

Krystyna Mruczek-Nasieniewska

EXTERNALLY COMPATIBLE ABELIAN GROUPS OF THE TYPE $(2, 1, 0)$

Abstract. In [4] the lattice of all subvarieties of the variety \mathcal{G}_{Ex}^n defined by so called externally compatible identities of Abelian groups together with the identity $x^n \approx y^n$, for any $n \in \mathcal{N}$ and $n \geq 1$ was described. In that paper classes of models of the type $(2, 1)$ were considered. It appears that diagrams of lattices of subvarieties defined by externally compatible identities satisfied in a given equational theory depend on the language of the considered class of algebras.

A question was asked to what extent the diagram of the lattice of subvarieties of the variety defined by externally compatible identities of a given variety will depend on changing the type of algebras. In general case, the answer to this question seems to be very complicated. In this paper we describe the variety of Abelian groups of exponent $p \cdot q$, where p, q are different primes of type $(2, 1, 0)$.

Keywords: Abelian groups, externally compatible identities, variety

2000 *Mathematics Subject Classification:* 03C05, 20A05

1. Preliminaries

We consider a given type of algebras $\tau: F \rightarrow \mathcal{N}$, where F is a set of fundamental operation symbols and \mathcal{N} is a set of non-negative integers. Let P be a partition of F . An identity $\varphi \approx \psi$ is called P -compatible (see [9]) iff it is of the form $x \approx x$ or of the form $f(\varphi_0, \dots, \varphi_{\tau(f)-1}) \approx g(\psi_0, \dots, \psi_{\tau(g)-1})$, where f and g belong to the same block $[f]_P$ of P and $\varphi_0, \dots, \varphi_{\tau(f)-1}, \psi_0, \dots,$

$\psi_{\tau(g)-1}$ are terms of type τ . An identity $\varphi \approx \psi$ is externally compatible (see [2]) iff it is P -compatible, where P contains singletons only. An identity $\varphi \approx \psi$ is normal (see [8] and [5]) iff it is of the form $x \approx x$ or neither φ nor ψ is a variable.

We denote the set of all P -compatible identities of type τ by $P(\tau)$ and the set of all identities of type τ by $Id(\tau)$. We will also denote by $Ex(\tau)$ the set of all externally compatible identities of type τ and by $N(\tau)$ the set of all normal identities of type τ . If V is a variety then $P(V)$ denotes the set of all P -compatible identities of type τ which are satisfied in V . We have $P(V) = P(\tau) \cap Id(V)$, where $Id(V)$ denotes the set of all identities satisfied in V .

Obviously, $Ex(V) \subseteq P(V) \subseteq N(V) \subseteq Id(V)$ for any partition P .

If Σ is a set of the identities of type τ then $Cn(\Sigma)$ denotes the deductive closure of Σ , i.e. $Cn(\Sigma)$ is closed under the rules of the inference ([1] and [10]). If $\Sigma = Cn(\Sigma)$ then Σ is called equational theory. It is easy to see that $P(\tau)$ and $P(V)$ are equational theories.

By $Mod(\Sigma)$ we denote the class of all models of Σ , that is the class of all algebras of type τ satisfying the identities from Σ . So, if $\Sigma = P(V)$ then $Mod(\Sigma) = V_P$. A variety V such that $Id(V) = P(V)$ is called P -compatible. Let $\mathcal{L}(V)$ be the lattice of all subvarieties of V .

Let Σ be an equational theory. It is well known that the lattice $\mathcal{L}(\Sigma)$ of all equational theories extending Σ is dually isomorphic to the lattice $\mathcal{L}(Mod(\Sigma))$ of all subvarieties of the variety $Mod(\Sigma)$ determined by the theory Σ .

2. The variety $\mathcal{G}_{Ex}^{p,q}$ of type τ_0

Let us fix the type τ_0 , where $\tau_0: \{\cdot, {}^{-1}\} \rightarrow \mathcal{N}$, $\tau_0(\cdot) = 2, \tau_0({}^{-1}) = 1$. Let x^0 and x^n denote $x \cdot x^{-1}$ and $x^{n-1} \cdot x$, respectively i.e. let $x^0 = x \cdot x^{-1}$, $x^n = x^{n-1} \cdot x$ for $n \in \mathcal{N}$.

Of course, in the case of the set $\{\cdot, {}^{-1}\}$ we have only two partitions: $Ex = \{\{\cdot\}, \{{}^{-1}\}\}$, $N = \{\{\cdot, {}^{-1}\}\}$.

Let \mathcal{G}^n denotes the variety of all Abelian groups of type τ_0 satisfying identity $x^n \approx y^0$, where $n \in \mathcal{N}$ and let \mathcal{G}_{Ex}^n denotes the variety defined by all externally compatible identities of Abelian groups of exponent n of type τ_0 . An identity $(x \cdot y)^{-1} \approx x^{-1} \cdot y^{-1}$ belongs to $Id(\mathcal{G}^n)$, but it is not externally compatible. It means that the variety \mathcal{G}_{Ex}^n is larger then \mathcal{G}^n and it may be interesting to characterise the lattice of all subvarieties of \mathcal{G}_{Ex}^n .

We will use the following notations:

$$\mathcal{P}_n^{\{k_1, \dots, k_s\}} = \text{Mod}(Ex(\mathcal{G}^n) \cup \bigcup_{i=1}^s \{x^0 \cdot x^{k_i} \approx ((x^{k_i})^{-1})^{-1}\}),$$

where $k_i \in \{0, 1, \dots, n-1\}$ for $i = 1, \dots, s$, and

$$C^n = \text{Mod}(Ex(\mathcal{G}^n) \cup \{x^0 \approx ((x^0)^{-1})^{-1}\}).$$

Let \mathcal{K}_n be the set of all natural divisors of n . In [4] it was proved the following theorem:

THEOREM 2.1. *Let n be a natural number such that $n > 0$. If V is a subvariety of \mathcal{G}_{Ex}^n of type τ_0 then V is one of the following classes: \mathcal{G}^r , \mathcal{G}_N^r , $\mathcal{P}_r^{\{k_1, \dots, k_s\}}$, C^r , \mathcal{G}_{Ex}^r , where $r \in \mathcal{K}_n$ and $k_1, \dots, k_s \in \mathcal{K}_r$.*

Using the above theorem we obtain the next corollary:

COROLLARY 2.1. *If V is a subvariety of the variety $\mathcal{G}_{Ex}^{p \cdot q}$, where p, q are prime numbers, then V is a one of the following classes: \mathcal{G}^r , \mathcal{G}_N^r , $\mathcal{P}_r^{\{k_1, \dots, k_s\}}$, C^r , \mathcal{G}_{Ex}^r , where $r \in \{1, p, q, p \cdot q\}$ and $k_1, \dots, k_s \in \mathcal{K}_r$.*

3. The variety $\mathcal{G}_{Ex}^{p \cdot q}$ of type τ_1

Let us fix the type $\tau_1 : \{\cdot, {}^{-1}, e\} \rightarrow \mathcal{N}$, where $\tau_1(\cdot) = 2$, $\tau_1({}^{-1}) = 1$, $\tau_1(e) = 0$.

The variety of Abelian groups of type τ_1 is denoted by \mathcal{G} and the variety of Abelian groups of type τ_1 satisfying identity $x^n \approx x \cdot x^{-1}$, for $n \in \mathcal{N}$ is denoted by \mathcal{G}^n .

In the current section we will describe the lattice of all subvarieties of the variety \mathcal{G}_{Ex}^n defined by externally compatible identities Abelian groups of exponent n of type τ_1 in the case $n = p \cdot q$, where p, q are prime numbers.

In [7] it was proved that:

THEOREM 3.1. *The following identities of the type τ_1 :*

$$(3.1) \quad (x \cdot y) \cdot z \approx x \cdot (y \cdot z),$$

$$(3.2) \quad x \cdot y \approx y \cdot x,$$

$$(3.3) \quad x \cdot y \cdot e \approx x \cdot y,$$

$$(3.4) \quad (x \cdot e)^{-1} \approx x^{-1},$$

$$(3.5) \quad x \cdot x^{-1} \approx e \cdot e$$

form an equational base of the class determined by all externally compatible identities of Abelian groups.

From that we deduce that the identities (3.1)–(3.5) together with the identity $x^n \approx x \cdot x^{-1}$ form an equational base of the variety \mathcal{G}_{Ex}^n .

It is known that to find all subvarieties of the variety \mathcal{G}_{Ex}^n , all of the classes $Mod(Cn(Ex(\mathcal{G}^n) \cup E))$, where $E \subseteq Id(\tau_1)$, must be taken into consideration.

In the next lemma we will give the canonical form of terms from the class \mathcal{G}_{Ex}^n . This lemma below is obvious so we skip its proof.

LEMMA 3.1. *Every term ϕ of type τ_1 on variables x_1, \dots, x_s is equivalent in \mathcal{G}_{Ex}^n to a term ϕ^* being one of the following form: $e, x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s}, ((x_1^{k_1} \cdot \dots \cdot x_s^{k_s})^{-1})^{-1}, x_i$, where $i \in \{1, \dots, s\}, k_1, \dots, k_s \in \{0, 1, \dots, n-1\}$.*

It follows that any identity of the type τ_1 is equivalent, on the basis of the theory $Ex(\mathcal{G}^n)$, to one of the following identities:

$$(3.6) \quad x_j \approx x_i,$$

$$(3.7) \quad e \approx x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s},$$

$$(3.8) \quad e \approx ((x_1^{k_1} \cdot \dots \cdot x_s^{k_s})^{-1})^{-1},$$

$$(3.9) \quad x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s} \approx x_j,$$

$$(3.10) \quad ((x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s})^{-1})^{-1} \approx x_j,$$

$$(3.11) \quad x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s} \approx x_1^0 \cdot x_1^{l_1} \cdot \dots \cdot x_s^{l_s},$$

$$(3.12) \quad ((x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s})^{-1})^{-1} \approx ((x_1^0 \cdot x_1^{l_1} \cdot \dots \cdot x_s^{l_s})^{-1})^{-1},$$

$$(3.13) \quad x_1^0 \cdot x_1^{l_1} \cdot \dots \cdot x_s^{l_s} \approx ((x_1^0 \cdot x_1^{k_1} \cdot \dots \cdot x_s^{k_s})^{-1})^{-1},$$

where $j, i \in \{1, \dots, s\}, k_1, \dots, k_s, l_1, \dots, l_s \in \{0, 1, \dots, n-1\}$.

For $k_1, \dots, k_s \in \mathcal{N}$, where $k_1^2 + \dots + k_s^2 > 0$, let (k_1, \dots, k_s) be the greatest common divisor of k_1, \dots, k_s .

THEOREM 3.2. *For any identity $\phi \approx \varphi$ of the type τ_1 , there is a finite set E_1 of identities of one variable such that $Cn(Ex(\mathcal{G}) \cup \{\phi \approx \varphi\}) = Cn(Ex(\mathcal{G}) \cup E_1)$.*

PROOF. Let us consider the following identities of the type τ_0 :

$$(3.14) \quad x \approx x^0 \cdot x^k,$$

$$(3.15) \quad x \approx ((x^k)^{-1})^{-1},$$

$$(3.16) \quad ((x^k)^{-1})^{-1} \approx ((x^l)^{-1})^{-1},$$

$$(3.17) \quad x^0 \cdot x^k \approx x^0 \cdot x^l,$$

$$(3.18) \quad x^0 \cdot x^k \approx ((x^l)^{-1})^{-1},$$

where $k, l \in \{0, 1, \dots, n-1\}$

In [3] it was shown that for any finite subset E of the set $Id(\tau_0)$ there is a set E_1 of identities of the form (3.14)–(3.18), such that $Cn(Ex(\mathcal{G}^n) \cup E) = Cn(Ex(\mathcal{G}^n) \cup E_1)$.

It follows that if to the set E belong only identities (3.6), (3.9)–(3.13), there is a set E_1 of identities of one variable such that $Cn(Ex(\mathcal{G}^n) \cup E) = Cn(Ex(\mathcal{G}^n) \cup E_1)$. Thus, we have to consider the following theories: $Cn(Ex(\mathcal{G}) \cup \{(3.7)\})$ and $Cn(Ex(\mathcal{G}) \cup \{(3.8)\})$.

We will show that if $k_1 = k_2 = \dots k_{s-1} = k_s = 0$, then $Cn(Ex(\mathcal{G}) \cup \{(3.7)\}) = Cn(Ex(\mathcal{G}) \cup \{e \approx x_1^0\})$ and for $k_1^2 + k_2^2 + \dots + k_s^2 > 0$, we have $Cn(Ex(\mathcal{G}) \cup \{(3.7)\}) = Cn(Ex(\mathcal{G}) \cup \{x_1^0 \cdot x_1^d \approx e\})$, where $d = (k_1, \dots, k_s)$. Indeed, to prove that $Cn(Ex(\mathcal{G}) \cup \{(e \approx x_1^0 \cdot x_1^0 \cdot \dots \cdot x_s^0)\}) = Cn(Ex(\mathcal{G}) \cup \{e \approx x_1^0\})$ it is enough to notice that the identity $x_1^0 \approx x_1^0 \cdot \dots \cdot x_s^0$ belongs to the set $Ex(\mathcal{G})$. Now, we consider the theory $Cn(Ex(\mathcal{G}) \cup \{(3.7)\})$ for $k_1^2 + k_2^2 + \dots + k_s^2 > 0$. By the fact that $d = (k_1, \dots, k_s)$, we see that there are integers p_1, \dots, p_s , such that $d = k_1 \cdot p_1 + \dots + k_s \cdot p_s$. By the substitution in the identity (3.7) of the term $x_1^{p_i}$ for variable x_i for every $i = 1, \dots, s$, we conclude $x_1^0 \cdot x_1^d \approx e \in Cn(Ex(\mathcal{G}) \cup \{(3.7)\})$. For the proof of the reverse inclusion it is enough to substitute the term $x_1^{\frac{k_1}{d}} \cdot \dots \cdot x_s^{\frac{k_s}{d}}$ for the variable x_1 in the identity $x_1^0 \cdot x_1^d \approx e$. Similarly we prove that if $k_1 = k_2 = \dots = k_{s-1} = k_s = 0$, then $Cn(Ex(\mathcal{G}) \cup \{(3.8)\}) = Cn(Ex(\mathcal{G}) \cup \{e \approx (x_1^0)^{-1}\})^{-1}$, and if $k_1^2 + k_2^2 + \dots + k_s^2 > 0$, then $Cn(Ex(\mathcal{G}) \cup \{(3.8)\}) = Cn(Ex(\mathcal{G}) \cup \{((x_1^d)^{-1})^{-1} \approx e\})$, for $d = (k_1, \dots, k_s)$. \square

We introduce the following notation:

$$(3.19) \quad L(n) = \{V \in \mathcal{L}(\mathcal{G}_{Ex}^n); \mathcal{G}^n \subseteq V\}, \text{ for } n \in \mathcal{N}.$$

LEMMA 3.2. *Let $k, n \in \mathcal{N}$.*

1. $(L(n); \subseteq)$ is a sublattice of the lattice $\mathcal{L}(\mathcal{G}_{Ex}^n)$.
2. If $k \neq n$, then $L(k) \cap L(n) = \emptyset$.

PROOF. 1. The proof follows directly from (3.19).

2. Assume for contradiction that $k \neq n$ and $L(k) \cap L(n) \neq \emptyset$. Without a loss of generality we can assume that $k < n$. Since $L(k) \cap L(n) \neq \emptyset$, then there is a variety V_1 , which belongs to $L(k) \cap L(n)$. By (3.19) it follows that $V_1 \in \mathcal{L}(\mathcal{G}_{Ex}^k)$, $\mathcal{G}^k \subseteq V_1$ and $V_1 \in \mathcal{L}(\mathcal{G}_{Ex}^n)$, $\mathcal{G}^n \subseteq V_1$. By the fact that $V_1 \in \mathcal{L}(\mathcal{G}_{Ex}^k)$ and by condition $\mathcal{G}^n \subseteq V_1$ we have that the class of algebras \mathcal{G}^n belongs to $\mathcal{L}(\mathcal{G}_{Ex}^k)$. Thus, $\mathcal{G}^n \subseteq \mathcal{G}_{Ex}^k$. But the last statement is equivalent to the following one $Ex(\mathcal{G}^k) \subseteq Id(\mathcal{G}^n)$. It is obvious that the identity $x^0 \approx x^0 \cdot x^k$ belongs to the set $Ex(\mathcal{G}^k)$. Therefore $x^0 \approx x^0 \cdot x^k \in Id(\mathcal{G}^n)$. Also the identity $x^0 \approx x^0 \cdot x^n$ is fulfilled in the variety \mathcal{G}^n , by the well known fact that if a and b are integers not equal 0, then there is an integer being a solution of the identity $a \cdot x + b \cdot y = (a, b)$. By the last two identities we have that $x^0 \approx x^0 \cdot x^{(n,k)} \in Id(\mathcal{G}^n)$. Since $(k, n) < n$, therefore $x^0 \approx x^0 \cdot x^{(n,k)} \notin Id(\mathcal{G}^n)$, which is a contradiction. \square

THEOREM 3.3. *If V is a subvariety of the variety \mathcal{G}_{Ex}^n , then there is a unique natural number $d \in \mathcal{K}_n$ such that $V \in L(d)$.*

PROOF. By the theorem (3.2) we have that every subvariety of the variety \mathcal{G}_{Ex}^n is generated by an identity of one variable. So, if $V \in \mathcal{L}(\mathcal{G}_{Ex}^n)$, then there is a subset E of the set of the form (3.14)–(3.18), $e \approx x_1$, $e \approx x^0 \cdot x^k$ or $e \approx ((x^k)^{-1})^{-1}$ such that $V = Mod(Ex(\mathcal{G}^n) \cup E)$. Now we introduce the following conventions: a number k will be called an exponent of the term $x^0 \cdot x^k$ and of the term $((x^k)^{-1})^{-1}$, 1 will be treated as the exponent of the term x , and finally let 0 be an exponent of the term e . To any identity $\phi \approx \varphi \in E$ we assign the absolute value of difference of exponents of ϕ and φ . Let d be the greatest common divisor of the number n and of all absolute values of differences of exponents of ϕ and φ , for any $\phi \approx \varphi \in E$. We will show that $V \in L(d)$. One can see that the identity $x^0 \approx x^0 \cdot x^d$ is fulfilled in the class of algebras V . Since every externally compatible identity of Abelian groups is fulfilled in the variety V , thus, we obtain that $V \subseteq \mathcal{G}_{Ex}^d$. To prove that $\mathcal{G}^d \subseteq V$, it is enough to show that every identity of the set E and every base identity of the equational theory $Ex(\mathcal{G}^n)$ belong to the set $Id(\mathcal{G}^d)$. It is obvious that every externally compatible identity of Abelian groups is fulfilled in the variety \mathcal{G}^d . Since d is a divisor of n , so the identity

$x^0 \approx x^0 \cdot x^n$ belongs to the set $Id(\mathcal{G}^d)$. If $\phi \approx \varphi \in E$, then d divides the absolute value of difference of exponents of terms ϕ and φ , therefore we have that $\phi \approx \varphi$ belongs to $Id(\mathcal{G}^d)$. We have proved that $\mathcal{G}^d \subseteq V$. From that and by the fact that $V \subseteq \mathcal{G}_{Ex}^d$, by (3.19) it follows that $V \in L(d)$. By the choice of d , we see that $d \in \mathcal{K}_n$. Uniqueness of d follows from the lemma (3.2). \square

Now we will describe sublattices $L(n)$ for any natural number of the form $n = p \cdot q$, where p and q are prime.

Since $1, p, q, p \cdot q$ are the only natural divisors of the number $p \cdot q$, to find all subvarieties of the variety $\mathcal{G}_{Ex}^{p \cdot q}$ it is enough to describe lattices: $L(1), L(p), L = (p \cdot q)$.

There are the following partitions of the set $\{\cdot, ^{-1}, e\}$:

$$P_0 = \{\{\cdot\}, \{-1\}, \{e\}\},$$

$$P_1 = \{\{\cdot, ^{-1}\}, \{e\}\},$$

$$P_2 = \{\{\cdot, e\}, \{-1\}\},$$

$$P_3 = \{\{-1, e\}, \{\cdot\}\},$$

$$P_4 = \{\{\cdot, ^{-1}, e\}\}.$$

The partition P_0 we will traditionally denote by Ex , and the partition P_4 by N .

Let us put:

$$(3.20) \quad C^n = Mod(Cn(Ex(\mathcal{G}^n) \cup \{e \cdot e \approx e^{-1}\})),$$

$$(3.21) \quad C_N^n = Mod(Cn(Ex(\mathcal{G}^n) \cup \{e \approx e \cdot e, e \approx e^{-1}\})).$$

In [6] was prove the following theorem:

THEOREM 3.4. *The lattice of subvarieties of the variety \mathcal{G}_{Ex}^n , where n is prime, looks like the diagram in Figure 1.*

From this follows:

COROLLARY 3.1. *The lattice $L(1)$ looks like the diagram in Figure 2.*

COROLLARY 3.2. *If n is prime number then the lattice $L(n)$ looks like the diagram in Figure 3.*

Let us put:

$$(3.22) \quad \mathcal{P}_{p \cdot q}^{\{k_1, \dots, k_s\}} = Mod(Cn(Ex(\mathcal{G}^{p \cdot q}) \cup \bigcup_{i=1}^s \{x^0 \cdot x^{k_i} \approx ((x^{k_i})^{-1})^{-1}\})),$$

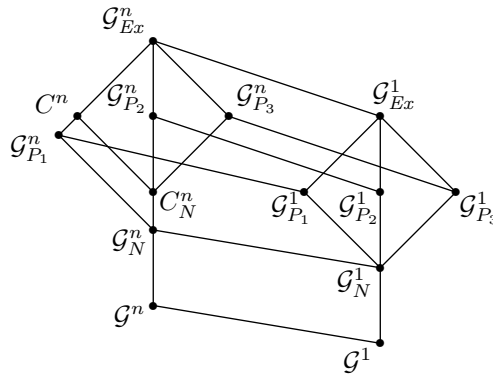


Figure 1. The lattice $\mathcal{L}(\mathcal{G}_{Ex}^n)$, where n is a prime number

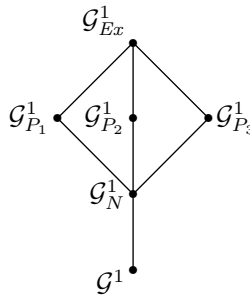


Figure 2. The lattice $L(1)$

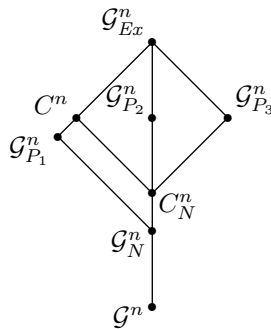


Figure 3. The lattice $L(n)$, where n is prime

$$(3.23) \quad \mathcal{P}_{p \cdot q, N}^{\{k_1, \dots, k_s\}} = \text{Mod}(Cn(Ex(\mathcal{G}^{p \cdot q}) \cup \{e \approx e \cdot e, e \approx e^{-1}\} \cup \bigcup_{i=1}^s \{x^0 \cdot x^{k_i} \approx ((x^{k_i})^{-1})^{-1}\})),$$

where $k_1, \dots, k_s \in \{0, 1, \dots, p \cdot q\}$.

We have the following lemma:

LEMMA 3.3. For any no-empty set $\{k_1, \dots, k_s\} \subseteq \{0, 1, \dots, p \cdot q\}$ there is a set $\{l_1, \dots, l_s\} \subseteq \mathcal{K}_{p \cdot q}$, such that $\mathcal{P}_{p \cdot q}^{\{k_1, \dots, k_s\}} = \mathcal{P}_{p \cdot q}^{\{l_1, \dots, l_s\}}$ and $\mathcal{P}_{p \cdot q, N}^{\{k_1, \dots, k_s\}} = \mathcal{P}_{p \cdot q, N}^{\{l_1, \dots, l_s\}}$.

PROOF. It is enough to take $(p \cdot q, k_i)$ as $l_i \in \{l_1, \dots, l_s\}$. □

From now on the classes (3.22) and (3.22) we will be indexed with subsets of the set $\mathcal{K}_{p \cdot q}$.

LEMMA 3.4. If $1 \in \{k_1, \dots, k_s\}$, then $\mathcal{P}_{p \cdot q}^{\{k_1, \dots, k_s\}} = \mathcal{G}_{P_1}^{p \cdot q}$.

PROOF. In [7] it was proved that the identities: (3.1), (3.2), (3.3), (3.5) and

$$(3.24) \quad x^{-1} \cdot e \approx x^{-1}$$

form an equational base of the variety defined by all P_1 -compatible identities of Abelian groups. To show that $\mathcal{P}_{p \cdot q}^{\{k_1, \dots, k_s\}} \subseteq \mathcal{G}_{P_1}^{p \cdot q}$, it is enough to observe that the identity (3.24) belongs to the set $Cn(Ex(\mathcal{G}^{p \cdot q}) \cup \bigcup_{i=1}^s \{x^0 \cdot x^{k_i} \approx ((x^{k_i})^{-1})^{-1}\})$. By substitution of the term $(x^{-1})^{-1}$ in (3.24) for variable x and by the fact that identities $x^{-1} \approx (x^{-1})^{-1})^{-1}$ and $e \cdot x^{-1} \approx x^0 \cdot x^{-1}$ belong to the set $Ex(\mathcal{G}^{p \cdot q})$ we get that $\mathcal{P}_{p \cdot q}^{\{k_1, \dots, k_s\}} \subseteq \mathcal{G}_{P_1}^{p \cdot q}$. The reverse inclusion is obvious. □

One can see that the following lemma is easy to prove:

LEMMA 3.5. If $k_1 = \dots = k_s = p \cdot q$, then $\mathcal{P}_{p \cdot q}^{\{k_1, \dots, k_s\}} = C^{p \cdot q}$.

THEOREM 3.5. The lattice $L(p \cdot q)$ has the following diagram:

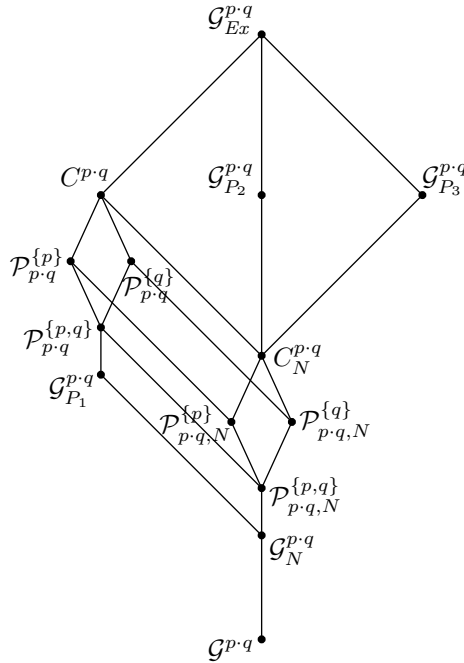


Figure 4. The Lattice $L(p \cdot q)$.

PROOF. By the definition of varieties from the above theorem it follows that every such variety belongs to $L(p \cdot q)$. Similarly as in the theorem (3.4), we prove the correctness of mutual placement of varieties: $\mathcal{G}^{p \cdot q}$, $\mathcal{G}_N^{p \cdot q}$, $\mathcal{G}_{P_1}^{p \cdot q}$, $\mathcal{G}_{P_2}^{p \cdot q}$, $\mathcal{G}_{P_3}^{p \cdot q}$, $\mathcal{G}_{Ex}^{p \cdot q}$, $C^{p \cdot q}$, $C_N^{p \cdot q}$. From the definition of each of classes $\mathcal{P}_{p \cdot q}^{\{p\}}$, $\mathcal{P}_{p \cdot q}^{\{q\}}$, $\mathcal{P}_{p \cdot q,N}^{\{p\}}$, $\mathcal{P}_{p \cdot q,N}^{\{q\}}$, $\mathcal{P}_{p \cdot q,N}^{\{p,q\}}$ it follows that $\mathcal{P}_{p \cdot q}^{\{p,q\}} \subseteq \mathcal{P}_{p \cdot q}^{\{p\}} \subseteq C^{p \cdot q}$ and $\mathcal{P}_{p \cdot q,N}^{\{p,q\}} \subseteq \mathcal{P}_{p \cdot q,N}^{\{p\}} \subseteq C_N^{p \cdot q}$. We will prove that indicated classes are the only elements of $L(p \cdot q)$. By theorems (3.2) and (3.3) it follows that to describe the lattice $L(p \cdot q)$ one have to consider all classes of models defined by theories $Cn(Ex(\mathcal{G}^{p \cdot q}) \cup E)$, where E is a subset of the set of identities of the form:

$$(3.25) \quad e \approx x^0 \cdot x^0,$$

$$(3.26) \quad e \approx ((x^0)^{-1})^{-1},$$

$$(3.27) \quad x^0 \cdot x^k \approx ((x^k)^{-1})^{-1},$$

$$(3.28) \quad x \approx x^0 \cdot x,$$

$$(3.29) \quad x \approx (x^{-1})^{-1},$$

where $k \in \{0, 1, \dots, p \cdot q - 1\}$. If to E belongs an identity of the form: (3.28) or (3.29), then $Cn(Ex(\mathcal{G}^{p \cdot q}) \cup E) = Id(\mathcal{G}^{p \cdot q})$.

From lemmas (3.3), (3.4) and (3.5) it follows that essential ones are only those subsets of the set of identities (3.25)–(3.29), to which belong only identities of the form: (3.25), (3.26), $x^0 \cdot x^p \approx ((x^p)^{-1})^{-1}$ or $x^0 \cdot x^q \approx ((x^q)^{-1})^{-1}$. However, it appears that while considering all non-empty subsets of the set whose elements are identities (3.25), (3.26), $x^0 \cdot x^p \approx ((x^p)^{-1})^{-1}$ or $x^0 \cdot x^q \approx ((x^q)^{-1})^{-1}$ we obtain the class of models being one of the following varieties: $C^{p \cdot q}$, $C_N^{p \cdot q}$, $\mathcal{P}_{p \cdot q}^{\{p\}}$, $\mathcal{P}_{p \cdot q}^{\{q\}}$, $\mathcal{P}_{p \cdot q}^{\{p, q\}}$, $\mathcal{P}_{p \cdot q, N}^{\{p\}}$, $\mathcal{P}_{p \cdot q, N}^{\{q\}}$, $\mathcal{P}_{p \cdot q, N}^{\{p, q\}}$.

By constructing the appropriate one-element free algebras in each of classes of the lattice $L(p \cdot q)$ one can show that these classes are mutually different. \square

References

- [1] G. Birkhoff, *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society 31 (1935), 433–454.
- [2] W. Chromik, “On externally compatible identities of algebras”, *Demonstratio Mathematica* 23(2) (1990), 345–355.
- [3] K. Gajewska-Kurdziel and K. Mruczek, “On some sets of identities satisfied in Abelian groups”, *Demonstratio Mathematica* 35(3) (2002), 447–453.
- [4] K. Gajewska-Kurdziel and K. Mruczek-Nasieniewska, “The lattice of subvarieties of the variety defined by externally compatible identities of Abelian groups of exponent n ”, *Studia Logica*. To appear.
- [5] I. J. Mel’nik, “Nilpotent shifts of varieties, (in Russian)”, *Mat. Zametki*, 14(5) (1973). English translation: *Math. Notes* 14 (1973), 962–966.
- [6] K. Mruczek, “On some lattice of varieties related to changes of the type”, in: K. Denecke K. and H.-J. Vogel (eds.), *General Algebra and Applications*, Proceedings of the 59th Workshop on General Algebra, Shaker Verlag, Aachen 2000, pp. 147–153.
- [7] K. Mruczek-Nasieniewska, “ P -compatible Abelian Groups”, *Logic and Logical Philosophy* 14(2) (2005), 253–263.
- [8] J. Płonka, “On the subdirect product of some equational classes of algebras”, *Math. Nachr.* 63 (1974), 303–305.
- [9] J. Płonka, “ P -compatible identities and their applications to classical algebras”, *Math. Slovaca* 40(1) (1990), 21–30.

- [10] A. Tarski, “Equational logic and equational theories of algebras”, in: H. A. Schmidt, K. Schütte, and H. J. Thiele (eds.), *Contributions to Mathematical Logic*, North Holland Publ. Co., Amsterdam 1968, pp. 275–288.

KRYSTYNA MRUCZEK-NASIEŃIEWSKA

Department of Logic,

Nicolaus Copernicus University,

ul. Asnyka 2,

87–100 Toruń, Poland

mruczek@uni.torun.pl