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RELEVANCE LOGICS, PARADOXES OF CONSISTENCY AND THE K RULE II
A Non-constructive Negation

Abstract. The logic B_+ is Routley and Meyer's basic positive logic. We define the logics B_{K+} and $B_{K'+}$ by adding to B_+ the K rule and to B_{K+} the characteristic S4 axiom, respectively. These logics are endowed with a relatively strong non-constructive negation. We prove that all the logics defined lack the K axiom and the standard paradoxes of consistency.

Keywords: Implicative paradoxes, Substructural Logics, Relevance Logics, Ternary Relational Semantics.

1. Introduction

In [9] it is studied the effect of adding the K rule to relevance logics in the presence of a constructive negation and in respect of the paradoxes of consistency. The aim of this paper is to study the effect of adding the same rule to the same group of logics now in the presence of a non-constructive negation.

As it is known, paradoxes of implication are generally classified in "paradoxes of material implication" and "paradoxes of strict implication". Charles I. Lewis, the first author in drawing this classification, accurately distinguishes both classes ([4], p. 511).

> In material implication, the key paradoxes, implicating all the others are: A false proposition implies any proposition; a true proposition

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> is implied by any; any two false propositions are equivalent; any two true propositions are equivalent. Correspondingly, the key paradoxes of strict implication are: A contradictory (self inconsistent) proposition implies any proposition; an analytic proposition is implied by any; any two contradictory propositions are equivalent; any two analytic propositions are equivalent.

Characteristic exemplars of the class of paradoxes of material implication are the K axiom

(i) $\vdash A \to (B \to A)$

or the following two versions of the EFQ axiom ("*E falso quodlibet*" axiom)

(ii) $\vdash \neg A \to (A \to B)$

and

(iii) $\vdash A \to (\neg A \to B)$

Typical members of the class of paradoxes of strict implication are the K rule

(iv) $\vdash A \Rightarrow \ \vdash B \to A$

or the ECQ ("*E contradictione quodlibet*") axiom

(v) $\vdash (A \wedge \neg A) \to B$

Both classes of paradoxes are, of course, paradoxes (fallacies) of relevance in Anderson and Belnap's sense (see [1]).

But we are here also interested in another way of classifying the paradoxes of implication that cut across Lewis's classification. In [2] (p. 349), Urquhart note

> To those who have taken the trouble to read the literature on relevance logic rather than fulminate against it, it has been a familiar fact that there are two conceptually distinct classes of "paradoxes of material implication". The archetype of the first class (paradox of consistency) is $(A \ \& \sim A) \to B$. The archetype of the second (paradox of relevance) is $A \to (B \to A)$.

In addition to the schemes pointed out by Urquhart, scheme (iv) noted above is a characteristic paradox of relevance (in fact, the rule originates a potentially infinite number of them); on the other hand, typical members of paradoxes of consistency are (ii), (iii) and (v).

(It should be noted, in passing, that Lewis—in so many ways a precursor of relevance logics—was not unaware of the distinction as it is readily deducible from the following remark on the paradoxes of strict implication ([4], p. 513).

> It remains to suggest why these paradoxes of strict implication are paradoxical. Let us observe that they concern two questions: what is to be taken as consequence of an assumption which, being self contradictory, could not possibly be the case; and what is to be taken as sufficient premise for that which being analytic, could not possibly fail to be the case).

Note that, as a matter of fact, paradoxes of consistency constitute a subclass of the class of paradoxes of relevance: that formed by those theses stating, in one way or another, that given a contradiction (a proposition and its negation), any proposition whatsoever is derivable. Relevance logicians have been especially interested (and so are we) in the possibilities of adding paradoxes of consistency to relevance logics without having, in general, the "other class", i.e., paradoxes of relevance. On the part of relevance logicians this interest is included in a more general one: that of exploring the frontiers between relevance and non-relevance logics, a task pursued since the beginning of the relevance enterprise. Notorious examples of these borderline cases are the well known logic R Mingle (see [1]) or the logics KR, CR and CE (see [5], [7], [8], [10]). The logic KR is the result of adding the axiom ECQ (v) to the logic of relevance R, and, on the other hand, the logic CR and the logic CE are obtained by adding a boolean negation to R and to the logic of entailment E, respectively. The present investigation can be viewed in a similar way. As it was remarked above, it is shown in [9] what kind of logics we have when the K rule is added to relevance logics in the context of constructive negation. The purpose of this paper is to study what happens if the context is a relatively strong non-constructive negation. Now, B_{+} is Routley and Meyer's basic positive logic (see [10]). Then B_{K+} is obtained by adding the K rule to B_{+} and $\mathrm{B}_{K'+}$ is an S4-type extension of B_{K+}. Next, the logics B_{Kc} and $\mathrm{B}_{K'c}$ are the extensions of B_{K+} and $\mathrm{B}_{K'+}$ with the contraposition axioms

(vi) $\quad (A \to \neg B) \to (B \to \neg A)$

and

(vii) $\quad (\neg A \to B) \to (\neg B \to A)$

respectively. Finally, the logics B_{Kci} and $\mathrm{B}_{K'ci}$ are defined by adding the axiom

(viii) $(A \to B) \vee \neg(A \to B)$

to B_{Kc} and $\mathrm{B}_{K'c}$, respectively. Well, we shall prove that the theses (i), (ii), (iii) and (v) are not provable in any of these logics. Moreover, the logics here defined are subsystems of S4 as axiomatized by Hacking (see [3]). Therefore, they all are modal logics (the arrow is some kind of strict implication).

So, let us try to formalize the definition of "paradox of strict implication" given by Lewis in the passage quoted above. In a standard modal logic (and in the logics treated in this paper as well) in which

(ix) $A \leftrightarrow \neg\neg A$

is a theorem, an implicative formula $A \to B$ is a "paradox of strict implication" iff

a. B is a theorem ($A \neq B$) or

b. A is the negation of a theorem.

Now, given the K rule and the contraposition axioms (vi) and (vii) (also present in any standard modal logic and in the logics present in this paper), we have for any theorems A, B

(a) $\vdash A \leftrightarrow B$

(b) $\vdash \neg A \leftrightarrow \neg B$

But, as it was noted above, in our logics (v), i.e. '$(A \wedge \neg A) \to B$', (which is provable even in Lewis's S1) is not derivable. Therefore, though

(c) If B is a theorem, $\vdash \neg B \leftrightarrow (A \wedge \neg A)$

is provable in all Lewis's systems, it is, however, not derivable in any of the logics we study here. In consequence, for the logics in this paper, we have:

a. None of them contains paradoxes of material implication: all are included in Lewis's S4 (as axiomatized by Hacking [3]) which does not contain this class of paradoxes.

b. All of them have, of course, standard paradoxes of strict implication (they all have the K rule) except the ECQ axiom (v).

c. All of them have paradoxes of relevance: they all have the K rule.

d. None of them has, of course, the standard paradoxes of consistency of material implication (ii) and (iii). Moreover,

e. None has the standard paradox of strict implication (v).

But note that,

f. In B_{Kci} and $\mathrm{B}_{K'ci}$, (v) is derivable when A is an implicative formula (see §9) and, most of all, if formulas of the form $A \to B$ (A is the negation of a theorem) are considered paradoxes of consistency of strict implication, all have paradoxes of consistency of strict implication.

The structure of the paper is as follows. In sections 2–5, the logics B_+, B_{K+} and $\mathrm{B}_{K'+}$ are studied. Sections 6–8 are devoted to B_{Kc} and 9–10 to B_{Kci}. Finally, in sections 11–12 some possibilities for extending the logics previously defined are treated. A final Appendix provides simple matrix proofs of some important facts claimed throughout the paper.

2. The positive logic B_{K+}

B_{K+} is axiomatized with

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of derivation are

Modus ponens (MP): $(\vdash A \;\&\; \vdash A \to B) \Rightarrow \;\vdash B$

Adjunction (Adj.): $(\vdash A \;\&\; \vdash B) \Rightarrow \;\vdash A \wedge B$

Suffixing (Suf.): $\vdash A \to B \Rightarrow \;\vdash (B \to C) \to (A \to C)$

Prefixing (Pref.): $\vdash A \to B \Rightarrow \;\vdash (C \to A) \to (C \to B)$

K: $\vdash A \Rightarrow \;\vdash B \to A$

Therefore, B_{K+} is B_+ with the addition of the K rule.

3. Semantics for $\mathbf{B}_{K+}$

A B_{K+} *model* is a triple $\langle K, R, \vDash\rangle$ where K is a non-empty set, and R is a ternary relation on K subject to the following definitions and postulates for all a, b, c, $d \in K$ with quantifiers ranging over K:

d1. $a \leq b =_{df} \exists xRxab$

d2. $R^2abcd =_{df} \exists x(Rabx \ \&\ Rxcd)$

P1. $a \leq a$

P2. $(a \leq b \ \&\ Rbcd) \Rightarrow Racd$

P3. $(b \leq d \ \&\ Radc) \Rightarrow Rabc$

Finally, $\vDash$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositional variables p, wff A, B and $a \in K$:

(i) $(a \leq b \ \&\ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \rightarrow B$ iff for all b, $c \in K$, $(Rabc \ \&\ b \vDash A) \Rightarrow c \vDash B$

A formula A is B_{K+} *valid* ($\vDash_{B_{k+}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

Note that the postulates

P4. $Rabc \Rightarrow b \leq c$

P5. $(a \leq b \ \&\ b \leq c) \Rightarrow a \leq c$

and

P6. $R^2abcd \Rightarrow Rbcd$

are immediate in all B_{K+} models.

Regarding semantic consistency (soundness), the proof that all theorems of B_{K+} are valid is left to the reader (see, for example, [2] or [6] for a general strategy).

A final note. As it is known, there is a set of "designated points" in the standard semantics for relevance logics (see the two items just quoted above). It is in respect of this set that d1 is introduced and wff are evaluated. The absence of this set in B_{K+} semantics (and the corresponding changes in d1 and the definition of validity are the only (but crucial) differences between B_{+} models and B_{K+} models.

4. Completeness of B_{K+}

We begin by recalling some definitions:

A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory a is *regular* iff all the theorems of B_{K+} belong to a. Finally, a is *null* iff no wff belong to a.

Now, we define the B_{K+} canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and a, b, $c \in K^T$, $R^T abc$ iff if $A \rightarrow B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime non-null theories and R^C be the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the B_{K+} canonical model is the triple $\langle K^C, R^C, \vDash^C \rangle$.

Next, we sketch a proof of the completeness theorem.

Lemma 1. *If a is a non-null theory, then a is regular.*

Proof. Let $A \in a$ and B be a theorem. By the K rule, $A \rightarrow B$ is a theorem. So, $B \in a$. □

Lemmas 2–6 below are an easy adaptation of the corresponding B_+ lemmas (see, e.g., [6]) to the case of non-null theories (as it is known, theories are not necessarily non-null in the B_+ canonical model and, in fact, in the canonical model of any standard relevance logic).

Lemma 2. *Let A be any wff, a a non-null element in K^T and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 3. *Let a be a non-null element in K^T, $b \in K^T$ and c a prime member in K^T such that $R^T abc$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 4. *Let $a \in K^T$, b a non-null element in K^T and c a prime member in K^T such that $R^T abc$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Now, we set

Definition 1. Let a, $b \in K^T$. Then, $a \leq^T b$ iff $R^T xab$ and $x \in K^C$.

We have

Lemma 5. *$a \leq^T b$ iff $a \subseteq b$.*

And consequently,

Lemma 6. *$a \leq^C b$ iff $a \subseteq b$.*

Note that b and c in Lemma 3 and a and c in lemma 4 need not be non-null. On the other hand, Lemma 7 below follows immediately from Lemma 2.

Lemma 7. *If $\nvdash_{B_{K+}} A$, then there is some $x \in K^C$ such that $A \notin x$.*

Lemma 8. *Let a, b be non-null theories. The set $x = \{B \mid \exists A[A \to B \in a$ and $A \in b]\}$ is a non-null theory such that $R^T abx$.*

Proof. It is easy to prove that x is a theory such that $R^T abx$. We prove that x is non-null. Let $A \in b$. By Lemma 1, $A \to A \in a$. So, $A \in x$ by $R^T abx$. □

The following three lemmas are proved similarly as in the standard semantics (use Lemma 8 in the proof of the canonical adequacy of clause (iv)).

Lemma 9. *The canonical postulates hold in the* B_{K+} *canonical model.*

Lemma 10. *$\vDash^C$ is a valuation relation satisfying conditions (i)–(iv) above.*

Lemma 11. *The canonical model* B_{K+} *is in fact a model.*

By lemmas 7 and 11, we have

Theorem 1 (Completeness of B_{K+}). *If $\vDash_{B_{K+}} A$, then $\vdash_{B_{K+}} A$.*

5. The logic $\mathbf{B}_{K'+}$

The logic $\mathrm{B}_{K'+}$ is the result of adding the axiom

A7. $(A \to B) \to [C \to (A \to B)]$

to B_{K+} (we note that B_{K+} and $\mathrm{B}_{K'+}$ are different logics. See Appendix). A $\mathrm{B}_{K'+}$ model is defined similarly as a B_{K+} model save for the addition of the postulate

P7. $R^2 abcd \Rightarrow Racd$

In order to prove semantic consistency, it remains to prove that A7 is valid (use P7). On the other hand, to prove completeness, it remains to prove that P7 is canonically valid. So, suppose $R^2 abcd$, i.e., $R^C abx$ and $R^C xcd$ for some $x \in K^C$. Further, suppose $A \to B \in a$, $A \in c$ for some wff A, B. We have to prove $B \in d$. Now, let $C \in b$. By A7, $C \to (A \to B) \in a$. So, $A \to B \in x$ ($R^C abx$, $C \in b$). Therefore, $B \in d$ ($R^C xcd$, $A \in c$).

6. The logic B_{Kc}

The logic B_{Kc} (B_{K+} with the contraposition axioms) is the result of adding

A8. $(A \to \neg B) \to (B \to \neg A)$

and

A9. $(\neg A \to B) \to (\neg B \to A)$

to B_{K+}. Some theorems of B_{Kc} are, for example, the following

T1.	$A \to \neg\neg A$	A1, A8
T2.	$\neg\neg A \to A$	A1, A9
T3.	$(A \to B) \to (\neg B \to \neg A)$	A8, T1
T4.	$(\neg A \to \neg B) \to (B \to A)$	A9, T1
T5.	$\neg (A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	A8, T3
T6.	$\neg (A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	A9, T3
T7.	$(A \vee B) \leftrightarrow \neg (\neg A \wedge \neg B)$	A8, A9, T5
T8.	$(A \wedge B) \leftrightarrow \neg (\neg A \vee \neg B)$	A8, A9, T6

We also note the rule

Recq. $\vdash A \Rightarrow \ \vdash \neg A \to B$ K, A9

7. Semantics for B_{Kc}

A B_{Kc} *model* is a quadruple $\langle K, R, *, \vDash\rangle$, where $*$ is an operation on K, and K, R and $\vDash$ are defined like in B_{K+} models except that the following clause and postulates are added

(v). $a \vDash \neg A$ iff $a* \nvDash A$

P8. $a = a**$

P9. $Rabc \Rightarrow Rac * b*$

$\vDash_{\mathrm{B}_{Kc}} A$ (A is B_{Kc} valid) iff $a \vDash A$ for all $a \in K$ in all models.

The easy proof of semantic consistency (soundness) is left to the reader (see, e.g., [2]).

8. Completeness of B_{Kc}

The main difference between the proof to be developed and the standard ones in normal relevance logics can be stated as follows. As it is known, canonical points need not be complete or consistent in normal relevance logics. Though not necessarily complete, they *must* be consistent, however, in B_{Kc} and any logic that includes it as in some of the exemplars we develop in the following sections.

The canonical model is the quadruple $\langle K^C, R^C, *^C, \vDash^C\rangle$, where R^C and $\vDash^C$ are defined similarly as in the B_{K+} canonical model and K^C is the set of all consistent prime non-null theories (a theory is inconsistent if it contains the negation of a theorem). Finally, let us define $*^T$ as follows: for any $a \in K^T$, $a*^T = \{A : \neg A \notin a\}$. Well, $*^C$ is the restriction of $*^T$ to K^C.

Now, let us define:

Definition 2. a is a degenerate theory iff every wff belongs to a.

We have:

Proposition 1. *a is a degenerate theory iff a is inconsistent.*

Proof. By Recq. □

We note that, given that all theories are regular, if a is inconsistent, it contains a contradiction. The converse, however, is not provable.

Now, it is clear that to prove the completeness of B_{Kc}, some of the lemmas in Section 4 must be modified. In fact, lemmas 2–5 and the canonical adequacy of clause (iv) in Lemma 10 must be modified. Well, by using Proposition 1, it is not difficult to prove the required modifications. Let us, for example, prove Lemma 2. It would now read:

Lemma 12. *Let A be any wff, a, a consistent non-null theory in K^T and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Proof. By Zorn's Lemma there is a maximal consistent non-null theory x such that $a \subseteq x$ and $A \notin x$. If x is not prime, then $B \vee C \in x$, $B \notin x$, $C \notin x$ for some wff B, C. Define then

$$[x, B] = \{D : \exists E[E \in x \ \& \ \vdash_{B_{Kc}} (B \wedge D) \to E]\}$$

Define $[x, C]$ similarly. It is not difficult to prove that $[x, B]$ and $[x, C]$ are theories strictly including x. By the maximality of x there are three possible situations:

1. $[x, B]$ and $[x, C]$ are inconsistent.

2. $A \in [x, B]$ and $A \in [x, C]$

3. $[x, B]$ is inconsistent and $A \in [x, C]$ or $[x, C]$ is inconsistent and $A \in [x, B]$.

By using Proposition 1, it is proved that from any of these possibilities Situation 2 follows. But then it is easy to prove $A \in x$ which contradicts the hypothesis. □

Next, we have

PROPOSITION 2. *For any wff A and $a \in K$, $\neg A \in a*^T$ iff $A \notin a$ (for any wff A and $a \in K$, $\neg A \in a*^C$ iff $A \notin a$).*

PROOF. Definitions and T1, T2. □

PROPOSITION 3. *If a is a consistent theory, then $a*^C$ is a non-null theory.*

PROOF. Let A be a theorem and $A \notin a*^C$. Then, $\neg A \in a$ contradicting the consistency of a. □

PROPOSITION 4. *If a is a regular theory, then $a*^C$ is a consistent theory.*

PROOF. Let A be a theorem. Suppose $\neg A \in a*^C$. Then $A \notin a$ contradicting the regularity of a. □

The following proposition is proved similarly as in the standard semantics for relevance logics (see, e.g., [2]).

PROPOSITION 5. *If a is a prime theory, then so is $a*^C$.*

PROOF. By using T3, T5 and T6. □

From propositions 3–5 we have

PROPOSITION 6. *$*^C$ is an operation on K, that is, if a is a consistent prime non-null theory, then so is $a*^C$.*

In order to establish the completeness of B_{Kc}, it remains to prove that postulates P7 and P8 and clause (v) are valid when read canonically. Well, it can be accomplished as in standard semantics for relevance logics (see, e.g., [2]).

9. The logic $\mathbf{B}_{\boldsymbol{Kci}}$

In trying to extend B_{Kc} and $\mathrm{B}_{K'c}$ with more negation principles, the addition of the "reductio axioms" such as

(a) $(A \to B) \to [(A \to \neg B) \to \neg A]$

(b) $(\neg A \to B) \to [(\neg A \to \neg B) \to A]$

(c) $(A \to \neg A) \to \neg A$

(d) $(\neg A \to A) \to A$

suggests itself by similarity with relevance logics. Unfortunately, the standard paradox of consistency

(v) $(A \wedge \neg A) \to B$

would then be immediate. Moreover, (v) would still be derivable if instead of (a)–(d), we introduce the considerable more weak

(e) $A \vee \neg A$

(f) $\neg(A \wedge \neg A)$

(g) If $\vdash B \to A$ and $\vdash B \to \neg A$, then $\vdash \neg B$

(h) If $\vdash A \to \neg A$, then $\vdash \neg A$

(i) If $\vdash \neg B \to A$ and $\vdash \neg B \to \neg A$, then $\vdash B$

(j) If $\vdash \neg A \to A$, then $\vdash A$

(k) If $\vdash A \to B$ and $\vdash \neg A \to B$, then $\vdash B$

In fact, they are equivalent to (v) given B_{Kc}. Nevertheless, we prove that if (e)–(k) are restricted to the case where A is an implicative formula (A is *implicative* if A is of the form $B \to C$), then they can be added to B_{Kc}, the paradox of consistency (v) being unprovable. Thus, the logic B_{Kci} is the result of adding the axiom

A10 $(A \to B) \vee \neg (A \to B)$

to B_{Kc}. In addition to T1–T8 and Recq, the following theses are also theorems of B_{Kci}.

T10. $\neg [(A \to B) \wedge \neg (A \to B)]$ A10, T1, T2, T5

T11. $[(A \to B) \wedge \neg (A \to B)] \to C$ K, T4, T10

T12. If $\vdash A \to (B \to C)$ and $\vdash A \to \neg(B \to C)$, then $\vdash \neg A$ T3, T10

T13. If $\vdash (A \to B) \to \neg(A \to B)$, then $\vdash \neg(A \to B)$ T12

T14. If $\vdash \neg A \to (B \to C)$ and $\vdash \neg A \to \neg(B \to C)$, then $\vdash A$ T2, T12

T15. If $\vdash \neg(A \to B) \to (A \to B)$, then $\vdash (A \to B)$ T2,T14

T16. If $\vdash (A \to B) \to C$ and $\vdash \neg(A \to B) \to C$, then $\vdash C$ A10

We note that A10, T10, T12–T16 are, respectively, (e)–(k) restricted to the case where A is an implicative formula; T11 is (v) restricted in the same way. Finally, we remark that A10 and T10–T16 are equivalent given B_{Kc} (proof is left to the reader).

10. Semantics of B_{Kci}

A B_{Kci} *model* is similar as a B_{Kc} model save for the addition of the postulate

P10. $Rabc \Rightarrow Ra * bc$

$\vDash_{\mathrm{B}_{Kci}} A$ (A is B_{Kci} valid) iff $a \vDash A$ for all $a \in K$ in all models.

The proof that B_{Kci} is semantically consistent is left to the reader (A10 is proved valid with P10). Regarding completeness, it is obvious that we have to prove only that P10 is valid when read canonically. This follows immediately from

Proposition 7. *For any prime non-null a in K^T and b, $c \in K^T$, $R^T abc \Rightarrow R^T a * bc$.*

Proof. Let $R^T abc$ for some prime non-null a in K^T. Suppose further $A \to B \in a*^T$, $A \in b$ and $B \notin c$ for some wff A, B, C. By Proposition 2 and definitions, $\neg(A \to B) \notin a$, $\neg A \notin b*^T$, $\neg B \in c*^T$; by P9, $R^T ac *^T b*^T$. By A10 and the primeness of a, $A \to B \in a$ or $\neg(A \to B) \in a$. So, $A \to B \in a$. By T3, $\neg B \to \neg A \in a$, whence, by $R^T ac *^T b*^T$ and $\neg A \notin b*^T$, we have $\neg B \notin c*$ contradicting $\neg B \in c*^T$ supra. □

11. The logics $\mathrm{B}_{K'c}$ and $\mathrm{B}_{K'ci}$

The logics $\mathrm{B}_{K'c}$ and $\mathrm{B}_{K'ci}$ are axiomatized by adding A7 to the logics B_{Kc} and B_{Kci}, respectively. $\mathrm{B}_{K'c}$ models and $\mathrm{B}_{K'ci}$ models are defined similarly as B_{Kc} and B_{Kci} models, respectively, save for the addition of P7.

12. Strengthening the logics

The logic B_{Kci} can be strengthened without the K axiom and the different versions of ECQ being derivable. Let us briefly discuss some possibilities.

Consider the axioms prefixing

A11. $(B \to C) \to [(A \to B) \to (A \to C)]$

suffixing

A12. $(A \to B) \to [(B \to C) \to (A \to C)]$

contraction

A13. $[A \to (A \to B)] \to (A \to B)$

and the rule of derivation assertion

A14. If $\vdash A$, then $\vdash (A \to B) \to B$

We note the following proposition, whose proof is left to the reader.

Proposition 8. *Given A11,* B_{K+} *and* $\mathrm{B}_{K'+}$ *are equivalent. So,* B_{Kc} *and* $\mathrm{B}_{K'c}$, B_{Kci} *and* $\mathrm{B}_{K'ci}$ *are equivalent.*

On the other hand, we recall the following positive logics. TW_+ ("Contractionless positive Ticket Entailment") is B_+ plus A11 and A12. The logic T_+ ("Positive Ticket Entailment") is TW_+ plus A13 and the logic E_+ ("Positive Logic of Entailment") is T_+ plus A14 (cf. about these logics [2]). So, TW_{K+}, T_{K+}, E_{K+} are TW_+, T_+ and E_+ plus the K rule, respectively.

Next, the logics TW_{Kc}, T_{Kc} and E_{Kc} are defined by adding the axioms A8 and A9 to TW_{K+}, T_{K+} and E_{K+}, respectively. And on the other hand, the logics TW_{Kci}, T_{Kci} and E_{Kci} are the result of adding A10 to TW_{Kc}, T_{Kc} and E_{Kc}, respectively.

Now, though TW_{Kc} and TW_{Kci} are different logics (see Appendix), we remark the following

Proposition 9. T_{Kc} *and* T_{Kci} *(and so,* E_{Kc} *and* E_{Kci}*) are equivalent logics.*

Proof. Derive $(A \to B) \to [\neg(A \to B) \to C]$ (A7, A9). Then, T11 follows by $[A \to (B \to C)] \to [(A \wedge B) \to C]$. But T11 and A10 are equivalent given B_{Kc}. □

Let us define now the semantics. Consider the following postulates (see [2])

P11. $R^2abcd \Rightarrow (\exists x \in K)\,(Rbcx \;\&\; Raxd)$

P12. $R^2abcd \Rightarrow (\exists x \in K)\,(Racx \;\&\; Rbxd)$

P13. $Rabc \Rightarrow R^2abbc$

P14. $(\exists x \in K)\,Raxa$

Given B_{K+} semantics, the postulates P11, P12, P13 and P14 are the corresponding postulates for A11, A12, A13 and A14 (that is, the axiom is proved valid with the respective postulate, and the postulate is proved canonically valid with the respective axiom, given the logic B_{K+} and B_{K+} semantics). Consequently, TW_{Kc} models are defined similarly as B_{Kc} models save for the addition of postulates P11 and P12. And TW_{Kci} models, T_{Kci} models and E_{Kci} models are defined similarly as B_{Kci} models except for the addition of the postulates P11 and P12, the postulate P13, and the postulate P14, respectively. Therefore, soundness and completeness of TW_{Kc}, TW_{Kci}, T_{Kci} and E_{Kci} are immediate from those of B_{Kc} or B_{Kci} and the fact that P11, P12, P13 and P14 are the corresponding postulates for A11, A12, A13 and A14 respectively.

13. Appendix

1. Consider the following set of matrices where the only designated value is 3.

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	2
2	2	2	3	3	1
3	0	2	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

This set satisfies the axioms and rules of B_{Kci} but falsifies A7 (e.g., $v(A) = 2$, $v(B) = 1$, $v(C) = 3$) thus showing that B_{Kci} and $\mathrm{B}_{K'ci}$ are different systems.

2. Consider the following set of matrices where the only designated value is 2.

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	1
2	0	1	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

This set satisfies the axioms and rules of TW_{Kc} but falsifies $A \vee \neg A$ when $v(A) = 1$ thus proving that TW_{Kc} and TW_{Kci} are different systems.

3. Consider the following set of matrices where the only designated value is 2.

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	0	2	2	1
2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

This set satisfies the axioms and rules of E_{Kci} but falsifies $A \to (B \to A)$ when $v(A) = 1$, $v(B) = 2$; $(A \wedge \neg A) \to B$, $A \to (\neg A \to B)$, $\neg A \to (A \to B)$ when $v(A) = 1$, $v(B) = 0$; and $A \vee \neg A$ when $v(A) = 1$.

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