langle X, A, \Phi\rangle$ be any tableau. We say that $\langle X, A, \Phi\rangle$ is *closed* iff:

1. it is complete,
2. every $\langle X, A, \Phi\rangle$ complete branch is closed.

DEFINITION 19. Let $\langle X, A, \Phi\rangle$ be any tableau. We say that $\langle X, A, \Phi\rangle$ is *open* iff it is not closed.

FACT 9. *Let X, $\{A\} \subseteq$ For and X be finite. Then, there is at least one complete tableau $\langle X, A, \Phi\rangle$.*

PROOF. Obvious, by the definition of a tableau, a complete tableau, and Fact 5. □

Now, we have another helpful lemmas.

LEMMA 5. *Let X, $\{A\} \subseteq$ For and X be finite. Then, (1) there is a closed tableau $\langle X, A, \Phi'\rangle$ iff (2) every complete tableau $\langle X, A, \Phi''\rangle$ is closed.*

PROOF. Let X, $\{A\} \subseteq$ For and X be finite. We assume (2). By the previous fact, for every finite $X \cup \{A\} \subset$ For, there is a complete tableau $\langle X, A, \Phi\rangle$. Hence, by (2), there is a closed tableau $\langle X, A, \Phi'\rangle$. We assume (1), so we have a closed tableau $\langle X, A, \Phi'\rangle$. We take any $\langle X, A, \Phi''\rangle$ which is complete and any $\langle X, A, \Phi''\rangle$ complete branch ϕ. It is either closed or not. We consider the case, when it is not closed $(*)$. Let $\phi = X_1, \ldots, X_n$, where $n \geqslant 1$. Consequently, $(**)$ X_n is closed under the rules of extending in this sense, that after one of them is applied, neither extension X_{n+1} different than X_n, such that $\phi \cup \{X_{n+1}\}$ is still a branch, is obtained. Moreover, we know that for any closed branch in $\langle X, A, \Phi'\rangle$, $Y_1 = X \cup \{\neg(A)\} \subseteq X_n$. If Y_1 is not contradictory, then there can be two branches ψ', $\psi'' \in \langle X, A, \Phi'\rangle$ with extensions $Y_2' \neq Y_2''$. Hence, by $(**)$, Y_2' or $Y_2'' \subseteq X_n$. We assume that there is a branch $\psi_l \in \langle X, A, \Phi'\rangle$ such that $Y_k^l \in \psi_l$ and $Y_k^l \subseteq X_n$. If Y_k^l is not contradictory, then there can be two branches ψ_l', $\psi_l'' \in \langle X, A, \Phi'\rangle$ with $Y_{k+1}^{l'} \neq Y_{k+1}^{l''}$. Hence, by $(**)$, $Y_{k+1}^{l'}$ or $Y_{k+1}^{l''} \in X_n$. In consequence, for some closed tableau $\langle X, A, \Phi'\rangle$ and its closed branch $\psi = Y_1, \ldots, Y_i$ with $1 \leqslant i$, a contradictory set Y_i is contained in X_n. Consequently, ϕ is closed, which contradicts the assumption $(*)$. Therefore, every complete branch in $\langle X, A, \Phi''\rangle$ is closed, so $\langle X, A, \Phi''\rangle$ is closed. □

Lemma 6. *Let X be a subset of* For *and A be a formula. Then, there is a finite $Y \subseteq X$, such that every complete tableau $\langle Y, A, \Phi\rangle$ is closed iff $X \rhd A$.*

Proof. We take some $X \subseteq \mathsf{For}$, $A \in \mathsf{For}$, starting the proof from right to left. Let $X \rhd A$. Hence, there is a finite $Y \subseteq X$, such that every complete branch of the form: $X_1 = Y \cup \{\neg(A)\}$, $X_2, \ldots, X_n$, where $n \geqslant 1$, is closed. Therefore, every complete tableau $\langle Y, A, \Phi\rangle$ is closed, otherwise there would be at least one complete, but not closed tableau $\langle Y, A, \Phi'\rangle$, and hence, at least one complete, but not closed branch $X_1 = Y \cup \{\neg(A)\}$, $X_2, \ldots, X_n$, where $n \geqslant 1$. Now, we assume that there is a finite $Y_0 \subseteq X$ and every complete tableau $\langle Y_0, A, \Phi''\rangle$ is closed. If not $X \rhd A$, then for each finite $Y \subseteq X$ there is a complete and open branch. But this contradicts the assumption, because for Y_0 at least one of tableaux must be complete and not closed. □

Going further, we have a theorem that really simplifies the process of checking whether an inference is correct. By the above lemmas:

Theorem 3. *Let $X \subseteq \mathsf{For}$ and $A \in \mathsf{For}$. Then there is a finite $Y \subseteq X$ and a complete, closed tableau $\langle Y, A, \Phi\rangle$ iff $X \rhd A$.*

Thanks to it, to check the correctness of an inference it is enough to build only one complete and closed tableau.

The presented approach can be also applied to first order logic as well as to modal logic, and especially to first order modal logic. However, in these cases we should cope with the problem of infinite branches, because we add additional symbols (called labels) to designate elements of domain, e.g. possible worlds. Hence, the application must be more general and the tableaux for propositional logic are only a specialization of a wider pattern. This pattern is a subject of a further presentation [1].

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