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Janusz Ciuciura

ON THE DA COSTA, DUBIKAJTIS AND KOTAS' SYSTEM OF THE DISCURSIVE LOGIC, D_2^*

Abstract. In the late forties, Stanisław Jaśkowski published two papers on the *discursive* (or *discussive*) *sentential calculus*, D_2. He provided a definition of it by an interpretation in the language of *S5* of Lewis. The known axiomatization of D_2 with discursive connectives as primitives was introduced by da Costa, Dubikajtis and Kotas in 1977. It turns out, however, that one of the axioms they used is not a thesis of the *real* Jaśkowski's calculus. In fact, they built a new system, D_2^* for short, that differs from D_2 in many respects. The aim of this paper is to introduce a direct *Kripke-type* semantics for the system, axiomatize it in a new way and prove soundness and completeness theorems. Additionally, we present labelled tableaux for D_2^*.[1]

Keywords: discursive (discussive) logic, D_2, paraconsistent logic, labelled tableaux.

1. Introduction

The language of D_2 is not simply formed by extending, for example, the classical propositional calculus with an extra operator (or operators) as it is in a modal case, but by replacing some of the classical connectives with their discursive counterparts, more explicitly:

[1]The main ideas of this paper were presented at the Logic-Philosophical Workshop, Bierzgłowo Teutonic Castle near Toruń, September 5–8, 2005.

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DEFINITION 1. Let *var* be a non-empty set of all propositional variables. The symbols: $\sim, \vee, \wedge_{\mathrm{d}}, \rightarrow_{\mathrm{d}}$ denote negation, disjunction, discursive conjunction and discursive implication, respectively. For_{D_2} is defined to be the least set such that:

(i) $\alpha \in var \Rightarrow \alpha \in For_{D_2}$

(ii) $\alpha \in For_{D_2} \Rightarrow \sim\alpha \in For_{D_2}$

(iii) $\alpha \in For_{D_2}$ and $\beta \in For_{D_2} \Rightarrow \alpha \bullet \beta \in For_{D_2}$, where $\bullet \in \{\vee, \wedge_{\mathrm{d}}, \rightarrow_{\mathrm{d}}\}$.[2]

It seems very *exotic* at first sight that Jaśkowski applied a translation procedure instead of just giving a *direct* semantics or a set of the syntactical rules for D_2. His choice, however, was not accidental.[3]

To give an insight into the procedure, we determine a translation function of the language of D_2 into the language of *S5* of Lewis, $f : For_{D_2} \Rightarrow For_{S5}$, as follows:

(i) $f(p_i) = p_i \; if p_i \in var$ and $i = \{1, 2, 3, \dots\}$

(ii) $f(\sim\alpha) = \sim f(\alpha)$

(iii) $f(\alpha \vee \beta) = f(\alpha) \vee f(\beta)$

(iv) $f(\alpha \wedge_{\mathrm{d}} \beta) = f(\alpha) \wedge \Diamond f(\beta)$

(v) $f(\alpha \rightarrow_{\mathrm{d}} \beta) = \Diamond f(\alpha) \rightarrow f(\beta)$

and additionally:

(vi) $\forall_{\alpha \in For_{D_2}} : \alpha \in D_2 \Leftrightarrow \Diamond f(\alpha) \in S5$.

By way of illustration, we demonstrate how the mechanism works in practice. Suppose then that we check whether the formula $\sim(\sim(\alpha \wedge_{\mathrm{d}} \beta) \vee \gamma) \rightarrow_{\mathrm{d}} (\alpha \wedge_{\mathrm{d}} \sim(\sim\beta \vee \gamma))$ is valid in D_2. As a result, we are made to apply the translation procedure and check if the formula $\Diamond(\Diamond\sim(\sim(\alpha \wedge \Diamond\beta) \vee \gamma) \rightarrow (\alpha \wedge \Diamond\sim(\sim\beta \vee \gamma)))$ is valid in *S5*. Unfortunately, it is a bit inconvenient to use the translation rules whenever we want to check out if a formula is valid in D_2 or it is not.[4]

The question arises: Is D_2 a finitely axiomatizable system? The year 1977 was a turning point. The well-known axiomatization presented by da

[2]The discursive equivalence is introduced as an abbreviation: $\alpha \leftrightarrow_{\mathrm{d}} \beta = (\alpha \rightarrow_{\mathrm{d}} \beta) \wedge_{\mathrm{d}} (\beta \rightarrow_{\mathrm{d}} \alpha)$.

[3]For details, see [2], [3], [9] and [11].

[4]We solved this problem in [4] and [5].

Costa, Dubikajtis and Kotas consists of the following axiom schemata and rules:

(A_1) $\alpha \to_{\mathrm{d}} (\beta \to_{\mathrm{d}} \alpha)$
(A_2) $(\alpha \to_{\mathrm{d}} (\beta \to_{\mathrm{d}} \gamma)) \to_{\mathrm{d}} ((\alpha \to_{\mathrm{d}} \beta) \to_{\mathrm{d}} (\alpha \to_{\mathrm{d}} \gamma))$
(A_3) $((\alpha \to_{\mathrm{d}} \beta) \to_{\mathrm{d}} \alpha) \to_{\mathrm{d}} \alpha$
(A_4) $\alpha \wedge_{\mathrm{d}} \beta \to_{\mathrm{d}} \alpha$
(A_5) $\alpha \wedge_{\mathrm{d}} \beta \to_{\mathrm{d}} \beta$
(A_6) $\alpha \to_{\mathrm{d}} (\beta \to_{\mathrm{d}} (\alpha \wedge_{\mathrm{d}} \beta))$
(A_7) $\alpha \to_{\mathrm{d}} \alpha \vee \beta$
(A_8) $\beta \to_{\mathrm{d}} \alpha \vee \beta$
(A_9) $(\alpha \to_{\mathrm{d}} \gamma) \to_{\mathrm{d}} ((\beta \to_{\mathrm{d}} \gamma) \to_{\mathrm{d}} (\alpha \vee \beta \to_{\mathrm{d}} \gamma))$
(A_{10}) $\alpha \to_{\mathrm{d}} \sim\sim\alpha$
(A_{11}) $\sim\sim\alpha \to_{\mathrm{d}} \alpha$
(A_{12}) $\sim(\alpha \vee \sim\alpha) \to_{\mathrm{d}} \beta$
(A_{13}) $\sim(\alpha \vee \beta) \to_{\mathrm{d}} \sim(\beta \vee \alpha)$
(A_{14}) $\sim(\alpha \vee \beta) \to_{\mathrm{d}} (\sim\alpha \wedge_{\mathrm{d}} \sim\beta)$
(A_{15}) $\sim(\sim\sim\alpha \vee \beta) \to_{\mathrm{d}} \sim(\alpha \vee \beta)$
(A_{16}) $(\sim(\alpha \vee \beta) \to_{\mathrm{d}} \gamma) \to_{\mathrm{d}} ((\sim\alpha \to_{\mathrm{d}} \beta) \vee \gamma))$
(A_{17}) $\sim((\alpha \vee \beta) \vee \gamma) \to_{\mathrm{d}} \sim(\alpha \vee (\beta \vee \gamma))$
(A_{18}) $\sim((\alpha \to_{\mathrm{d}} \beta) \vee \gamma) \to_{\mathrm{d}} (\alpha \wedge_{\mathrm{d}} \sim(\beta \vee \gamma))$
(A_{19}) $\sim((\alpha \wedge_{\mathrm{d}} \beta) \vee \gamma) \to_{\mathrm{d}} (\alpha \to_{\mathrm{d}} \sim(\beta \vee \gamma))$
(A_{20}) $\sim(\sim(\alpha \vee \beta) \vee \gamma) \to_{\mathrm{d}} (\sim(\sim\alpha \vee \gamma) \vee \sim(\sim\beta \vee \gamma))$
(A_{21}) $\sim(\sim(\alpha \to_{\mathrm{d}} \beta) \vee \gamma) \to_{\mathrm{d}} (\alpha \to_{\mathrm{d}} \sim(\sim\beta \vee \gamma))$
(A_{22}) $\sim(\sim(\alpha \wedge_{\mathrm{d}} \beta) \vee \gamma) \to_{\mathrm{d}} (\alpha \wedge_{\mathrm{d}} \sim(\sim\beta \vee \gamma))$
$(\mathrm{MP})^*$ $\alpha, \alpha \to_{\mathrm{d}} \beta / \beta$
$(R_{\mathrm{d}}1)$ $\alpha \leftrightarrow_{\mathrm{d}} \beta = (\alpha \to_{\mathrm{d}} \beta) \wedge_{\mathrm{d}} (\beta \to_{\mathrm{d}} \alpha)$
$(R_{\mathrm{d}}2)$ $\alpha \to \beta = \sim\alpha \vee \beta$
$(R_{\mathrm{d}}3)$ $\circ\alpha = \sim(\alpha \vee \sim\alpha)$
$(R_{\mathrm{d}}4)$ $\Box\alpha = \sim\alpha \to_{\mathrm{d}} \circ\alpha$
$(R_{\mathrm{d}}5)$ $\Diamond\alpha = \sim\Box\sim\alpha$
$(R_{\mathrm{d}}6)$ $\alpha \wedge \beta = \sim(\sim\alpha \vee \sim\beta)$
$(R_{\mathrm{d}}7)$ $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$.[5]

It is amazing that their construction has been widely recognized as a *real* axiomatization of D_2. To shed some light on the point, take the axiom schema:

(A_{19}) $\sim((\alpha \wedge_{\mathrm{d}} \beta) \vee \gamma) \to_{\mathrm{d}} (\alpha \to_{\mathrm{d}} \sim(\beta \vee \gamma))$

[5] See [1], [6] and [12]

apply the translation procedure to obtain:

$$\Diamond(\Diamond \sim((\alpha \wedge \Diamond\beta) \vee \gamma) \rightarrow (\Diamond\alpha \rightarrow \sim(\beta \vee \gamma)))$$

and check if the translated formula is valid in *S5* of Lewis.

COROLLARY 1. *The formula is not valid in S5 of Lewis (for every* $\alpha, \beta, \gamma \in For_{S5}$*).*

We solve the problem defining a new function $f^* : For_{D_2} \Rightarrow For_{S5}$ in the following way:

(i)' $f^*(p_i) = p_i \; if p_i \in var$ and $i = \{1, 2, 3, \ldots\}$

(ii)' $f^*(\sim \alpha) = \sim f^*(\alpha)$

(iii)' $f^*(\alpha \vee \beta) = f^*(\alpha) \vee f^*(\beta)$

(iv)' $f^*(\alpha \wedge_{\mathrm{d}} \beta) = \Diamond f^*(\alpha) \wedge f^*(\beta)$

(v)' $f^*(\alpha \rightarrow_{\mathrm{d}} \beta) = \Diamond f^*(\alpha) \rightarrow f^*(\beta)$

and introducing the key definition:

(vi)' $\forall_{\alpha \in For_{D_2}} : \alpha \in D_2 \Leftrightarrow \Diamond f^*(\alpha) \in S5.$

Let D_2^* denote the system defined by the new translation.

COROLLARY 2. *All of the axiom schemata are valid in* D_2^* *and* (MP)* *preserves validity.*

Note that despite their superficial similarities, the two systems (D_2 and D_2^*) are slightly different.[6]

2. Kripke-type Semantics for D_2^*

Although we depicted how to translate any discursive formula into its modal counterpart, the procedures introduced in Section 1 were a little unhandy and time-consuming to handle in practice. The inconvenience results in the search for a new semantic tool we could use trying to avoid passing through the translation rules. In aid of it we present here a *Kripke-type* semantics for D_2^*.

A frame (D_2^*-frame) is a pair $\langle W, R\rangle$ where W is a non-empty set (of possible worlds) and R is a binary relation on W. Moreover, R is subject to the conditions:

[6] See [4].

(i) $\forall_{x \in W}(xRx)$

(ii) $\forall_{x,y \in W}(xRy \Rightarrow yRx)$

(iii) $\forall_{x,y,z \in W}(xRy \text{ and } yRz \Rightarrow xRz)$.

The conditions define R as being the equivalence relation on W.

A model (D_2^*-model) is a triple $\langle W, R, v\rangle$ where v is a mapping from propositional variables to sets of worlds, $v : var \Rightarrow 2^W$. The satisfaction relation $\models_m$ is inductively defined:

$$
\begin{array}{llcl}
(var) & x \models_m p_i & \Leftrightarrow & x \in v(p_i) \text{ and } i = \{1, 2, 3, \dots\} \\
(\sim) & x \models_m \sim\alpha & \Leftrightarrow & x \not\models_m \alpha \\
(\vee) & x \models_m \alpha \vee \beta & \Leftrightarrow & x \models_m \alpha \text{ or } x \models_m \beta \\
(\wedge_{\mathrm{d}}) & x \models_m \alpha \wedge_{\mathrm{d}} \beta & \Leftrightarrow & \exists_{y \in W}(xRy \text{ and } y \models_m \alpha) \text{ and } x \models_m \beta \\
(\rightarrow_{\mathrm{d}}) & x \models_m \alpha \rightarrow_{\mathrm{d}} \beta & \Leftrightarrow & \text{if } \exists_{y \in W}(xRy \text{ and } y \models_m \alpha) \text{ then } x \models_m \beta.
\end{array}
$$

We define the notion of a valid sentence as follows:

$$\models \alpha \quad \Leftrightarrow \quad \text{for any model } \langle W, R, v\rangle, \forall_{x \in W}, \exists_{y \in W}(xRy \text{ and } y \models_m \alpha).$$

Notice that the non-standard definition is a direct result of (vi)'. Furthermore, not only is the discursive equivalence definable in our semantics:

$$\alpha \leftrightarrow_{\mathrm{d}} \beta = (\alpha \rightarrow_{\mathrm{d}} \beta) \wedge_{\mathrm{d}} (\beta \rightarrow_{\mathrm{d}} \alpha),$$

but also the discursive implication can be eliminated:

$$\alpha \rightarrow_{\mathrm{d}} \beta = \sim(\alpha \wedge_{\mathrm{d}} (p_1 \vee \sim p_1)) \vee \beta.$$

Now we can establish a link between the translation rules and the semantics in question.

COROLLARY 3. $\forall_{\alpha \in For_{D_2^*}} :\models \alpha \Leftrightarrow \alpha \in D_2^*$ $(\Leftrightarrow \Diamond f^*(\alpha) \in S5)$.

PROOF. By induction. First, we have to prove that for every model $\langle W, R, v\rangle$ and every $x \in W$ it is true that $x \models_m \alpha \Leftrightarrow x \models^{\#} f^*(\alpha)$, where $\models^{\#} \subseteq W \times For_{S5}$ is the satisfaction relation defined in any $S5$-model $\langle W, R, v\rangle$.
Case (1): $\alpha = p_i, i = \{1, 2, 3, \dots\}$.

$x \models_m p_i \Leftrightarrow x \in v(p_i) \Leftrightarrow x \models^{\#} p_i \Leftrightarrow x \models^{\#} f^*(p_i)$.

Case (2): $\alpha = \sim\gamma$.

$x \models_m \sim\gamma \Leftrightarrow x \not\models_m \gamma \Leftrightarrow x \not\models^{\#} f^*(\gamma) \Leftrightarrow x \models^{\#} \sim f^*(\gamma) \Leftrightarrow x \models^{\#} f^*(\sim\gamma)$.

Case (3): $\alpha = \gamma \vee \delta$.

$x \models_m \gamma \vee \delta \Leftrightarrow [x \models_m \gamma$ or $x \models_m \delta] \Leftrightarrow [x \models^{\#} f^*(\gamma)$ or $x \models^{\#} f^*(\delta)] \Leftrightarrow$
$\Leftrightarrow x \models^{\#} f^*(\gamma) \vee f^*(\delta) \Leftrightarrow x \models^{\#} f^*(\gamma \vee \delta)$.

Case (4): $\alpha = \gamma \wedge_{\mathrm{d}} \delta$,

$x \models_m \gamma \wedge_{\mathrm{d}} \delta \Leftrightarrow [(\exists_{y \in W}(xRy$ and $y \models_m \gamma)$ and $x \models_m \delta)] \Leftrightarrow$
$\Leftrightarrow [\exists_{y \in W}(xRy$ and $y \models^{\#} f^*(\gamma))$ and $x \models^{\#} f^*(\delta))] \Leftrightarrow$
$\Leftrightarrow [x \models^{\#} \Diamond f^*(\gamma)$ and $x \models^{\#} f^*(\delta)] \Leftrightarrow x \models^{\#} \Diamond f^*(\gamma) \wedge f^*(\delta) \Leftrightarrow$
$\Leftrightarrow x \models^{\#} f^*(\gamma \wedge_{\mathrm{d}} \delta)$.

Next we show that

$$\begin{aligned} \models \alpha \quad &\Leftrightarrow \quad \text{in any model } \langle W, R, v\rangle, \forall_{x \in W} \exists_{y \in W}(xRy \text{ and } y \models_m \alpha) \\ &\Leftrightarrow \quad \text{in any model } \langle W, R, v\rangle, \forall_{x \in W} \exists_{y \in W}(xRy \text{ and } y \models^{\#} f^*(\alpha)) \\ &\Leftrightarrow \quad \text{in any model } \langle W, R, v\rangle, \forall_{x \in W}(x \models^{\#} \Diamond f^*(\alpha)) \\ &\Leftrightarrow \quad \Diamond f^*(\alpha) \in S5 \\ &\Leftrightarrow \quad \alpha \in D_2. \end{aligned}$$

□

The translation procedure became redundant and we succeeded in constructing a new (*direct*) semantics for D_2^*. All the axiom schemata (A_1)–(A_{22}) are valid in the modified semantics (and (MP)* preserves validity).

Since the accessibility relation defined on D_2^*-frames is reflexive, symmetric and transitive, it implies that any world is accessible from any other and we might well considere the relation to be complete. Consequently, the notion of D_2^*-model can be simplified to the form:

A model (D_2^*-model) is a pair $\langle W, v\rangle$ where W is a non-empty set (of possible worlds, points, etc.) and v is a function that each pair consisting of a formula and a point assigns an element of $\{1, 0\}$, $v : For_{D_2^*} \times W \Rightarrow \{1, 0\}$, defined as follows:

$$\begin{array}{llcl} (\sim) & v(\sim\alpha, x) = 1 & \Leftrightarrow & v(\alpha, x) = 0 \\ (\vee) & v(\alpha \vee \beta, x) = 1 & \Leftrightarrow & v(\alpha, x) = 1 \text{ or } v(\beta, x) = 1 \\ (\wedge_{\mathrm{d}}) & v(\alpha \wedge_{\mathrm{d}} \beta, x) = 1 & \Leftrightarrow & \exists_{y \in W}(v(\alpha, y) = 1) \text{ and } v(\beta, x) = 1 \\ (\rightarrow_{\mathrm{d}}) & v(\alpha \rightarrow_{\mathrm{d}} \beta, x) = 1 & \Leftrightarrow & \forall_{y \in W}(v(\alpha, y) = 0) \text{ or } v(\beta, x) = 1. \end{array}$$

The notion of a valid sentence also needs to be modified:

$$\models \alpha \quad \Leftrightarrow \quad \text{in any model } \langle W, v\rangle, \exists_{y \in W}(v(\alpha, y) = 1).$$

It is worth mentioning that the most of the *notorious*, in a very *real* paraconsistent sense, formulas are not valid in D_2^*, for instance:

(1) $p \rightarrow_{\mathrm{d}} (\sim p \rightarrow_{\mathrm{d}} q)$

(2) $p \rightarrow_{\mathrm{d}} (\sim p \rightarrow_{\mathrm{d}} \sim q)$

(3) $(p \rightarrow_{\mathrm{d}} q) \rightarrow_{\mathrm{d}} (\sim q \rightarrow_{\mathrm{d}} \sim p)$

(4) $(\sim p \rightarrow_{\mathrm{d}} \sim q) \rightarrow_{\mathrm{d}} (q \rightarrow_{\mathrm{d}} p)$

(5) $(p \rightarrow_{\mathrm{d}} q) \rightarrow_{\mathrm{d}} (\sim(p \rightarrow_{\mathrm{d}} q) \rightarrow_{\mathrm{d}} r)$

(6) $p \rightarrow_{\mathrm{d}} (\sim p \rightarrow_{\mathrm{d}} (\sim\sim p \rightarrow_{\mathrm{d}} q))$

(7) $(p \wedge_{\mathrm{d}} \sim p) \rightarrow_{\mathrm{d}} q$.

3. New Axiomatization of D_2^*

In this section, we present a new axiomatization of D_2^* making use of the discursive connectives occurring *directly* in a set of axiom schemata. The role of axiom schemata of D_2^* can be taken on by the following:

(A_1) $\alpha \rightarrow_{\mathrm{d}} (\beta \rightarrow_{\mathrm{d}} \alpha)$
(A_2) $(\alpha \rightarrow_{\mathrm{d}} (\beta \rightarrow_{\mathrm{d}} \gamma)) \rightarrow_{\mathrm{d}} ((\alpha \rightarrow_{\mathrm{d}} \beta) \rightarrow_{\mathrm{d}} (\alpha \rightarrow_{\mathrm{d}} \gamma))$
(A_3) $((\alpha \rightarrow_{\mathrm{d}} \beta) \rightarrow_{\mathrm{d}} \alpha) \rightarrow_{\mathrm{d}} \alpha$
(A_4) $\alpha \wedge_{\mathrm{d}} \beta \rightarrow_{\mathrm{d}} \alpha$
(A_5) $\alpha \wedge_{\mathrm{d}} \beta \rightarrow_{\mathrm{d}} \beta$
(A_6) $\alpha \rightarrow_{\mathrm{d}} (\beta \rightarrow_{\mathrm{d}} (\alpha \wedge_{\mathrm{d}} \beta))$
(A_7) $\alpha \rightarrow_{\mathrm{d}} \alpha \vee \beta$
(A_8) $\beta \rightarrow_{\mathrm{d}} \alpha \vee \beta$
(A_9) $(\alpha \rightarrow_{\mathrm{d}} \gamma) \rightarrow_{\mathrm{d}} ((\beta \rightarrow_{\mathrm{d}} \gamma) \rightarrow_{\mathrm{d}} (\alpha \vee \beta \rightarrow_{\mathrm{d}} \gamma))$.
(A_9) $\alpha \vee \sim \alpha$
(A_{10}) $\alpha \rightarrow_{\mathrm{d}} \sim(\sim(\alpha \vee \beta) \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha)$
(A_{11}) $\sim(\sim(\alpha \vee \beta) \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha) \rightarrow_{\mathrm{d}} \sim(\sim(\alpha \vee \beta \vee \gamma) \wedge_{\mathrm{d}} \sim \gamma \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha)$
(A_{12}) $\sim(\sim(\alpha \vee \gamma \vee \beta) \wedge_{\mathrm{d}} \sim \gamma \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \sim(\sim(\alpha \vee \beta \vee \gamma) \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \gamma \wedge_{\mathrm{d}} \sim \alpha)$
(A_{13}) $\sim(\sim(\alpha \vee \beta) \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha) \rightarrow_{\mathrm{d}} ((\alpha \vee \sim \beta) \rightarrow_{\mathrm{d}} \alpha)$
(A_{14}) $\sim(\sim(\alpha \vee \beta \vee \gamma) \wedge_{\mathrm{d}} \sim \gamma \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha) \rightarrow_{\mathrm{d}} ((\alpha \vee \beta \vee \sim \gamma) \rightarrow_{\mathrm{d}} (\alpha \vee \beta))$
(A_{15}) $\sim(\sim(\alpha \vee \beta \vee \gamma) \wedge_{\mathrm{d}} \sim \gamma \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} (\sim(\sim(\alpha \vee \beta \vee \sim \gamma) \wedge_{\mathrm{d}} \sim\sim \gamma \wedge_{\mathrm{d}} \sim \beta \wedge_{\mathrm{d}} \sim \alpha) \rightarrow_{\mathrm{d}} \sim(\sim \beta \wedge_{\mathrm{d}} \sim \alpha))$
(A_{16}) $\sim(\sim \alpha \wedge_{\mathrm{d}} \sim \beta) \rightarrow_{\mathrm{d}} (\alpha \vee \beta)$
(A_{17}) $(\alpha \vee \sim\sim \beta) \rightarrow_{\mathrm{d}} (\alpha \vee \beta)$
(A_{18}) $(\alpha \vee \beta) \rightarrow_{\mathrm{d}} (\alpha \vee \sim\sim \beta)$

The sole rule of inference is *Detachment Rule*

(MP)* $\alpha, \alpha \rightarrow_{\mathrm{d}} \beta / \beta$

The consequence relation $\vdash_{D_2^*}$ is determined by the set of axioms and (MP)*.

Observe that $(A_1), (A_2)$ are axiom schemata of D_2^* and our system is closed under the detachment rule. It immediately follows that the proof of the deduction theorem is standard.

THEOREM 1. $\Phi \vdash_{D_2^*} \alpha \rightarrow_d \beta \Leftrightarrow \Phi \cup \{\alpha\} \vdash_{D_2^*} \beta$, *where* $\alpha, \beta \in For_{D_2^*}, \Phi \subseteq For_{D_2^*}$.

COROLLARY 4. *The formulas listed below are provable in* D_2^**:*

(T_1) $(\alpha \vee \alpha) \rightarrow_d \alpha$
(T_2) $(\alpha \vee \beta) \leftrightarrow_d (\beta \vee \alpha)$
(T_3) $((\alpha \vee \beta) \vee \gamma) \leftrightarrow_d (\alpha \vee (\beta \vee \gamma))$
(T_4) $(\alpha \vee (\beta \rightarrow_d \gamma)) \leftrightarrow_d ((\alpha \vee \beta) \rightarrow_d (\alpha \vee \gamma))$
(T_5) $\alpha \vee (\alpha \rightarrow_d \beta)$
(T_6) $(\alpha \rightarrow_d \beta) \rightarrow_d ((\gamma \vee \alpha) \rightarrow_d (\gamma \vee \beta))$
(T_7) $(\alpha \rightarrow_d (\alpha \rightarrow_d \beta)) \rightarrow_d \beta$
(T_8) $(\beta \vee \alpha \vee \beta) \rightarrow_d (\alpha \vee \beta)$
(T_9) $\sim(\sim(\alpha \vee \beta) \wedge_d \sim\beta \wedge_d \sim\alpha) \rightarrow_d$
$\rightarrow_d (\sim(\sim(\alpha \vee \sim\beta) \wedge_d \sim\sim\beta \wedge_d \sim\alpha) \rightarrow_d \alpha)$

and the set of $\{\alpha :\vdash_{D_2^*} \alpha\}$ *is closed under the rules:*

(R_1) $\alpha, \beta \ / \ \alpha \wedge_d \beta$
(R_2) $\alpha \wedge_d \beta \ / \ \alpha \ (\beta)$
(R_3) $\alpha \ (\beta) \ / \ \alpha \vee \beta$.

PROOF. We prove $(T_1) - (T_8)$ in much the same way as it is in the (positive) classical case. (T_9):

1.	$\sim(\sim(\alpha \vee \beta) \wedge_d \sim\beta \wedge_d \sim\alpha)$	by deduction theorem
2.	$\sim(\sim(\alpha \vee \sim\beta) \wedge_d \sim\sim\beta \wedge_d \sim\alpha)$	by deduction theorem
3.	$(\alpha \vee \sim\beta) \vee \sim\beta \vee \alpha$	(A_{16}), 2 and (MP)*
4.	$\alpha \vee \sim\beta$	(T_8), (T_3), 3 and (MP)*
5.	α	(A_{13}), 1, 4 and (MP)*

(R_1)–(R_3) are obvious due to $(A_6), (A_5), (A_4), (A_7), (A_8)$ and (MP)*. □

COROLLARY 5. *Each of the axiom schemata of* D_2^*, (A_1)–(A_{18}), *becomes a schema of the thesis of the classical propositional calculus after replacing in* A_i, *where* $i \in \{1, \ldots, 18\}$, *all the discursive connectives with their classical counterparts (i.e.* $\rightarrow_d \ / \rightarrow$ *and* $\wedge_d / \wedge$*).*[7] *The rule* (MP)* *becomes an*

[7] (A_9) can already be treated as a thesis of *CPC*.

admissible rule of CPC after replacing $\rightarrow_{\mathrm{d}}$ with $\rightarrow$.

Let $(D_2^*) = \{\alpha : \vdash_{(D_2^*)} \alpha\}$ be the system described in Corollary 5 and $CPC = \{\alpha : \vdash_{CPC} \alpha\}$.

Corollary 6. *$(D_2^*) \subset CPC$.*

4. Soundness and Completeness

Theorem 2 (Soundness). *$\vdash_{D_2^*} \alpha \Rightarrow \models \alpha$.*

Proof. By induction. All that needs to be checked is that (A_1)–(A_{18}) are valid and (MP)* preserves validity. □

Theorem 3 (Completeness). *$\models \alpha \Rightarrow \vdash_{D_2^*} \alpha$*

Proof. (Outline). Assume that $\nvdash_{D_2^*} \alpha$ (by contraposition) and $\models \alpha$. Define a sequence of all the formulas of D_2^* as follows:

$\Gamma = \gamma_1, \gamma_2, \gamma_3, \ldots$ where $\gamma_1 = \alpha$.

Define the family of (finite) subsequences of Γ:

$\Delta_1 = \delta_1$ where $\delta_1 = \gamma_1 = \alpha$

$\Delta_2 = \delta_1, \delta_2$ where $\delta_1 = \gamma_1 = \alpha$ and $\delta_2 = \gamma_i$ iff $\nvdash_{D_2^*} \delta_1 \vee \gamma_i$, otherwise take the very next formula(s) occurring in Γ, γ_j for short, and check if $\nvdash_{D_2^*} \delta_1 \vee \gamma_j$

$\Delta_3 = \delta_1, \delta_2, \delta_3$ where $\delta_1 = \gamma_1 = \alpha, \delta_2 = \gamma_i$ and $\delta_3 = \gamma_{i+n}$ iff $\nvdash_{D_2^*} \delta_1 \vee \delta_2 \vee \gamma_{i+n}$, otherwise go on testing the very next formulas of the sequence Γ

$\vdots$

$\Delta_n = \delta_1, \delta_2, \delta_3, \ldots, \delta_n$

$\vdots$

Next define:

$$\nabla_1 = \underbrace{\delta_1}_{\Delta_1}, \underbrace{\delta_1, \delta_2}_{\Delta_2}, \underbrace{\delta_1, \delta_2, \delta_3}_{\Delta_3}, \ldots, \underbrace{\delta_1, \delta_2, \delta_3, \ldots, \delta_n}_{\Delta_n}, \ldots$$

$$\nabla_2 = \underbrace{\delta_1, \delta_2}_{\Delta_2}, \underbrace{\delta_1, \delta_2, \delta_3}_{\Delta_3}, \ldots, \underbrace{\delta_1, \delta_2, \delta_3, \ldots, \delta_n}_{\Delta_n}, \ldots$$

$$\nabla_3 = \underbrace{\delta_1, \delta_2, \delta_3}_{\Delta_3}, \ldots, \underbrace{\delta_1, \delta_2, \delta_3, \ldots, \delta_n}_{\Delta_n}, \ldots$$

$\vdots$

$$\nabla_n = \underbrace{\delta_1, \ldots, \delta_n}_{\Delta_n}, \ldots, \underbrace{\delta_1, \delta_2, \delta_3, \ldots, \delta_{n+k}}_{\Delta_{n+k}}, \ldots$$

$\vdots$

Observe that all the sequences are *infinite*. □

From now on we use ∇_i, where $i = \{1, 2, 3, \ldots\}$, to denote both the i-sequence and the set of formulas which contains all the elements of the i-sequence. Additionally, let $\nabla = \{\nabla_1, \nabla_2, \ldots \nabla_i, \ldots, \nabla_n, \ldots\}$.

Lemma 1. *(i)* $\nvdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_n$, *for any* $n \in N$

(ii) if $\beta \notin \nabla_i$, *then* $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \beta$, *for some* $k \in N$.

Proof. Apply the definition of ∇_i, where $i = \{1, 2, 3, \ldots\}$. □

Definition 2. $\nabla_i \mathbf{R} \nabla_k \Leftrightarrow (\nabla_i = \nabla_k)$, for every $\nabla_i, \nabla_k \in \nabla$.

Lemma 2. $\mathbf{R}$ *is the equivalence relation on* ∇.

Proof. Immediately from Definition 2. □

In Section 2, we mentioned that the connectives of $\leftrightarrow_{\mathrm{d}}$ and $\rightarrow_{\mathrm{d}}$ were redundant. This fact simplifies a proof of the next lemma.

Lemma 3. $\forall_{\beta,\gamma \in For_{D_2^*}}, \forall_{\nabla_i, \nabla_k \in \nabla}$:

(i) $\beta \vee \gamma \in \nabla_i \Leftrightarrow \beta \in \nabla_i$ *and* $\gamma \in \nabla_i$

(ii) $\beta \wedge_{\mathrm{d}} \gamma \in \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla}(\nabla_i \mathbf{R} \nabla_k \Rightarrow \beta \in \nabla_k)$ *or* $\gamma \in \nabla_i$

(iii) $\sim\beta \in \nabla_i \Leftrightarrow \beta \notin \nabla_i$.

Proof. We only show (ii) and (iii).

(ii) $\Rightarrow$. Let (1) $\beta \wedge_{\mathrm{d}} \gamma \in \nabla_i$, (2)$\exists_{\nabla_k \in \nabla}(\nabla_i \mathbf{R} \nabla_k$ and $\beta \notin \nabla_k)$ and $\gamma \notin \nabla_i$. Then, due to (2), we obtain (3) $\nabla_i \mathbf{R} \nabla_k$, (4) $\beta \notin \nabla_k$ and (5) $\gamma \notin \nabla_i$. By Definition 2 and (4), we have (6) $\beta \notin \nabla_i$ and consequently (7) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \beta$, for some $k \in N$ (Lemma 1(ii) and (6)), (8) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_r \vee \gamma$, for some $r \in N$ (Lemma 1 (ii) and (5)). Suppose that $k \geq r$ (we prove the second case, i.e. $r > k$, on much the same way as $k \geq r$). Apply (R_3), (T_2), (T_3), (MP)* to (8), to get (9) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \gamma$. Now use (R_1) to obtain (10) $\vdash_{D_2^*} (\delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \beta) \wedge_{\mathrm{d}} (\delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \gamma)$ and finally, (T_4) to get (11) $\vdash_{D_2^*} (\delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k) \vee (\beta \wedge_{\mathrm{d}} \gamma)$. Obviously, $\delta_1, \delta_2, \ldots, \delta_k, \beta \wedge_{\mathrm{d}} \gamma \in \nabla_i$. A contradiction due to Lemma 1(i).

(ii) $\Leftarrow$. Assume that (1) $\forall_{\nabla_k \in \nabla}(\nabla_i \mathbf{R} \nabla_k \Rightarrow \beta \in \nabla_k)$ or $\gamma \in \nabla_i$ and (2) $\beta \wedge_{\mathrm{d}} \gamma \notin \nabla_i$. Subcase (a): if (1) $\forall_{\nabla_k \in \nabla}(\nabla_i \mathbf{R} \nabla_k \Rightarrow \beta \in \nabla_k), (2) \beta \wedge_{\mathrm{d}} \gamma \notin \nabla_i$, then (3) $\beta \in \nabla_i$ (by $\mathbf{R}$) and (4) $\vdash_{D_2^*} (\delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k) \vee (\beta \wedge_{\mathrm{d}} \gamma)$, for some $k \in N$ (Lemma 1(ii) and (2)). Now apply (T_4) to get (5) $\vdash_{D_2^*}$ $(\delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \beta) \wedge_{\mathrm{d}} (\delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \gamma)$ and (R_2) to obtain (6) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_1 \vee \cdots \vee \delta_k \vee \beta$, but $\delta_1, \ldots, \delta_k, \beta \in \nabla_i$. A contradition due to Lemma 1(i). Subcase (b): (1) $\gamma \in \nabla_i$ and (2) $\beta \wedge_{\mathrm{d}} \gamma \notin \nabla_i$. Now proceed analogously to the subcase (a).

(iii) $\Rightarrow$. Assume that $\sim\beta \in \nabla_i$ and $\beta \in \nabla_i$. It means the formula $\beta \vee \sim\beta$ is not a thesis of D_2^* (Lemma 1 (i)). A contradiction due to (A_9).

(iii) $\Leftarrow$. Let ∇_i be a sequence $i = \{1, 2, 3, \ldots\}$. For every ∇_i define:

$$\nabla_i^* = \delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*, \ldots$$

where

(a) $\delta_1^* = \delta_1 = \gamma_1 = \alpha$

(b) for every $\delta_n \in \nabla_i : (\delta_n = \delta_k^*) \Leftrightarrow \nvdash_{D_2^*} \sim(\sim(\delta_1^* \vee \ldots \vee \delta_k^*) \wedge_{\mathrm{d}} \sim\delta_k^* \wedge_{\mathrm{d}} \ldots \wedge_{\mathrm{d}} \sim\delta_1^*)$.

Definition 3. We call a formula β *classical* if it does not include constant symbols other than $\sim$ and $\vee$. We call a formula β *discursive* if it contains at least one discursive connective. A formula β is a *discursive thesis* if it is a thesis and discursive.

Corollary 7. *(i)* $\nabla_i^* \subseteq \nabla_i$, *for every* $i \in \{1, 2, 3, \ldots\}$

(ii) $\nvdash_{D_2^*} \sim(\sim(\delta_1^* \vee \cdots \vee \delta_n^*) \wedge_{\mathrm{d}} \sim\delta_n^* \wedge_{\mathrm{d}} \ldots \wedge_{\mathrm{d}} \sim\delta_1^*)$, *for every* $n \in N$

(iii) If β *is not a discursive thesis,* $\beta \notin \nabla_i$, *then* $\vdash_{D_2^*} \sim(\sim(\delta_1^* \vee \cdots \vee \delta_k^* \vee \beta) \wedge_{\mathrm{d}} \sim\beta \wedge_{\mathrm{d}} \sim\delta_k^* \wedge_{\mathrm{d}} \ldots \wedge_{\mathrm{d}} \sim\delta_1^*)$, *for some* $k \in N$.

Now assume that (1) $\sim\beta \notin \nabla_i$ and (2) $\beta \notin \nabla_i$. Apply Lemma 1(ii), to get (3) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_m \vee \sim\beta$ and (4) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_n \vee \beta$, for some $m, n \in N$. Suppose that $m \geq n$ (the case $n > m$ is similar to $m \geq n$). Use (R_3), (T_2), (T_3), (MP)* to (4), to obtain (5) $\vdash_{D_2^*} \delta_1 \vee \cdots \vee \delta_m \vee \beta$. If $\sim\beta \notin \nabla_i, \beta \notin \nabla_i$ and $\nabla_i^* \subseteq \nabla_i$, then (6) $\sim\beta \notin \nabla_i^*$, (7) $\beta \notin \nabla_i^*$. We have to consider three subcases:

(A) neither β nor $\sim\beta$ is a *discursive thesis*

(B) β *is a discursive thesis*, but $\sim\beta$ is *not* a discursive thesis

(C) $\sim\beta$ *is a discursive thesis*, but β is *not* a discursive thesis.

Note that the fourth subcase (both β and $\sim\beta$ is a *discursive thesis*) is impossible due to *Soundness.*

Subcase (A).

Let $m = 1$. (8) $\vdash_{D_2^*} \sim(\sim(\delta_1^* \vee \beta) \wedge_d \sim\beta \wedge_d \sim\delta_1^*)$, Corollary 7 (iii) and (2), (9) $\vdash_{D_2^*} \sim(\sim(\delta_1^* \vee \sim\beta) \wedge_d \sim\sim\beta \wedge_d \sim\delta_1^*)$, Corollary 7 (iii) and (1). Apply (T_9) to (8) and (9), to get (10) $\vdash_{D_2^*} \delta_1^*$, but $\delta_1^* = \delta_1 = \gamma_1 = \alpha$. A contradiction. Let $m > 1$. (8)' $\vdash_{D_2^*} \sim(\sim(\delta_1^* \vee \cdots \vee \delta_p^* \vee \beta) \wedge_d \sim\beta \wedge_d \sim\delta_p^* \wedge_d \ldots \wedge_d \sim\delta_1^*)$, for some $p \in N$, (9)' $\vdash_{D_2^*} \sim(\sim(\delta_1^* \vee \cdots \vee \delta_r^* \vee \sim\beta) \wedge_d \sim\sim\beta \wedge_d \sim\delta_r^* \wedge_d \ldots \wedge_d \sim\delta_1^*)$, for some $r \in N$. Note that $p \geq r$ or $r > p$. If $p \geq r$, then apply (A_{11}), (A_{12}) and (MP)* to (9)', to get (10)' $\vdash_{D_2^*} \sim(\sim(\delta_1^* \vee \cdots \vee \delta_p^* \vee \sim\beta) \wedge_d \sim\sim\beta \wedge_d \sim\delta_p^* \wedge_d \ldots \wedge_d \sim\delta_1^*)$, for some $r \in N$. Now consider (8)', (10)' and use (A_{15}) and (MP)*, to obtain (11)' $\vdash_{D_2^*} \sim(\sim\delta_p^* \wedge_d \ldots \wedge_d \sim\delta_1^*)$. Apply (A_{16}) to (11)', to get (12)' $\vdash_{D_2} \delta_1^* \vee \cdots \vee \delta_p^*$. Since $\nabla_i^* \subseteq \nabla_i$, (R_3) is an admissible rule in D_2^* and we have (T_2), (T_3), then (13)' $\vdash_{D_2} \delta_1 \vee \cdots \vee \delta_p$ (where $\delta_1^* = \delta_1$, $\delta_2^* = \delta_2$, ... , $\delta_p^* = \delta_p$). Clearly, $\delta_1, \ldots, \delta_p \in \nabla_i$. A contradition due to Lemma 1(i).

We prove the subcases (B) and (C) in a very similar way. Make use of (A_{11}), (A_{12}) (A_{13}), (A_{14}), (A_{17}) and (A_{18}). □

Now we construct a canonical model for D_2^* that will falsify any non-theorem (and invalidate a non-derivable rule). Let $M_C = \langle \nabla, \mathbf{R}, v_c \rangle$ be such a model. The canonical valuation $v_c : For_{D_2^*} \times \nabla \Rightarrow \{1, 0\}$ is defined:

$$v_c(\beta, \nabla_i) = \begin{cases} 1, & \text{if } \beta \notin \nabla_i \\ 0, & \text{if } \beta \in \nabla_i. \end{cases}$$

We have to show:

Case (1): $\beta = \sigma \vee \tau$

(i) $v_c(\sigma \vee \tau, \nabla_i) = 1 \Leftrightarrow \sigma \vee \tau \notin \nabla_i \Leftrightarrow \sigma \notin \nabla_i$ or $\tau \notin \nabla_i \Leftrightarrow v_c(\sigma, \nabla_i) = 1$ or $v_c(\tau, \nabla_i) = 1$

(ii) $v_c(\sigma \vee \tau, \nabla_i) = 0 \Leftrightarrow \sigma \vee \tau \in \nabla_i \Leftrightarrow \sigma \in \nabla_i$ and $\tau \in \nabla_i \Leftrightarrow v_c(\sigma, \nabla_i) = 0$ and $v_c(\tau, \nabla_i) = 0$.

Case (2): $\beta = \sigma \wedge_d \tau$

(i) $v_c(\sigma \wedge_d \tau, \nabla_i) = 1 \Leftrightarrow \sigma \wedge_d \tau \notin \nabla_i \Leftrightarrow \exists_{\nabla_k \in \nabla}(\nabla_i \mathbf{R} \nabla_k$ and $\sigma \notin \nabla_k)$ and $\tau \notin \nabla_i \Leftrightarrow \exists_{\nabla_k \in \nabla}(\nabla_i \mathbf{R} \nabla_k$ and $v_c(\sigma, \nabla_k) = 1)$ and $v_c(\tau, \nabla_i) = 1$

(ii) $v_c(\sigma \wedge_d \tau, \nabla_i) = 0 \Leftrightarrow \sigma \wedge_d \tau \in \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla}$ (if $\nabla_i \mathbf{R} \nabla_k$ then $\sigma \in \nabla_k)$ or $\tau \in \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla}$ (if $\nabla_i \mathbf{R} \nabla_k$ then $v_c(\sigma, \nabla_k) = 0)$ or $v_c(\tau, \nabla_i) = 1$.

Case (3): $\beta = \sim\sigma$

(i) $v_c(\sim\sigma, \nabla_i) = 1 \Leftrightarrow \sim\sigma \notin \nabla_i \Leftrightarrow \sigma \in \nabla_i \Leftrightarrow v_c(\sigma, \nabla_i) = 0$

(ii) $v_c(\sim\sigma, \nabla_i) = 0 \Leftrightarrow \sim\sigma \in \nabla_i \Leftrightarrow \sigma \notin \nabla_i \Leftrightarrow v_c(\sigma, \nabla_i) = 1$.

To finish the proof, recall $\nvdash_{D_2^*} \alpha$, but $\models \alpha$. Notice, however, that the formula α is the very first element of all the sequences ∇_i, where $i \in \{1, 2, 3, \dots\}$. Since $\alpha \in \nabla_i$, then the formula is not valid in $\langle \nabla, \mathbf{R}, v_c \rangle$, and consequently $\not\models \alpha$. A contradiction.

5. Labelled Tableaux for D_2^*

In what follows, we will use *signed labelled formulas* such as $\sigma :: T\mathrm{P}$ (or $\sigma :: F\mathrm{P}$), where σ is a label and $T\mathrm{P}$ (or $F\mathrm{P}$) is a *signed formula* (i.e. a formula prefixed with a "T" or "F"). The phrase $\sigma :: T\mathrm{P}$ is read as "P is true at the world σ" and $\sigma :: F\mathrm{P}$ as "P is false at the world σ". By *label*, we understand a natural number. We call ρ *root label* and always assume that $\rho = 1$. A tableau for a labelled formula P is a downward rooted tree, where each of the nodes contains a signed labelled formula, constructed using the branch extension rules defined below.

Non-discursive rules:

The rules for disjunction and negation are identical to the ones used in classical case.

$$(\boldsymbol{T}\vee)\quad \frac{\sigma :: T\mathrm{P} \vee \mathrm{Q}}{\sigma :: T\mathrm{P} \mid \sigma :: T\mathrm{Q}} \qquad (\boldsymbol{F}\vee)\quad \frac{\sigma :: F\mathrm{P} \vee \mathrm{Q}}{\begin{array}{c}\sigma :: F\mathrm{P}\\ \sigma :: F\mathrm{Q}\end{array}}$$

$$(\boldsymbol{T}\sim)\quad \frac{\sigma :: T\sim\mathrm{P}}{\sigma :: F\,\mathrm{P}} \qquad (\boldsymbol{F}\sim)\quad \frac{\sigma :: F\sim\mathrm{P}}{\sigma :: T\,\mathrm{P}}$$

The rules $(\boldsymbol{F}\vee)$, $(\boldsymbol{F}\sim)$ and $(\boldsymbol{T}\sim)$ are linear, but $(\boldsymbol{T}\vee)$ is branching.

Discursive rules:

$$(\boldsymbol{T}\wedge_{\mathrm{d}})\quad \frac{\sigma :: T\ \mathrm{P} \wedge_{\mathrm{d}} \mathrm{Q}}{\begin{array}{c}\tau :: T\ \mathrm{P}\\ \sigma :: T\ \mathrm{Q}\end{array}} \qquad (\boldsymbol{F}\wedge_{\mathrm{d}})\quad \frac{\sigma :: F\ \mathrm{P} \wedge_{\mathrm{d}} \mathrm{Q}}{\sigma' :: F\ \mathrm{P} \mid \sigma :: F\ \mathrm{Q}}$$

Notice that τ, for $(T\wedge_{\mathrm{d}})$, is a label that is *new* to the branch, but σ', for $(F\wedge_{\mathrm{d}})$, is a label that has been *already used* in the branch.

$$(\boldsymbol{T}{\to_\mathrm{d}})\quad \frac{\sigma :: T\,\mathrm{P} \to_\mathrm{d} \mathrm{Q}}{\sigma' :: F\,\mathrm{P} \mid \sigma :: T\,\mathrm{Q}} \qquad (\boldsymbol{F}{\to_\mathrm{d}})\quad \frac{\sigma :: F\,\mathrm{P} \to_\mathrm{d} \mathrm{Q}}{\begin{array}{c}\tau :: T\,\mathrm{P}\\ \sigma :: F\,\mathrm{Q}\end{array}}$$

where σ', for $(T \to_\mathrm{d})$, has been *already used* in the branch and τ, for $(F \to_\mathrm{d})$, is a label that is *new* to the branch.

Closure rule:

A branch of a tableau is closed if we can apply the rule:

$$(\mathbf{C})\quad \frac{\begin{array}{c}\sigma :: T\ \mathrm{P}\\ \sigma :: F\ \mathrm{P}\end{array}}{\text{closed}}$$

Otherwise the branch is open. A tableau is closed if all of its branches are closed, otherwise the tableau is open.

Special rule:

$$(\mathbf{S})\quad \frac{\rho :: F\ \mathrm{P}}{\sigma' :: F\ \mathrm{P}}$$

ρ is a root label and σ' is a label that has been *already used* in the branch. The application of the rule is always limited to root labels.

Let P be a formula. By a D_2^**-tableau proof* of P we mean a closed tableau with $1 :: F\mathrm{P}$.

Now, we give a few examples to illustrate how the rules we defined work.

EXAMPLE 1. Closed tableau for the second Clavius' law.

(a)	$1 :: F\ ({\sim}\,\mathrm{P} \to_\mathrm{d} \mathrm{P}) \to_\mathrm{d} \mathrm{P}$	(start)
(b)	$2 :: T\ {\sim}\,\mathrm{P} \to_\mathrm{d} \mathrm{P}$	$(F \to_\mathrm{d})$, (a)
(c)	$1 :: F\ \mathrm{P}$	$(F \to_\mathrm{d})$, (a)
1st branch		
(d)	$1 :: F\ {\sim}\,\mathrm{P}$	$(T \to_\mathrm{d})$, (b)
(e)	$1 :: T\ \mathrm{P}$	$(F\,{\sim})$, (d)
	Closed	(C), (c), (e)
2nd branch		
(d)'	$2 :: T\ \mathrm{P}$	$(T \to_\mathrm{d})$, (b)
(e)'	$2 :: F\ ({\sim}\,\mathrm{P} \to_\mathrm{d} \mathrm{P}) \to_\mathrm{d} \mathrm{P}$	(S), (a)
(f)'	$3 :: T\ {\sim}\,\mathrm{P} \to_\mathrm{d} \mathrm{P}$	$(F \to_\mathrm{d})$, (e)'
(g)'	$2 :: F\ \mathrm{P}$	$(F \to_\mathrm{d})$, (e)'
	Closed	(C), (d)', (g)'

In our example, we applied one of the branching rules, i.e. $(T \rightarrow_{\mathrm{d}})$, to the line (b) and used the notions 1^{st} *branch* and 2^{nd} *branch* to indicate that the (*new*) branches were opened.

In the next example, we will generate an infinite tableau for a *notorious* law of CPC.

EXAMPLE 2. Infinite tableau for the Duns Scotus thesis

(a)	$1 :: F\ \mathrm{P} \rightarrow_{\mathrm{d}} (\sim \mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q})$	(start)
(b)	$2 :: T\ \mathrm{P}$	$(F \rightarrow_{\mathrm{d}})$, (a)
(c)	$1 :: F\ {\sim}\mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q}$	$(F \rightarrow_{\mathrm{d}})$, (a)
(d)	$3 :: T\ {\sim}\mathrm{P}$	$(F \rightarrow_{\mathrm{d}})$, (c)
(e)	$1 :: F\ \mathrm{Q}$	$(F \rightarrow_{\mathrm{d}})$, (c)
(f)	$3 :: F\ \mathrm{P}$	$(T \sim)$, (d)
(g)	$2 :: F\ \mathrm{P} \rightarrow_{\mathrm{d}} (\sim \mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q})$	(S), (a)
(h)	$4 :: T\ \mathrm{P}$	$(F \rightarrow_{\mathrm{d}})$, (g)
(i)	$2 :: F\ {\sim}\mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q}$	$(F \rightarrow_{\mathrm{d}})$, (g)
(j)	$5 :: T\ {\sim}\mathrm{P}$	$(F \rightarrow_{\mathrm{d}})$, (i)
(k)	$2 :: F\ \mathrm{Q}$	$(F \rightarrow_{\mathrm{d}})$, (i)
(l)	$5 :: F\ \mathrm{P}$	$(T \sim)$, (j)
(m)	$3 :: F\ \mathrm{P} \rightarrow_{\mathrm{d}} (\sim \mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q})$	(S), (a)
(n)	$6 :: T\ \mathrm{P}$	$(F \rightarrow_{\mathrm{d}})$, (m)
(o)	$3 :: F\ {\sim}\mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q}$	$(F \rightarrow_{\mathrm{d}})$, (m)
(p)	$7 :: T\ {\sim}\mathrm{P}$	$(F \rightarrow_{\mathrm{d}})$, (o)
(r)	$3 :: F\ \mathrm{Q}$	$(F \rightarrow_{\mathrm{d}})$, (o)
(s)	$7 :: F\ \mathrm{P}$	$(T \sim)$, (p)
(t)	$4 :: F\ \mathrm{P} \rightarrow_{\mathrm{d}} (\sim \mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q})$	(S), (a)
	$\vdots$	

The procedure goes on ad infinitum.

THEOREM 4. *A formula* P *has a* D_2^**-tableau proof* $\Leftrightarrow$ P *is valid in* D_2^*.

PROOF. See [5]. □

6. Unsigned Labelled Tableaux for D_2^*

Now, we give a new set of tableau rules for our system We will work with *labelled formulas* such as σ :: P, where σ is a label (being viewed as a natural number) and P is a formula. The notation $\sigma :: \mathrm{P}$ intuitively means "P holds in world σ".

D_2^**-tableau* is a tree of labelled formulas with root label ρ (we always assume that $\rho = 1$) and all the nodes of a tree are obtained by the rules schematically described in Table 1. A branch of D_2^*-tableau is closed if it contains $\bot$, otherwise it is open. A D_2^*-tableau is closed if all of the branches it contains are closed, otherwise it is open. By a D_2^**-tableau proof* of P we mean a closed tableau with $1 :: {\sim}\mathrm{P}$.

Classical rules:

$$(\vee)\quad \frac{\sigma :: \mathrm{P} \vee \mathrm{Q}}{\sigma :: \mathrm{P} \mid \sigma :: \mathrm{Q}} \qquad ({\sim}\vee)\quad \frac{\sigma :: {\sim}(\mathrm{P} \vee \mathrm{Q}}{\begin{array}{c}\sigma :: {\sim}\mathrm{P}\\ \sigma :: {\sim}\mathrm{Q}\end{array}}$$

$$({\sim}{\sim})\quad \frac{\sigma :: {\sim}{\sim}\mathrm{P}}{\sigma :: \mathrm{P}}$$

Discursive rules:

$$(\wedge_{\mathrm{d}})\quad \frac{\sigma :: \mathrm{P} \wedge_{\mathrm{d}} \mathrm{Q}}{\begin{array}{c}\tau :: \mathrm{P}\\ \sigma :: \mathrm{Q}\\ (\text{for } \tau \text{ \textit{new}})\end{array}} \qquad ({\sim}\wedge_{\mathrm{d}})\quad \frac{\sigma :: {\sim}(\mathrm{P} \wedge_{\mathrm{d}} \mathrm{Q})}{\begin{array}{c}\sigma' :: {\sim}\mathrm{P} \mid \sigma :: {\sim}\mathrm{Q}\\ (\text{for } \sigma' \text{ \textit{used}})\end{array}}$$

$$(\rightarrow_{\mathrm{d}})\quad \frac{\sigma :: \mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q}}{\begin{array}{c}\sigma' :: {\sim}\mathrm{P} \mid \sigma :: \mathrm{Q}\\ (\text{for } \sigma' \text{ \textit{used}})\end{array}} \qquad ({\sim}\rightarrow_{\mathrm{d}})\quad \frac{\sigma :: {\sim}(\mathrm{P} \rightarrow_{\mathrm{d}} \mathrm{Q})}{\begin{array}{c}\tau :: \mathrm{P}\\ \sigma :: {\sim}\mathrm{Q}\\ (\text{for } \tau \text{ \textit{new}})\end{array}}$$

Closing rule: **Special rule:**

$$\textbf{(C)}\quad \frac{\begin{array}{c}\sigma :: \mathrm{P}\\ \sigma :: {\sim}\mathrm{P}\end{array}}{\bot} \qquad \textbf{(S)}\quad \frac{\rho :: {\sim}\mathrm{P}}{\sigma' :: {\sim}\mathrm{P}} \quad \begin{array}{c}(\text{for \textit{root label} } \rho)\\ (\text{for } \sigma' \text{ \textit{used}})\end{array}$$

Table 1. Unsigned Labelled Tableaux for D_2^*

Here is an example of a tableau proof of $\sim\sim P \rightarrow_d P$.

Example 3. Closed tableau for the law of double negation.

(a)	$1 :: \sim(\sim\sim P \rightarrow_d \sim P)$	(start)
(b)	$2 :: \sim\sim P$	$(\sim \rightarrow_d)$, (a)
(c)	$1 :: \sim P$	$(\sim \rightarrow_d)$, (a)
(d)	$2 :: P$	$(\sim\sim)$, (b)
(e)	$2 :: \sim(\sim\sim P \rightarrow_d \sim P)$	(S), (a)
(f)	$3 :: \sim\sim P$	$(\sim \rightarrow_d)$, (e)
(g)	$2 :: \sim P$	$(\sim \rightarrow_d)$, (e)
(i)	$\perp$	(d), (g)

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JANUSZ CIUCIURA
Department of Logic
University of Łódź
Kopcińskiego 16/18
90-232 Łódź
Poland
janciu@uni.lodz.pl