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THE AXIOMATIZATION OF HORST WESSEL'S STRICT LOGICAL CONSEQUENCE RELATION*

1. Introduction

In his book from 1984 Horst Wessel presents the system of strict logical consequence $\mathbf{F}^{\mathrm{s}}$ (see also (Wessel, 1979)). The author maintained that this system axiomatized the relation $\models_{\mathrm{s}}$ of strict logical consequence between formulas of Classical Propositional Calculi (CPC). Let $\models$ be the classical consequence relation in CPC. The relation $\models_{\mathrm{s}}$ is defined as follows:

$$\begin{aligned}\varphi \models_{\mathrm{s}} \psi \text{ iff } \varphi \models \psi, &\text{ every variable from } \psi \text{ occurs in } \varphi \text{ and}\\ &\text{neither } \varphi \text{ is a contradiction nor } \psi \text{ is a tautology.}\end{aligned}$$

Clearly, if $\varphi \models_{\mathrm{s}} \psi$, then neither φ is a tautology nor ψ is a contradiction.

Intuitions connected with the relation $\models_{\mathrm{s}}$ were presented in (Wessel, 1984). The analysis of the relation $\models_{\mathrm{s}}$ is also carried out in (Pietruszczak, 2004). In the present paper we will show that the system $\mathbf{F}^{\mathrm{s}}$ is not a complete axiomatization of the relation $\models_{\mathrm{s}}$. Moreover, we will present the system $\mathbf{VF}^{\mathrm{s}}$ that is an «extension to completeness» of the $\mathbf{F}^{\mathrm{s}}$.[1]

*This is a corrected version of the Polish paper (Pietruszczak, 1997) and its German version (Pietruszczak, 1998). Translation from Polish by Rafał Gruszczyński (authorized).

[1]The book (Wessel, 1999) is a revised edition of (Wessel, 1984). Wessel replaced the system $\mathbf{F}^{\mathrm{s}}$ with $\mathbf{VF}^{\mathrm{s}}$ (under the old name '$\mathbf{F}^{\mathrm{s}}$'). In (Wessel, 1999) some of the theorems are given without proofs, in particular the Completenness Theorem. Wessel refers readers to our paper (Pietruszczak, 1998), cited several times in (Wessel, 1999).

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2. The calculus $\mathbf{F}^{\mathrm{s}}$

Let $\mathcal{L} = \langle \mathsf{L}, \vee, \wedge, \neg \rangle$ be a propositional language. Formulas of $\mathcal{L}$ (i.e., elements of the set L) are built in a standard way from propositional variables from the countable set $\mathsf{V} := \{p_1, p_2, p_3, \ldots\}$, brackets and functors $\vee$, $\wedge$ and $\neg$ understood respectively as truth-value connectives of disjunction, conjunction and negation.

First three variables in examples will be denoted by, respectively, 'p', 'q' and 'r'. Let $\mathsf{V}(\varphi)$ be a set of variables occurring in a formula φ.

Let T (resp. F) be the set of all tautologies (resp. contradictions) of CPC. We say that a given formula is *contingent* iff it is neither tautology nor contradiction. Let K be the set of all contingent formulas, i.e., $\mathsf{K} := \mathsf{L} \setminus (\mathsf{T} \cup \mathsf{F})$. Directly form definitions for all $\varphi, \psi \in \mathsf{L}$ we obtain:

$$\begin{aligned} \varphi \models \psi \;\&\; \varphi \in \mathsf{T} &\implies \psi \in \mathsf{T}\,, \\ \varphi \models \psi \;\&\; \psi \in \mathsf{F} &\implies \varphi \in \mathsf{F}\,, \\ \varphi \models \psi \;\&\; \varphi \notin \mathsf{F} \;\&\; \psi \notin \mathsf{T} &\implies \varphi, \psi \in \mathsf{K}\,. \end{aligned} \tag{2.1}$$

In our terminology, for all $\varphi, \psi \in \mathsf{L}$ we have:

$$\begin{aligned} \varphi \models_{\mathrm{s}} \psi &\overset{\mathrm{df}}{\iff} \varphi \models \psi \;\&\; \mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi) \;\&\; \varphi \notin \mathsf{F} \;\&\; \psi \notin \mathsf{T}\,, \\ \varphi \models_{\mathrm{s}} \psi &\iff \varphi \models \psi \;\&\; \mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi) \;\&\; \varphi, \psi \in \mathsf{K}\,. \end{aligned} \tag{2.2}$$

Let $\{\ulcorner \varphi \vdash \psi \urcorner : \varphi, \psi \in \mathsf{L}\}$ be a set of sequents. The sign '$\vdash$' do not mark any binary relation on L. A sequent $\ulcorner \varphi \vdash \psi \urcorner$ is a «new formula» that render the argument with the assumption φ and the claim ψ. The formula φ is called the *antecedent* and ψ is called the *succedent* of the sequent $\ulcorner \varphi \vdash \psi \urcorner$. A sequent $\ulcorner \varphi \vdash \psi \urcorner$ is called *correct* iff $\varphi \models_{\mathrm{s}} \psi$.

The calculus $\mathbf{F}^{\mathrm{s}}$ is a deductive system (with the standard notion of proof) built in the set of all sequents.

An *axiom* of the system is this and only this sequent that satisfies the following three conditions:

(*E1*) neither antecedent of this sequent is a contradiction nor its succedent is a tautology;

(*E2*) every variable occurring in a succedent of this sequent occurs in its antecedent as well;

(*E3*) the sequent has one of the following nine forms:

$$\varphi \vdash \neg\neg\varphi \tag{A1}$$

$$\neg\neg\varphi \vdash \varphi \tag{A2}$$

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(A3) $$\varphi \wedge \psi \vdash \varphi$$

(A4) $$\varphi \wedge \psi \vdash \psi \wedge \varphi$$

(A5) $$\neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$$

(A6) $$\neg\varphi \vee \neg\psi \vdash \neg(\varphi \wedge \psi)$$

(A7) $$(\varphi \vee \psi) \wedge \chi \vdash (\varphi \wedge \chi) \vee \psi$$

(A8) $$(\varphi \wedge \chi) \vee (\psi \wedge \chi) \vdash (\varphi \vee \psi) \wedge \chi$$

(A9) $$\varphi \vdash \varphi \wedge (\psi \vee \neg\psi)$$

Moreover, the system $\mathbf{F}^{\mathrm{s}}$ has three *rules of inference*:

(R1) $$\frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \chi}$$

(R2) $$\frac{\varphi \vdash \psi \qquad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

(R3) $$\frac{\varphi \vdash \psi \qquad \psi \vdash \varphi}{\chi \vdash \chi(\varphi/\psi)} \qquad \text{dla } \chi \notin \mathsf{F} \text{ i } \chi(\varphi/\psi) \notin \mathsf{T}$$

A given sequent is a *thesis* of $\mathbf{F}^{\mathrm{s}}$ iff it is derivable in a finite number of steps from the axioms by application of the rules of inference.

Wessel proves:

Theorem on the Correctness 2.1 (Wessel, 1984, cf. *MT1*, *MT2* and *MT3*, p. 170). *If a sequent* $\ulcorner\varphi \vdash \psi\urcorner$ *is a thesis of the calculus* $\mathbf{F}^{\mathrm{s}}$*, then it is correct, i.e.,* $\varphi \models_{\mathrm{S}} \psi$. □

3. Incompleteness of $\mathbf{F}^{\mathrm{s}}$

In (Wessel, 1984, p. 172) one can find the completeness metatheorem *MT7*, which says that all correct sequents are theses of $\mathbf{F}^{\mathrm{s}}$, i.e., if $\varphi \models_{\mathrm{S}} \psi$, then $\ulcorner\varphi \vdash \psi\urcorner$ is a thesis of $\mathbf{F}^{\mathrm{s}}$. Yet we will show that this theorem does not hold. For example, the correct sequent

$$(p \wedge (\neg q \vee q)) \wedge (\neg r \vee r) \vdash p \wedge ((\neg q \vee q) \wedge (\neg r \vee r))$$

is not a thesis of $\mathbf{F}^{\mathrm{s}}$, since it does not fulfill the following criterion:

Criterion. *If a sequent* $\ulcorner\varphi \vdash \psi\urcorner$ *is a thesis of the calculus* $\mathbf{F}^{\mathrm{s}}$ *and a tautology* $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ *is a subformula of* ψ*, then* $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ *is also a subformula of* φ.

Proof. Induction on complexity of proofs of theses. Proofs of the axioms are of complexity zero; proofs of theses derivable directly from the axioms by means of the rules of inference are of complexity one; proofs of theses derivable form the theses whose proofs are of complexity zero or one are of complexity two; etc.

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(I) Clearly, the axioms of the form (A1)–(A6), (A8), (A9) satisfy the above criterion. Similarly the axioms of the form (A7), since a tautology $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ has to be a subformula of formulas that are mentioned in the above schema. A succedent cannot have the form $\ulcorner(\tau_1 \wedge \tau_2) \vee \psi\urcorner$, since it would be a tautology (in case of the axioms of the form (A4) and(A6) the argument is similar).

(II) As inductive hypothesis, let us assume that the criterion holds for the sequents whose proof is of complexity less then n. Let a sequent $\ulcorner\varphi \vdash \psi\urcorner$ has a proof of complexity n and let a tautology $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ be a subformula of a formula ψ.

(i) If the sequent $\ulcorner\varphi \vdash \psi\urcorner$ was derived by means of the formula (R1), then for some χ the sequents $\ulcorner\varphi \vdash \chi\urcorner$ and $\ulcorner\chi \vdash \psi\urcorner$ have a proof of complexity less then n. Thus, by inductive hypothesis, the tautology $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ is a subformula of φ.

(ii) If the sequent $\ulcorner\varphi \vdash \psi\urcorner$ was derived by means of the rule (R2), then for some $\chi_1 \neq \tau_1$ and $\chi_2 \neq \tau_2$ we have $\psi = \ulcorner\chi_1 \wedge \chi_2\urcorner$, where the sequents $\ulcorner\varphi \vdash \chi_1\urcorner$ and $\ulcorner\varphi \vdash \chi_2\urcorner$ have a proof of complexity lees than n. Thus the tautology $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ is a subformula of χ_1 or χ_2. Hence, by inductive hypothesis, this tautology also is a subformula of φ.

(iii) If sequent $\ulcorner\varphi \vdash \psi\urcorner$ was derived by means of the rule (R3), then for some φ' and ψ' we have $\psi = \varphi(\varphi'/\psi')$, where the sequents $\ulcorner\varphi' \vdash \psi'\urcorner$ and $\ulcorner\psi' \vdash \varphi'\urcorner$ have a proof of complexity less than n. Since the tautology $\ulcorner(\tau_1 \wedge \tau_2)\urcorner$ is a subformula of ψ, then at least one of the following two cases holds: (a) this tautology is a subformula of φ, (b) this tautology is a subformula of ψ' and the substitution φ'/ψ' was essential (i.e., φ' occurred in φ). In the case (b), by inductive hypothesis, this tautology is also a subformula of φ'. Therefore, it is also a subformula of φ. □

In (Wessel, 1984, p. 167) one can find a proof of the fact, that a sequent $\ulcorner(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)\urcorner$ is a thesis of $\mathbf{F}^{\mathrm{s}}$ (cf. *T4*), without any additional restrictions put on formulas φ, ψ and χ except for (*E1*) and (*E2*). Yet this proof does not take into account cases in which φ, ψ or χ are tautologies, but $\ulcorner(\varphi \wedge \psi) \wedge \chi\urcorner$ and $\ulcorner\varphi \wedge (\psi \wedge \chi)\urcorner$ are contingent. Let us analyze a derivation of a sequent $\ulcorner(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)\urcorner$ taking one additional assumption, that $\varphi, \psi, \chi \notin \mathsf{T}$. In this derivation we will apply the following thesis of $\mathbf{F}^{\mathrm{s}}$, for any $\varphi, \psi \in \mathsf{L}$ such that $\ulcorner\varphi \wedge \psi\urcorner \notin \mathsf{F}$ and $\psi \notin \mathsf{T}$:

$$\varphi \wedge \psi \vdash \psi\,. \tag{3.1}$$

1.	$\varphi \wedge \psi \vdash \psi \wedge \varphi$	(A4), by hypothesis $\ulcorner\psi \wedge \varphi\urcorner \in \mathsf{K}$
2.	$\psi \wedge \varphi \vdash \psi$	(A3)
3.	$\varphi \wedge \psi \vdash \psi$	1, 2 and (R1)

We have the following derivation of a sequent $\ulcorner(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)\urcorner$ (such that $\ulcorner(\varphi \wedge \psi) \wedge \chi\urcorner \in \mathsf{K}$) with additional assumptions:

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1a.	$\varphi \notin \mathsf{T}$	additional assumption
2a.	$\psi \notin \mathsf{T}$	additional assumption
3a.	$\chi \notin \mathsf{T}$	additional assumption
4a.	$(\varphi \wedge \psi) \wedge \chi \vdash \chi$	(3.1), (*E1*), 3a
5a.	$(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge \psi$	(A3), (*E1*), 1a, 2a
6a.	$\varphi \wedge \psi \vdash \varphi$	(A3), (*E1*), 1a
7a.	$\varphi \wedge \psi \vdash \psi$	(3.1), (*E1*), 2a
8a.	$(\varphi \wedge \psi) \wedge \chi \vdash \varphi$	5a, 6a and (R1)
9a.	$(\varphi \wedge \psi) \wedge \chi \vdash \psi$	5a, 7a and (R1)
10a.	$(\varphi \wedge \psi) \wedge \chi \vdash \psi \wedge \chi$	9a, 4a and (R2)
11a.	$(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$	8a, 10a and (R2)

From the CRITERION one can see that assumptions 1a–3a were essential.

Similar gaps can be found in the derivations of the theses *T5* and *T12* (Wessel, 1984, p. 167, 168). Moreover, in some derivations we can find gaps of different kinds. For example:

- In the proofs of the theses *T6–T8*, $\ulcorner\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi\urcorner$, $\ulcorner\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)\urcorner$ and $\ulcorner\varphi \vee \psi \vdash \psi \vee \varphi\urcorner$, while applying the rule (R3), it is being assumed that theses of the systemu $\mathbf{F^s}$ are sequents $\varphi \vdash \neg\neg\varphi$ and $\neg\neg\varphi \vdash \varphi$, although it is not ruled out that φ is not in K (similarly for ψ).
- In the derivation of the thesis *T18*, while applying the rule (R3), it is being assumed that the thesis of $\mathbf{F^s}$ is a sequent $\neg(\psi \vee \neg\psi) \vdash \neg\psi \wedge \psi$ which has a contradictory antecedent.
- In the proof of a sequent $\varphi \vee (\neg\psi \wedge \psi) \vdash \varphi$ the axiom (A9) $\ulcorner\varphi \vdash \varphi \wedge (\psi \vee \neg\psi)\urcorner$ is being applied. Yet we may use this axiom only if $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi)$ (see the condition *E2*). But it does not have to obtain for the sequent being proved.

4. The calculus $\mathbf{VF^s}$

We will build a new system $\mathbf{VF^s}$ in the set of sequents. It will have six *rules of inference*: to the rules of the system $\mathbf{F^s}$ we will add rules:

(R4) $$\frac{\varphi \vdash \psi}{\varphi \vdash \psi \wedge \tau} \quad \text{if } \tau \in \mathsf{T} \text{ and } \mathsf{V}(\tau) \subseteq \mathsf{V}(\varphi)$$

(R5) $$\frac{\varphi \vdash \psi}{\varphi \vdash \psi \vee \phi} \quad \text{if } \phi \in \mathsf{F} \text{ and } \mathsf{V}(\phi) \subseteq \mathsf{V}(\varphi)$$

(R6) $$\frac{\varphi \vdash \psi}{\varphi \vdash \phi \vee \psi} \quad \text{if } \phi \in \mathsf{F} \text{ and } \mathsf{V}(\phi) \subseteq \mathsf{V}(\varphi)$$

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By definitions we obtain:

LEMMA 4.1. *Six rules of inference of the calculus* $\mathbf{VF^s}$ *are correct in the following sense: when applied to correct sequents these yield a correct sequent.*

PROOF. The relation $\models_s$ is transitive, by (2.2) (since $\models$ is transitive). Therefore the rule (R1) preserves strict consequence.

If $\varphi \models_s \psi$ and $\varphi \models_s \chi$, then $\psi \wedge \chi \notin \mathsf{T}$ and $\varphi \models \psi \wedge \chi$. Thus the rule (R2) also preserves strict consequence relation.

If $\varphi \models_s \psi$ and $\psi \models_s \varphi$, then $\varphi \mathrel{\rightleftarrows} \psi$, $\mathsf{V}(\psi) = \mathsf{V}(\varphi)$ and $\varphi, \psi \in \mathsf{K}$, by (2.2). Thus, by the extensionality of CPC and (2.2), the rule (R3) preserves strict consequence.

Finally, the rules (R4), (R5) and (R6) preserve strict consequence, since $\varphi \models \psi$ entails $\varphi \models \psi \wedge \tau$, $\varphi \models \psi \vee \phi$ and $\varphi \models \phi \vee \psi$. □

The *axiom* of $\mathbf{VF^s}$ is this and only this sequent that fulfills conditions (*E1*) and (*E2*) for the axioms of $\mathbf{F^s}$ and

(*E3′*) the sequent in question is a specification of one of the following ten schemas: (A1)–(A8) and

(A10) $$\varphi \vee \phi \vdash \varphi$$

(A11) $$\phi \vee \varphi \vdash \varphi$$

where ϕ is a contradiction.

By the definition of axioms we obtain:

LEMMA 4.2. *All axioms of the calculus* $\mathbf{VF^s}$ *are correct sequents.*

PROOF. For any axiom $\ulcorner\varphi \vdash \psi\urcorner$ we have $\varphi \models \psi$. Moreover, from the conditions (*E1*) and (*E2*) we have $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi)$, $\varphi \notin \mathsf{F}$ and $\psi \notin \mathsf{T}$. Thus $\varphi \models_s \psi$. □

From lemmas 4.1 and 4.2 we obtain:

THEOREM ON THE CORRECTNESS 4.1. *All theses of* $\mathbf{VF^s}$ *are correct sequents, i.e., if a sequent* $\ulcorner\varphi \vdash \psi\urcorner$ *is a thesis of the calculus* $\mathbf{VF^s}$*, then* $\varphi \models_s \psi$.

PROOF. As we showed, all axioms of $\mathbf{VF^s}$ are correct sequent. Moreover, all rules of $\mathbf{VF^s}$ always lead from correct sequents to correct sequents. Thus, by induction over $\mathbf{VF^s}$, we see that every derivable sequent is correct. □

We will show that for $\mathbf{VF^s}$ Theorem on the Adequacy holds, i.e., the sequent $\ulcorner\varphi \vdash \psi\urcorner$ will be thesis of $\mathbf{VF^s}$ iff $\varphi \models_s \psi$. We will derive auxiliary theses of $\mathbf{VF^s}$ necessary to prove that (cf. Section 6, p. 138):

COMPLETENNESS THEOREM 4.2. *All correct sequents are theses of the calculus* $\mathbf{VF^s}$.

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5. Some auxiliary theses of VF^s

By means of (A1), (A2) and (R1) for any $\varphi \in \mathsf{K}$ we will derive the sequent:

(5.1) $$\varphi \vdash \varphi$$

From this, by means of the rules (R2), (R5), (R6) and (R4), respectively, for all $\varphi \in \mathsf{K}$, $\phi \in \mathsf{F}$ and $\tau \in \mathsf{T}$ such that $\mathsf{V}(\phi), \mathsf{V}(\tau) \subseteq \mathsf{V}(\varphi)$ we derive the following sequents

(5.2) $$\varphi \vdash \varphi \wedge \varphi$$

(5.3) $$\varphi \vdash \varphi \vee \phi$$

(5.4) $$\varphi \vdash \phi \vee \varphi$$

(5.5) $$\varphi \vdash \varphi \wedge \tau$$

By (5.5) all sequents of the schema (A9), that satisfy conditions (*E1*) and (*E2*), are theses of **VF**^{s}. Hence we have:

Fact 5.1. *All theses of* **F**^{s} *are theses of* **VF**^{s}. □

We will infer a derivable rule of the system **VF**^{s} (and of **F**^{s} as well):

(R7) $$\frac{\chi_1 \vdash \chi_2 \qquad \varphi_1 \vdash \psi_1 \qquad \psi_1 \vdash \varphi_1 \qquad \varphi_2 \vdash \psi_2 \qquad \psi_2 \vdash \varphi_2}{\chi_1(\varphi_1/\psi_1) \vdash \chi_2(\varphi_2/\psi_2)}$$

1.	$\chi_1 \vdash \chi_2$	assumption
2.	$\varphi_1 \vdash \psi_1$	assumption
3.	$\psi_1 \vdash \varphi_1$	assumption
4.	$\varphi_2 \vdash \psi_2$	assumption
5.	$\psi_2 \vdash \varphi_2$	assumption
6.	$\chi_2 \vdash \chi_2(\varphi_2/\psi_2)$	4, 5, (R3)
7.	$\chi_1 \vdash \chi_2(\varphi_2/\psi_2)$	1, 6, (R1)
8.	$\chi_1(\varphi_1/\psi_1) \vdash \chi_1$	$\chi_1 = \chi_1(\varphi_1/\psi_1).(\psi_1/\varphi_1)$, 2, 3, (R3)
9.	$\chi_1(\varphi_1/\psi_1) \vdash \chi_2(\varphi_2/\psi_2)$	8, 7, (R1)

Farther in this paper all auxiliary theses of **VF**^{s} will satisfy conditions (*E1*) and (*E2*). This fact will not be mentioned separately.

By means of the new rule (R4) we can «finish the proof» concerning the sequents of the form

(5.6) $$(\varphi \wedge \psi) \wedge \psi \vdash \varphi \wedge (\psi \wedge \chi)$$

(5.7) $$\varphi \wedge (\psi \wedge \chi) \vdash (\varphi \wedge \psi) \wedge \chi$$

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1b.	$\varphi \notin \mathsf{T}$	additional assumption
2b.	$\psi \notin \mathsf{T}$	additional assumption
3b.	$\chi \in \mathsf{T}$	additional assumption
4b.	$(\varphi \wedge \psi) \wedge \chi \vdash \psi \wedge \chi$	2b=2a, 3b, 9a, (R4)
5b.	$(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$	1b=1a, 8a, 4b, (*R2*)
1c.	$\varphi \notin \mathsf{T}$	additional assumption
2c.	$\psi \in \mathsf{T}$	additional assumption
3c.	$\chi \in \mathsf{T}$	additional assumption
4c.	$(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$	1c=1a, 8a, 2c, 3c, (R4)
1d.	$\varphi \in \mathsf{T}$	additional assumption
2d.	$\psi \notin \mathsf{T}$	additional assumption
3d.	$\chi \in \mathsf{T}$	additional assumption
4d.	$(\varphi \wedge \psi) \wedge \chi \vdash (\psi \wedge \chi) \wedge \varphi$	2d=2b, 3d=3b, 4b, 1d, (R4)
5d.	$(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$	(A4), (R7)

Analogously we will analyze remaining alternative cases. The similar proof is carried out for the sequent (5.7).

Let us prove that some sequents are theses of the calculus $\mathbf{VF}^{\mathrm{s}}$.

(5.8) $$\varphi \vdash \varphi \vee \varphi$$

1.	$\neg\varphi \wedge \neg\varphi \vdash \neg\varphi$	(A3)
2.	$\neg\varphi \vdash \neg\varphi \wedge \neg\varphi$	(5.2), (R3)
3.	$\neg\neg\varphi \vdash \neg(\neg\varphi \wedge \neg\varphi)$	1, 2 and (R3) for $\chi = \ulcorner\neg\neg\varphi\urcorner$
4.	$\neg\neg\varphi \vdash \neg\neg\varphi \vee \neg\neg\varphi$	3, (A5), (A6), (R7)
5.	$\varphi \vdash \varphi \vee \varphi$	4, (A1), (A2), (R7)

(5.9) $$\varphi \vee \varphi \vdash \varphi$$

1.	$\neg\neg\varphi \vdash \neg\neg\varphi$	(5.1)
2.	$\neg(\neg\varphi \wedge \neg\varphi) \vdash \varphi$	1, (A3), (5.2), (A1), (A2), (R7)
3.	$\neg\neg\varphi \vee \neg\neg\varphi \vdash \varphi$	2, (A5), (A6), (R7)
4.	$\varphi \vee \varphi \vdash \varphi$	3, (A1), (A2), (R7)

(5.10) $$\varphi \vee \psi \vdash \neg(\neg\varphi \wedge \neg\psi)$$

1.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \neg(\neg\varphi \wedge \neg\psi)$	(A6)
2a.	$\varphi \notin \mathsf{F}$	additional assumption
3a.	$\psi \notin \mathsf{F}$	additional assumption
4a.	$\varphi \vee \psi \vdash \neg\neg\varphi \vee \neg\neg\psi$	2a, 3a, (5.1), (A1), (A2), (R7)
2b.	$\varphi \notin \mathsf{F}$	additional assumption
3b.	$\psi \in \mathsf{F}$	additional assumption

4b.	$\varphi \vee \psi \vdash \varphi$	2b, 3b, (A10)
5b.	$\varphi \vee \psi \vdash \varphi \vee \neg\neg\psi$	4b, (R5)
6b.	$\varphi \vee \psi \vdash \neg\neg\varphi \vee \neg\neg\psi$	5b, 2b, (A1), (A2), (R7)
2c.	$\varphi \in \mathsf{F}$	additional assumption
3c.	$\psi \notin \mathsf{F}$	additional assumption
4c.	$\varphi \vee \psi \vdash \psi$	2b, 3b, (A11)
5c.	$\varphi \vee \psi \vdash \neg\neg\varphi \vee \psi$	4c, (R6)
6c.	$\varphi \vee \psi \vdash \neg\neg\varphi \vee \neg\neg\psi$	5c, 2c, (A1), (A2), (R7)
7.	$\varphi \vee \psi \vdash \neg(\neg\varphi \wedge \neg\psi)$	4a=6b=6c, 1, (R1)

(5.11) $$\neg(\neg\varphi \wedge \neg\psi) \vdash \varphi \vee \psi$$

1.	$\neg(\neg\varphi \wedge \neg\psi) \vdash \neg\neg\varphi \vee \neg\neg\psi$	(A5)
2a.	$\varphi \notin \mathsf{F}$	additional assumption
3a.	$\psi \notin \mathsf{F}$	additional assumption
4a.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \varphi \vee \psi$	2a, 3a, (5.1), (A1), (A2), (R7)
2b.	$\varphi \notin \mathsf{F}$	additional assumption
3b.	$\psi \in \mathsf{F}$	additional assumption
4b.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \neg\neg\varphi$	2b, 3b, (A10)
5b.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \neg\neg\varphi \vee \psi$	4b, (R5)
6b.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \varphi \vee \psi$	5b, 2b, (A1), (A2), (R7)
2c.	$\varphi \in \mathsf{F}$	additional assumption
3c.	$\psi \notin \mathsf{F}$	additional assumption
4c.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \neg\psi$	2b, 3b, (A11)
5c.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \varphi \vee \neg\neg\psi$	4c, (R6)
6c.	$\neg\neg\varphi \vee \neg\neg\psi \vdash \varphi \vee \psi$	5c, 2c, (A1), (A2), (R7)
7.	$\neg(\neg\varphi \wedge \neg\psi) \vdash \varphi \vee \psi$	4a=6b=6c, 1, (R1)

(5.12) $$\varphi \vee \psi \vdash \psi \vee \varphi$$

1.	$\varphi \vee \psi \vdash \neg(\neg\varphi \wedge \neg\psi)$	(5.10)
2.	$\neg(\neg\varphi \wedge \neg\psi) \vdash \neg(\neg\psi \wedge \neg\varphi)$	(A4), (*R3*)
3.	$\neg(\neg\psi \wedge \neg\varphi) \vdash \psi \vee \varphi$	(5.11)
4.	$\varphi \vee \psi \vdash \psi \vee \varphi$	1–3, (*R1*)

(5.13) $$\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi$$

1.	$\neg(\varphi \vee \psi) \vdash \neg\neg(\neg\varphi \wedge \neg\psi)$	(5.10, (5.11), (*R3*)
2.	$\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi$	1, (A2), (R1)

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(5.14) $$\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$$

1.	$\neg\neg(\neg\varphi \wedge \neg\psi) \vdash \neg(\varphi \vee \psi)$	(5.10), (5.11), (*R3*)
2.	$\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$	(A1), 1, (R1)

(5.15) $$(\varphi \vee \psi) \wedge \chi \vdash (\varphi \wedge \chi) \vee (\psi \wedge \chi)$$

1.	$(\varphi \vee \psi) \wedge \chi \vdash (\varphi \wedge \chi) \vee \psi$	(A7)
2a.	$\chi \notin \mathsf{T}$	additional assumption
3a.	$(\varphi \vee \psi) \wedge \chi \vdash \chi$	2a, (3.1)
4a.	$(\varphi \vee \psi) \wedge \chi \vdash ((\varphi \wedge \chi) \vee \psi) \wedge \chi$	1, 3a, (R2)
2b.	$\chi \in \mathsf{T}$	additional assumption
3b.	$(\varphi \wedge \chi) \vee \psi \vdash ((\varphi \wedge \chi) \vee \psi) \wedge \chi$	2b, (5.5)
4b.	$(\varphi \vee \psi) \wedge \chi \vdash ((\varphi \wedge \chi) \vee \psi) \wedge \chi$	1, 3b, (R1)
5.	$((\varphi \wedge \chi) \vee \psi) \wedge \chi \vdash (\psi \vee (\varphi \wedge \chi)) \wedge \chi$	(5.12), (R3)
6.	$(\psi \vee (\varphi \wedge \chi)) \wedge \chi \vdash (\psi \wedge \chi) \vee (\varphi \wedge \chi \wedge \chi)$	(A7)
7.	$(\psi \vee (\varphi \wedge \chi)) \wedge \chi \vdash (\psi \wedge \chi) \vee (\varphi \wedge \chi)$	(A3), (5.2), (R7)
8.	$(\psi \wedge \chi) \vee (\varphi \wedge \chi) \vdash (\varphi \wedge \chi) \vee (\psi \wedge \chi)$	(5.12)
9.	$(\varphi \vee \psi) \wedge \chi \vdash (\varphi \wedge \chi) \vee (\psi \wedge \chi)$	4a=4b, 5, 7, 8 (R1)

(5.16) $$(\varphi \vee \psi) \vee \chi \vdash \varphi \vee (\psi \vee \chi)$$

1a.	$\varphi \vee \psi \in \mathsf{K}$	additional assumption
2a.	$(\varphi \vee \psi) \vee \chi \vdash \neg\neg((\varphi \vee \psi) \vee \chi)$	(A1)
3a.	$(\varphi \vee \psi) \vee \chi \vdash \neg((\neg\varphi \wedge \neg\psi) \wedge \neg\chi)$	1a, 2a, (5.13), (5.14), (R7)
4a.	$(\varphi \vee \psi) \vee \chi \vdash \neg(\neg\varphi \wedge (\neg\psi \wedge \neg\chi))$	3a, (5.6), (5.7), (R7)
5aa.	$\psi \vee \chi \in \mathsf{K}$	additional assumption
6aa.	$(\varphi \vee \psi) \vee \chi \vdash \neg\neg(\varphi \vee (\psi \vee \chi))$	5aa, 4a, (5.13), (5.14), (R7)
7aa.	$(\varphi \vee \psi) \vee \chi \vdash \varphi \vee (\psi \vee \chi)$	6aa, (A2), (R1)
5ab.	$\psi \vee \chi \in \mathsf{F}$	additional assumption
6ab.	$\psi \in \mathsf{F}$	5ab
7ab.	$\chi \in \mathsf{F}$	5ab
8ab.	$\varphi \in \mathsf{K}$	1a, 6ab
9ab.	$(\varphi \vee \psi) \vee \chi \vdash \varphi$	8ab, 6ab, 7ab, (A10), (R1)
10ab.	$(\varphi \vee \psi) \vee \chi \vdash \varphi \vee (\psi \vee \chi)$	5ab, 9ab, (R5)
1b.	$\varphi \vee \psi \in \mathsf{F}$	additional assumption
2b.	$\varphi \in \mathsf{F}$	1b
3b.	$\psi \in \mathsf{F}$	1b
4b.	$\chi \in \mathsf{K}$	1b, (*E1*)

5b.	$(\varphi \vee \psi) \vee \chi \vdash \chi$	1b, 4b, (A11)
6b.	$(\varphi \vee \psi) \vee \chi \vdash \psi \vee \chi$	3b, 5b, (R6)
7b.	$(\varphi \vee \psi) \vee \chi \vdash \varphi \vee (\psi \vee \chi)$	2b, 6b, (R6)

The sequent

$$\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi \tag{5.17}$$

will be derived in an analogous way. We will need a couple of auxiliary theses.

$$(\varphi \wedge \psi) \vee \chi \vdash (\varphi \vee \chi) \wedge (\psi \vee \chi) \tag{5.18}$$

1.	$(\varphi \wedge \psi) \vee \chi \vdash \neg\neg((\varphi \wedge \psi) \vee \chi)$	(A1)
2.	$(\varphi \wedge \psi) \vee \chi \vdash \neg(\neg(\varphi \wedge \psi) \wedge \neg\chi)$	1, (5.13), (5.14), (R7)
3a.	$\varphi \wedge \psi \in \mathsf{K}$	additional assumption
4a.	$(\varphi \wedge \psi) \vee \chi \vdash \neg((\neg\varphi \vee \neg\psi) \wedge \neg\chi)$	3a, 2, (A5), (A6), (R7)
3b.	$\varphi \wedge \psi \in \mathsf{F}$	additional assumption
4b.	$\neg(\varphi \wedge \psi) \wedge \neg\chi \vdash \neg\chi$	3b, (3.1)
5b.	$\neg(\varphi \wedge \psi) \wedge \neg\chi \vdash (\neg\varphi \vee \neg\psi) \wedge \neg\chi$	4b, (R4),(A4), (R7)
6b.	$(\neg\varphi \vee \neg\psi) \wedge \neg\chi \vdash \neg(\varphi \wedge \psi) \wedge \neg\chi$	(3.1), 3b, (R4), (A4), (R7)
7b.	$(\varphi \wedge \psi) \vee \chi \vdash \neg((\neg\varphi \vee \neg\psi) \wedge \neg\chi)$	5b, 6b, 2, (R7)
8.	$(\varphi \wedge \psi) \vee \chi \vdash \neg((\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi))$	4a=7b, (5.15), (A8), (R7)
9a.	$\varphi \vee \chi \in \mathsf{K}$	additional assumption
10a.	$\psi \vee \chi \in \mathsf{K}$	additional assumption
11a.	$(\varphi \wedge \psi) \vee \chi \vdash \neg(\neg(\varphi \vee \chi) \vee \neg(\psi \vee \chi))$	9a, 10a, 8, (5.13), (5.14), (R7)
9b.	$\varphi \vee \chi \in \mathsf{K}$	additional assumption
10b.	$\psi \vee \chi \in \mathsf{T}$	additional assumption
11b.	$(\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi) \vdash \neg\varphi \wedge \neg\chi$	9b, 10b, (A10)
12b.	$(\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi) \vdash (\neg\varphi \wedge \neg\chi) \vee \neg(\psi \vee \chi)$	10b, 11b, (R5)
13b.	$(\neg\varphi \wedge \neg\chi) \vee \neg(\psi \vee \chi) \vdash (\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi)$	9b, 10b, (A10), (R5)
14b.	$(\varphi \wedge \psi) \vee \chi \vdash \neg((\neg\varphi \wedge \neg\chi) \vee \neg(\psi \vee \chi))$	12b, 13b, 8, (R7)
15b.	$(\varphi \wedge \psi) \vee \chi \vdash \neg(\neg(\varphi \vee \chi) \vee \neg(\psi \vee \chi))$	14b, 10b, (5.13), (5.14), (R7)
9c.	$\varphi \vee \chi \in \mathsf{T}$	additional assumption
10c.	$\psi \vee \chi \in \mathsf{K}$	additional assumption
11c.	$(\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi) \vdash \neg\psi \wedge \neg\chi$	9c, 10c, (A11)
12c.	$(\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi) \vdash \neg(\varphi \vee \chi) \vee (\neg\psi \wedge \neg\chi)$	9c, 11c, (R6)
13c.	$\neg(\varphi \vee \chi) \vee (\neg\psi \wedge \neg\chi) \vdash (\neg\varphi \wedge \neg\chi) \vee (\neg\psi \wedge \neg\chi)$	9c, 10c, (A11), (R6)
14c.	$(\varphi \wedge \psi) \vee \chi \vdash \neg((\neg\varphi \wedge \neg\chi) \vee \neg(\psi \vee \chi))$	12c, 13c, 8, (R7)
15c.	$(\varphi \wedge \psi) \vee \chi \vdash \neg(\neg(\varphi \vee \chi) \vee \neg(\psi \vee \chi))$	14c, 10c, (5.13), (5.14), (R7)
16.	$(\varphi \wedge \psi) \vee \chi \vdash \neg\neg((\varphi \vee \chi) \wedge (\psi \vee \chi))$	10a=15b-c, (A5), (A6), (R7)
17.	$(\varphi \wedge \psi) \vee \chi \vdash (\varphi \vee \chi) \wedge (\psi \vee \chi)$	16, (A2), (R1)

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Analogously we will derive the sequent:

$$(\varphi \vee \chi) \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi \tag{5.19}$$

Finally we will derive the sequent:

$$\varphi \vdash \varphi \vee \psi \tag{5.20}$$

1.	$\varphi \vdash (\psi \wedge \neg\psi) \vee \varphi$	(5.4)
2.	$\varphi \vdash (\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$	1, (5.12), (5.18), (5.19), (R7)
3.	$\varphi \vdash \varphi \vee \psi$	2, (A3), (R1)

6. Completeness of the calculus $\mathbf{VF}^{\mathrm{s}}$

We will prove Completeness Theorem 4.2 for the system $\mathbf{VF}^{\mathrm{s}}$. The proof of this theorem will consist of a series of auxiliary lemmas.

For the beginning we will need a generalized form of a couple of previously proved theorems.

LEMMA 6.1. *If* $\ulcorner(\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi\urcorner, \ulcorner\varphi_i \wedge \psi\urcorner \in \mathsf{K}$ *for* $i = 1, \ldots, n$, *then the following sequents are theses of* $\mathbf{VF}^{\mathrm{s}}$*:*

$$(\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi \vdash (\varphi_1 \wedge \psi) \vee \cdots \vee (\varphi_n \wedge \psi)$$
$$(\varphi_1 \wedge \psi) \vee \cdots \vee (\varphi_n \wedge \psi) \vdash (\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi$$

PROOF. For $n = 1$ the lemma holds by (5.1). As inductive hypothesis, let us assume that the lemma is true for $n - 1$. From (5.12), (A8), (5.15) and (R7) we will derive theses: $\ulcorner(\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi \vdash ((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi) \vee (\varphi_n \wedge \psi)\urcorner$ and $\ulcorner((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi) \vee (\varphi_n \wedge \psi) \vdash (\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi\urcorner$. Notice that (by assumption): $\ulcorner(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi\urcorner \in \mathsf{K}$. Indeed, if $\ulcorner(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi\urcorner \notin \mathsf{K}$, then $\ulcorner(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi\urcorner \in \mathsf{F}$, thus also $\varphi_i \wedge \psi \in \mathsf{F}$ for $i \leqslant n - 1$, contrary to the assumption. Thus we can apply the inductive hypothesis. Hence, applying (A8), (5.15) and (R7), we will get both sequents. □

LEMMA 6.2. *If* $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_n\urcorner \in \mathsf{K}$, *then the following sequents are theses of* $\mathbf{VF}^{\mathrm{s}}$*:*

$$\neg(\varphi_1 \wedge \cdots \wedge \varphi_n) \vdash \neg\varphi_1 \vee \cdots \vee \neg\varphi_n$$
$$\neg\varphi_1 \vee \cdots \vee \neg\varphi_n \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$$

PROOF. For $n = 1$ the lemma holds by (5.1). Let $n > 1$. Then, by means of (A5), (A6) and (R7) we will derive: (a) $\ulcorner\neg((\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \wedge \varphi_n) \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \vee \neg\varphi_n\urcorner$ and (b) $\ulcorner\neg(\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \vee \neg\varphi_n \vdash \neg((\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \wedge \varphi_n)\urcorner$. As inductive hypothesis, let us assume that the condition holds for $n-1$. Thus in case if $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1}\urcorner \in \mathsf{K}$, by inductive hypothesis and from (R7), we get the thesis. In case if $\ulcorner\varphi_1 \wedge \cdots \wedge$

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$\varphi_{n-1}\urcorner \in \mathsf{T}$, we get that $\varphi_1, \ldots, \varphi_{n-1} \in \mathsf{T}$ and $\varphi_n \in \mathsf{K}$. Thus from (a), (A11) and (R1) we will derive the sequent $\ulcorner\neg((\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \wedge \varphi_n) \vdash \neg\varphi_n\urcorner$. From this, applying (R6), we get the first of the sequents being proved. Similarly, from (A11) and (R6) we will derive the sequent $\ulcorner\neg\varphi_1 \vee \cdots \vee \neg\varphi_{n-1} \vee \neg\varphi_n \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_{n-1}) \vee \neg\varphi_n\urcorner$. From this and from (b), applying (R1) we get the second of the sequents being proved.

The above reasoning can be repeated for an arbitrary combination of brackets, applying respectively (5.6) and (5.7) or (5.16) and (5.17). □

Lemma 6.3. *If* $\ulcorner\varphi_1 \vee \cdots \vee \varphi_n\urcorner \in \mathsf{K}$, *then the following sequents are theses of* $\mathbf{VF^s}$:

$$\neg(\varphi_1 \vee \cdots \vee \varphi_n) \vdash \neg\varphi_1 \wedge \cdots \wedge \neg\varphi_n$$
$$\neg\varphi_1 \wedge \cdots \wedge \neg\varphi_n \vdash \neg(\varphi_1 \vee \cdots \vee \varphi_n)$$

Proof. For $n = 1$ the lemma holds by (5.1). Let $n > 1$. Then, by means of (5.13), (5.14) and (R7) we will derive sequents: (a) $\ulcorner\neg((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \vee \varphi_n) \vdash \neg(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \neg\varphi_n\urcorner$ and (b) $\ulcorner\neg(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \neg\varphi_n \vdash \neg((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \vee \varphi_n)\urcorner$. As inductive hypothesis, let us assume that the condition holds for $n-1$. Thus in case if $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1}\urcorner \in \mathsf{K}$, by inductive hypothesis and by (R7), we get the thesis. In case if $\ulcorner\varphi_1 \vee \cdots \vee \varphi_{n-1}\urcorner \in \mathsf{F}$, we get that $\varphi_1, \ldots, \varphi_{n-1} \in \mathsf{F}$ and $\varphi_n \in \mathsf{K}$. Thus from (a), (3.1) and (R1) we will derive the sequent $\ulcorner\neg((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \vee \varphi_n) \vdash \neg\varphi_n\urcorner$. From this, applying (A4) and (R4), we get the first of the sequents being proved. Similarly, from (3.1), (A4) and (R4) we will derive the sequent $\ulcorner\neg\varphi_1 \wedge \cdots \wedge \neg\varphi_{n-1} \wedge \neg\varphi_n \vdash \neg(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \neg\varphi_n\urcorner$. From this, (b) and (R1) we will get the second of the sequents being proved. □

Let $\mathsf{E}^{\wedge}$ be the set of *elementary conjunctions*. These will include variables and their negations, and conjunctions built from variables and their negations. Moreover, let $\mathsf{L}^{\vee\wedge}$ be the set of all conjunctions from $\mathsf{E}^{\wedge}$ and all disjunctions of these conjunctions. Thus all members of $\mathsf{L}^{\vee\wedge}$ have a *disjunctive–conjunctive normal form*. We will prove a couple of lemmas concerning the formulas from $\mathsf{L}^{\vee\wedge}$.

Lemma 6.4. *For every* $\kappa \in \mathsf{E}^{\wedge} \cap \mathsf{K}$ *there are such* $\varphi \in \mathsf{L}^{\vee\wedge} \cap \mathsf{K}$, *that theses of* $\mathbf{VF^s}$ *are sequents:* $\ulcorner\neg\kappa \vdash \varphi\urcorner$ *and* $\ulcorner\varphi \vdash \neg\kappa\urcorner$.

Proof. By Lemma 6.2, (A1) (A2) and (R7). □

Lemma 6.5. *Let* $\varphi_1, \ldots, \varphi_n \in \mathsf{L}^{\vee\wedge}$ *for* $n > 0$. *If* $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_n\urcorner \in \mathsf{K}$, *then there is such* $\psi \in \mathsf{L}^{\vee\wedge} \cap \mathsf{K}$, *that the following sequents are theses of* $\mathbf{VF^s}$:

$$\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \psi$$
$$\psi \vdash \varphi_1 \wedge \cdots \wedge \varphi_n$$

Proof. By induction on n. (I) For $n = 1$: by (5.1) and the assumption take $\psi = \varphi_1$.

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(II) For $n = 2$: assume that $\varphi_1 = \ulcorner\kappa_1 \vee \cdots \vee \kappa_m\urcorner$ and $\varphi_2 = \ulcorner\lambda_1 \vee \cdots \vee \lambda_l\urcorner$, where $m, l > 0$ and $\kappa_i, \lambda_i \in \mathsf{E}^\wedge$. We will consider three cases.

(i) Let $l = 1 = m$. Then by the assumption $\ulcorner\kappa_1 \wedge \lambda_1\urcorner \in \mathsf{E}^\wedge \cap \mathsf{K} \subseteq \mathsf{L}^{\vee\wedge}$. Hence, by (5.1), we can set $\psi = \ulcorner\kappa_1 \wedge \lambda_1\urcorner$.

(ii) Let $m + l = k > 1$ and $m > 1$. Then by the assumption, (5.1), (5.6), (5.7) and (R7) we get sequents: $\ulcorner\varphi_1 \wedge \varphi_2 \vdash (\kappa_1 \vee (\kappa_2 \vee \cdots \vee \kappa_m)) \wedge \varphi_2\urcorner$ and $\ulcorner(\kappa_1 \vee (\kappa_2 \vee \cdots \vee \kappa_m)) \wedge \varphi_2 \vdash \varphi_1 \wedge \varphi_2\urcorner$. From these and from (A8) and (5.15) by application (R7) we get sequents: (a) $\ulcorner\varphi_1 \wedge \varphi_2 \vdash (\kappa_1 \wedge \varphi_2) \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2)\urcorner$ and (b) $\ulcorner(\kappa_1 \wedge \varphi_2) \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2) \vdash \varphi_1 \wedge \varphi_2\urcorner$.

As inductive hypothesis, let us assume that for $n = 2$ the lemma is true for all m and l such that $m + l < k$. By the assumption and the Theorem 4.1 one of the following three subcases holds:

(a) $\ulcorner\kappa_1 \wedge \varphi_2, (\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2\urcorner \in \mathsf{K}$. By inductive hypothesis, there are such $\psi_1, \psi_2 \in \mathsf{L}^{\vee\wedge} \cap \mathsf{K}$, that sequents $\ulcorner\kappa_1 \wedge \varphi_2 \vdash \psi_1\urcorner$, $\ulcorner\psi_1 \vdash \kappa_1 \wedge \varphi_2\urcorner$, $\ulcorner(\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2 \vdash \psi_2\urcorner$ and $\ulcorner\psi_2 \vdash (\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2\urcorner$ are theses of $\mathbf{VF^s}$. Hence, applying (R7) to sequents (a) and (b) we get: $\ulcorner\varphi_1 \wedge \varphi_2 \vdash \psi_1 \vee \psi_2\urcorner$ and $\ulcorner\psi_1 \vee \psi_2 \vdash \varphi_1 \wedge \varphi_2\urcorner$.

(b) $\ulcorner\kappa_1 \wedge \varphi_2\urcorner \in \mathsf{K}$ and $\ulcorner(\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2\urcorner \in \mathsf{F}$. By inductive hypothesis, there are such $\psi \in \mathsf{L}^{\vee\wedge} \cap \mathsf{K}$, that sequents: $\ulcorner\kappa_1 \wedge \varphi_2 \vdash \psi\urcorner$ and $\ulcorner\psi \vdash \kappa_1 \wedge \varphi_2\urcorner$ are theses of $\mathbf{VF^s}$. Hence from sequents (a) and (b) applying (R7) we will derive sequents: (a') $\ulcorner\varphi_1 \wedge \varphi_2 \vdash \psi \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2)\urcorner$ and (b') $\ulcorner\psi \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2) \vdash \varphi_1 \wedge \varphi_2\urcorner$. From (a'), by application of (A10) and (R1), we get a sequent: $\ulcorner\varphi_1 \wedge \varphi_2 \vdash \psi\urcorner$. Hence, applying (R5), we will derive $\ulcorner\varphi_1 \wedge \varphi_2 \vdash \psi \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_j} \wedge \neg p_{i_j})\urcorner$, where $p_{i_1}, \ldots, p_{i_j}$ are all variables from the set $\mathsf{V}(\varphi_1 \wedge \varphi_2) \setminus \mathsf{V}(\psi)$. Moreover, the sequent: $\ulcorner\psi \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_j} \wedge \neg p_{i_j}) \vdash \psi\urcorner$ is a particular instance of the axiom (A10). From this, applying (R6), we will get $\ulcorner\psi \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_j} \wedge \neg p_{i_j}) \vdash \psi \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2)\urcorner$. From this and from (b'), applying (R1), we will get $\ulcorner\psi \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_j} \wedge \neg p_{i_j}) \vdash \varphi_1 \wedge \varphi_2\urcorner$.

(c) $\ulcorner\kappa_1 \wedge \varphi_2\urcorner \in \mathsf{F}$ and $\ulcorner(\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2\urcorner \in \mathsf{K}$. Analogously to (b).

(iii) Let $m + l > 2$ and $l > 1$. Analogously to the case (ii).

(III) For $n > 2$: as inductive hypothesis, let us assume that the lemma being proved is true for all $m < n$. Consider two cases:

(i) $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1}\urcorner \in \mathsf{K}$. Then, by inductive hypothesis, there is such $\psi' \in \mathsf{L}^{\vee\wedge}$, that sequents $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \vdash \psi'\urcorner$ and $\ulcorner\psi' \vdash \varphi_1 \wedge \cdots \wedge \varphi_{n-1}\urcorner$ are theses of $\mathbf{VF^s}$. From this and from (5.1), by the assumption and (R7), we get sequents: $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \psi' \wedge \varphi_n\urcorner$ and $\ulcorner\psi' \wedge \varphi_n \vdash \varphi_1 \wedge \cdots \wedge \varphi_n\urcorner$. Applying inductive hypothesis, again we get such $\psi \in \mathsf{L}^{\vee\wedge}$, that sequents $\ulcorner\psi' \wedge \varphi_n \vdash \psi\urcorner$ and $\ulcorner\psi \vdash \psi' \wedge \varphi_n\urcorner$ are theses of $\mathbf{VF^s}$. Thus by application of (R1) we get the theses being proved.

(ii) $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1}\urcorner \notin \mathsf{K}$. Then $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1}\urcorner \in \mathsf{T}$ and $\varphi_n \in \mathsf{K}$. Thus, applying (R5), from the thesis $\ulcorner\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \wedge \varphi_n \vdash \varphi_n\urcorner$ of the schema (3.1) we will derive

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the sequent $\ulcorner\varphi_1\wedge\cdots\wedge\varphi_{n-1}\wedge\varphi_n \vdash \varphi_n\vee(p_{i_1}\wedge\neg p_{i_1})\vee\cdots\vee(p_{i_j}\wedge\neg p_{i_j})\urcorner$, where $p_{i_1},\ldots,$ p_{i_j} are all variables from the set $\mathsf{V}(\varphi_1\wedge\cdots\wedge\varphi_{n-1})\setminus\mathsf{V}(\varphi_n)$. Moreover, the particular instance of the axiom (A10) is the sequent: $\ulcorner\varphi_n\vee(p_{i_1}\wedge\neg p_{i_1})\vee\cdots\vee(p_{i_j}\wedge\neg p_{i_j})\vdash\varphi_n\urcorner$. From this, applying (R4), we will get $\ulcorner\varphi_n\vee(p_{i_1}\wedge\neg p_{i_1})\vee\cdots\vee(p_{i_j}\wedge\neg p_{i_j})\vdash$ $\varphi_n\wedge(\varphi_1\wedge\cdots\wedge\varphi_{n-1})\urcorner$. From this and from (A4), applying (R7), we will get $\ulcorner\varphi_n\vee(p_{i_1}\wedge\neg p_{i_1})\vee\cdots\vee(p_{i_j}\wedge\neg p_{i_j})\vdash\varphi_1\wedge\cdots\wedge\varphi_n\urcorner$.

Considerations from (III) are repeated for an arbitrary combination of brackets. □

Lemma 6.6. *For every $\varphi\in\mathsf{L}^{\vee\wedge}\cap\mathsf{K}$ there is such $\psi\in\mathsf{L}^{\vee\wedge}\cap\mathsf{K}$, that sequents $\ulcorner\neg\varphi\vdash\psi\urcorner$ and $\ulcorner\psi\vdash\neg\varphi\urcorner$ are theses of $\mathbf{VF^s}$.*

Proof. Assume that $\varphi=\ulcorner\kappa_1\vee\cdots\vee\kappa_n\urcorner$, where $\kappa_i\in\mathsf{E}^{\wedge}$ dla $i=1,\ldots,n\geqslant 1$. By the Lemma 6.3 we get theses $\ulcorner\neg(\kappa_1\vee\cdots\vee\kappa_n)\vdash\neg\kappa_1\wedge\cdots\wedge\neg\kappa_n\urcorner$ and $\ulcorner\neg\kappa_1\wedge\cdots\wedge\neg\kappa_n\vdash$ $\neg(\kappa_1\vee\cdots\vee\kappa_n)\urcorner$.

Let $\kappa_{i_1},\ldots,\kappa_{i_m}\in\mathsf{K}$ dla $0<m\leqslant n$. By Lemma 6.4 there are such $\varphi_{i_1},\ldots,$ $\varphi_{i_m}\in\mathsf{L}^{\vee\wedge}\cap\mathsf{K}$, that for $j=1,\ldots,m$ sequents $\ulcorner\neg\kappa_{i_j}\vdash\varphi_{i_j}\urcorner$ and $\ulcorner\varphi_{i_j}\vdash\neg\kappa_{i_j}\urcorner$ are theses of $\mathbf{VF^s}$.

Let us notice that since $\varphi,\neg\kappa_{i_j}\in\mathsf{K}$ for $1\leqslant j\leqslant m$, so $\ulcorner\neg\kappa_{i_1}\wedge\cdots\wedge\neg\kappa_{i_m}\urcorner\in\mathsf{K}$. Hence, beginning with the thesis $\neg\kappa_{i_1}\wedge\cdots\wedge\neg\kappa_{i_m}\vdash\neg\kappa_{i_1}\wedge\cdots\wedge\neg\kappa_{i_m}\urcorner$ and applying (R7), we get theses: (a) $\ulcorner\neg\kappa_{i_1}\wedge\cdots\wedge\neg\kappa_{i_m}\vdash\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\urcorner$ and (b) $\ulcorner\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\vdash$ $\neg\kappa_{i_1}\wedge\cdots\wedge\neg\kappa_{i_m}\urcorner$.

From (A3), (3.1) (5.6), (5.7) and (R1) we will derive the sequent $\ulcorner\neg\varphi\vdash\neg\kappa_{i_1}\wedge$ $\cdots\wedge\neg\kappa_{i_m}\urcorner$. From this and from (a) we get the sequent $\ulcorner\neg\varphi\vdash\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\urcorner$. From this, applying (R4), we derive (a') $\ulcorner\neg\varphi\vdash\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\wedge(p_{k_1}\vee\neg p_{k_1})\wedge\cdots\wedge(p_{k_l}\vee$ $\neg p_{k_l})\urcorner$, where $p_{k_1},\ldots,p_{k_l}$ are all variables from the set $\mathsf{V}(\varphi)\setminus\mathsf{V}(\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m})$.

From (b), (A3) and (R1) we will derive $\ulcorner\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\wedge(p_{k_1}\vee\neg p_{k_1})\wedge\cdots\wedge(p_{k_l}\vee$ $\neg p_{k_l})\vdash\neg\kappa_{i_1}\wedge\cdots\wedge\neg\kappa_{i_m}\urcorner$. Since remaining elementary conjunctions occurring in φ are in F, so applying (R4) we get $\ulcorner\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\wedge(p_{k_1}\vee\neg p_{k_1})\wedge\cdots\wedge(p_{k_l}\vee\neg p_{k_l})\vdash$ $\neg\kappa_1\wedge\cdots\wedge\neg\kappa_n\urcorner$. From this, applying (R1), we derive (b') $\ulcorner\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\wedge(p_{k_1}\vee$ $\neg p_{k_1})\wedge\cdots\wedge(p_{k_l}\vee\neg p_{k_l})\vdash\neg\varphi\urcorner$.

By application of the Lemma 6.5 to the conjunction $\ulcorner\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_m}\wedge(p_{k_1}\vee$ $\neg p_{k_1})\wedge\cdots\wedge(p_{k_l}\vee\neg p_{k_l})\urcorner$, there is such $\psi\in\mathsf{L}^{\vee\wedge}$, that—applying the rule (R7) to (a') and (b')—we get theses: $\ulcorner\neg\varphi\vdash\psi\urcorner$ and $\ulcorner\psi\vdash\neg\varphi\urcorner$. □

Lemma 6.7. *For every $\varphi\in\mathsf{K}$ there is such $\varphi^{\mathrm{a}}\in\mathsf{L}^{\vee\wedge}\cap\mathsf{K}$, that sequents $\ulcorner\varphi\vdash\varphi^{\mathrm{a}}\urcorner$ and $\ulcorner\varphi^{\mathrm{a}}\vdash\varphi\urcorner$ are theses of $\mathbf{VF^s}$.*

Proof. Induction on the complexity of the formula φ.

(I) $\varphi\in\mathsf{V}$. Then $\varphi\in\mathsf{L}^{\vee\wedge}\cap\mathsf{K}$ and $\ulcorner\varphi\vdash\varphi\urcorner$ is a thesis of the form (5.1).

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(II) $\varphi = \ulcorner\neg\psi\urcorner$. Then $\psi \in \mathsf{K}$. As inductive hypothesis, let us assume that for ψ the lemma being proved holds, i.e., there is such $\psi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge} \cap \mathsf{K}$, that sequents $\ulcorner\psi \vdash \psi^{\mathrm{a}}\urcorner$ and $\ulcorner\psi^{\mathrm{a}} \vdash \psi\urcorner$ are theses of $\mathbf{VF^s}$. Thus, applying rule (R7) to $\ulcorner\varphi \vdash \varphi\urcorner$, we get sequents $\ulcorner\neg\psi \vdash \neg\psi^{\mathrm{a}}\urcorner$ and $\ulcorner\neg\psi^{\mathrm{a}} \vdash \neg\psi\urcorner$. Applying Lemma 6.6 to ψ^{a} we get such $\varphi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge}$, that theses of $\mathbf{VF^s}$ are sequents $\ulcorner\neg\psi^{\mathrm{a}} \vdash \varphi^{\mathrm{a}}\urcorner$ and $\ulcorner\varphi^{\mathrm{a}} \vdash \neg\psi^{\mathrm{a}}\urcorner$. Thus, by (R1), we get the theses being proved.

(III) $\varphi = \ulcorner\psi \vee \chi\urcorner$. Consider three cases. (i) $\psi,\chi \in \mathsf{K}$. By inductive hypothesis, there are such $\psi^{\mathrm{a}},\chi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge}$, that we will derive theses: $\ulcorner\psi \vdash \psi^{\mathrm{a}}\urcorner$, $\urcorner\psi^{\mathrm{a}} \vdash \psi\urcorner$, $\ulcorner\chi \vdash \chi^{\mathrm{a}}\urcorner$ and $\ulcorner\chi^{\mathrm{a}} \vdash \chi\urcorner$. From these, by (R7), we have: $\ulcorner\varphi \vdash \psi^{\mathrm{a}} \vee \chi^{\mathrm{a}}\urcorner$ and $\ulcorner\psi^{\mathrm{a}} \vee \chi^{\mathrm{a}} \vdash \varphi\urcorner$. Clearly, $\ulcorner\psi^{\mathrm{a}} \vee \chi^{\mathrm{a}}\urcorner \in \mathsf{L}^{\vee\wedge}$. (ii) $\psi \in \mathsf{K}$ and $\chi \in \mathsf{F}$. Then by (A10), we have $\ulcorner\varphi \vdash \psi\urcorner$. By inductive hypothesis, there is such $\psi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge}$, that sequents $\ulcorner\psi \vdash \psi^{\mathrm{a}}\urcorner$ and $\ulcorner\psi^{\mathrm{a}} \vdash \psi\urcorner$ are theses of $\mathbf{VF^s}$. By (R1) and (R5), we have the thesis $\ulcorner\varphi \vdash \psi^{\mathrm{a}} \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_1} \wedge \neg p_{i_k})\urcorner$, where $p_{i_1}, \ldots, p_{i_k}$ are all variables from the set $\mathsf{V}(\varphi) \setminus \mathsf{V}(\psi)$. Moreover, applying the axiom $\ulcorner\psi^{\mathrm{a}} \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_1} \wedge \neg p_{i_k}) \vdash \psi^{\mathrm{a}}\urcorner$ of the form (A10) and the rules (R1) and (R5) we will get thesis: $\ulcorner\psi^{\mathrm{a}} \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_1} \wedge \neg p_{i_k}) \vdash \psi \vee \chi\urcorner$. (iii) $\psi \in \mathsf{F}$ and $\chi \in \mathsf{K}$. Analogously to (ii), applying (A11) and (R6).

(IV) $\varphi = \ulcorner\psi \wedge \chi\urcorner$. Consider two cases. (i) $\psi,\chi \in \mathsf{K}$. By inductive hypothesis, for some $\psi^{\mathrm{a}},\chi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge}$ we have theses: $\ulcorner\psi \vdash \psi^{\mathrm{a}}\urcorner$, $\ulcorner\psi^{\mathrm{a}} \vdash \psi\urcorner$, $\ulcorner\chi \vdash \chi^{\mathrm{a}}\urcorner$ and $\ulcorner\chi^{\mathrm{a}} \vdash \chi\urcorner$. From these, by (R7), we have theses: $\ulcorner\varphi \vdash \psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}}\urcorner$ and $\ulcorner\psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}} \vdash \varphi\urcorner$. By Lemma 6.5, there is $\varphi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge}$ such that sequents $\ulcorner\psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}} \vdash \varphi^{\mathrm{a}}\urcorner$ and $\ulcorner\varphi^{\mathrm{a}} \vdash \psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}} \vdash \varphi^{\mathrm{a}}\urcorner$ are theses $\mathbf{VF^s}$. From (R1) we thus get sequents $\ulcorner\varphi \vdash \varphi^{\mathrm{a}}\urcorner$ and $\ulcorner\varphi^{\mathrm{a}} \vdash \varphi\urcorner$. (ii) $\psi \in \mathsf{K}$ and $\chi \in \mathsf{T}$. Then, by inductive hypothesis, for some $\psi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge}$ we get theses: $\ulcorner\psi \vdash \psi^{\mathrm{a}}\urcorner$ and $\ulcorner\psi^{\mathrm{a}} \vdash \psi\urcorner$. Now from the axiom $\ulcorner\varphi \vdash \psi\urcorner$ of the form (A3) and (R1), we have $\ulcorner\varphi \vdash \psi^{\mathrm{a}}\urcorner$. Hence applying rules (R1) and (R5) we get thesis $\ulcorner\varphi \vdash \psi^{\mathrm{a}} \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_1} \wedge \neg p_{i_k})\urcorner$, where $p_{i_1}, \ldots, p_{i_k}$ are all variables from the set $\mathsf{V}(\varphi) \setminus \mathsf{V}(\psi)$. Moreover, applying axiom $\ulcorner\psi^{\mathrm{a}} \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_1} \wedge \neg p_{i_k}) \vdash \psi^{\mathrm{a}}\urcorner$ of the form (A10) and the rules (R1) and (R4) we will get thesis: $\ulcorner\psi^{\mathrm{a}} \vee (p_{i_1} \wedge \neg p_{i_1}) \vee \cdots \vee (p_{i_1} \wedge \neg p_{i_k}) \vdash \psi \wedge \chi\urcorner$. (iii) $\psi \in \mathsf{T}$ and $\chi \in \mathsf{K}$. Analogously to (ii), applying (3.1) and (A4). □

For a formula φ from K by φ° we will denote its *canonical disjunctive-conjunctive normal form* (cf. (Asser, 1959)). Let $\mathsf{V}(\varphi) = \{p_{i_1}, \ldots, p_{i_n}\}$ for $i_1 < \cdots < i_n$. Then $\varphi^{\circ} := \kappa^{\mathrm{k}}_1 \vee \cdots \vee \kappa^{\mathrm{k}}_m$, where $\kappa^{\mathrm{k}}_i = p^{b_{i_1}}_{i_1} \wedge \cdots \wedge p^{b_{i_n}}_{i_n}$, the evaluation $\mathsf{V}(\varphi) \ni p_{i_j} \mapsto b_{i_j} \in \{0, 1\}$ satisfies the formula φ and $p^{b_i}_{i_j}$ is p_{i_j}, if $b_{i_j} = 1$, otherwise it is $\neg p_{i_j}$. Moreover, the order of elementary conjunctions in φ° is determined by an increasing order of numbers $b_{i_1} \ldots b_{i_n}$ in binary notation.

LEMMA 6.8. *If* $\varphi \in \mathsf{K}$, *then sequents* $\ulcorner\varphi \vdash \varphi^{\circ}\urcorner$ *and* $\ulcorner\varphi^{\circ} \vdash \varphi\urcorner$ *are theses of* $\mathbf{VF^s}$.

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Proof. For $\varphi \in \mathsf{K}$ let $\mathsf{V}(\varphi) = \{p_{i_1}, p_{i_2}, \ldots, p_{i_n}\}$, where $i_1 < i_2 < \cdots < i_n$.

By Lemma 6.7, there is such $\varphi^{\mathrm{a}} \in \mathsf{L}^{\vee\wedge} \cap \mathsf{K}$, that sequents $\ulcorner \varphi \vdash \varphi^{\mathrm{a}} \urcorner$ and $\ulcorner \varphi^{\mathrm{a}} \vdash \varphi \urcorner$ are theses of $\mathbf{VF}^{\mathrm{s}}$. Let $\kappa_1, \ldots, \kappa_m$ (for $m > 0$) be all elementary conjunctions in φ^{a} that are elements of K.

For $1 \leqslant i \leqslant m$, by (A4), (5.6), (5.7), (A3), (5.2), (R1) and (R7), we can derive sequents $\ulcorner \kappa_i \vdash \kappa'_i \urcorner$ and $\ulcorner \kappa'_i \vdash \kappa_i \urcorner$, where $\mathsf{V}(\kappa'_i) = \mathsf{V}(\kappa_i)$ and κ'_i differs from κ_i only in that, that no element of the conjunction κ'_i repeats and these are ordered according to an increasing order of indexes of variables. Let now $l_1 < \cdots < l_j$ and $\{p_{l_1}, \ldots, p_{l_j}\} = \mathsf{V}(\varphi) \setminus \mathsf{V}(\kappa_i)$. By (A4), (5.15), (A8), (5.6), (5.7) and (R7), we get sequents $\ulcorner \kappa'_i \wedge (p_{l_1} \vee \neg p_{l_1}) \vdash (\kappa'_i)^1_1 \vee (\kappa'_i)^0_1 \urcorner$ and $\ulcorner \kappa^1_1 \vee \kappa^0_1 \vdash \kappa'_i \wedge (p_{l_1} \vee \neg p_{l_1}) \urcorner$, where $(\kappa'_i)^1_1$ and $(\kappa'_i)^0_1$ differ from $\ulcorner \kappa'_i \wedge p_{l_1} \urcorner$ and $\ulcorner \kappa'_i \wedge \neg p_{l_1} \urcorner$, respectively, only in that, that their elements are ordered according to increasing numbers of indexes of variables. In a second step, for a variable p_{l_2}, in an analogous way, we get sequents $\ulcorner (\kappa'_i)^1_1 \wedge (p_{l_2} \vee \neg p_{l_2}) \vdash (\kappa'_i)^{11}_{12} \vee (\kappa'_i)^{10}_{12} \urcorner$, $\ulcorner (\kappa'_i)^{11}_{12} \vee (\kappa'_i)^{10}_{12} \vdash (\kappa'_i)^1_1 \wedge (p_{l_2} \vee \neg p_{l_2}) \urcorner$, $\ulcorner (\kappa'_i)^0_1 \wedge (p_{l_2} \vee \neg p_{l_2}) \vdash (\kappa'_i)^{11}_{12} \vee (\kappa'_i)^{10}_{12} \urcorner$ and $\ulcorner (\kappa'_i)^{11}_{12} \vee (\kappa'_i)^{10}_{12} \vdash (\kappa'_i)^0_1 \wedge (p_{l_2} \vee \neg p_{l_2}) \urcorner$. Thus, by (5.15), (A8), (R1) and (R7), we have sequents $\ulcorner \kappa'_i \wedge (p_{l_1} \vee \neg p_{l_1}) \wedge (p_{l_2} \vee \neg p_{l_2}) \vdash (\kappa'_i)^{11}_{12} \vee (\kappa'_i)^{10}_{12} \vee (\kappa'_i)^{01}_{12} \vee (\kappa'_i)^{00}_{12} \urcorner$ and $\ulcorner (\kappa'_i)^{11}_{12} \vee (\kappa'_i)^{10}_{12} \vee (\kappa'_i)^{01}_{12} \vee (\kappa'_i)^{00}_{12} \vdash \kappa'_i \wedge (p_{l_1} \vee \neg p_{l_1}) \wedge (p_{l_2} \vee \neg p_{l_2}) \urcorner$. These steps are repeated for j and thus we get sequents $\ulcorner \kappa'_i \wedge (p_{l_1} \vee \neg p_{l_1}) \wedge \cdots \wedge (p_{l_j} \vee \neg p_{l_j}) \vdash (\kappa'_i)^{11\ldots1}_{12\ldots j} \vee (\kappa'_i)^{11\ldots0}_{12\ldots j} \vee \ldots \vee (\kappa'_i)^{00\ldots0}_{12\ldots j} \urcorner$ and $\ulcorner (\kappa'_i)^{11\ldots1}_{12\ldots j} \vee (\kappa'_i)^{11\ldots0}_{12\ldots j} \vee \ldots \vee (\kappa'_i)^{00\ldots0}_{12\ldots j} \vdash \kappa'_i \wedge (p_{l_1} \vee \neg p_{l_1}) \wedge \cdots \wedge (p_{l_j} \vee \neg p_{l_j}) \urcorner$. Let us denote derived 2^j elementary disjunction by κ^{ka}_i. By construction, $\mathsf{V}(\kappa^{\mathrm{ka}}_i) = \mathsf{V}(\varphi)$.

Notice that by (A10), (A11), (5.12), (5.16), (5.17), (R1), (R7) and previously proved theses, we will derive a sequent $\ulcorner \varphi^{\mathrm{a}} \vdash \kappa'_1 \vee \cdots \vee \kappa'_m \urcorner$. From this, applying (R4), (5.12), (5.16), (5.17), (A8), (5.15), (A3), (3.1), (R1), (R7) and Lemma 6.1, we get sequent $\ulcorner \varphi^{\mathrm{a}} \vdash \kappa^{\mathrm{ka}}_1 \vee \cdots \vee \kappa^{\mathrm{ka}}_m \urcorner$.

Moreover, applying (5.18), (5.19), (A3), (3.1), (5.12), (5.16), (5.17), (R1) and (R7), we will derive the sequent $\ulcorner \kappa^{\mathrm{ka}}_1 \vee \cdots \vee \kappa^{\mathrm{ka}}_m \vdash \kappa'_1 \vee \cdots \vee \kappa'_m \urcorner$. From this and previously proved theses, by (5.12), (5.16), (5.17), (R5), (R6) and (R7), we get the thesis $\ulcorner \kappa^{\mathrm{ka}}_1 \vee \cdots \vee \kappa^{\mathrm{ka}}_m \vdash \varphi^{\mathrm{a}} \urcorner$.

Now, by (5.16), (5.17), (5.8), (5.9) and (5.12) from disjunction $\ulcorner \kappa^{\mathrm{ka}}_1 \vee \cdots \vee \kappa^{\mathrm{ka}}_m \urcorner$ we can delete elementary conjunctions that repeat and arrange it in order proper for a formula φ°.

It remains to show that the above disjunction is really the formula φ°.

Since this disjunction is equivalent, within CPC, to the formula φ, thus an arbitrary 0-1 evaluation $p_{i_1} \mapsto b_{i_1}, \ldots, p_{i_n} \mapsto b_{i_n}$ satisfies the formula φ iff it satisfies the disjunction in question, i.e., it satisfies at least one of its members. Hence it follows, firstly, that the disjunction contains all elementary conjunctions determined by evaluations satisfying the formula φ; secondly, that only such conjunctions. $\square$

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Now we can finish the proof of the Completeness Theorem.

PROOF OF COMPLETENNESS THEOREM 4.2. Assume that $\varphi \models_s \psi$. Thus $\varphi \models \psi$, $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi)$ and $\varphi, \psi \in \mathsf{K}$. Moreover, set $\psi' = \ulcorner \psi \wedge (p_{i_1} \vee \neg p_{i_1}) \wedge \cdots \wedge (p_{i_n} \vee \neg p_{i_n}) \urcorner$, where $p_{i_1}, \ldots, p_{i_n}$ are all variables from the set $\mathsf{V}(\varphi) \setminus \mathsf{V}(\psi)$. Thus $\mathsf{V}(\psi') = \mathsf{V}(\varphi)$.

By Lemma 6.8 we get theses $\ulcorner \varphi \vdash \varphi^\circ \urcorner$ and $\ulcorner \psi'^\circ \vdash \psi' \urcorner$. Since $\varphi \models \psi$ and ψ' is equivalent, within CPC, to ψ, thus $\varphi^\circ \models \psi'^\circ$. Therefore all members of disjunction φ° occur also in a disjunction ψ'°. Hence by (5.20) and (R1) we will derive thesis $\ulcorner \varphi^\circ \vdash \psi'^\circ \urcorner$. Moreover from (A3) and (R1) we will derive thesis $\ulcorner \psi' \vdash \psi \urcorner$. Thus, applying the rule (R1) to the theses already proved, we will derive $\ulcorner \varphi \vdash \psi \urcorner$. $\square$

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