The consequence relation $\models$ of CPC preserves truth solely by the meaning of truth-value connectives and the structure of sentences that is a result of properties of these connectives. It is not the case yet that it always preserves logical information (in a sense explained farther in the paper). The similar situation holds as well for different non-classical logical consequence relations defined in the set of formulas of CPC (with classical meanings of connectives).

For example, in (Zinov'ev, 1971) the author analyzes the relation $\models^*$ of *strong logical consequence* satisfying the following condition: $\varphi \models^* \psi$ iff (*1*) $\varphi \models \psi$ and (*2*) every variable of ψ is a variable of φ. The latter condition is connected with the notion of "sense units" of a given formula, which are its propositional variables. Thus additional condition (*2*) says that all "sense units" of a formula ψ are at the same time "sense units" of a formula φ. In our opinion (*2*) is too weak to «correct» the classical consequence relation and free it from «paradoxes». It is because a sense of a formula is given not only by its "sense units" but also by the meaning of logical connectives it contains and its structure. The relation $\models^*$ generates «paradoxes» as well. E.g. $p \not\models^* p \vee q$, but $p \wedge \neg q \models^* p \vee q$, $p \wedge (q \vee \neg q) \models^* p \vee q$ and $p \vee (q \wedge \neg q) \models^* p \vee q$. One can ask: if $p \vee q$ is not a consequence of p itself, then what is an influence of $\neg q$, $q \vee \neg q$ or $q \wedge \neg q$ on it? In the first case, it seems that lack of a "sense unit" q is «less noticeable» then lack of a "sense unit" represented by $\neg q$. If the "sense unit" p itself is not enough to conclude $p \vee q$, then the "sense" contained in $p \wedge \neg q$ cannot be enough all the more. And similarly for the two remaining cases.

In our opinion, if we accept that $p \vee q$ is not a consequence of p, then this is not because of a variable q but p. It is not that q is in the succedent, but not in the antecedent. The point is that the succedent does not say «the whole truth» that the antecedent says about p. In other words, the succedent is «informationally weaker» by p than the antecedent. For the same reasons we maintain that $p \vee q$ is not a consequence of $p \wedge q$ (since the succedent is «informationally weaker» than the antecedent by both p and q). We think, that in an acceptable consequence relation between φ and ψ (where ψ is a consequence of φ) that «presereves information», the succedent ψ can be «poorer in content» then the antecedent φ, yet ψ should extract «maximal content» from φ about the "sense units" of φ.

The axiomatization of $\models^*$ can be found in (Zinov'ev, 1971) and (Wessel, 1984). Let us notice that $\models^*$ remains in connection with Epstein's calculus D. If in an implication $\ulcorner \varphi \to \psi \urcorner$ of the language of the calculus D subformulas of φ and ψ are Boolean combinations of variables (i.e., $\varphi \to \psi$ is a first degree implication), then the following theorem holds: $\ulcorner \varphi \to \psi \urcorner \in D$ iff $\varphi \models^* \psi$ (cf. Epstein, 1990, p. 141).

As Wessel noticed in (Wessel, 1984), the relation $\models^*$ has some residues of the «paradoxes» of the classical consequence relation: "everything that is built from variables occurring in a contradiction is a consequence of this contradiction" and "every tautology is a consequence of everything that includes all variables of this tautology". Wessel defines the so called *strict logical consequence* $\models^{**}$ that eliminates these paradoxes. He accepts that: $\varphi \models^{**} \psi$ iff (*1*), (*2*) and (*3*) neither φ is a contradiction nor ψ is a tautology of CPC (cf. (*1*) and (*2*) in the definition of $\models^{*}$).[3]

The «paradoxes» of the relation $\models^{*}$, that we analyzed earlier, in the text apply to $\models^{**}$ as well. Others, applying to both of them, are presented in (Pietruszczak, 1992).

In Section 5 of this paper we will define the consequence relation $\models_{\mathrm{i}}$ that preserves information:

$\varphi \models_{\mathrm{i}} \psi$ iff (*a*) neither φ is a contradiction nor ψ is a tautology, and
(*b*) information contained in ψ is a part of information contained in φ.

The notions of *a part of information* and *information contained in a formula* that occur in (*b*) will be so defined, that (*b*) will entail (*1*) occurring in the definitions of the relations $\models^{*}$ and $\models^{**}$. Yet the condition (*2*) of those definitions will not hold. Namely, the consequence of (*b*) will only be the fact that the set of so called *essential variables* in ψ is not empty and is a subset of the set of essential variables of φ. «Paradoxes» we mentioned before will not concern the relation $\models_{\mathrm{i}}$ (see Example 5.1). In Section 6 we give a different definition of $\models_{\mathrm{i}}$, and in Section 7 we axiomatize this relation.

2. Some facts of classical propositional calculi (CPC)

Let $\mathcal{L} = \langle \mathsf{L}, \vee, \wedge, \neg \rangle$ be a propositional language. The formulas of $\mathcal{L}$ (i.e., elements of the set L) are composed in a standard way from propositional variables being elements of the denumerable set $\mathsf{V} := \{p_0, p_1, p_2, \ldots\}$, brackets and functors $\vee$, $\wedge$ and $\neg$ understood, respectively, as truth-value connectives of disjunction, conjunction and negation. From formal point of view $\mathcal{L}$ is an absolutely free algebra where V is its set of free generators.

In examples first three variables will be denoted by, respectively, 'p', 'q' and 'r'. The set of propositional variables of a formula $\varphi \in \mathsf{L}$ will be denoted by $\mathsf{V}(\varphi)$.

Let $\mathcal{B}_2$ be a two-element Boolean algebra in the set $\{0, 1\}$, with operations of max, min and subtraction from 1. By homomorphism from $\mathcal{L}$ to $\mathcal{B}_2$ we mean every function h from L to $\{0, 1\}$, such that: $h(\neg\varphi) = 1 - h(\varphi)$, $h(\varphi \vee \psi) = \max\{h(\varphi), h(\psi)\}$ and $h(\varphi \wedge \psi) = \min\{h(\varphi), h(\psi)\}$. Let $\mathrm{Hom}(\mathcal{L}, \mathcal{B}_2)$ be the set of all such homomorphisms.

[3] Regarding a sequent calculus for the relation $\models^{**}$ see footnote 17 and (Pietruszczak, 2004).

By an *evaluation of variables* we mean any function e on the set V, taking values in $\{0, 1\}$. Every homomorphism $h \in \mathrm{Hom}(\mathcal{L}, \mathcal{B}_2)$ is unambiguously determined by an evaluation $h|_{\mathsf{V}} \colon \mathsf{V} \to \{0, 1\}$. And conversely, every evaluation $e \colon \mathsf{V} \to \{0, 1\}$ determines uniquely some homomorphism h^e from $\mathrm{Hom}(\mathcal{L}, \mathcal{B}_2)$.

A formula $\varphi \in \mathsf{L}$ is a *tautology* (resp. a *contradiction*) of CPC iff for every h in $\mathrm{Hom}(\mathcal{L}, \mathcal{B}_2)$ we have $h(\varphi) = 1$ (resp. $h(\varphi) = 0$). Let T (resp. F) be the set of all tautologies (resp. contradictions) of CPC. We say that a given formula is *contingent* iff it is neither tautology nor contradiction. Let K be the set of all contingent formulas, i.e., $\mathsf{K} \coloneqq \mathsf{L} \setminus (\mathsf{T} \cup \mathsf{F})$.

A formula ψ is said to be a *consequence* of a formula φ (according to CPC) iff for every $h \in \mathrm{Hom}(\mathcal{L}, \mathcal{B}_2)$ we have $h(\varphi) \leq h(\psi)$. If ψ is a consequence of φ (according to CPC), we write: $\varphi \models \psi$. For all $\varphi, \psi \in \mathsf{L}$ we have:

$$\varphi \models \psi \ \&\ \varphi \in \mathsf{T} \implies \psi \in \mathsf{T}\,,$$

$$\varphi \models \psi \ \&\ \psi \in \mathsf{F} \implies \varphi \in \mathsf{F}\,,$$

$$\varphi \models \psi \ \&\ \varphi \notin \mathsf{F} \ \&\ \psi \notin \mathsf{T} \implies \varphi, \psi \in \mathsf{K}\,. \tag{2.1}$$

A formula φ is said to be *equivalent* to a formula ψ (according to CPC) iff for every $h \in \mathrm{Hom}(\mathcal{L}, \mathcal{B}_2)$ we have $h(\varphi) = h(\psi)$. If φ is equivalent to ψ, we write: $\varphi \mathrel{\Dashv\models} \psi$. In other words, $\varphi \mathrel{\Dashv\models} \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$. The relation $\Dashv\models$ is a congruence of the algebra $\mathcal{L}$.

We say that a given variable is *essential* in a given formula iff the logical value of the formula can be changed by changing the logical value of this variable. To define this notion in a formal way we introduce some auxiliary operation in the set E of all evaluations. For $\alpha \in \mathsf{V}$ and $e \in \mathrm{E}$ we define the operation $e^-_\alpha \colon \mathsf{V} \to \{0, 1\}$ by the following equation:

$$e^-_\alpha(\beta) \coloneqq \begin{cases} 1 - e(\beta) & \text{if } \alpha = \beta \\ e(\beta) & \text{if } \alpha \neq \beta \end{cases}$$

i.e., e^-_α differs from e exactly on a variable α. For $\varphi \in \mathsf{L}$ and $\alpha \in \mathsf{V}$ we accept that

$$\alpha \text{ is essential in } \varphi \overset{\text{df}}{\iff} \exists_{e \in \mathrm{E}}\ h^e(\varphi) \neq h^{e^-_\alpha}(\varphi)\,.$$

Let $\mathsf{V_e}(\varphi) \coloneqq \{\alpha \in \mathsf{V} : \alpha \text{ is essential in } \varphi\}$. Let us notice that

$$\mathsf{V_e}(\varphi) \subseteq \mathsf{V}(\varphi)\,, \tag{2.2}$$

$$\mathsf{V_e}(\varphi) = \emptyset \iff \varphi \in \mathsf{T} \text{ or } \varphi \in \mathsf{F}\,, \tag{2.3}$$

$$e_1 \text{ and } e_2 \text{ coincide on } \mathsf{V_e}(\varphi) \implies h^{e_1}(\varphi) = h^{e_2}(\varphi)\,, \tag{2.4}$$

$$\varphi \mathrel{\Dashv\models} \psi \implies \mathsf{V_e}(\varphi) = \mathsf{V_e}(\psi)\,. \tag{2.5}$$

3. Logical informations in CPC

As we are interested in logical information connected only with the meaning of truth-value connectives, thus—in case of analyzing sentential schemata from L—it deals solely with logical values assigned to variables from V. For example, a statement whose schema is '$p \wedge q$' gives logical information that both statements, represented by 'p' and 'q', are true. A statement whose schema is '$p \vee q$'—that at least one of statements represented by 'p' and 'q' is true. Finally, a statement whose schema is '$\neg p$'—that a statement represented by 'p' is false.

3.1. Information states

We will make use of functions characterized on finite subsets of V and taking values in $\{0, 1\}$. We will call these functions *information states* and their set will be denoted by 'IS':

$$s \in \mathrm{IS} \overset{\mathrm{df}}{\iff} s \in \mathrm{Fun}(V; \{0, 1\}) \text{ for some finite subset } V \text{ of } \mathsf{V}.$$

The empty set $\emptyset$ is an element of IS ($\emptyset$ is a function with an empty domain, i.e., $\emptyset \colon \emptyset \to \{0, 1\}$). If $\emptyset$ stands as an «empty information state» we will denote it by 'ø'.

Information states from IS are to represent information about the assignment of logical values to propositional variables in the domains of these states. «Empty» information state represents «the lack of knowledge about every variable» (we know nothing about logical values assigned to variables).

In examples—in order to shorten and emphasize notation—an information state s with a domain $\{\alpha_1, \dots, \alpha_n\}$ will be denoted by the formal sequence $\alpha_1^* \dots \alpha_n^*$, such that for $i = 1, \dots, n$

$$\alpha_i^* := \begin{cases} \alpha_i & \text{if } s(\alpha_i) = 1 \\ \overline{\alpha_i} & \text{if } s(\alpha_i) = 0 \end{cases}$$

The order of the elements of such a sequence is of no importance, i.e., sequences that have the same elements but differ in order represent the same function from IS.

It is accidental that information states and evaluations of variables are represented by mathematical objects of the same structure. It is connected with considered scope of information; with «weak strength of expression» of formulas from L.

Every function from IS is a binary relation in $\mathsf{V} \times \{0, 1\}$, i.e., for every s in IS, $s \subseteq \mathsf{V} \times \{0, 1\}$. A relation $r \subseteq \mathsf{V} \times \{0, 1\}$ is not a function iff for some variable α both $\langle \alpha, 1 \rangle \in r$ and $\langle \alpha, 0 \rangle \in r$. For a function s instead of $\langle \alpha, i \rangle \in s$ we write $s(\alpha) = i$. In the set of all relations in $\mathsf{V} \times \{0, 1\}$, as the power set $\wp(\mathsf{V} \times \{0, 1\})$, the following set-theoretical operations are performable: product $\cap$, sum $\cup$, complement $-$ and subtraction $\setminus$. Moreover, the set $\wp(\mathsf{V} \times \{0, 1\})$ is partially ordered by $\subseteq$.

The set IS is closed under $\cap$ (i.e., if $s_1, s_2 \in \mathrm{IS}$, then $s_1 \cap s_2 \in \mathrm{IS}$). Generally: if $s \in \mathrm{IS}$ and $r \subseteq \mathsf{V} \times \{0,1\}$, then $s \cap r \in \mathrm{IS}$ and $s \setminus r \in \mathrm{IS}$. Yet the set IS is not closed under $\cup$ (i.e., the sum of two functions may not be a function).

Remind that the domain of the function $s \in \mathrm{IS}$ is the set $\mathrm{dm}(s) = \{\alpha \in \mathsf{V} : \exists_{i\in\{0,1\}}\langle\alpha, i\rangle \in s\}$, i.e., the set of those variables for which the value of the function s is determined. By a restriction of a function $s \in \mathrm{IS}$ to a set $V \subseteq \mathsf{V}$ we mean the function $s|_V$, whose domain is a set $\mathrm{dm}(s) \cap V$ and which takes the same values as s for the variables from the new domain. Formally,

$$s|_V := s \cap (V \times \{0,1\}) .$$

Of course, if $\mathrm{dm}(s) \cap V = \emptyset$, then $s|_V = \varnothing$. Thus $\varnothing|_V = \varnothing$ and $s|_\emptyset = \varnothing$. For any function $s, t \in \mathrm{IS}$ we have

$$s \subseteq t \iff \mathrm{dm}(s) \subseteq \mathrm{dm}(t) \ \ \& \ \ \forall_{\alpha\in\mathrm{dm}(s)}\ s(\alpha) = t(\alpha) ,$$
$$s \subseteq t \iff t|_{\mathrm{dm}(s)} = s .$$

We say that an information state s from IS *is compatible* with an evaluation e from E iff $s \subset e$, i.e., s assigns the same values to variables as e.

Let us notice that the empty information state $\varnothing$ is compatible with every evaluation from E.[4]

3.2. Alternatives of information states

By an *alternative of information states* we mean any finite subset of IS. Their totality will be denoted by A, i.e.,

$$\mathsf{A} := \{A \subset \mathrm{IS} : \mathrm{Card}\, A < \aleph_0\} .$$

Intuitively, a set A from A «informs» that at least one of the states in A is compatible with a given evaluation.

---

[4] The triple $\langle \mathrm{IS}, \supseteq, \varnothing\rangle$ is Cohen's forcing (cf. Bell, 1977, p. 44). We say that a partially ordered set $\langle P, \leq, \mathit{1}\rangle$ with unity $\mathit{1}$ is a *forcing*, if it satisfies a polarization condition: $x \nleq y \Rightarrow \exists_{z\leq x}\ z \perp y$, where $z \perp y \overset{\mathrm{df}}{\iff} {\sim}\exists_s(s \leq z \ \&\ s \leq y)$ expresses an incompatibility condition. In case when P is—for some infinite set X—a set $C(X)$ of all functions with finite domains being subsets of X and taking values in the set $\{0,1\}$ and relation $\leq$ is an inverse inclusion in $C(X)$, a forcing is called the *Cohen's forcing* (Cohen himself used simple inclusion; in later works the order was inverted for technical reasons; cf. [2, p. 47]). In our case $X = \mathsf{V}$, $C(X) = \mathrm{IS}$ and $\mathit{1} = \varnothing$. The incompatibility condition $s \perp t$ says that there is no function in IS such that would include both s and t. Such situation holds iff for some variable α from $\mathrm{dm}(s) \cap \mathrm{dm}(t)$, $s(\alpha) \neq t(\alpha)$.

The "forcing relation" $\Vdash$, that is included in $\mathrm{IS} \times \mathsf{L}$, can be defined by the following condition: $s \Vdash \varphi \overset{\mathrm{df}}{\iff} \forall_{e\in\mathrm{E}}(s \subset e \Rightarrow h^e(\varphi) = 1)$. We will not deal with forcing farther in this paper.

We say that an alternative of information states A *is true* for an evaluation e from E (in symbols: $e \models A$) iff at least one information state in A is compatible with e. Formally,

$$e \models A \overset{\text{df}}{\iff} \exists_{s \in A}\, s \subset e\,.$$

Because $\emptyset \in \mathsf{A}$, so it is an «alternative» of information states as well. As such we will denote $\emptyset$ by 'Λ'.[5] There is no evaluation for which «the empty alternative» Λ is true.

By a *set of variables of an alternative A* we will mean a sum of all domains of information states being in A, i.e.,

$$\mathsf{V}(A) := \bigcup_{s \in A} \mathrm{dm}(s)\,.$$

3.3. The set of logical information

We identify alternatives of information states that are true for the same evaluations and we maintain that they transmit the same logical information. In the set A we define an equivalence relation:

$$A \cong B \overset{\text{df}}{\iff} \forall_{e \in \mathrm{E}}\big(e \models A \Leftrightarrow e \models B\big)\,.$$

The relation $\cong$ is reflexive, symmetrical and transitive.

Example 3.1. (a) $\{ø, \ldots\} \cong \{ø\} \cong \{p, \bar{p}, \ldots\}$.

(b) $\{p, q\} \cong \{pq, p\bar{q}, \bar{p}q\} \cong \{p, q, pq\} \cong \{p, \bar{p}q\} \cong \{q, p\bar{q}\}$.

(c) $\{\overline{pq}, qr\} \cong \{\overline{pq}, \bar{p}r, qr\} \cong \{\overline{pq}r, \overline{pqr}, \bar{p}qr, \overline{pq}r, pqr, \bar{p}qr\}$.
Indeed, if $e \models \bar{p}r$, then $e(p) = 0$, $e(r) = 1$ and either $e(q) = 1$ or $e(q) = 0$. In the first case $e \models qr$. In the second case $e \models \overline{pq}$.

(d) $\{\overline{pq}, p\bar{r}, qr\} \cong \{\overline{pq}r, \overline{pqr}, pq\bar{r}, p\overline{qr}, pqr, \bar{p}qr\} \cong \{pq, \bar{p}r, \overline{qr}\}$. □

An equivalence class of an alternative A in relation $\cong$ we will denote by $[A]$:

$$[A] := \{B \in \mathsf{A} \;:\; A \cong B\}\,.$$

We assume that the set of logical information is the quotient set $\mathsf{A}/_{\cong}$. The set of logical information we will denote by $\mathbb{INF}$, i.e.,

$$\mathbb{INF} := \mathsf{A}/_{\cong} := \{\,[A] \;:\; A \in \mathsf{A}\,\}\,.$$

[5] The fact that $\emptyset$ represents two different notions ($ø$ and Λ) will not cause any formal difficulties. The context will disambiguate which of the two notions is being represented.

Remark 3.1. A question arises: could we use, as an explication of the notion of logical information, some representatives (i.e., some alternatives of information states) chosen in some «canonical» way, instead of using equivalence classes of these states? This way would have to abide by some sensible criterion of «minimal complexity» of these alternatives. Such sensible criteria of complexity of members of A could be for example: either a minimal number of tokens of propositional variables occurring in a given alternative or just a number of its elements. Yet according to these two criteria the following equivalent alternatives are «equally minimal»: $\{\overline{pq}, p\overline{r}, qr\}$ and $\{pq, \overline{p}r, \overline{qr}\}$ (see Example 3.1d). □

Directly from the definition we have that the relation $\cong$ is a congruence with respect to $\models$. Therefore we can say that logical information I *is true* at evaluation e (in symbols: $e \models I$) iff some (arbitrary) representative of the class I is true in e. We can express it in the following different way:

$$e \models [A] \overset{\text{df}}{\iff} e \models A\,.$$

We single out two of all logical information: logically empty $\curlyvee$ and logically contradictory $\curlywedge$.

By *logically empty information* we mean information that is true at every evaluation. Clearly, there exists exactly one such information which we denote by $\curlyvee$. It will be a counterpart of logical information contained in tautologies of CPC. Representatives of the class $\curlyvee$ are, among others, the following alternatives: $\{\text{ø}\}$ and every alternative whose element is ø; $\{\alpha, \overline{\alpha}, \alpha_1, \ldots, \alpha_n\}$; a set of functions $\mathrm{Fun}(V; \{0,1\})$ for every finite set of variables V. Thus in particular the following holds: $\curlyvee = [\{\text{ø}\}] = [\{p, \overline{p}\}]$.

By *logically contradictory information* we mean the one that is not true for every evaluation. Clearly, there exists exactly one such information that we denote by $\curlywedge$. It will be a counterpart of logical information contained in contradictions of CPC. The class $\curlywedge$ has exactly one element which is Λ. Therefore $\curlywedge = [\Lambda] = \{\Lambda\}$.

We fix the following notational convention: while writing particular examples of logical information we omit «brackets of a set», i.e., instead of $[\{s_1, s_2, \ldots, s_n\}]$ we write $[s_1, s_2, \ldots, s_n]$. E.g., the class $[\{\text{ø}\}]$ will be written down as $[\text{ø}]$, $[\{pq, \overline{r}\}]$ as $[pq, \overline{r}]$, etc.

Example 3.2 (see Example 3.1). (a) $\curlyvee = [\text{ø}, \ldots] = [\text{ø}] = [p, \overline{p}, \ldots]$.

(b) $[p, q] = [pq, p\overline{q}, \overline{p}q] = [p, q, pq] = [p, \overline{p}q] = [q, p\overline{q}]$.

(c) $[\overline{pq}, qr] = [\overline{pq}, \overline{p}r, qr] = [\overline{pq}r, \overline{pqr}, \overline{p}qr, \overline{pq}r, pqr, \overline{p}qr]$.

(d) $[\overline{pq}, p\overline{r}, qr] = [\overline{pq}r, \overline{pqr}, pq\overline{r}, p\overline{qr}, pqr, \overline{p}qr] = [pq, \overline{p}r, \overline{qr}]$. □

3.4. Domain of information

Now we will define the notion of *an essential variable in an alternative of information states*. This definition if similar to the definition of a corresponding notion in the set L:

$$\alpha \text{ is essential in } A \overset{\text{df}}{\iff} \sim\forall_{e\in\mathrm{E}}(e \models A \Leftrightarrow e^{-}_{\alpha} \models A)\,.$$

Let $\mathsf{V_e}(A) := \{\alpha \in \mathsf{V} \ :\ \alpha \text{ is essential in } A\}$. Clearly, $\mathsf{V_e}(A) \subseteq \mathsf{V}(A)$.

To introduce a notion of *a domain of information* let us notice that directly from the definition we have:

FACT 3.1. *For all* $A, B \in \mathsf{A}$*:* $A \cong B \implies \mathsf{V_e}(A) = \mathsf{V_e}(B)$.

PROOF. Let $A \cong B$. Then for all $\alpha \notin \mathsf{V_e}(A)$ and $e \in \mathrm{E}$: $e \models B$ iff $e \models A$ iff $e^{-}_{\alpha} \models A$ iff $e^{-}_{\alpha} \models B$. So $\alpha \notin \mathsf{V_e}(B)$, i.e., $\mathsf{V_e}(B) \subseteq \mathsf{V_e}(A)$. Similarly, $\mathsf{V_e}(A) \subseteq \mathsf{V_e}(B)$. □

Therefore the relation $\cong$ is a congruence with respect to a function $\mathsf{V_e}(\cdot)$. So we can define a domain of information I (in symbols: $\mathrm{dm}(I)$) as a set of essential variables of its (arbitrary) representative:

$$\mathrm{dm}([A]) := \mathsf{V_e}(A)\,.$$

Obviously, $\mathrm{dm}(\wedge) = \emptyset = \mathrm{dm}(\vee)$.

3.5. Restriction of logical information

By *a restriction* of an alternative $A \in \mathsf{A}$ to a set V of variables in V we mean an alternative $A|_V$ defined by the equation:

$$A|_V := \{\, s|_V \ :\ s \in A \,\}\,.$$

Clearly, $\Lambda|_V = \Lambda$, $\{\varnothing\}|_V = \{\varnothing\}$ and for $A \neq \Lambda$ we have $A|_\emptyset = \{\varnothing\}$.

Let us prove that a restriction of an alternative is invariant with respect to $\cong$.

FACT 3.2. *For every subset* V *of* V *it holds that:*

$$A \cong B \implies A|_V \cong B|_V\,.$$

PROOF. Let $A \cong B$ and let e be an arbitrary evaluation satisfying $e \models A|_V$. Then for some s from $A|_V$, $s \subset e$. Let s' be an arbitrary element of a set A such that $s = s'|_V$ and let e' be an arbitrary evaluation such that $s' \subset e'$ (clearly, there are such s' and e'). By hypotheses $e|_{\mathrm{dm}(s)} = s = s'|_{\mathrm{dm}(s)} = e'|_{\mathrm{dm}(s)}$. Now we are constructing a new evaluation e'': for α from V we set $e''(\alpha) := e(\alpha)$; for α from $\mathrm{dm}(s')$ we set $e''(\alpha) := e'(\alpha)$; for the remaining variables it may be anything one likes. The function e'' is well defined, since $\mathrm{dm}(s) = \mathrm{dm}(s') \cap V$. Thus for α in $\mathrm{dm}(s') \cap V$ we have $e(\alpha) = s(\alpha) = s'(\alpha) = e'(\alpha)$. Since $e''|_{\mathrm{dm}(s')} = e'|_{\mathrm{dm}(s')} = s'$, so $s' \subset e''$, i.e., $e'' \models A$. Moreover $e'' \models B$, i.e., $s'' \subset e''$ for some s'' in B. Hence $s''|_V \subseteq e''|_V = e|_V \subseteq e$, that is $e \models B|_V$. The converse is proved analogously. □

Thanks to the above fact we can introduce operation of *restriction of information I* to a set of variables V:

$$[A]|_V := [A|_V]\,.$$

Notice that $\curlyvee|_V = \curlyvee$ and $\curlywedge|_V = \curlywedge$ for any $V \subseteq \mathsf{V}$.

FACT 3.3. *For any $I \in \mathbb{INF}$: $I = I|_{\mathrm{dm}(I)}$.*

PROOF. It is enough to show that for an arbitrary A from A, $A \cong A|_{\mathsf{V_e}(A)}$. Let e be an arbitrary evaluation. If $s \subset e$, for some s from A, then $s|_{\mathsf{V_e}(A)} \subseteq s \subset e$ as well. Conversely, let $e \models A|_{\mathsf{V_e}(A)}$, i.e., $s \subset e$, for some s from $A|_{\mathsf{V_e}(A)}$. Let s' be an arbitrary element of A such that $s = s'|_{\mathsf{V_e}(A)}$ and let e' be an arbitrary evaluation, for which $s' \subset e'$. Let us notice, that $\mathrm{dm}(s) = \mathrm{dm}(s') \cap \mathsf{V_e}(A)$ and $e|_{\mathrm{dm}(s)} = s = s'|_{\mathrm{dm}(s)} = e'|_{\mathrm{dm}(s)}$. If evaluations e and e' coincide on the set $\mathrm{dm}(s')$ too, then $s' \subset e$, that is $e \models A$. Otherwise from the set $\mathrm{dm}(s') \setminus \mathrm{dm}(s)$ we take all variables $\alpha_1, \ldots, \alpha_n$, which take different values for e than for e'. Hence e coincide with an evaluation $(...(e'^{-}_{\alpha_1})...)^{-}_{\alpha_n}$ on the set $\mathrm{dm}(s')$. From this and from the fact that $\alpha_1, \ldots, \alpha_n$ are not essential in A we achieve: $e' \models A \Leftrightarrow e'^{-}_{\alpha_1} \models A \Leftrightarrow \cdots \Leftrightarrow (...(e'^{-}_{\alpha_1})...)^{-}_{\alpha_n} \models A \Leftrightarrow e \models A$. □

3.6. Operations on logical information

In the set $\mathbb{INF}$ we will introduce operations of denial, convolution and alternation of information, which will correspond to the respective connectives of: negation, conjunction and disjunction. These operations will be induced from some operations in the set A that are invariant with respect to the relation $\cong$.

The operation of denial $\ominus\colon \mathsf{A} \to \mathsf{A}$ is defined as follows: for n-element ($n \geqslant 0$) set $A = \{s_1, \ldots, s_n\}$ we take

$$t \in \ominus A \overset{\mathrm{df}}{\iff} t \in \mathrm{IS} \text{ and for } i = 1\ldots, n \text{ there are such } \alpha_i \in \mathrm{dm}(s_i), \text{ that } \mathrm{dm}(t) = \{\alpha_1, \ldots \alpha_n\} \text{ i } t(\alpha_i) = 1 - s(\alpha_i)\,.$$

Therefore the elements of the set $\ominus A$ are these and only these functions which arise in the following way: for every $1 \leqslant i \leqslant n$ from the domain of s_i we chose one element α_i. If a set of pairs $\{\langle \alpha_1, 1 - s(\alpha_1)\rangle, \ldots, \langle \alpha_1, 1 - s(\alpha_1)\rangle\}$ is a function (i.e., only one value: 0 or 1 is assigned to every variable α_i), then $\alpha_i \in \ominus A$.

Example 3.3. (a) $\ominus\{\varnothing\} = \Lambda$ and $\ominus\Lambda = \{\varnothing\}$.

(b) $\ominus\{p, \overline{p}\} = \Lambda$.

(c) $\ominus\{p, q\} = \ominus\{pq, p\overline{q}, \overline{p}q\} = \{\overline{pq}\}$ and $\ominus\{\overline{pq}\} = \{p, q\}$.

(d) $\ominus\{pqr, \overline{qr}\} = \{\overline{p}q, \overline{p}r, \overline{q}r, q\overline{r}\}$.

(e) $\ominus\{p\overline{q}, pr\} = \{\overline{p}, \overline{pr}, \overline{p}q, q\overline{r}\}$. □

It is provable that for every $e \in \mathrm{E}$ it holds that:

$$e \models \ominus A \iff e \not\models A\,. \tag{3.1}$$

The immediate conclusion is

$$A \cong B \iff \ominus A \cong \ominus B\,. \tag{3.2}$$

Therefore the operation of denial of information may be defined as follows:

$$\ominus[A] := [\ominus A]\,.$$

From the above facts we conclude that:

$$e \models \ominus I \iff e \not\models I\,. \tag{3.3}$$

Moreover, $\ominus\vee = \wedge$, $\ominus\wedge = \vee$ and generally: $\ominus\ominus I = I$.

Conjunction will be connected with a binary operation of "convolution". We define the operation $\owedge\colon \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ by means of the following equation:

$$A \owedge B := \{\, s \cup t \in \mathrm{IS} \;:\; s \in A \,\&\, t \in B \,\}\,.$$

Example 3.4. (a) $\{p\} \owedge \{q\} = \{pq\}$; $\{p, q\} \owedge \{p, q\} = \{p, pq, q\}$.

(b) $\{p, q\} \owedge \{\overline{p}, q\} = \{pq, \overline{p}q, q\}$ and $\{\overline{pq}\} \owedge \{p\overline{q}, r\} = \{\overline{pqr}\}$.

(c) $\{p\} \owedge \{\overline{p}\} = \Lambda$ and generally $\{s\} \owedge \ominus\{s\} = \Lambda$.

(d) $\Lambda \owedge A = \Lambda$ and $\{\varnothing\} \owedge A = A$. □

It is easily provable that

$$e \models A \owedge B \iff e \models A \,\&\, e \models B\,. \tag{3.4}$$

The immediate conclusion is:

$$A \cong B \implies A \owedge C \cong B \owedge C\,. \tag{3.5}$$

Therefore the operation of a *convolution of information* $\owedge\colon \mathbb{INF}\times\mathbb{INF}\to\mathbb{INF}$ may be defined as follows:

$$[A]\owedge[B] \;\coloneqq\; [A\owedge B]\,.$$

It holds that

$$e\models I\owedge J \iff e\models I \;\&\; e\models J\,. \tag{3.6}$$

This operation is idempotent, symmetrical and associative, i.e., $I\owedge I=I$, $I\owedge J=J\owedge I$ and $I\owedge(J\owedge K)=(I\owedge J)\owedge K$. Besides $I\owedge\curlyvee=I$ and $I\owedge\curlywedge=\curlywedge$.

The set-theoretical sum of sets from A is an operation that corresponds to the connective of disjunction. It is idempotent, symmetrical and associative, besides it satisfies the condition:

$$e\models A\cup B \iff e\models A \text{ or } e\models B\,. \tag{3.7}$$

The direct conclusion of this fact is that:

$$A\cong B \implies A\cup C\cong B\cup C\,. \tag{3.8}$$

Thus the operation $\cup:\mathsf{A}\times\mathsf{A}\to\mathsf{A}$ induces a binary operation in a set $\mathbb{INF}$. We call it the *alternation of information*:[6]

$$[A]\ovee[B] \;\coloneqq\; [A\cup B]\,.$$

Alternation of information satisfies the condition:

$$e\models I\ovee J \iff e\models I \text{ or } e\models J\,. \tag{3.9}$$

Besides the operation $\ovee$ is idempotent, symmetrical and associative, i.e., $I\ovee I=I$, $I\ovee J=J\ovee I$ and $I\ovee(J\ovee K)=(I\ovee J)\ovee K$. Moreover $I\ovee\curlywedge=I$ and $I\ovee\curlyvee=\curlyvee$.

Let us notice that distributivity of $\owedge$ and $\ovee$ holds: $I\ovee(J\owedge K)=(I\ovee J)\owedge(I\ovee K)$ and $I\owedge(J\ovee K)=(I\owedge J)\ovee(I\owedge K)$.

The above equations imply that $\langle\mathbb{INF},\owedge,\ovee,\ominus,\curlyvee,\curlywedge\rangle$ is a Boolean algebra with sum $\owedge$, product $\ovee$, complement $\ominus$, zero $\curlyvee$ and unity $\curlywedge$. A standard Boolean partial order $\leq$ determined in this algebra by definition:

$$I\leq J \overset{\text{df}}{\iff} I\owedge J=J$$

satisfies the condition:

$$I\leq J \iff \forall_{e\in\mathrm{E}}\big(e\models J\Rightarrow e\models I\big)\,. \tag{3.10}$$

Yet relation $\leq$ is not proper for formalization of the notion of *being a part of information*. For example $[p,q]\leq[p]$, $[p,q]\leq[p\overline{q}]$ etc.

[6]Yet we must use a different symbol to denote it, since set theoretical sum of equivalence classes is a standard notion characterized by the equation: $[A]\cup[B]=\{C : C\cong A \text{ or } C\cong B\}$.

3.7. The relation of being a part of information

We will introduce the binary relation $\sqsubseteq$ *being a part of* in the set $\mathbb{INF}$ that will be properly included in the relation $\leq$. We say that information I is a part of information J (in symbols: $I \sqsubseteq J$) iff information J restricted to a set $\mathrm{dm}(I)$ is identical with I. Formally,

$$I \sqsubseteq J \overset{\mathrm{df}}{\iff} J|_{\mathrm{dm}(I)} = I\,.$$

Example 3.5. $[p,q] \not\sqsubseteq [p]$, but $[p,q] \leq [p]$. □

FACT 3.4. *The relation $\sqsubseteq$ partially orders the set $\mathbb{INF}$, i.e., it is reflexive, transitive and antisymmetrical:*

$$I \sqsubseteq I\,,$$
$$I \sqsubseteq J \;\&\; J \sqsubseteq K \implies I \sqsubseteq K\,,$$
$$I \sqsubseteq J \;\&\; J \sqsubseteq I \implies I = J\,.$$

PROOF. Reflexivity follows from Fact 3.3. Let $I \sqsubseteq J$ and $J \sqsubseteq K$. Then $I = K|_{\mathrm{dm}(I)\cap\mathrm{dm}(J)}$. Therefore, by Fact 3.1, $\mathrm{dm}(I) = \mathrm{dm}(K)\cap\mathrm{dm}(I)\cap\mathrm{dm}(J)$, so $\mathrm{dm}(I) \subseteq \mathrm{dm}(I) \cap \mathrm{dm}(J)$, and hence $\mathrm{dm}(I) = \mathrm{dm}(I) \cap \mathrm{dm}(J)$. Therefore $I = K|_{\mathrm{dm}(I)}$, i.e., $I \sqsubseteq K$. Finally let $I \sqsubseteq J$ and $J \sqsubseteq I$. Then $I = I|_{\mathrm{dm}(I)\cap\mathrm{dm}(J)}$. Hence, by Fact 3.1, $\mathrm{dm}(I) = \mathrm{dm}(I) \cap \mathrm{dm}(J)$, i.e., $\mathrm{dm}(I) \subseteq \mathrm{dm}(J)$. In an analogous way we show that $\mathrm{dm}(J) \subseteq \mathrm{dm}(I)$. Thus $\mathrm{dm}(I) = \mathrm{dm}(J)$. Hence, by Fact 3.3, we have that $I = J|_{\mathrm{dm}(I)} = J|_{\mathrm{dm}(J)} = J$. □

It is easily provable that contradictory information $\curlywedge$ is not a part of any other information, and that it is the only its part.

FACT 3.5. *For any $I \in \mathbb{INF}$: $\curlywedge \sqsubseteq I \iff I = \curlywedge \iff I \sqsubseteq \curlywedge$.*

PROOF. Let $I = \curlywedge$. Then $\curlywedge \sqsubseteq I$ and $I \sqsubseteq \curlywedge$, by Fact 3.4. Overwise, let $I \neq \curlywedge$ and $I = [A]$ for some $\Lambda \neq A \in \mathsf{A}$. Then $A|_\emptyset = \{\varnothing\}$. So $I|_\emptyset = [\{\varnothing\}] = \curlyvee \neq \curlywedge$, i.e., $\curlywedge \not\sqsubseteq I$. Moreover, $\curlywedge|_\emptyset = \{\Lambda|_\emptyset\} = \{\Lambda\} = \curlywedge \neq I$, so $I \not\sqsubseteq \curlywedge$. □

Equally easy we show that the empty information $\curlyvee$ is a part of every other information different from $\curlywedge$, and that it is not a part of information $\curlywedge$. Moreover, the empty information is the only part of its own.

FACT 3.6. *For any $I \in \mathbb{INF}$:*

$$I \neq \curlywedge \iff \curlyvee \sqsubseteq I\,,$$
$$I \sqsubseteq \curlyvee \iff \curlyvee = I\,.$$

PROOF. Let $I \neq \curlywedge$ and $I = [A]$ for some $\Lambda \neq A \in \mathsf{A}$. Then $A|_\emptyset = \{\varnothing\}$. Therefore $I|_\emptyset = [\{\varnothing\}] = \curlyvee$, i.e., $\curlyvee \sqsubseteq I$. Overwise, let $\curlyvee \sqsubseteq I$. Since $\curlyvee \neq \curlywedge$, so $\curlyvee \not\sqsubseteq \curlywedge$, by Fact 3.5. Hence $I \neq \curlywedge$.

Moreover, let $I \sqsubseteq \curlyvee$, i.e., $\curlyvee|_\emptyset = I$. Hence $I = [\varnothing|_\emptyset] = [\varnothing] = \curlyvee$. □

We say that an information I is a *proper part* of an information J (in symbols: $I \sqsubset J$) iff $I \sqsubseteq I$ and $I \neq J$.

Example 3.6. (a) $[p] \sqsubset [pq]$ and $[q] \sqsubset [pq]$;

(b) $[p] \sqsubset [pq, pr]$ and $[q, r] \sqsubset [pq, pr]$;

(c) $[p_2, p_3, p_4] \sqsubset [pp_2, pp_3, qp_4]$ and $[p, q] \sqsubset [pp_2, pp_3, qp_4]$;

(d) $[\overline{p}, r] \sqsubset [\overline{pq}, qr]$ (see examples 3.1c and 3.2c);

(e) $[\overline{p}, q] = [\overline{pq}, q] \sqsubset [\overline{pq}, qr]$;

(f) $[\overline{p}, qr] \not\sqsubseteq [\overline{pq}, qr]$.

Finally we present some facts:

$$I \sqsubseteq J \Longrightarrow \mathrm{dm}(I) \subseteq \mathrm{dm}(J)\,, \tag{3.11}$$

$$I \sqsubset J \Longrightarrow \mathrm{dm}(I) \subset \mathrm{dm}(J)\,, \tag{3.12}$$

$$\forall_{V \subseteq \mathsf{V}}\ I|_V \sqsubseteq I\,, \tag{3.13}$$

$$I \sqsubseteq J \iff \forall_{V \subseteq \mathsf{V}}\ I|_V \sqsubseteq J|_V \iff \forall_{V \subseteq \mathrm{dm}(I)}\ I|_V \sqsubseteq J|_V\,. \tag{3.14}$$

It is provable that $\sqsubseteq$ is included in $\leq$, i.e.,

$$I \sqsubseteq J \implies I \leq J\,. \tag{3.15}$$

It is a proper inclusion (see Example 3.5).

3.8. Logical information of formulas from L

As it is known, values of variables occurring in some formula for a given evaluation are the only determiners of a value of this formula. More precisely, for an arbitrary formula $\varphi \in \mathsf{L}$: if evaluations e_1 and e_2 are exactly the same on the set $\mathsf{V}(\varphi)$, then $h^{e_1}(\varphi) = h^{e_2}(\varphi)$. Hence for every $s \in \mathrm{IS}$ such that $\mathsf{V}(\varphi) \subseteq \mathrm{dm}(s)$ we can assume that $s(\varphi)$ is a shared value $h^e(\varphi)$ for all these evaluations e, for which $s \subseteq e$.

Let us define function $\mathbf{I}$ on the set L and taking values in $\mathbb{INF}$:

$$\mathbf{I}(\varphi) := \big[\{\, s \in \mathrm{IS} \;:\; \mathrm{dm}(s) = \mathsf{V}(\varphi) \;\&\; s(\varphi) = 1 \,\}\big]\,.$$

The class $\mathbf{I}(\varphi)$ is called *a logical information of a formula* φ. It is an equivalence class of these and only these evaluations of variables occurring in φ, for which φ is true. Clearly, the following equality holds:

$$\mathbf{I}(\varphi) \;=\; \big[\{\, s \in \mathrm{IS} \;:\; \mathrm{dm}(s) = \mathsf{V_e}(\varphi) \;\&\; s(\varphi) = 1 \,\}\big] .$$

It follows directly from the definition that:

(3.16) $$e \models \mathbf{I}(\varphi) \iff h^e(\varphi) = 1 ,$$

(3.17) $$\mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi) \implies \varphi \models \psi ,$$

(3.18) $$\varphi \models \psi \iff \mathbf{I}(\psi) \leq \mathbf{I}(\varphi) ,$$

(3.19) $$\varphi \mathrel{=\!\!|\!\models} \psi \iff \mathbf{I}(\varphi) = \mathbf{I}(\psi) ,$$

(3.20) $$\varphi \in \mathsf{T} \iff \mathbf{I}(\varphi) = \curlyvee ,$$

(3.21) $$\varphi \in \mathsf{F} \iff \mathbf{I}(\varphi) = \curlywedge ,$$

(3.22) $$\mathbf{I}(\neg\varphi) = \ominus\mathbf{I}(\varphi) ,$$

(3.23) $$\mathbf{I}(\varphi \wedge \psi) = \mathbf{I}(\varphi) \owedge \mathbf{I}(\psi) ,$$

(3.24) $$\mathbf{I}(\varphi \vee \psi) = \mathbf{I}(\varphi) \ovee \mathbf{I}(\psi) ,$$

(3.25) $$\mathrm{dm}(\mathbf{I}(\varphi)) = \mathsf{V_e}(\varphi) .$$

Example 3.7. (a) $\mathbf{I}(p) = [p]$;

(b) $\mathbf{I}(\neg p) = [\bar{p}]$ and $\mathbf{I}(\neg\neg p) = [p]$;

(c) $\mathbf{I}(p \wedge q) = [pq] = \mathbf{I}(p \wedge (\neg p \vee q))$;

(d) $\mathbf{I}(p \vee q) = [p, q]$;

(e) $\mathbf{I}(p \supset r) = [\bar{p}, r]$,[7]

(f) $\mathbf{I}((p \supset q) \wedge (q \supset r)) = [\overline{pq}, qr]$;
so $\mathbf{I}(p \supset r) \sqsubset \mathbf{I}((p \supset q) \wedge (q \supset r))$ (see Example 3.6d)
and $\mathbf{I}(p \supset q) \sqsubset \mathbf{I}((p \supset q) \wedge (q \supset r))$ (see Example 3.6e),[8]
but $\mathbf{I}(p \supset q) \wedge (p \supset r)) \not\sqsubseteq \mathbf{I}((p \supset q) \wedge (q \supset r))$ (see Example 3.6f). □

4. Logical information and Boolean normal forms

We will compare the structure of logical information with some Boolean normal forms of formulas from L. This comparison will be useful while proving soundness for the axiomatization of the relation preserving logical information.

---

[7]Farther in this paper $\ulcorner(\varphi \supset \psi)\urcorner$ is an abbreviation of $\ulcorner(\neg\varphi \vee \psi)\urcorner$.

[8]It is not always the case that $\mathbf{I}(\varphi) \sqsubseteq \mathbf{I}(\varphi \wedge \psi)$ for $\varphi, \psi \in \mathsf{K}$. For example: $\mathbf{I}(p \vee q) = [p, q] \not\sqsubseteq [p] = \mathbf{I}((p \vee q) \wedge p)$. Another example is: $\mathbf{I}((p \vee q) \wedge r) = [pr, qr] \not\sqsubseteq [pr] = \mathbf{I}(((p \vee q) \wedge r) \wedge p)$.

4.1. Disjunctive-conjunctive normal forms

A formula φ from L is called *generalized conjunction* (resp. *generalized disjunction*) iff there are such formulas $\psi_1, \ldots, \psi_n$ ($n > 0$), that $\varphi = \psi_1 \wedge \cdots \wedge \psi_n$ (resp. $\varphi = \psi_1 \vee \cdots \vee \psi_n$).[9] If so, the formulas $\psi_1, \ldots, \psi_n$ are called the elements of a formula φ (if $n = 1$, then $\varphi = \psi_1$).

Generalized conjunction φ is called *elementary* iff the elements of φ are solely propositional variables or their negations, with reservation that the elements of φ are not both a variable and its negation.[10] Let ek be the set of all elementary conjunctions.

A formula φ has its *disjunctive–conjunctive normal form* (in symbols: $\varphi \in \mathsf{AN}$) iff φ is a generalized disjunction whose all elements are elementary conjunctions. Clearly, $\mathsf{ek} \subsetneq \mathsf{AN}$.

4.2. Boolean (disjunctive) normal forms

Let us agree that for every $i \geqslant 0$ and $b \in \{0, 1\}$:

$$p_i^b := \begin{cases} \neg p_i & \text{if } b = 0 \\ p_i & \text{if } b = 1 \end{cases}$$

Moreover for every increasing sequence of natural numbers $\vec{\imath} = \langle i_1, \ldots, i_n \rangle$ (i.e., $0 \leqslant i_1 < \cdots < i_n$ when $n > 0$) and for every sequence of 0s and 1s $\langle b_1, \ldots, b_n \rangle \in \{0, 1\}^n$ let $\bigwedge_{\vec{\imath}}^k$ be an elementary conjunction:

$$\bigwedge\nolimits_{\vec{\imath}}^k := p_{i_1}^{b_1} \wedge \cdots \wedge p_{i_n}^{b_n} ,$$

where $k = \sum_{i=1}^n b_i \cdot 2^{n-i}$ (i.e., $b_1 b_2 \ldots b_n$ is a binary notation of a number k).

Let φ be an arbitrary formula from L such that $\mathsf{V}(\varphi) = \{p_{i_1}, p_{i_2}, \ldots, p_{i_n}\}$, where $\vec{\imath} = \langle i_1, \ldots, i_n \rangle$ is an increasing sequence ($n > 0$). We say that a sequence of 1s and 0s, $\langle b_1, b_2 \ldots, b_n \rangle$, *satisfies* the formula φ iff $h^e(\varphi) = 1$ for every evaluation e such that $e(p_{i_1}) = b_1, \ldots, e(p_{i_n}) = b_n$.

For every formula $\varphi \notin \mathsf{F}$ we will construct its *Boolean (disjunctive) normal form* φ° (cf. Asser, 1959). A formula φ° will be generalized disjunction whose elements are all *canonical elementary conjunctions* (cf. Asser, 1959) built from variables of a formula φ and determined by sequences of 0s and 1s satisfying the formula φ. The order of the conjunctions in a disjunction φ° coincide with the order of numbers whose sequences of binary expansions satisfy the formula φ.

---

[9] By convention, ⌜$\psi_1 \wedge \cdots \wedge \psi_n$⌝ is an abbreviation of a formula ⌜$(\ldots(\psi_1 \wedge \psi_2) \wedge \cdots \wedge \psi_{n-1}) \wedge \psi_n$⌝. An analogical convention applies to a disjunction.

[10] We will not use so called *contradictory* elementary conjunctions, whose elements are some variable and its negation, e.g. '$r \wedge \neg p \wedge q \wedge p$'.

Let $\varphi \notin \mathsf{F}$ and let $k_1 < k_2 < \cdots < k_s$ $(0 < s \leqslant 2^n)$ be an increasing sequence of natural numbers such that $k_l = \sum_{i=1}^{n} b_i^{k_l} \cdot 2^{n-i}$ for $l = 1, \ldots, s$, where $\langle b_1^{k_1}, \ldots, b_n^{k_1}\rangle$, $\ldots, \langle b_1^{k_s}, \ldots, b_n^{k_s}\rangle$ are all sequences satisfying the formula φ. Take

$$\varphi^\circ \coloneqq \bigwedge\nolimits_{\vec{\imath}}^{k_1} \vee \cdots \vee \bigwedge\nolimits_{\vec{\imath}}^{k_s} .$$

It is obvious that $\varphi^\circ \in \mathsf{AN}$ and $\varphi^\circ \mathrel{\vDash\!\!\!\Dash} \varphi$ for $\varphi \notin \mathsf{F}$. It follows directly from the definition that for $\varphi, \psi \notin \mathsf{F}$: $\varphi \mathrel{\vDash\!\!\!\Dash} \psi$ & $\mathsf{V}(\varphi) = \mathsf{V}(\psi) \iff \varphi^\circ = \psi^\circ$.

Now let us take an arbitrary formula $\varphi \in \mathsf{K}$ such that $\mathsf{V}(\varphi) = \{p_{i_1}, p_{i_2}, \ldots, p_{i_n}\}$, where $\vec{\imath} = \langle i_1, \ldots, i_n\rangle$ is an increasing sequence and $\mathsf{V_e}(\varphi) = \{p_{j_1}, p_{j_2}, \ldots, p_{j_m}\}$, where $\vec{\jmath} = \langle j_1, \ldots, j_m\rangle$ is a (increasing) subsequence of a sequence $\vec{\imath}$ $(0 < m \leqslant n)$. We say that 0-1 sequence $\langle b_1, b_2 \ldots, b_m\rangle$ *essentially satisfies* the formula φ iff for every evaluation e such that $e(p_{j_1}) = b_1$, $e(p_{j_2}) = b_2$, $\ldots$, $e(p_{j_m}) = b_m$ we have $h^e(\varphi) = 1$.

Taking a subsequence $\vec{\jmath}$ instead of $\vec{\imath}$ and using the notion of essential satisfaction instead of satisfaction, for φ we define—analogously as φ°—a formula $\varphi^\bullet$ from AN. In other words, for $\varphi \in \mathsf{K}$ the formula $\varphi^\bullet$ is a generalized disjunction, whose elements are all canonical elementary conjunctions built from essential variables of the formula φ and determined by 0-1 sequences essentially satisfying the formula φ. The order of these conjunctions in disjunction $\varphi^\bullet$ is the same as the order of numbers whose sequences of binary expansions essentially satisfy the formula φ.

Clearly, for $\varphi, \psi \in \mathsf{K}$ we have $\varphi^\bullet \mathrel{\vDash\!\!\!\Dash} \varphi \mathrel{\vDash\!\!\!\Dash} \varphi^\circ$ and $\mathsf{V_e}(\varphi) = \mathsf{V}(\varphi^\bullet)$ and $\varphi \mathrel{\vDash\!\!\!\Dash} \psi \iff \varphi^\bullet = \psi^\bullet$.

Let $\kappa \in \mathsf{ek}$ and V be a subset of V such that $\mathsf{V}(\kappa) \cap V \neq \emptyset$. By *A restriction of a conjunction* κ to a set V we mean an elementary conjunction $\kappa|_V$ formed from κ by removing all its elements that are variables from beyond V or negations of variables from beyond V.

Let $\varphi \notin \mathsf{F}$ and $\varphi^\circ = \kappa_1 \vee \cdots \vee \kappa_n$, where $\kappa_i \in \mathsf{ek}$. Clearly, $\mathsf{V}(\varphi) = \mathsf{V}(\kappa_1) = \cdots = \mathsf{V}(\kappa_n)$. For every set V such that $\mathsf{V}(\varphi) \cap V \neq \emptyset$ we take $\varphi^\circ|_V \coloneqq \kappa_1|_V \vee \cdots \vee \kappa_n|_V$. It is easily provable that for every formula φ from K, $\varphi^\bullet \mathrel{\vDash\!\!\!\Dash} \varphi^\circ|_{\mathsf{V_e}(\varphi)}$ (the equality not necessarily holds since some elements in $\varphi^\circ|_{\mathsf{V_e}(\varphi)}$ may repeat or be in improper order).

For φ from K we define $\varphi^\bullet|_V$ in an analogous way.

4.3. A comparison of formulas from AN with sets from A

We can construct «canonical embedding» of the set of formulas ek into the set of functions IS. For the notation from the paragraph 4.1, it will be a function which for $b_1, \ldots, b_n \in \{0,1\}$ and different $\alpha_1, \ldots, \alpha_n \in \mathsf{V}$

$$\mathsf{ek} \ni \alpha_1^{b_1} \wedge \cdots \wedge \alpha_n^{b_n} \overset{*}{\longmapsto} \left(\begin{smallmatrix} \alpha_1 & \ldots & \alpha_n \\ b_1 & \ldots & b_n \end{smallmatrix}\right) \in \mathrm{IS}\,.$$

Obviously, it is not an «invertible» function. It is neither injection (to conjunctions that are different solely in order of elements we assign the same state from IS) nor surjection (the state ø is not assigned to any elementary conjunction).

The above embedding $(\cdot)^*$ is extended on the set AN by setting:

$$\mathsf{AN} \ni \kappa_1 \vee \cdots \vee \kappa_m \overset{*}{\longmapsto} \{\kappa_1^*, \ldots, \kappa_m^*\} \in \mathsf{A}\,.$$

Clearly, it also is not injection nor surjection (Λ is not among its values). For every $\varphi, \psi \in \mathsf{AN}$ the following equality holds:

$$\varphi \dashv\vdash \psi \iff \varphi^* \cong \psi^*\,. \tag{4.1}$$

It is also obvious that for every $\varphi \notin \mathsf{F}$

$$(\varphi^\circ)^* = \left\{\left(\begin{smallmatrix} \alpha_1 & \ldots & \alpha_n \\ b_1 & \ldots & b_n \end{smallmatrix}\right) \in \mathrm{IS} \;:\; \mathsf{V}(\varphi) = \{\alpha_1, \ldots, \alpha_n\} \;\&\; \langle b_1, \ldots, b_n\rangle \models \varphi\right\},$$

where $\models$ is the satisfaction defined in 4.2 (p. 104). Similarly for $\varphi \in \mathsf{K}$

$$(\varphi^\bullet)^* = \left\{\left(\begin{smallmatrix} \alpha_1 & \ldots & \alpha_n \\ b_1 & \ldots & b_n \end{smallmatrix}\right) \in \mathrm{IS} \;:\; \mathsf{V_e}(\varphi) = \{\alpha_1, \ldots, \alpha_n\} \;\&\; \langle b_1, \ldots, b_n\rangle \Vdash \varphi\right\},$$

where $\Vdash$ is the relation of essential satisfaction from Section 4.2 (p. 105). Thus the function $(\cdot)^*$ assigns representatives of information $\mathbf{I}(\varphi)$ to formulas φ° and $\varphi^\bullet$.

4.4. Comparison of the set $\mathsf{AN}/_{\dashv\vdash}$ with the set $\mathbb{INF}$

From (4.1) it follows that the function $(\cdot)^*$ induces one-to-one assignment from the set $\mathsf{AN}/_{\dashv\vdash}$ to $\mathbb{INF}$:

$$\|\varphi\|^* \;:=\; [\varphi^*]\,,$$

where $\varphi \in \mathsf{AN}$ and $\|\varphi\| := \{\psi \in \mathsf{AN} \;:\; \psi \dashv\vdash \varphi\}$. Thus for every noncontradictory formula φ we have $\mathbf{I}(\varphi) = \|\varphi^\circ\|^*$.[11]

Finally let us notice that from the above facts it follows that:

$$\forall_{\varphi,\psi \in \mathsf{K}}\Big(\mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi) \iff \psi^\bullet \dashv\vdash \varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}\Big), \tag{4.2}$$

withal the disjunction $\psi^\bullet$ may differ from $\varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$, at the most, that the second one contains the iterative elements.

[11] Yet if, while constructing a set of logical information, instead of the set A, we would like to use the set AN, then we should extend it with «an empty elementary conjunction» (the counterpart of the empty information state ø). Otherwise, the operation of restriction of an elementary conjunction would not be performable on every set of variables (cf. Section 4.2). Moreover, we should extend the set AN with an object being the counterpart of the contradictory alternative of information states Λ.

5. The definition of the relation $\models_{\mathrm{i}}$ by means of the notion of information

Let $\models_{\mathrm{i}}$ be the consequence relation preserving logical information such that it is included in $\mathsf{L} \times \mathsf{L}$:

$$\varphi \models_{\mathrm{i}} \psi \overset{\mathrm{df}}{\iff} \varphi \notin \mathsf{F} \;\;\&\;\; \psi \notin \mathsf{T} \;\;\&\;\; \mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi)\,.$$

From above definition and Fact 3.4 we have: $\varphi \models_{\mathrm{i}} \varphi$ iff $\varphi \in \mathsf{K}$. Thus the relation $\models_{\mathrm{i}}$ is reflexive on the set K. Moreover, by Fact 3.4, the relation $\models_{\mathrm{i}}$ is transitive.

The relation $\dashv\models_{\mathrm{i}}$ is the superposition of the relation $\models_{\mathrm{i}}$ and its converse, i.e.,

$$\varphi \dashv\models_{\mathrm{i}} \psi \overset{\mathrm{df}}{\iff} \varphi \models_{\mathrm{i}} \psi \;\;\&\;\; \psi \models_{\mathrm{i}} \varphi\,.$$

From the definition of $\models_{\mathrm{i}}$ and by (3.17) we have:

(5.1) $$\varphi \models_{\mathrm{i}} \psi \implies \varphi \models \psi\,.$$

Hence, by (2.1), we have:

(5.2) $$\varphi \models_{\mathrm{i}} \psi \implies \varphi, \psi \in \mathsf{K}\,.$$

Hence, by (2.3), (3.11) and (3.25), we have:

(5.3) $$\varphi \models_{\mathrm{i}} \psi \implies \emptyset \neq \mathsf{V_e}(\psi) \subseteq \mathsf{V_e}(\varphi)\,.$$

From the definitions of $\dashv\models_{\mathrm{i}}$ and $\models_{\mathrm{i}}$, and by Fact 3.4 and (3.19), it follows that,

(5.4) $$\varphi \dashv\models_{\mathrm{i}} \psi \iff \varphi \dashv\models \psi \;\;\&\;\; \varphi, \psi \in \mathsf{K}\,.$$

Indeed, $\varphi \dashv\models_{\mathrm{i}} \psi$ iff $\varphi \models_{\mathrm{i}} \psi \;\&\; \psi \models_{\mathrm{i}} \varphi$ iff $\varphi \notin \mathsf{F} \;\&\; \psi \notin \mathsf{T} \;\&\; \mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi) \;\&\; \psi \notin \mathsf{F} \;\&\; \varphi \notin \mathsf{T} \;\&\; \mathbf{I}(\varphi) \sqsubseteq \mathbf{I}(\psi)$ iff $\mathbf{I}(\varphi) = \mathbf{I}(\psi) \;\&\; \varphi, \psi \in \mathsf{K}$ iff $\varphi \dashv\models \psi \;\&\; \varphi, \psi \in \mathsf{K}$.

Example 5.1. By examples 3.5–3.7:

(a) $p \wedge (p \supset q) \models_{\mathrm{i}} q$. Indeed, $\mathbf{I}(q) = [q] \sqsubset [pq] = \mathbf{I}(p \wedge (\neg p \vee q))$.

(b) $(p \supset q) \wedge (q \supset r) \models_{\mathrm{i}} p \supset q$.

(c) $p \supset q \dashv\models_{\mathrm{i}} \neg q \supset \neg p$. Indeed, $\mathbf{I}(\neg p \vee q) = [\bar{p}q] = \mathbf{I}(\neg\neg q \vee \neg p)$.[12]

(d) $p \not\models_{\mathrm{i}} p \vee q$.

(e) $p \wedge \neg q \not\models_{\mathrm{i}} p \vee q$.

(f) $p \wedge (q \vee \neg q) \not\models_{\mathrm{i}} p \vee q$.

(g) $p \vee (q \wedge \neg q) \not\models_{\mathrm{i}} p \vee q$.

---

[12] Yet $\varphi \models_{\mathrm{i}} \psi$ does not imply $\neg\psi \models_{\mathrm{i}} \neg\varphi$. Counterexample: φ = '$p \wedge q$' and ψ = 'p'.

6. A definition of the relation $\models_{\mathrm{i}}$ by means of evaluations

In (Pietruszczak, 1992) we put forward a definition of some relation, that was meant to be a modification of the relation $\models^{**}$. We will show that this definition in nothing but different characterization of the relation $\models_{\mathrm{i}}$.

THEOREM 6.1. *For every* $\varphi, \psi \in \mathsf{L}$

$$\varphi \models_{\mathrm{i}} \psi \iff \varphi \models \psi \;\&\; \varphi \notin \mathsf{F} \;\&\; \psi \notin \mathsf{T} \;\&\; \forall_{e\in \mathrm{E}}\big(h^e(\psi) = 1 \Rightarrow \exists_{e'\in \mathrm{E}}(e'|_{\mathsf{V_e}(\psi)} = e|_{\mathsf{V_e}(\psi)} \;\&\; h^{e'}(\varphi) = 1)\big).$$

In words: $\varphi \models_{\mathrm{i}} \psi$ *iff* $\varphi, \psi \in \mathsf{K}$*,* ψ *is a classical consequence of* φ *and for every evaluation satisfying* ψ *there is another one that coincides with the first one on a set of essential variables of a formula* ψ *and satisfies* φ*.*

PROOF. "$\Rightarrow$" Let $\varphi \models_{\mathrm{i}} \psi$. Then $\varphi \notin \mathsf{F}$, $\psi \notin \mathsf{T}$ and $\varphi \models \psi$, by (5.1). Since $\mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi)$, so $\psi^\bullet \;\mathrel{=}\!\!\!\mid\!\!\models\; \varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$, by (4.2). Set an arbitrary evaluation e such that $h^e(\psi) = 1$. Then in $\varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$ there is an elementary conjunction κ, for which $h^e(\kappa) = 1$. Let κ' be such an elementary conjunction in $\varphi^\bullet$ that $\kappa = \kappa'|_{\mathsf{V_e}(\psi)}$. Clearly, there is an evaluation e' that satisfies κ' and coincides with e on the set $\mathsf{V_e}(\psi)$.

"$\Leftarrow$" In (Pietruszczak, 1992) it was proved that the right-handed side of the equivalence entails the condition $\emptyset \neq \mathsf{V_e}(\psi) \subseteq \mathsf{V_e}(\varphi)$. For $\mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi)$, by (4.2), it is enough to show that $\psi^\bullet \;\mathrel{=}\!\!\!\mid\!\!\models\; \varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$. Let $h^e(\psi^\bullet) = 1$. Then also $h^e(\psi) = 1$. Hence, by assumption, there is an evaluation e' such that $h^{e'}(\varphi) = 1$ and $e'|_{\mathsf{V_e}(\psi)} = e|_{\mathsf{V_e}(\psi)}$. It entails that $h^e(\varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}) = 1$. Conversely, let $h^e(\varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}) = 1$. Then there is such an e', that $e'|_{\mathsf{V_e}(\psi)} = e|_{\mathsf{V_e}(\psi)}$ and $h^{e'}(\varphi^\bullet) = 1$. Since $\varphi^\bullet \models \psi^\bullet$, so $h^{e'}(\psi^\bullet) = 1$. Thus $h^e(\psi^\bullet) = 1$, because $\mathsf{V_e}(\psi) = \mathsf{V}(\psi^\bullet)$. □

COROLLARY 6.1. *If* $\mathsf{V_e}(\varphi) = \mathsf{V_e}(\psi)$ *and* $\varphi \models_{\mathrm{i}} \psi$*, then* $\varphi \;\mathrel{=}\!\!\!\mid\!\!\models_{\mathrm{i}}\; \psi$*.*

PROOF. Let $\mathsf{V_e}(\varphi) = \mathsf{V_e}(\psi)$ and $\varphi \models_{\mathrm{i}} \psi$. Then $\varphi \models \psi$, by (5.1); $\varphi, \psi \in \mathsf{K}$, by (5.2); and $\mathsf{V_e}(\varphi) = \mathsf{V_e}(\psi) \neq \emptyset$, by (2.3). Hence, $\psi \models \varphi$, by Theorem 6.1. So $\varphi \;\mathrel{=}\!\!\!\mid\!\!\models\; \psi$. Hence, $\varphi \;\mathrel{=}\!\!\!\mid\!\!\models_{\mathrm{i}}\; \psi$, by (5.4). □

7. The axiomatization of the relation $\models_{\mathrm{i}}$

Now we define some class $\mathfrak{I}$ of consequence relations preserving information. All members of the class $\mathfrak{I}$ included in $\mathsf{L} \times \mathsf{L}$. After that we will show that the relation $\models_{\mathrm{i}}$ is the smallest in the class $\mathfrak{I}$.

A relation $\succ$ is a member of $\mathfrak{I}$ iff for all $\varphi, \psi, \chi \in \mathsf{L}$ the following axioms are satisfied (we will use a symmetrical relation $\gtrless$ as superposition of $\succ$ with its converse $\prec$[13]):

$$\text{(a1)} \quad \varphi \in \mathsf{K} \implies \varphi \gtrless \neg\neg\varphi\,,$$

$$\text{(a2)} \quad \curlyvee \neq \mathbf{I}(\varphi) \sqsubseteq \mathbf{I}(\varphi \wedge \psi) \neq \curlywedge \implies \varphi \wedge \psi \succ \varphi\,,$$ [14]

$$\text{(a3)} \quad \varphi \wedge \psi \in \mathsf{K} \implies \varphi \wedge \psi \succ \psi \wedge \varphi\,,$$

$$\text{(a4)} \quad (\varphi \wedge \psi) \wedge \chi \in \mathsf{K} \implies (\varphi \wedge \psi) \wedge \chi \gtrless \varphi \wedge (\psi \wedge \chi)\,,$$

$$\text{(a5)} \quad \varphi \wedge \psi \in \mathsf{K} \implies \neg(\varphi \wedge \psi) \gtrless \neg\varphi \vee \neg\psi\,,$$

$$\text{(a6)} \quad (\varphi \vee \psi) \wedge \chi \in \mathsf{K} \implies (\varphi \vee \psi) \wedge \chi \gtrless (\varphi \wedge \chi) \vee (\psi \wedge \chi)\,,$$

$$\text{(a7)} \quad \varphi \in \mathsf{K} \,\&\, \tau \in \mathsf{T} \implies \varphi \succ \varphi \wedge \tau\,,$$

$$\text{(a8)} \quad \varphi \in \mathsf{K} \,\&\, \phi \in \mathsf{F} \implies \varphi \gtrless \varphi \vee \phi\,,$$

$$\text{(a9)} \quad \varphi \in \mathsf{K} \,\&\, \phi \in \mathsf{F} \implies \varphi \gtrless \phi \vee \varphi\,,$$

$$\text{(a10)} \quad \varphi \succ \psi \,\&\, \psi \succ \chi \implies \varphi \succ \chi\,,$$

$$\text{(a11)} \quad \mathbf{I}(\varphi \wedge \psi) \sqsubseteq \mathbf{I}(\chi) \,\&\, \chi \succ \varphi \,\&\, \chi \succ \psi \implies \chi \succ \varphi \wedge \psi\,,$$ [15]

$$\text{(a12)} \quad \chi \notin \mathsf{F} \,\&\, \chi(\varphi/\psi) \notin \mathsf{T} \,\&\, \varphi \gtrless \psi \implies \chi \succ \chi(\varphi/\psi)\,.$$

Remark 7.1. Formally, (a1)–(a9) have the following forms (for all $\pi, \sigma \in \mathsf{L}$):

(a1′) If either $\pi \in \mathsf{K}$ and $\sigma = \ulcorner\neg\neg\pi\urcorner$, or $\sigma \in \mathsf{K}$ and $\pi = \ulcorner\neg\neg\sigma\urcorner$, then $\pi \succ \sigma$.

(a2′) If $\pi = \ulcorner\sigma \wedge \alpha\urcorner$ for some α, and $\curlyvee \neq \mathbf{I}(\sigma) \sqsubseteq \mathbf{I}(\pi) \neq \curlywedge$, then $\pi \succ \sigma$.

(a3′) $\pi \in \mathsf{K}$, $\pi = \ulcorner\alpha \wedge \beta\urcorner$ for some α, β, and $\sigma = \ulcorner\beta \wedge \alpha\urcorner$, then $\pi \succ \sigma$.

(a4′) If for some α, β, γ: either $\pi = \ulcorner(\alpha \wedge \beta) \wedge \gamma\urcorner$, $\pi \in \mathsf{K}$ and $\sigma = \ulcorner\beta \wedge (\alpha \wedge \gamma)\urcorner$, or $\pi = \ulcorner\beta \wedge (\alpha \wedge \gamma)\urcorner$, $\sigma = \ulcorner(\alpha \wedge \beta) \wedge \gamma\urcorner$ and $\sigma \in \mathsf{K}$, then $\pi \succ \sigma$.

(a5′) If for some α, β: $\ulcorner\alpha \wedge \beta\urcorner \in \mathsf{K}$ and either $\pi = \ulcorner\neg(\alpha \wedge \beta)\urcorner$ and $\sigma = \ulcorner\neg\alpha \vee \neg\beta\urcorner$, or $\pi = \ulcorner\neg\alpha \vee \neg\beta\urcorner$ and $\sigma = \ulcorner\neg(\alpha \wedge \beta)\urcorner$, then $\pi \succ \sigma$.

(a6′) If for some α, β, γ: either $\pi = \ulcorner(\alpha \vee \beta) \wedge \gamma\urcorner$, $\pi \in \mathsf{K}$ and $\sigma = \ulcorner(\beta \wedge \gamma) \vee (\alpha \wedge \gamma)\urcorner$, or $\pi = \ulcorner(\beta \wedge \gamma) \vee (\alpha \wedge \gamma)\urcorner$, $\sigma = \ulcorner(\alpha \vee \beta) \wedge \gamma\urcorner$ and $\sigma \in \mathsf{K}$, then $\pi \succ \sigma$.

(a7′) If $\pi \in \mathsf{K}$ and $\sigma = \ulcorner\pi \wedge \tau\urcorner$ for some $\tau \in \mathsf{T}$, then $\pi \succ \sigma$.

(a8′) If either $\pi \in \mathsf{K}$ and $\sigma = \ulcorner\pi \vee \tau\urcorner$ for some $\phi \in \mathsf{F}$, or $\sigma \in \mathsf{K}$ and $\pi = \ulcorner\sigma \vee \tau\urcorner$ for some $\phi \in \mathsf{F}$, then $\pi \succ \sigma$.

(a9′) If either $\pi \in \mathsf{K}$ and $\sigma = \ulcorner\tau \vee \pi\urcorner$ for some $\phi \in \mathsf{F}$, or $\sigma \in \mathsf{K}$ and $\pi = \ulcorner\tau \vee \sigma\urcorner$ for some $\phi \in \mathsf{F}$, then $\pi \succ \sigma$. □

[13] I.e., for all $\varphi, \psi \in \mathsf{L}$: $\varphi \gtrless \psi$ iff $\varphi \succ \psi$ & $\psi \succ \varphi$.

[14] See Footnote 8.

[15] $\mathbf{I}(\varphi) \sqsubseteq \mathbf{I}(\chi)$ and $\mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\chi)$ does not entail $\mathbf{I}(\varphi \wedge \psi) \sqsubseteq \mathbf{I}(\chi)$. Counterexample: φ = '$p \supset q$', ψ = '$p \supset r$' and χ = '$(p \supset q) \wedge (q \supset r)$' (see Example 3.7f).

Obviously, the class $\mathfrak{I}$ is not empty, since the full relation $\mathsf{L} \times \mathsf{L}$ is a member of $\mathfrak{I}$. The other member of $\mathfrak{I}$ is the relation $\models_{\mathrm{i}}$.

THEOREM ON THE CORRECTNESS 7.1. *The relation* $\models_{\mathrm{i}}$ *is a member of* $\mathfrak{I}$.

PROOF. We must prove that the relation $\models_{\mathrm{i}}$ satisfies axioms (a1)–(a12). For (a1), (a3)–(a9) and (a12) the fact follows from (5.4). For (a2) from (3.20) and (3.21). For (a10) from Fact 3.4. The axiom (a11) follows from the fact that if $\varphi, \psi \notin \mathsf{T}$, then $\varphi \wedge \psi \notin \mathsf{T}$. □

We put

$$\succ_{\circ} := \bigcap \mathfrak{I} := \{\langle \varphi, \psi \rangle \in \mathsf{L} \times \mathsf{L} : \forall_{\succ \in \mathfrak{I}}\ \varphi \succ \psi\},$$

By our definitions we have.

FACT 7.1. $\succ_{\circ} \in \mathfrak{I}$, *ie., the relation* $\succ_{\circ}$ *is the smallest one in the class* $\mathfrak{I}$. □

Moreover, by our definitions and by Theorem on the Correctness 7.1 we obtain the Corollary on the Correctness.

COROLLARY 7.1 (on the Correctness). $\succ_{\circ} \subseteq \models_{\mathrm{i}}$, *i.e., if* $\varphi \succ_{\circ} \psi$ *then* $\varphi \models_{\mathrm{i}} \psi$. □

The remaining part of this paper is devoted to proving the inverse inclusion, that is $\models_{\mathrm{i}} \subseteq \succ_{\circ}$. In Section 10 we prove:

COMPLETENNESS THEOREM 7.2. $\models_{\mathrm{i}} \subseteq \succ_{\circ}$, *i.e.,* $\varphi \models_{\mathrm{i}} \psi$ *then* $\varphi \succ_{\circ} \psi$.

From these the adequacy will follow:

THEOREM ON THE ADEQUACY 7.3. $\models_{\mathrm{i}} = \succ_{\circ}$, *i.e., the relation* $\models_{\mathrm{i}}$ *is the smallest in* $\mathfrak{I}$.

Further we will profit from the below lemma that we obtain in a standard way.

LEMMA 7.1. $\varphi \succ_{\circ} \psi$ *iff there is a finite sequence of pairs of formulas* $\langle \pi_1, \sigma_1 \rangle, \ldots, \langle \pi_n, \sigma_n \rangle$ *such that* $\pi_n = \varphi$ *and* $\sigma_n = \psi$, *and for* $i = 1, \ldots, n$ *at least one of the following conditions holds:*

1. *for* $\pi = \pi_i$ *and* $\sigma = \sigma_i$ *an antecedent of some implication from* (a1′)–(a9′) *is true;*
2. *there are* $j, k < i$ *such that* $\pi_i = \pi_j$, $\sigma_i = \sigma_k$ *and* $\sigma_j = \pi_k$;
3. $\mathbf{I}(\sigma_i) \sqsubseteq \mathbf{I}(\pi_i)$ *and there are* $j, k < i$ *such that* $\pi_i = \pi_j = \pi_k$ *and* $\sigma_i = \sigma_j \wedge \sigma_k$;
4. $\pi_i \notin \mathsf{F}$, $\sigma_i \notin \mathsf{T}$ *and there are* $j, k < i$ *such that* $\pi_j = \sigma_k$ *and* $\pi_k = \sigma_j$, *and* $\pi_i = \chi$ *and* $\sigma_i = \chi(\pi_j/\sigma_j)$ *for some formula* χ.[16] □

[16] For conditions 2–4 see axioms (a10)–(a12).

8. Auxiliary properties of all relations from the class $\mathfrak{I}$

Let $\succ$ be an arbitrary relation in the class $\mathfrak{I}$.

Let us notice that (a1) and (a10) entail that the relation $\succ$ is reflexive on K:

(8.1) $$\varphi \in \mathsf{K} \implies \varphi \succ \varphi\,.$$

Moreover, from (a3), (3.19), (3.20), (3.21, (a2) and (a10) it follows that

(8.2) $$\vee \neq \mathbf{I}(\psi) \sqsubseteq \mathbf{I}(\varphi \wedge \psi) \neq \wedge \implies \varphi \wedge \psi \succ \psi\,,$$

and (3.20), (3.21), (a2), (8.1) and (a11) entail idempotence for $\wedge$:

(8.3) $$\varphi \in \mathsf{K} \implies \varphi \divideontimes \varphi \wedge \varphi\,.$$

Let us notice also that from (a2) and (a7) follows

(8.4) $$\varphi \in \mathsf{K} \;\&\; \tau \in \mathsf{T} \implies \varphi \divideontimes \varphi \wedge \tau\,,$$

because $\vee \neq \mathbf{I}(\varphi) = \mathbf{I}(\varphi \wedge \tau) \neq \wedge$.

We prove a theorem that will be useful farther:

(8.5) $$\chi_1 \succ \chi_2 \;\;\&\;\; \varphi_1 \divideontimes \psi_1 \;\;\&\;\; \varphi_2 \divideontimes \psi_2 \implies \chi_1(\varphi_1/\psi_1) \succ \chi_2(\varphi_2/\psi_2)\,.$$

Indeed, from the antecedent of the implication, by (a12), we have $\chi_2 \succ \chi_2(\varphi_2/\psi_2)$, and from this, by (a10), $\chi_1 \succ \chi_2(\varphi_2/\psi_2)$. Since $\chi_1 = \chi_1(\varphi_1/\psi_1).(\psi_1/\varphi_1)$, so from the antecedent of the implication and (a12), we derive $\chi_1(\varphi_1/\psi_1) \succ \chi_1$. Thus, using (a10), we have $\chi_1(\varphi_1/\psi_1) \succ \chi_2(\varphi_2/\psi_2)$.

The relation $\succ$ is idempotent for $\vee$ in K:

(8.6) $$\varphi \in \mathsf{K} \implies \varphi \divideontimes \varphi \vee \varphi\,.$$

Indeed, by (8.3) and (a12), $\neg(\neg\varphi \wedge \neg\varphi) \succ \neg\neg\varphi$ and $\neg\neg\varphi \succ \neg(\neg\varphi \wedge \neg\varphi)$. From this (a5) and (8.5), $\neg\neg\varphi \divideontimes \neg\neg\varphi \vee \neg\neg\varphi$. And, by (a1) and (8.5), we get $\varphi \divideontimes \varphi \vee \varphi$.

We prove that

(8.7) $$\varphi \vee \psi \in \mathsf{K} \implies \varphi \vee \psi \divideontimes \neg(\neg\varphi \wedge \neg\psi)\,.$$

Indeed, by (a5), $\neg\neg\varphi \vee \neg\neg\psi \divideontimes \neg(\neg\varphi \wedge \neg\psi)$. Assume additionally that $\varphi, \psi \in \mathsf{K}$. Then, by (a1), (8.1) and (8.5), we get $\varphi \vee \psi \divideontimes \neg\neg\varphi \vee \neg\neg\psi$. From this and from (a10), $\varphi \vee \psi \divideontimes \neg(\neg\varphi \wedge \neg\psi)$. Assume now that $\varphi \in \mathsf{K}$ and $\psi \in \mathsf{F}$. Then, by (a8), (a1), (a5) and (8.5), we get respectively: $\varphi \vee \psi \divideontimes \varphi \divideontimes \varphi \vee \neg\neg\psi \divideontimes \neg\neg\varphi \vee \neg\neg\psi \divideontimes \neg(\neg\varphi \wedge \neg\psi)$. The case $\varphi \in \mathsf{F}$ and $\psi \in \mathsf{K}$ is similar – we take (a9) instead of (a8).

Now from (8.7), (a3) and (8.5) we have:

(8.8) $$\varphi \vee \psi \in \mathsf{K} \implies \varphi \vee \psi \divideontimes \psi \vee \varphi\,.$$

Similarly, from (8.7), (a1), (a12) and (a10) we get:

(8.9) $$\varphi \vee \psi \in \mathsf{K} \implies \neg(\varphi \vee \psi) \divideontimes \neg\varphi \wedge \neg\psi\,.$$

We will show that relation $\divideontimes$ «preserves» commutativity for $\vee$ in K:

(8.10) $$(\varphi \vee \psi) \vee \chi \in \mathsf{K} \implies (\varphi \vee \psi) \vee \chi \divideontimes \varphi \vee (\psi \vee \chi)\,.$$

Indeed, in case if $\varphi\vee\psi \in \mathsf{K}$ we have: $(\varphi\vee\psi)\vee\chi \divideontimes \neg\neg((\varphi\vee\psi)\vee\chi) \divideontimes \neg((\neg\varphi\wedge\neg\psi)\wedge\neg\chi) \divideontimes \neg(\neg\varphi\wedge(\neg\psi\wedge\neg\chi))$. Now if (i) $\psi\vee\chi \in \mathsf{K}$ we have: $\ldots \divideontimes \neg\neg(\varphi\vee(\psi\vee\chi)) \divideontimes (\varphi\vee(\psi\vee\chi))$. In case of (ii), if $\psi\vee\chi \in \mathsf{F}$ the following holds: $(\varphi\vee\psi)\vee\chi \divideontimes \varphi\vee\psi \divideontimes \varphi \divideontimes \varphi\vee(\psi\vee\chi)$. In case if $\varphi\vee\psi \in \mathsf{F}$ we have: $(\varphi\vee\psi)\vee\chi \divideontimes \chi \divideontimes \psi\vee\chi \divideontimes \varphi\vee(\psi\vee\chi)$.

Finally

(8.11) $$(\varphi \wedge \psi) \vee \chi \in \mathsf{K} \implies (\varphi \wedge \psi) \vee \chi \divideontimes (\varphi \vee \chi) \wedge (\psi \vee \chi)\,.$$

Indeed, let us notice that $(\varphi\wedge\psi)\vee\chi \divideontimes \neg\neg((\varphi\wedge\psi)\vee\chi) \divideontimes \neg(\neg(\varphi\wedge\psi)\wedge\neg\chi)$. Now, if $\varphi\wedge\psi \in \mathsf{K}$, then $\ldots \divideontimes \neg((\neg\varphi\vee\neg\psi)\wedge\neg\chi)$. Yet if $\varphi\wedge\psi \in \mathsf{F}$, then $\neg(\varphi\wedge\psi)\wedge\neg\chi \divideontimes \chi \divideontimes (\neg\varphi\vee\neg\psi)\wedge\neg\chi$ that is we also get $\ldots \divideontimes \neg((\neg\varphi\vee\neg\psi)\wedge\neg\chi)$. Thus in both cases we have: $\ldots \divideontimes \neg((\neg\varphi\wedge\neg\chi)\vee(\neg\psi\wedge\neg\chi))$. Now we have to consider three cases. (A) when $\varphi\vee\chi \in \mathsf{K}$ and $\psi\vee\chi \in \mathsf{K}$. Then $\ldots \divideontimes \neg(\neg(\varphi\vee\chi)\vee\neg(\psi\vee\chi))$. (B) when $\varphi\vee\chi \in \mathsf{K}$ and $\psi\vee\chi \in \mathsf{T}$ we can prove that $(\neg\varphi\wedge\neg\chi)\vee(\neg\psi\wedge\neg\chi) \divideontimes \varphi \divideontimes (\neg\varphi\wedge\neg\chi)\vee\neg(\psi\vee\chi)$. Thus we also get: $\ldots \divideontimes \neg((\neg\varphi\wedge\neg\chi)\vee\neg(\psi\vee\chi)) \divideontimes \neg(\neg(\varphi\vee\chi)\vee\neg(\psi\vee\chi))$. (C) when $\varphi\vee\chi \in \mathsf{T}$ and $\psi\vee\chi \in \mathsf{K}$, then we will show analogously that $\ldots \divideontimes \neg(\neg(\varphi\vee\chi)\vee(\neg\psi\wedge\neg\chi)) \divideontimes \neg(\neg(\varphi\vee\chi)\vee\neg(\psi\vee\chi))$. Thus in all three cases we have: $\ldots \divideontimes \neg\neg((\varphi\vee\chi)\wedge(\psi\vee\chi)) \divideontimes (\varphi\vee\chi)\wedge(\psi\vee\chi)$.

We will need a couple of more generalized theorems that we proved earlier. The axiom (a6) will be used farther in the following form:

(8.12) $$(\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi \in \mathsf{K},\ \varphi_i \wedge \psi \in \mathsf{K} \implies (\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi \divideontimes (\varphi_1 \wedge \psi) \vee \cdots \vee (\varphi_n \wedge \psi)\,.$$

Indeed, as inductive hypothesis, let us assume that the condition is true for $n-1$. By (a6) we get: $(\varphi_1 \vee \cdots \vee \varphi_n) \wedge \psi \divideontimes ((\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi) \vee (\varphi_n \wedge \psi)$. Let us notice that by the antecedent of implication: $(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi \in \mathsf{K}$. Indeed, if $(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi \notin \mathsf{K}$, then $(\varphi_1 \vee \cdots \vee \varphi_{n-1}) \wedge \psi \in \mathsf{F}$, so $\varphi_i \wedge \psi \in \mathsf{F}$ for every $i \leqslant n$, contrary to the assumption. Therefore we can apply inductive hypothesis.

Conditions (a5) and (8.9) will also be inductively generalized:

(8.13) $$\varphi_1 \wedge \dots \wedge \varphi_n \in \mathsf{K} \implies \neg(\varphi_1 \wedge \dots \wedge \varphi_n) \divideontimes \neg\varphi_1 \vee \dots \vee \neg\varphi_n\,,$$

(8.14) $$\varphi_1 \vee \dots \vee \varphi_n \in \mathsf{K} \implies \neg(\varphi_1 \vee \dots \vee \varphi_n) \divideontimes \neg\varphi_1 \wedge \dots \wedge \neg\varphi_n\,.$$

For (8.13): by induction on n. For $n = 1$, by (8.1), we have $\neg\varphi_1 \succ \neg\varphi_1$. For $n > 1$, by (a5), it holds that $\neg((\varphi_1 \wedge \dots \wedge \varphi_{n-1}) \wedge \varphi_n) \divideontimes \neg(\varphi_1 \wedge \dots \wedge \varphi_{n-1}) \vee \neg\varphi_n$. Assume inductively that the condition holds for $n-1$. Thus in case if $\varphi_1 \wedge \dots \wedge \varphi_{n-1} \in \mathsf{K}$, by inductive hypothesis, we get $\neg(\varphi_1 \wedge \dots \wedge \varphi_{n-1}) \divideontimes \neg\varphi_1 \vee \dots \vee \neg\varphi_{n-1}$. Hence, by (a10) and (a12), we get the thesis. In case if $\varphi_1 \wedge \dots \wedge \varphi_{n-1} \in \mathsf{T}$, we have $\varphi_1, \dots, \varphi_{n-1} \in \mathsf{T}$ and $\varphi_n \in \mathsf{K}$. Hence, by (8.4) and (a3), we have $(\varphi_1 \wedge \dots \wedge \varphi_{n-1}) \wedge \varphi_n \divideontimes \varphi_n$. Now, by (a12), we get $\neg(\varphi_1 \wedge \dots \wedge \varphi_n) \divideontimes \neg\varphi_n$. Moreover, $\neg\varphi_1 \vee \dots \vee \neg\varphi_{n-1} \in \mathsf{F}$ and $\neg\varphi_n \in \mathsf{K}$. Hence, by (a8) and (8.8), we have $(\neg\varphi_1 \vee \dots \vee \neg\varphi_{n-1}) \vee \neg\varphi_n \divideontimes \neg\varphi_n$. Finally, we use (a10).

The above reasoning can be carried out for an arbitrary combination of brackets in a given conjunction.

For (8.14): analogously as for (8.13). Instead of (a5) we apply (8.9).

9. Auxiliary properties of the relation $\succ_\circ$

Obviously, the relation $\succ_\circ$ has properties from Section 8. Moreover, we will prove that this relation has some additional properties that are indispensable while proving the completeness theorem 7.3.

For a start let us remark that since $\succ_\circ \subseteq \models_{\mathrm{i}}$, so by (5.1) and (5.2),

(9.1) $$\varphi \succ_\circ \psi \implies \varphi \models \psi \;\;\&\;\; \varphi, \psi \in \mathsf{K}\,.$$

From axioms (a1), (a3), (a10) and (a12), and from (9.1) and (8.8) it follows that

(9.2) $$\varphi \divideontimes_\circ \psi \Rightarrow \neg\varphi \divideontimes_\circ \neg\psi\,,$$

(9.3) $$\varphi \vee \chi \notin \mathsf{T} \implies \varphi \divideontimes_\circ \psi \Rightarrow \varphi \vee \chi \divideontimes_\circ \psi \vee \chi\,,$$

(9.4) $$\varphi_1 \vee \psi_2 \notin \mathsf{T} \implies \varphi_1 \divideontimes_\circ \varphi_2 \;\&\; \psi_1 \divideontimes \psi_2 \Rightarrow \varphi_1 \vee \psi_1 \divideontimes_\circ \varphi_2 \vee \psi_2\,,$$

(9.5) $$\varphi \wedge \chi \notin \mathsf{F} \implies \varphi \divideontimes_\circ \psi \Rightarrow \varphi \wedge \chi \divideontimes_\circ \psi \wedge \chi\,,$$

(9.6) $$\varphi_1 \wedge \psi_1 \notin \mathsf{F} \implies \varphi_1 \divideontimes_\circ \varphi_2 \;\&\; \psi_1 \divideontimes_\circ \psi_2 \Rightarrow \varphi_1 \wedge \psi_1 \divideontimes_\circ \varphi_2 \wedge \psi_2\,.$$

The conditions (9.4) and (9.6) can be easily generalized in an inductive way:

(9.7) $$\varphi_1 \vee \dots \vee \varphi_n \notin \mathsf{T} \implies \varphi_1 \divideontimes_\circ \psi_1 \;\&\dots\&\; \varphi_n \divideontimes_\circ \psi_n \Rightarrow \varphi_1 \vee \dots \vee \varphi_n \divideontimes_\circ \psi_1 \vee \dots \vee \psi_n\,,$$

(9.8) $$\varphi_1 \wedge \dots \wedge \varphi_n \notin \mathsf{F} \implies \varphi_1 \divideontimes_\circ \psi_1 \;\&\dots\&\; \varphi_n \divideontimes_\circ \psi_n \Rightarrow \varphi_1 \wedge \dots \wedge \varphi_n \divideontimes_\circ \psi_1 \wedge \dots \wedge \psi_n\,.$$

Now we will prove a couple of lemmas that will be necessary to prove completeness.

Lemma 9.1. *For all* $\kappa \in \mathsf{ek}$ *there is such* $\varphi \in \mathsf{AN}$ *that* $\mathsf{V}(\kappa) = \mathsf{V}(\varphi)$ *and* $\neg\kappa \divideontimes_{\circ} \varphi$.

Proof. By (8.13), (a1), (a10), (9.3) and the fact $\mathsf{ek} \subsetneq \mathsf{K}$. □

Lemma 9.2. *Let* $\varphi_1, \ldots, \varphi_n \in \mathsf{AN}$ *for* $n > 0$. *Let* $\varphi_1 \wedge \cdots \wedge \varphi_n \in \mathsf{K}$. *Then there is such* $\psi \in \mathsf{AN}$ *that* $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi_1 \wedge \cdots \wedge \varphi_n)$ *and* $(\varphi_1 \wedge \cdots \wedge \varphi_n) \divideontimes_{\circ} \psi$.

Proof. Induction on n. (I) For $n = 1$: by (8.1) we set that $\psi = \varphi_1$.

(II) For $n = 2$: assume that $\varphi_1 = \kappa_1 \vee \cdots \vee \kappa_m$, $\varphi_2 = \lambda_1 \vee \cdots \vee \lambda_l$, where $m, l > 0$ and $\kappa_i, \lambda_i \in \mathsf{ek} \subsetneq \mathsf{K}$. We will consider three cases:

(i) Let $l = 1 = m$. Then by hypothesis $\kappa_1 \wedge \lambda_1 \in \mathsf{ek} \subsetneq \mathsf{AN}$. Hence, by (8.1), we can set $\psi = \kappa_1 \wedge \lambda_1$.

(ii) Let $m+l = k > 1$ and $m > 1$. Then, by (a6), we get $(\kappa_1 \vee (\kappa_2 \vee \cdots \vee \kappa_m)) \wedge \varphi_2 \divideontimes_{\circ} (\kappa_1 \wedge \varphi_2) \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2)$.

Assume that for $n = 2$ the lemma is true for all m and l such that $m + l < k$. By hypothesis and (9.1) one of the following three cases holds:

(a) $\kappa_1 \wedge \varphi_2, (\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2 \in \mathsf{K}$. By inductive hypothesis, there are such $\psi_1, \psi_2 \in \mathsf{AN}$ that $\kappa_1 \wedge \varphi_2 \divideontimes_{\circ} \psi_1$ and $(\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2 \divideontimes_{\circ} \psi_2$, and $\mathsf{V}(\psi_1) \subseteq \mathsf{V}(\kappa_1 \wedge \varphi_2)$ and $\mathsf{V}(\psi_2) \subseteq \mathsf{V}((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2)$. By hypothesis and (9.1), we can apply (9.4) to get $(\kappa_1 \wedge \varphi_2) \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2) \divideontimes_{\circ} \psi_1 \vee \psi_2$. Thus for $\psi = \psi_1 \vee \psi_2 \in \mathsf{AN}$, by (a10), we have $\varphi_1 \wedge \varphi_2 \divideontimes_{\circ} \psi$. Moreover, $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi_1 \wedge \varphi_2)$.

(b) $\kappa_1 \wedge \varphi_2 \in \mathsf{K}$ and $(\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2 \in \mathsf{F}$. By inductive hypothesis, there is such $\psi \in \mathsf{AN}$ that $\kappa_1 \wedge \varphi_2 \divideontimes_{\circ} \psi$ and $\mathsf{V}(\psi) \subseteq \mathsf{V}(\kappa_1 \wedge \varphi_2)$. By (9.3), $(\kappa_1 \wedge \varphi_2) \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2) \divideontimes_{\circ} \psi \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2)$, from (a8) we get $\psi \vee ((\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2) \divideontimes_{\circ} \psi$. Hence, by (a10), we get $\varphi_1 \wedge \varphi_2 \divideontimes_{\circ} \psi$. Moreover, $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi_1 \wedge \varphi_2)$.

(c) $\kappa_1 \wedge \varphi_2 \in \mathsf{F}$ and $(\kappa_2 \vee \cdots \vee \kappa_m) \wedge \varphi_2 \in \mathsf{K}$. Analogously to (b).

(iii) Let $m + l > 2$ and $l > 1$. Analogously like (ii).

(III) For $n > 2$: assume that the lemma in question is true for all $m < n$. Consider two cases:

(i) $\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \in \mathsf{K}$. Then, by inductive hypothesis, there is such $\psi' \in \mathsf{AN}$ that $\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \divideontimes_{\circ} \psi'$ and $\mathsf{V}(\psi') \subseteq \mathsf{V}(\varphi_1 \wedge \cdots \wedge \varphi_{n-1})$. Hence, by hypothesis and (9.5), we have $\varphi_1 \wedge \cdots \wedge \varphi_n \divideontimes_{\circ} \psi' \wedge \varphi_n$. By assumption and from (9.1), $\psi' \wedge \varphi_n \in \mathsf{K}$. Therefore, by inductive hypothesis, there is such $\psi \in \mathsf{AN}$ that $\psi' \wedge \varphi_n \divideontimes_{\circ} \psi$ and $\mathsf{V}(\psi) \subseteq \mathsf{V}(\psi' \wedge \varphi_n) \subseteq \mathsf{V}(\varphi_1 \wedge \cdots \wedge \varphi_n)$. By (a10) we get $\varphi_1 \wedge \cdots \wedge \varphi_n \divideontimes_{\circ} \psi$.

(ii) $\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \notin \mathsf{K}$. Then $\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \in \mathsf{T}$ and $\varphi_n \in \mathsf{K}$. Hence, by (8.4) and (a3), we get $\varphi_1 \wedge \cdots \wedge \varphi_n \divideontimes_{\circ} \varphi_n$.

We repeat considerations from (III) for an arbitrary combination of brackets. □

Lemma 9.3. *For every* $\varphi \in \mathsf{AN} \cap \mathsf{K}$ *there is such* $\psi \in \mathsf{AN} \cap \mathsf{K}$ *that* $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi)$ *and* $\neg\varphi \divideontimes_\circ \psi$.

Proof. Assume that $\varphi = \kappa_1 \vee \cdots \vee \kappa_n$, where $n \leqslant 1$ and $\kappa_i \in \mathsf{ek} \subsetneq \mathsf{K}$ for $i = 1, \ldots, n$. By (8.14) we get that $\neg(\kappa_1 \vee \cdots \vee \kappa_n) \divideontimes_\circ \neg\kappa_1 \wedge \cdots \wedge \neg\kappa_n$.

By Lemma 9.1 there are such $\varphi_1, \ldots, \varphi_n \in \mathsf{AN}$ that for $i = 1, \ldots, n$ we have $\mathsf{V}(\kappa_i) = \mathsf{V}(\varphi_i)$ and $\neg\kappa_i \divideontimes_\circ \varphi_i$.

Let us notice that since $\varphi, \neg\kappa_i \in \mathsf{K}$ for $1 \leqslant i \leqslant n$, so for every $m \leqslant n$ we have $\neg\kappa_1 \wedge \cdots \wedge \neg\kappa_m \in \mathsf{K}$. Hence by induction, applying (9.6), we can show that $\neg\kappa_1 \wedge \cdots \wedge \neg\kappa_n \divideontimes_\circ \varphi_1 \wedge \cdots \wedge \varphi_n$. By Lemma 9.2 there is such $\psi \in \mathsf{AN}$ that $\mathsf{V}(\psi) \subseteq \mathsf{V}(\varphi_1 \wedge \cdots \wedge \varphi_n)$ and $\varphi_1 \wedge \cdots \wedge \varphi_n \divideontimes_\circ \psi$. Thus, using (a10), we get $\neg\varphi \divideontimes_\circ \psi$. □

Lemma 9.4. *For every* $\varphi \in \mathsf{K}$ *there is such* $\varphi^{\mathrm{a}} \in \mathsf{AN}$ *that* $\mathsf{V}(\varphi^{\mathrm{a}}) \subseteq \mathsf{V}(\varphi)$ *and* $\varphi \divideontimes_\circ \varphi^{\mathrm{a}}$.

Proof. Induction on the construction of the formula φ.

(I) $\varphi \in \mathsf{V}$. Then $\varphi \in \mathsf{AN}$ and $\varphi \divideontimes_\circ \varphi$, by (8.1).

(II) $\varphi = \neg\psi$. Then $\psi \in \mathsf{K}$. As inductive hypothesis, let us assume that for ψ the theorem holds, i.e., there is such $\psi^{\mathrm{a}} \in \mathsf{AN}$ that $\mathsf{V}(\psi^{\mathrm{a}}) \subseteq \mathsf{V}(\psi)$ and $\psi \divideontimes_\circ \psi^{\mathrm{a}}$. Thus, using (9.2), we get $\neg\psi \divideontimes_\circ \neg\psi^{\mathrm{a}}$. From Lemma 9.3 we get such $\psi \in \mathsf{AN}$ that $\mathsf{V}(\psi) \subseteq \mathsf{V}(\psi^{\mathrm{a}})$ and $\neg\psi^{\mathrm{a}} \divideontimes_\circ \psi$. Hence, by (a10), we get $\varphi \divideontimes_\circ \psi$.

(III) $\varphi = \psi \vee \chi$. Consider three cases. (i) $\psi, \chi \in \mathsf{K}$. By inductive hypothesis, there are such $\psi^{\mathrm{a}}, \chi^{\mathrm{a}} \in \mathsf{AN}$ that $\mathsf{V}(\psi^{\mathrm{a}}) \subseteq \mathsf{V}(\psi)$ and $\mathsf{V}(\chi^{\mathrm{a}}) \subseteq \mathsf{V}(\chi)$, and $\psi \divideontimes_\circ \psi^{\mathrm{a}}$ and $\chi \divideontimes_\circ \chi^{\mathrm{a}}$. By (9.4), we have $\varphi \divideontimes_\circ \psi^{\mathrm{a}} \vee \chi^{\mathrm{a}}$. Clearly, $\psi^{\mathrm{a}} \vee \chi^{\mathrm{a}} \in \mathsf{AN}$ and $\mathsf{V}(\psi^{\mathrm{a}} \vee \chi^{\mathrm{a}}) \subseteq \mathsf{V}(\varphi)$. (ii) $\psi \in \mathsf{K}$ and $\chi \in \mathsf{F}$. Then by (a8), we have $\varphi \divideontimes_\circ \psi$. By inductive hypothesis, there is such, $\psi^{\mathrm{a}} \in \mathsf{AN}$ that $\mathsf{V}(\psi^{\mathrm{a}}) \subseteq \mathsf{V}(\psi)$ and $\psi \divideontimes_\circ \psi^{\mathrm{a}}$. By (a10), we have $\varphi \divideontimes_\circ \psi^{\mathrm{a}}$. (iii) $\psi \in \mathsf{F}$ and $\chi \in \mathsf{K}$. Analogously as for (ii).

(IV) $\varphi = (\psi \wedge \chi)$. Consider three cases. (i) $\psi, \chi \in \mathsf{K}$. By inductive hypothesis, there are such, $\psi^{\mathrm{a}}, \chi^{\mathrm{a}} \in \mathsf{AN}$ that $\mathsf{V}(\psi^{\mathrm{a}}) \subseteq \mathsf{V}(\psi)$ and $\mathsf{V}(\chi^{\mathrm{a}}) \subseteq \mathsf{V}(\chi)$, and $\psi \divideontimes_\circ \psi^{\mathrm{a}}$ and $\chi \divideontimes_\circ \chi^{\mathrm{a}}$. By (9.6), we have $\varphi \divideontimes_\circ \psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}}$. Clearly, $\mathsf{V}(\psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}}) \subseteq \mathsf{V}(\varphi)$. By (9.1) and Lemma 9.3, there is such $\psi \in \mathsf{AN}$ that $\mathsf{V}(\psi) \subseteq \mathsf{V}(\chi^{\mathrm{a}} \wedge \psi^{\mathrm{a}})$ and $\psi^{\mathrm{a}} \wedge \chi^{\mathrm{a}} \divideontimes_\circ \psi$. Thus from (a10) we get $\varphi \divideontimes_\circ \psi$. (ii) $\psi \in \mathsf{K}$ and $\chi \in \mathsf{T}$. Then by (8.4), we have $\varphi \divideontimes_\circ \psi$. By inductive hypothesis, there is such, $\psi^{\mathrm{a}} \in \mathsf{AN}$ that $\mathsf{V}(\psi^{\mathrm{a}}) \subseteq \mathsf{V}(\psi)$ and $\psi \divideontimes_\circ \psi^{\mathrm{a}}$. From (a10) we have $\varphi \divideontimes_\circ \psi^{\mathrm{a}}$. (iii) $\psi \in \mathsf{T}$ and $\chi \in \mathsf{K}$, analogously as (ii). □

Lemma 9.5. *Let* $i_1 < i_2 < \cdots < i_n$ *and for* $\kappa \in \mathsf{ek}$ *let* $\mathsf{V}(\kappa) = \{p_{j_1}, p_{j_2}, \ldots, p_{j_m}\} \subseteq \{p_{i_1}, p_{i_2}, \ldots, p_{i_n}\}$, *where* $j_1 < j_2 < \cdots < j_m$. *Then there are different* $\kappa_1, \ldots, \kappa_{2^{n-m}} \in \mathsf{ek}$ *such that* $\kappa \divideontimes_\circ \kappa_1 \vee \cdots \vee \kappa_{2^{n-m}}$ *and* $\mathsf{V}(\kappa_i) = \{p_{i_1}, p_{i_2}, \ldots, p_{i_n}\}$.

Proof. Let $k_1 < k_2 < \cdots < k_{n-m}$ and $\{k_1, k_2, \ldots, k_{n-m}\} = \{i_1, i_2, \ldots, i_n\} \setminus \{j_1, j_2, \ldots, j_m\}$. By (8.4), (a3), (a6) (a4), (9.6) and (9.4), we have $\kappa \divideontimes_\circ \kappa \wedge (p_{k_1} \vee \neg p_{k_1}) \divideontimes_\circ$

$(\kappa\wedge p_{k_1})\vee(\kappa\wedge\neg p_{k_1}) \Join_\circ \kappa_1^1\vee\kappa_1^0$, where κ_1^1 and κ_1^0 are different respectively from $\kappa\wedge p_{k_1}$ and $\kappa\wedge\neg p_{k_1}$ in this, that their elements are ordered according to increasing indexes of variables. In a second step, for a variable p_{k_2} we analogously get $\kappa_1^1 \Join_\circ \kappa_{12}^{11}\vee\kappa_{12}^{10}$ and $\kappa_1^0 \Join_\circ \kappa_{12}^{11}\vee\kappa_{12}^{10}$. Hence, by (9.4) and (a10), we have $\kappa \Join_\circ \kappa_{12}^{11}\vee\kappa_{12}^{10}\vee\kappa_{12}^{01}\vee\kappa_{12}^{00}$. These steps are repeated for $n-m$ and we get $\kappa \Join_\circ \kappa_{12\ldots n-m}^{11\ldots 1}\vee\kappa_{12\ldots n-m}^{11\ldots 0}\vee\cdots\vee\kappa_{12\ldots n-m}^{00\ldots 0}$. Finally we have 2^{n-m} of different elements . □

LEMMA 9.6. *If* $\varphi\in\mathsf{K}$, *then* $\varphi \Join_\circ \varphi^\circ$.

PROOF. For $\varphi\in\mathsf{K}$ let $\mathsf{V}(\varphi)=\{p_{i_1},p_{i_2},\ldots,p_{i_n}\}$, where $i_1<i_2<\cdots<i_n$.

By Lemma 9.4, there is such $\varphi^{\mathrm{a}}\in\mathsf{AN}$ that $\varphi \Join_\circ \varphi^{\mathrm{a}}$ and $\mathsf{V}(\varphi^{\mathrm{a}})\subseteq\mathsf{V}(\varphi)$. Thus $\varphi^{\mathrm{a}}=\kappa_1\vee\cdots\vee\kappa_m$, where $m>0$ and $\kappa_1,\ldots,\kappa_m\in\mathsf{ek}$. By (a3), (a4), (8.3), (9.5) and (a10), we can show that $\kappa_i \Join_\circ \kappa_i'$, where for $i\leqslant m$ we have $\mathsf{V}(\kappa_i')=\mathsf{V}(\kappa_i)$ and κ_i' are different from κ_i, only in this, that no element of a conjunction κ_i' repeats and all elements are ordered according to an increasing indexes of variables. Hence, by (9.7), we have $\varphi^{\mathrm{a}} \Join_\circ \kappa_1'\vee\cdots\vee\kappa_m'$.

Now, by Lemma 9.5, for every $i\leqslant m$ there is such $\varphi_i^{\mathrm{a}}\in\mathsf{AN}$ that $\mathsf{V}(\varphi_i^{\mathrm{a}})=\mathsf{V}(\varphi)$ and $\kappa_i' \Join_\circ \varphi_i^{\mathrm{a}}$. Hence, by (9.7), we have $\varphi \Join_\circ \varphi_1^{\mathrm{a}}\vee\cdots\vee\varphi_m^{\mathrm{a}}$. Next, by (8.10), (8.6) and (8.8), we can delete recurrent elementary conjunctions in a disjunction $\varphi_1^{\mathrm{a}}\vee\cdots\vee\varphi_k^{\mathrm{a}}$ and order it such that $\varphi \Join_\circ \varphi^{\mathrm{a}} \Join_\circ \kappa_1^{k_1}\vee\kappa_2^{k_2}\vee\cdots\vee\kappa_l^{k_l}$, where $k_1<k_2<\cdots<k_l$ and for $j\leqslant l$ number k_j is equal to the sum $\sum_{i=1}^n b_i^{k_j}\cdot 2^{n-i}$, in which $b_i^{k_j}=0$, if in $\kappa_j^{k_j}$ there is $\neg p_i$ and $b_i^{k_j}=1$, if in $\kappa_j^{k_j}$ there is p_i.

It remains to prove that $\varphi^\circ=\kappa_1^{k_1}\vee\cdots\vee\kappa_l^{k_l}$. By (9.1), $\varphi \mathrel{\text{⫤⊨}} \kappa_1^{k_1}\vee\cdots\vee\kappa_l^{k_l}$. Set an arbitrary 0-1 sequence $\langle b_1,\ldots,b_n\rangle$. It satisfies a formula φ iff it satisfies disjunction $\kappa_1^{k_1}\vee\cdots\vee\kappa_l^{k_l}$, i.e., for some $j\leqslant l$ the sequence satisfies conjunction $\kappa_j^{k_j}$. The last condition is equivalent to the fact that $b_i^{k_j}=b_i$ for all $i\leqslant n$. Hence it follows— from the one side—that in disjunction $\kappa_1^{k_1}\vee\cdots\vee\kappa_l^{k_l}$ there are all elementary conjunctions determined by sequences satisfying a formula φ—from the other—that only such conjunctions. □

LEMMA 9.7. *If* $\varphi\in\mathsf{K}$, *then* $\varphi^\circ \Join_\circ \varphi^\bullet$.

PROOF. If $\mathsf{V}(\varphi)=\mathsf{V}_{\mathrm{e}}(\varphi)$, then $\varphi^\circ=\varphi^\bullet$, so we get the thesis from (8.1).

Let $\mathsf{V}(\varphi)=\{p_{i_1},p_{i_2},\ldots,p_{i_n}\}$ for $i_1<i_2<\cdots<i_n$ and $p_{i_k}\notin\mathsf{V}_{\mathrm{e}}(\varphi)$ for some $k\leqslant n$. Thus for every evaluation $e\colon\mathsf{V}\to\{0,1\}$ we have $h^e(\varphi)=h^{e^-_{p_{i_k}}}(\varphi)$. Hence it follows that for every $\langle b_1,\ldots,b_n\rangle\in\{0,1\}^n$: $p_{i_1}^{b_1}\wedge\cdots\wedge p_{i_k}^{b_k}\wedge\cdots\wedge p_{i_n}^{b_n}$ occurs in φ° iff $p_{i_1}^{b_1}\wedge\cdots\wedge p_{i_k}^{1-b_k}\wedge\cdots\wedge p_{i_n}^{b_n}$ occurs in φ°. Therefore, either both conjunction occur in φ° or neither of them. Moreover, by (a3), (a4), (a6) and (8.4), we have

$(p_{i_1}^{b_1} \wedge \cdots \wedge p_{i_k}^{b_k} \wedge \cdots \wedge p_{i_n}^{b_n}) \vee (p_{i_1}^{b_1} \wedge \cdots \wedge p_{i_k}^{1-b_k} \wedge \cdots \wedge p_{i_n}^{b_n}) \asymp_\circ (p_{i_1}^{b_1} \wedge \cdots \wedge p_{i_{k-1}}^{b_{k-1}} \wedge p_{i_{k+1}}^{b_{k+1}} \wedge \cdots \wedge p_{i_n}^{b_n}) \wedge (p_{i_k}^{b_k} \vee p_{i_k}^{1-b_k}) \asymp_\circ (p_{i_1}^{b_1} \wedge \cdots \wedge p_{i_{k-1}}^{b_{k-1}} \wedge p_{i_{k+1}}^{b_{k+1}} \wedge \cdots \wedge p_{i_n}^{b_n})$.
By the above facts and (8.8) together with (9.7) we can «eliminate» from φ° a variable p_{i_k}, i.e., $\varphi^\circ \asymp_\circ \psi_1$, where ψ_1 is different from the restriction of φ° to the set $\{p_{i_k}, \dots, p_{i_{k-1}}, p_{i_{k+1}}, \dots, p_{i_n}\}$, in this, that it does not include recurring elements. Subsequently, in the same way, we eliminate the second inessential variable φ in ψ_1, getting such ψ_2 that $\varphi^\circ \asymp_\circ \psi_1 \asymp_\circ \psi_2$ (we now apply the above facts to ψ_1). In an analogous way we eliminate all inessential variables in φ, getting $\varphi^\circ \asymp_\circ \varphi^\bullet$, by (8.8) and (a10). □

Lemma 9.8. *Let $\varphi \in \mathsf{K}$ and let V be an arbitrary nonempty set included in $\mathsf{V}_\mathsf{e}(\varphi)$ such that $\varphi^\bullet|_V \notin \mathsf{T}$. Then $\varphi^\bullet \succ_\circ \varphi^\bullet|_V$.*

Proof. Let $\varphi^\bullet = \kappa_1 \vee \cdots \vee \kappa_n$. Proof by induction on a number of variables in the set $\mathsf{V}(\varphi^\bullet) \setminus V$. If it is empty, then $\varphi^\bullet = \varphi^\bullet|_V$.

(I) Let p_k be the only element of the set $\mathsf{V}(\varphi^\bullet) \setminus V$. Consider two possible cases:

(i) For some $b \in \{0, 1\}$ a formula p_k^b occurs in every κ_i for $i \leqslant n$. Then, by the equations proved so far, for every $i \leqslant n$ we have $\kappa_i \asymp_\circ \kappa_i' \wedge p_k^b$, where κ_i' is a κ_i with p_k^b deleted (i.e. we move p_k^b to the last place in κ_i). Since all assumptions in the antecedent of the implication (8.12) are satisfied, then we get $\varphi^\bullet \asymp_\circ (\kappa_1' \vee \cdots \vee \kappa_n') \wedge p_k^b$. Since $\kappa_1' \vee \cdots \vee \kappa_n' = \varphi^\bullet|_V$ and $\mathsf{v} \neq \mathbf{I}(\varphi^\bullet|_V) \sqsubseteq \mathbf{I}((\kappa_1' \vee \cdots \vee \kappa_n') \wedge p_k^b)$, thus we can apply (a2) and (a10) acquiring $\varphi^\bullet \succ_\circ \varphi^\bullet|_V$.

(ii) p_k occurs in $\kappa_{i_1}, \kappa_{i_2}, \dots, \kappa_{i_m}$, where $0 < m < n$ and $i_1 < i_2 < \cdots < i_m \leqslant n$, and $\neg p_k$ occurs $\kappa_{j_1}, \kappa_{j_2}, \dots, \kappa_{j_{n-m}}$, where $j_1 < j_2 < \cdots < j_{n-m} \leqslant n$. Clearly, $\varphi^\bullet \asymp_\circ (\kappa_{i_1} \vee \cdots \vee \kappa_{i_m}) \vee (\kappa_{j_1} \vee \cdots \vee \kappa_{j_{n-m}})$, and both elements of the second of disjunctions are in K. Therefore the antecedent of the implication is satisfied (8.12), thus, similarly to (i), by (9.4), we get $\varphi^\bullet \asymp_\circ ((\kappa_{i_1}' \vee \cdots \vee \kappa_{i_m}') \wedge p_k) \vee ((\kappa_{j_1}' \vee \cdots \vee \kappa_{j_{n-m}}') \wedge \neg p_k)$. By (8.11) we have $\varphi^\bullet \asymp_\circ \big((\kappa_{i_1}' \vee \cdots \vee \kappa_{i_m}') \vee ((\kappa_{j_1}' \vee \cdots \vee \kappa_{j_{n-m}}') \wedge \neg p_k)\big) \wedge \big(p_k \vee ((\kappa_{j_1}' \vee \cdots \vee \kappa_{j_{n-m}}') \wedge \neg p_k)\big)$. Again applying (8.11) together with (8.8) and (9.6), we get $\varphi^\bullet \asymp_\circ (\kappa_{i_1}' \vee \cdots \vee \kappa_{i_m}' \vee \kappa_{j_1}' \vee \cdots \vee \kappa_{j_{n-m}}') \wedge (\kappa_{i_1}' \vee \cdots \vee \kappa_{i_m}' \vee \neg p_k) \wedge (p_k \vee \kappa_{j_1}' \vee \cdots \vee \kappa_{j_{n-m}}') \wedge (p_k \vee \neg p_k)$. (The transformations were permitted since by the assumptions the whole formula and respective subformulas are in K.) Disjunction $\kappa_{i_1}' \vee \cdots \vee \kappa_{i_m}' \vee \kappa_{j_1}' \vee \cdots \vee \kappa_{j_{n-m}}'$ differs from $\varphi^\bullet|_V$ at the most in the order of elements. Since the antecedent in the implication (a2) is satisfied then, applying (8.8) we get $\varphi^\bullet \succ_\circ \varphi^\bullet|_V$.

(II) Applying inductive hypothesis, similarly to (I), we will prove the theorem for an arbitrary (finite) number of elements of the set $\mathsf{V}(\varphi^\bullet) \setminus V$. □

10. Proof of the fact: $\models_{\mathrm{i}} \subseteq \succ_\circ$

Applying (4.2) and lemmas from Section 9, we can prove that $\models_{\mathrm{i}} \subseteq \succ_\circ$.

Let $\varphi \models_{\mathrm{i}} \psi$. By (5.2), we have $\varphi, \psi \in \mathsf{K}$, and by (5.3), we get $\emptyset \neq \mathsf{V}_{\mathrm{e}}(\psi) \subseteq \mathsf{V}_{\mathrm{e}}(\varphi)$. Moreover, by (4.2), $\psi^\bullet \;\mathrel{\text{⫤⊨}}\; \varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$.

Now, by lemmas 9.6, 9.7 and 9.8, we have $\varphi \mathrel{\divideontimes_\circ} \varphi^\circ \mathrel{\divideontimes_\circ} \varphi^\bullet \succ \varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$ and $\psi^\bullet \mathrel{\divideontimes_\circ} \psi^\circ \mathrel{\divideontimes_\circ} \psi$.

Since $\psi^\bullet \;\mathrel{\text{⫤⊨}}\; \varphi^\bullet|_{\mathsf{V}_{\mathrm{e}}(\psi)}$, so $\varphi^\bullet|_{\mathsf{V}(\psi^\bullet)}$ can differ from $\psi^\bullet$ at the most in the fact that it contains recurring elements. Hence, by (8.6) and some other equations, we get $\varphi^\bullet|_{\mathsf{V}(\psi^\bullet)} \mathrel{\divideontimes_\circ} \psi^\bullet$. Thus, from (a10), we get $\varphi \succ_\circ \psi$.

11. A sequent calculus for the relation $\models_{\mathrm{i}}$

Let $\{\ulcorner \varphi \vdash \psi \urcorner : \varphi, \psi \in \mathsf{L}\}$ be a set of sequents. The sign '$\vdash$' does not mark binary relations on L. The sequent $\ulcorner \varphi \vdash \psi \urcorner$ is a «new formula» that renders the argument with assumption φ and claim ψ. The formula φ is called the *antecedent* and ψ is called the *succedent* of the sequent $\ulcorner \varphi \vdash \psi \urcorner$. A sequent $\ulcorner \varphi \vdash \psi \urcorner$ is called *correct* iff $\varphi \models_{\mathrm{i}} \psi$.

Let us build, in the set of all sequents, the calculus $\mathbf{C}^{\mathbf{i}}$ that will satisfy Theorem on the Adequacy 11.2: a sequent $\ulcorner \varphi \vdash \psi \urcorner$ is a thesis of $\mathbf{C}^{\mathbf{i}}$ iff $\varphi \models_{\mathrm{i}} \psi$, i.e., $\ulcorner \varphi \vdash \psi \urcorner$ is correct.[17]

Given sequent is an *axiom* of $\mathbf{C}^{\mathbf{i}}$ iff it satisfies the following two conditions:

(i) neither the antecedent of this sequent is a contradiction nor its succedent is a tautology;

(ii) the sequent has one of the following fifteen forms:

(A1) $\varphi \vdash \neg\neg\varphi$

(A2) $\neg\neg\varphi \vdash \varphi$

(A3) $\varphi \wedge \psi \vdash \varphi$ if $\mathbf{I}(\varphi) \sqsubseteq \mathbf{I}(\varphi \wedge \psi)$

(A4) $\varphi \wedge \psi \vdash \psi \wedge \varphi$

(A5) $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$

(A6) $\varphi \wedge (\psi \wedge \chi) \vdash (\varphi \wedge \psi) \wedge \chi$

(A7) $\neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$

[17] In (Wessel, 1984) in the set of all sequents Wessel gave an axiomatization for the relation $\models^{**}$ creating the calculus of *strict logical consequence* $\mathbf{F^s}$. Yet—as it has been proved in (Pietruszczak, 2004)—his axiomatization is too weak for $\models^{**}$. In (Pietruszczak, 2004) one can find the calculus $\mathbf{VF^s}$ for $\models^{**}$ that is complete. $\mathbf{VF^s}$ is an «extension to completeness» of the calculus $\mathbf{F^s}$, i.e., for all $\varphi, \psi \in \mathsf{L}$: $\varphi \models^{**} \psi$ iff the sequent $\ulcorner \varphi \vdash \psi \urcorner$ is a thesis of $\mathbf{VF^s}$.

(A8) $\neg\varphi \vee \neg\psi \vdash \neg(\varphi \wedge \psi)$

(A9) $(\varphi \vee \psi) \wedge \chi \vdash (\varphi \wedge \chi) \vee (\psi \wedge \chi)$

(A10) $(\varphi \wedge \chi) \vee (\psi \wedge \chi) \vdash (\varphi \vee \psi) \wedge \chi$

(A11) $\varphi \vdash \varphi \wedge \tau$

(A12) $\varphi \vdash \varphi \vee \phi$

(A13) $\varphi \vee \phi \vdash \varphi$

(A14) $\varphi \vdash \phi \vee \varphi$

(A15) $\phi \vee \varphi \vdash \varphi$

where $\tau \in \mathsf{T}$ and $\phi \in \mathsf{F}$.

LEMMA 11.1. *All axioms of the calculus* $\mathbf{C^i}$ *are correct sequents.*

PROOF. For the axiom (A3), by assumptions: $\varphi \wedge \psi \notin \mathsf{F}$, $\varphi \notin \mathsf{T}$ and $\mathbf{I}(\varphi) \sqsubseteq \mathbf{I}(\varphi \wedge \psi)$. So $\varphi \wedge \psi \models_{\mathrm{i}} \varphi$. For the others axioms: antecedents (A) and succedents (S) are members of K, and $A \dashv\models S$. Hence $A \dashv\models_{\mathrm{i}} S$, by (5.4). □

Moreover, the calculus $\mathbf{C^i}$ has three *rules of inference*:

(R1) $$\frac{\varphi \vdash \psi \qquad \psi \vdash \chi}{\varphi \vdash \chi}$$

(R2) $$\frac{\chi \vdash \varphi \qquad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \text{if } \mathbf{I}(\varphi \wedge \psi) \sqsubseteq \mathbf{I}(\chi)$$

(R3) $$\frac{\varphi \vdash \psi \qquad \psi \vdash \varphi}{\chi \vdash \chi(\varphi/\psi)} \quad \text{if } \chi \notin \mathsf{F} \text{ and } \chi(\varphi/\psi) \notin \mathsf{T}$$

These rules are *correct* in the following sense: when applied to correct sequents they yield a correct sequent.

LEMMA 11.2. *Three rules of inference of the calculus* $\mathbf{C^i}$ *are correct.*

PROOF. (R1): the relation $\models_{\mathrm{i}}$ is transitive. (R2): if $\chi \models_{\mathrm{i}} \varphi$ and $\chi \models_{\mathrm{i}} \psi$, then $\chi \notin \mathsf{F}$ and $\varphi, \psi \notin \mathsf{T}$. So $\varphi \wedge \psi \notin \mathsf{T}$ and $\chi \models_{\mathrm{i}} \varphi \wedge \psi$, by the additional assumption. (R3): if $\varphi \models_{\mathrm{i}} \psi$ and $\psi \models_{\mathrm{i}} \varphi$, then $\varphi, \psi \in \mathsf{K}$ and $\varphi \dashv\models \psi$. Thus $\chi \dashv\models \chi(\varphi/\psi)$ and $\chi, \chi(\varphi/\psi) \in \mathsf{K}$, by the additional assumption. Hence $\chi \dashv\models_{\mathrm{i}} \chi(\varphi/\psi)$. □

We say that a sequent is a *thesis* of calculus $\mathbf{C^i}$ iff there is its derivation, i.e., it is derivable in finite number of steps from the axioms by the rules of inference.

From lemmas 11.1 and 11.2 we obtain:

THEOREM ON THE CORRECTNESS 11.1. *All theses of* $\mathbf{C^i}$ *are correct sequents.*

PROOF. As we showed, all axioms of $\mathbf{C^i}$ are correct sequent. Moreover, all rules of $\mathbf{C^i}$ always lead from correct sequents to correct sequents. Thus, by induction over $\mathbf{C^i}$, we see that every derivable sequent is correct. □

From Remark 7.1 it follows that:

LEMMA 11.3. *A sequent* $\ulcorner \varphi \vdash \psi \urcorner$ *is a thesis of the calculus* $\mathbf{C^i}$ *iff* $\varphi \succ_\circ \psi$.

PROOF. A finite sequence of sequents $\ulcorner \pi_1 \vdash \sigma_1 \urcorner, \dots, \ulcorner \pi_n \vdash \sigma_n \urcorner$ is a derivation of the sequent $\ulcorner \varphi \vdash \psi \urcorner$ iff a finite sequence of pairs of formulas $\langle \pi_1, \sigma_1 \rangle, \dots, \langle \pi_n, \sigma_n \rangle$ satisfies the conditions from Remark 7.1. Thus, by Remark 7.1, a sequent $\ulcorner \varphi \vdash \psi \urcorner$ is a thesis of the calculus $\mathbf{C^i}$ iff $\varphi \succ_\circ \psi$. □

From Theorem on the Adequacy 7.3 and Lemma 11.3 it follows that:

THEOREM ON THE ADEQUACY 11.2. *A sequent is a thesis of* $\mathbf{C^i}$ *iff it is correct.*

PROOF. $\varphi \models_{\mathrm{i}} \psi$ iff $\varphi \succ_\circ \psi$ iff the sequent $\ulcorner \varphi \vdash \psi \urcorner$ is a thesis of the calculus $\mathbf{C^i}$. □

References

Asser, G., 1959, *Einführung in die matematische Logik*, Teil I, Leipzig.

Bell, J. L., 1977 *Boolean-Valued Models and Independence Proofs in Set Theory*, Oxford.

Epstein, R. L., 1990 *The semantic Foundations of Logic. Vol. 1: Propositional Logics*, Dordrecht.

Perzanowski, J., 1989, "Logika a filozofia. Uwagi o zasięgu analizy logicznej w naukach filozoficznych", pp. 229–261 in: *Jak filozofować*, J. Perzanowski (ed.), Warsaw.

Pietruszczak, A., 1992, "O ścisłym wynikaniu logicznym i jego modyfikacji", pp. 5–20 in: *Acta Universitatis Nicolai Copernici*, 255, Logika III, Toruń.

Pietruszczak, A., 1997, "Wynikanie zachowujące informację logiczną", pp. 251–280 in: *Byt, Logos, Matematyka. FLFL 1995*, J. Perzanowski and A. Pietruszczak (eds.), The NCU Press, Toruń.

Pietruszczak, A., 2004, "The axiomatization of Horst Wessel's strict logical consequence", this volume, pp. 121–138.

Wessel, H., 1984, *Logik*, Deutshe Verlag der Wissenschaften Berlin.

Zinov'ev, A. A., 1971, *Logika nauki*, Moskva.

ANDRZEJ PIETRUSZCZAK
Nicolaus Copernicus University
Department of Logic and Semiotics
ul. Asnyka 2
PL 87-100 Toruń, Poland
pietrusz@uni.torun.pl