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WHAT IS THE LOGIC OF PROPOSITIONAL IDENTITY?†

Abstract. Propositional identity is not expressed by a predicate. So its logic is not given by the ordinary first order axioms for identity. What are the logical axioms governing this concept, then? Some axioms in addition to those proposed by Arthur Prior are proposed.

Keywords: propositions, identity, entailment, canceling-out, Geach, Prior, Quine.

1. Arthur Prior and Propositions

A view sufficiently widely accepted to be called the *standard* view is as follows. Consider the sentence

(∗) John believes that snow is white.

On the standard view this sentence is seen as built up in the following stages:

	John believes that snow is white
1.	snow is white
2.	that snow is white
3.	John believes that snow is white

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At stage 1 the sentence ‘snow is white’ is built up from the substantive ‘snow’ and the predicate phrase ‘is white’. At stage 2 the sentence nominalization ‘that snow is white’ is constructed by affixing ‘that’ to the result of the stage 1 construction, ‘snow is white’. At stage 3 the verb ‘believes’ is flanked by the substantives ‘John’ and ‘that snow is white’.

The type of semantics that typically accompanies this syntactic analysis treats ‘believes’ as a two place predicate and both ‘John’ and ‘that snow is white’ as designating phrases. The former designates a person, the latter designates a special object. The key point is that on this view the semantic function of the word ‘that’ is to form a designator from a sentence.

Further, on the standard view (∗) has a more explicit paraphrase in

(∗∗) John believes the proposition that snow is white.

Here the phrase ‘the proposition that’ also serves to form a designator from a sentence — but in this case a designator which serves as well to indicate the type to which its designation belongs. On the view at hand, the operator ‘the proposition that’ makes it explicit that belief is a relation between persons and propositions. It is customarily supposed that the referent of a phase of the form ‘ the proposition that *p*’ is the sense of the sentence *p*, so that a sentence has a sense in virtue of standing in a relation to a certain type of object.

Prior’s view can be most easily understood in contrast with the above analysis. On Prior’s view (∗) has the following constructional history:

	John believes that snow is white
1.	snow is white
2.	John believes that snow is white

At stage 1 the sentence ‘snow is white’ is formed by affixing the substantive ‘snow’ to the predicate phrase ‘is white’. At stage 2 the phrase ‘believes that’ is flanked to the left by the substantive ‘John’ and to the right by the sentence ‘snow is white’. On Prior’s view ‘believes that’ is a predicate at one end and a connective at the other.

On this view belief is *not* a relation. Similar analyses are offered for such locutions as ‘hopes’, ‘fears’, ‘wishes’, ‘brings it about’ etc.

The key point for Prior is that the so-called nominalizing expressions like ‘that’ and ‘the proposition that’ do not serve to form designators from sentences. Rather, these expressions serve to so modify verb phrases as to allow their connection to sentences. That is, they serve to form connectives from verb phrases, not designators from sentences. (See, in particular, Prior 1963 and Chapter 2 of Prior 1971.)

In the *Tractatus*, Wittgenstein rejects Frege's view that sentences are names (3.143). A sentence is regarded as a type of picture that "represents a possible situation" (2.202). This situation is its sense (2.221) and cannot be named (3.144).

Following Wittgenstein, Prior held that what a sentence says (how a sentence says things are) cannot be signified except by a sentence or the utterance of a sentence. No sign not a sentence or, introduced via a sentence, can signify what a sentence says. In particular, no sign not a sentence can name what a sentence says. Since no sentence is a name, what a sentence says cannot be named. (See "Ramsey and the Later Wittgenstein" in Prior 1967.)

Prior's anti-nominalization view holds that each expression such as 'that' classically construed as a device of sentence nominalization *never* forms names from sentences. Prior thought instead that such expressions serve either as connectives or as devices for converting other expressions into connectives.

The anti-nominalization thesis follows from Prior's basic principle that only a sentence can signify what a sentence says. For if this is so, then what a sentence says cannot be named. Certainly a non-sentence such as 'that snow is white' cannot name what a sentence says because naming is a way of signifying. What about the sentence itself? Can it name what it says? That is also ruled out by the basic principle. Suppose, for example, that 'Snow is white' names what it says. Then 'the object named by 'Snow is white'' would also name what 'Snow is white' says. But that violates the basic principle, since 'the object named by 'Snow is white'' is a non-sentence.

This paper assumes that Prior's theory of propositions is essentially correct. (This assumption is defended in Hugly and Sayward 1996.)

2. Applying the Theory to Propositional Identity

Concerning the concept of propositional identity Prior writes:

> When we assert or deny, as we sometimes may do, that the proposition that p is the very same proposition as the proposition that q — for example, when we assert that the proposition that for all x, x is red, is the same proposition as the proposition that for all y, y is red, and when we deny that the proposition that

> grass is green is the same proposition as the proposition that it is not the case that it is not the case that grass is green — we are employing such a connective, and there is no need to take the component phrase "the proposition that *p*" at all seriously as a name of an object; the phrase "the proposition that — " is merely part of the connective, just as it is merely part of a connective when we say "The proposition that *p* is materially equivalent to the proposition that *q*", i.e., "*p* if and only if *q*" (in one sense of "if"). (Prior 1976, p. 151)

One of the consequences of Prior's theory of propositions is that such concepts as propositional identity, material equivalence, logical equivalence and mutual entailment are not concepts of relations holding between objects called 'propositions'. These are concepts expressed by sentential connectives, not predicates. (Other people are also clear about this point. See, for example, Suszko, 1968 as well as Bloom and Suszko, 1972.)

Consider material implication, for example. This is expressed, e.g., by

That ... materially implies that _______

which is just a fluffed up way of saying

... $\supset$_______

In *Mathematical Logic* W. V. Quine holds that the concept material implication is expressed by a binary predicate applying to sentences, *not* a binary sentential connective:

> [...] *material implication*, may be said to hold whenever the truth-functional conditional which has the one statement as antecedent and the other as consequent is true. Thus one statement materially implies another provided merely that the first is false or the second true. This relation is so broad as not to deserve the name of implication at all except by analogy. But — and this is the point usually missed — 'materially implies' is still a binary predicate, not a binary statement connective. It stands to '$\supset$' precisely as 'is false' stand to '$\sim$'. (Quine 1947, p. 29)

(By a statement Quine means a sentence with truth conditions.)

Does Quine contradict what Prior has to say? Not really. Prior's primary point is that material implication is not a relation between special objects called ‘propositions’. He does not deny that ‘materially implies’ may have a use to express a relation between sentences.

Prior distinguishes between a redundant use of ‘true’ from a semantical use of that word. In its semantical use it is used to form a predicate applicable to sentences. In its redundant use it is used to form a unary sentential connective (‘it is true that’). (Prior 1971, pp. 98–107) A similar distinction can be made with respect to ‘materially implies’. In its semantical use, on which Quine focuses, it is used as a predicate applying to sentences. When used to connect sentences, however, as in

> That snow is white materially implies that grass is green,

it is just fluff for the connective ‘$\supset$’:

> Snow is white $\supset$ grass is green.

With respect to the concept of propositional identity, Prior would have no problem with such locutions as

> is synonymous with,
>
> has the same force as,
>
> has the same content as,
>
> expresses the same proposition as,

being predicates applying to sentences. What he denies is that there is a relation of identity holding among special objects called ‘propositions’. The locution

> That . . . is the very same proposition as the proposition that _______

is a binary sentential connective, and, if Prior is right, one which does not include or contain any binary predicate.

3. The Logic of Propositional Identity

Authors often say such things as “All the errors in this book are my own”, thus conversationally implicating that there are at least some false statements among those they have written. Reflecting on this Prior proves

> If it is said in the book that something said in the book is false, then *something other than that something said in the book is false* is said in the book and is false

is a theorem of logic. In symbols,

$$\delta(\exists p(\delta p \wedge \neg p)) \;.\!\supset\!.\; \neg\exists p(p = (\exists q(\delta q \wedge \neg q)) \wedge \delta p \wedge \neg p)$$

where 'δ' is a unary sentential connective and '$=$' is a binary sentential connective ('δp' is read as 'it is said in the book that p' and '$p = q$' is read as 'that p is the very same proposition as that q'). (Prior 1971, pp. 86–87)

In proving this theorem (as well as others) Prior employs a system of sentential quantification and propositional identity. It has these features:

> [...] (*a*) ordinary propositional calculus, enriched with variables for expressions which form sentences from sentences, with quantifiers binding variables standing for sentences, and with an identity-function with sentences as arguments; (*b*) the ordinary theory of quantification applied to our special quantifiers; and (*c*) ordinary laws of identity applied to our special function. This is simply a fragment of the discipline which Leśniewski called protothetic, to which Tarski himself made important contributions in the early 1920s, and which is well known to be consistent.
>
> (Prior 1971, p. 101)

Prior thinks that the logic of propositional identity is given by all instances of

Axiom schema 1: $\quad A = A$

and all instances of

Axiom schema 2: $\quad A = B \;.\!\supset\!.\; C = C(A/B)$

where A, B, C stand for sentences, and where $C(A/B)$ stands for a sentence which differs from C in that at least one occurrence of B in C is replaced by the sentence A, and where '$=$' is the special operator for propositional identity.

The logic of ordinary object identity is given by all instances of

(1) $$t = t$$

and all instances of

(2) $$t = s \;.\!\supset\!.\; F \supset F(t/s)$$

where t and s stand for singular terms, and where $F(t/s)$ stands for sentence which differs from a sentence F in that at least one occurrence of the singular term s in F is replaced by the singular term t.

Is Prior right to so strictly parallel the logic of '=' in these syntactically quite different applications? After all, from the premise that the proposition that Bill believes p is the very same proposition as the proposition that Bill believes q, does it not *follow* that the proposition that p is the very same proposition as the proposition that q? And there are many similar cases. Take negation. It appears that

$$\neg p = \neg q \;.\!\supset\!.\; p = q$$

is valid. Similarly with disjunction and conjunction:

$$p \vee q = p \vee r \;.\!\supset\!.\; q = r$$

and

$$p \wedge q = p \wedge r \;.\!\supset\!.\; q = r$$

appear valid.

What these and other examples suggest is that

(*PAS:*) $$C = C(A/B) \;.\!\supset\!.\; A = B$$

function as a third axiom schema for propositional identity.

It is certainly not the case that the proposed schema holds in application to referential terms:

$$f(t) = f(s) \;.\!\supset\!.\; t = s$$

has many false instances. So, if *PAS*, the proposed axiom schema, is sound, there is a significant disanalogy between the logic of propositional identity and object identity, contrary to what Prior thinks.

4. Propositional Identity and Mutual Entailment

It can be proved that if propositional identity is equated with mutual entailment, *PAS* is unsound. The notion of mutual entailment I have in mind is not the same thing as logical equivalence formalized in first order logic. It is a concept formalized in relevance logic by Alan Ross Anderson and Nuel Belnap Jr. (Anderson and Belnap 1975)

Note that

$$p$$

and

$$p \wedge (p \wedge q)$$

entail each other. So do

$$p$$

and

$$p \wedge (p \wedge r)$$

Thus

$$p \wedge (p \wedge r)$$

and

$$p \wedge (p \wedge q)$$

entail each other. So if propositional identity and mutual entailment are the same and if *PAS* is sound, then

$$q = r$$

which is absurd.

So either propositional identity and mutual entailment are distinct or *PAS* is unsound, from which, of course, one cannot conclude that it is *PAS* which is unsound.

Prior has the following remarks to make in regard to propositional identity and other notions.

> If the proposition that p really *is* the same proposition as the proposition that q, then of course it follows, not only that if it is (or is not) the case that p then it equally is (or is not) the case that q, but also that if James says, thinks, wishes or brings it about that p he *ipso facto* says, thinks, wishes or brings it about that q; but this last does not follow from the mere material equivalence — or, I would add, from the strict equivalence, or from the mutual entailment — of the proposition that p and the proposition that q. To equate propositional identity with any of these other things has about as much justification as equating the identity of individuals with, say, having the same barber.
>
> (Prior 1976, pp. 151–152)

I think it is clear that Prior would have rejected equating mutual entailment in the Anderson and Belnap sense with propositional identity.

5. Canceling-Out

PAS says that if two sentences that say the same thing differ only in the interchange of sentences A and B, then A and B say the same thing.

A more general principle is what Peter Geach labels a "canceling-out" principle. This says that if two sentences say the same and differ only with the interchange of expressions E and F, then E and F mean the same. Geach shows by example that this canceling-out principle is fallacious:

> We just cannot infer that if two propositions verbally differ precisely in that one contains the expression E_1 and the other the expression E_2, then, if the total force of the two propositions is the same, we may cancel out the identical parts and say that E_1 here means the same as E_2. I shall call this sort of inference *the canceling-out fallacy*; we shall come across it more than once. A simple example of it would be: the predicables "_______ killed Socrates" and "_______ was killed by Socrates" must mean the same, because "Socrates killed Socrates" means the same as "Socrates was killed by Socrates". (Geach 1968, p. 61)

(By a proposition Geach means a sentence with truth conditions.)

Now from the fact that P is an instance of a more general principle Q and that Q is fallacious it does not follow that P is fallacious. For example, where 's' is read as 'the immediate successor of' and '=' stands for ordinary identity

$$\forall x \forall y (s(x) = s(y) \;.\!\supset\!.\; x = y)$$

is an axiom of arithmetic even though it is an instance of the more general principle '$\forall x \forall y (f(x) = f(y) \supset x = y)$', which is invalid. Still, Geach's analysis of the canceling-out principle should make us more suspicious of *PAS*.

6. Other Possible Axioms

Among the logical axioms for propositional identity are all instances of

$$A = A$$

and all instances of

$$A = B\ .\supset.\ C = C(A/B)$$

And if there are no other logical axioms for propositional identity then the logic of propositional identity is analogous to the logic of ordinary object identity.

What I have been considering is whether instances of the canceling-out principle

$$C = C(A/B)\ .\supset.\ A = B$$

should also be axioms.

Here are at least two other possibilities: association

$$A \wedge (B \wedge C)\ .=.\ (A \wedge B) \wedge C,$$

and idempotence

$$A\ .=.\ A \wedge A.$$

Now it is not possible to accept all of the latter three schemata. If one were to accept canceling-out, association *and* idempotence, absurdity would result.

By association we have

$$(p \wedge q) \wedge r\ .=.\ p \wedge (q \wedge r).$$

By idempotence we have

$$q\ .=.\ q \wedge q.$$

By axiom schema 2 it follows that

$$(p \wedge q) \wedge r\ .=.\ p \wedge ((q \wedge q) \wedge r).$$

By association and transitivity of '=' (which follows by axiom schemata 1 and 2 plus ordinary logic) we have

$$(p \wedge q) \wedge r\ .=.\ (p \wedge q) \wedge (q \wedge r).$$

From this we get

$$r\ .=.\ q \wedge r$$

by canceling-out. Since the result is not valid, not all three principles are sound principles of logic.

I think this shows that the proposed canceling-out axiom schema will not work. Consider the language Prior described above:

> [...] (a) ordinary propositional calculus, enriched with variables for expressions which form sentences from sentences, with quantifiers binding variables standing for sentences, and with an identity-function with sentences as arguments; (b) the ordinary theory of quantification applied to our special quantifiers; and (c) ordinary laws of identity applied to our special function.
> (Prior 1971, p. 101)

Let the primitive logical constants of this language be:

$$\neg, \wedge, =, \exists$$

In addition to (c) I would add:

Axiom schema 3: $(A \wedge B) \wedge C \;.{=}.\; A \wedge (B \wedge C)$;

Axiom schema 4: $A \wedge A \;.{=}.\; A$;

Axiom schema 5: $A \wedge B \;.{=}.\; B \wedge A$;

Axiom schema 6: $\neg A = \neg B \;.{\supset}.\; A = B$;

Call $\exists\alpha A$ and $\exists\beta B$ alphabetic variants if A and B differ at most in that (i) α is free in A wherever β is free in B, and (ii) β is free in B wherever α is free in A. Then:

Axiom schema 7: $\exists\alpha A \;.{=}.\; \exists\beta B$ if $\exists\alpha A$ and $\exists\beta B$ alphabetic variants.

What motivates the last axiom schema are such facts as this: Consider '$\exists p\, p$' and '$\exists q\, q$'. Both say the same thing, *viz.*, something is the case.

An obvious omission is

$$\neg\neg A \;.{=}.\; A.$$

It would not be wise to add such an axiom schema to our list in the light of the fact that the constructivist denies that

$$\neg\neg A \equiv A$$

is a logical law.

7. Concluding Remarks

Reasonable criteria for sentences having the same propositional content go from syntactic identity to logical equivalence. The equivalence classes produced by the different criteria differ with respect to the degree of syntactic similarity of their members.

On one view, sentences the same in informational content express the same proposition and any two logically equivalent sentences have the same informational content. On this view it is possible for sentences which are very dissimilar syntactically to express the same proposition.

A view which tunes propositional identity more finely to syntactic structure is the view that propositional identity is mutual entailment. The sentences thus classed are syntactically more similar than those grouped by logical equivalence.

A view which tunes propositional identity even more finely to syntactic structure embraces the canceling-out principle. By canceling-out one can prove various mutual entailments to be nonidentities. For example, given that

$$q \neq r$$

it follows by canceling-out that

$$p \wedge (p \wedge q) \neq p \wedge (p \wedge r)$$

even though both sides of the nonidentity entail each other.

The logic of propositional identity as conceived here is intermediate between canceling-out and mutual entailment. It tunes propositional identity less finely to syntactic structure than does canceling-out but more finely to syntactic structure than does mutual entailment.

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