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Andreas B. M. Brunner

FIRST-ORDER ANTI-INTUITIONISTIC LOGIC WITH APARTNESS

Abstract. In this paper we will develop a first-order anti-intuitionistic logic without and with *paraconsistent apartness*. We will give a system of Hilbert-type counter-axioms, that we show to be correct and complete with respect to a deictic Kripke semantics. Also we will illustrate some examples about objects being apart and not apart in some possible world.

1. Introduction

The notion of apartness was first introduced by L. E. J. Brouwer and formalized by A. Heyting. It is well known that in Heyting's intuitionistic logic the principle of the excluded third is not valid. As a consequence of this we have to develop direct or in the spirit of Brouwer *constructive* proofs for theorems. But there is a lack in this constructivism. Negated statements are not proved in the same rigorous constructive way as the positive ones.

If we want to demonstrate some sentence $\neg\varphi$ then we are showing that every constructive and hypothetical proof of the sentence φ converts to a constructive proof of $\bot$. This means that supposing φ as true we always obtain a contradiction. On the other side, there is no direct proof of the fact $\neg\varphi$, we have only shown that every constructive proof supposing φ, leads to a contradiction. Therefore we are saying that negated sentences are regular in intuitionistic logics in the sense that the double negated sentence is equivalent to the original sentence, i.e. they behave as in classical logic.

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In the same way we are proving the inequality of two objects, we are supposing that two objects are equal and if we can transform every hypothetical and constructive proof of this equality in a constructive proof of $\perp$, then we have shown the two objects unequal. This weak inequality notion has given the origin of the introduction of a positive notion of inequality by Brouwer, the intuitionistic *apartness* relation stronger than the inequality relation. For example, in constructive analysis it is not sufficient to know a number unequal to 0 in order to invert that number. For the construction of the inverse we have to know some natural number n such that the given number always has distance greater or equal than 2^{-n}, cf. [3]. It is not sufficient to know that supposing the equality to zero of this given number leads to a contradiction.

In this paper we are leading with first order anti-intuitionistic logics (the dual of Heyting's intuitionistic logic), obtained by the method of dualization introduced in [1]. Considering Heyting's calculus with *equality*, the question arises, what is the real dual of this calculus? Although, in general, in intuitionistic logic equality is not treated as a logical symbol (to the contrary of classical logic), we think that we have to dualize in Heyting's calculus, also the equality relation, obtaining a *paraconsistent apartness* relation in the dual calculus. If we do so we satisfy the abstract characterization for logical duality introduced in definitions 2.7 and 2.8 of [1], i.e., for Γ, Δ sets of sentences in L, and Γ^*, Δ^* their duals in L^*, we have

$$\Gamma \vdash_H \Delta \qquad \Longleftrightarrow \qquad \Gamma^* \dashv_{H^*} \Delta^* \qquad \Longleftrightarrow \qquad \Delta^* \vdash_{H^*} \Gamma^*.$$

Dealing with dualizing concepts, has been leading to the consideration of a first-order predicate logic with *paraconsistent apartness* #. Paraconsistent apartness is also a positive notion of inequality and therefore we are considering this relation as the dual of the usual equality in Heyting's intuitionistic logic. But paraconsistent apartness is not as strong as Brouwer's apartness. To the contrary, paraconsistent apartness is a very weak concept of inequality in the sense that we can have objects being apart and not apart at the same time generating paraconsistent situations.

Therefore, we can speak of three notions of inequality: Brouwer's strong intuitionistic *apartness*, the usual *inequality*—or equivalently, negated equality—and our weak *paraconsistent apartness*. Clearly, in classical logic we cannot make any difference between these notions.

To the best of the authors knowledge, in the literature there are only a few notes about anti-intuitionistic first-order logics. In [6] and [14], the authors dualize Heyting's first-order calculus *without* equality; in [9], the author considers a Heyting-Brouwer logic with equality but without going into details whether there can arise paraconsistent situations considering equality.

Using the methods of dualization introduced in [1], we show our system (without and with paraconsistent apartness) to be correct and complete, if we are considering a language *without* function symbols. We also will show that in our logic with paraconsistent apartness inconsistent situations containing apartness are possible and we illustrate these situations with a lot of examples.

Let us now consider the language L with the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\perp$ (bottom), $\exists$ (existential quantifier) and $\forall$ (universal quantifier). By $L_=$ let us denote the language L with equality $=$. By L^* we denote the dual language of L with the logical connectives $\vee$ (disjunction), $\wedge$ (conjunction), $-$ (pseudo-difference), $\top$ (top), $\forall$ (universal quantifier) and $\exists$ (existential quantifier). By $L^*_\#$ we mean L^* with paraconsistent apartness #.

We extend our *dualizing translation* introduced in section 2 of [1]

$$*\colon L_= \longrightarrow L^*_\#$$

by induction in the complexity

[atom]	let φ be an atomic formula without $=$, then $\varphi^* := \varphi, \quad \perp^* := \top$.
[con]	$(\varphi \wedge \psi)^* := \varphi^* \vee \psi^*, \quad (\varphi \vee \psi)^* := \varphi^* \wedge \psi^*, \quad (\varphi \rightarrow \psi)^* := \psi^* - \varphi^*$.
[quan]	$(\exists x\varphi)^* := \forall x\varphi^*, \quad (\forall x\varphi)^* := \exists x\varphi^*$.
[equal]	$(\tau_1 = \tau_2)^* := (\tau_1^* \# \tau_2^*) \quad$ for L-terms τ_1, τ_2.

Observe that we are defining the dual-term of a given L-term τ by induction in complexity as $\tau^* := \tau$. Also remember that a negated formula in L is defined as $\neg\varphi := \varphi \rightarrow \perp$ and therefore we have an anti-intuitionistic negation in L^* given by $\neg^*\varphi := \top - \varphi$.

2. The first-order logic without apartness

We begin with the Hilbert-type counter-axiomatization of Heyting's dual calculus H^* for first-order logic without function symbols and without apartness relation extending the propositional calculus in [1]. The elenctic (i.e. refutative) axiomatization will show us which sentences have to be rejected. As intuitionistic logic is "false by default" (in the sense that a sentence and its negation can both be taken to be false) anti-intuitionistic logic is "true by default" (in the sense that a sentence and its negation can both be taken to be true). In this sense our logic H^* is a liberal logic, because in the beginning we are accepting almost everything as true. Passing time and obtaining new informations and new objects we are going to reject certain conjectures, cf. [8]. And once rejected a conjecture we are sure that it cannot be verified anywhere.

For first-order anti-intuitionistic logic we obtain for all formulas φ, ψ and χ the following system of counter-axioms:

I. $(\varphi - \psi) - \varphi$;

II. $[(\chi - \varphi) - ((\chi - \psi) - \varphi)] - (\psi - \varphi)$;

III. $[(\varphi \vee \psi) - \varphi] - \psi$;

IV. $\varphi - (\psi \vee \varphi)$; $\psi - (\psi \vee \varphi)$;

V. $(\varphi \wedge \psi) - \varphi$; $(\varphi \wedge \psi) - \psi$;

VI. $[(\chi - (\varphi \wedge \psi)) - (\chi - \varphi)] - (\chi - \psi)$;

VII. $\varphi - \top$;

VIII. If τ is a L^*-term free for x in φ then

$$\text{(a) } \varphi(\tau) - \exists x\varphi(x) \quad \text{and} \quad \text{(b) } \forall x\varphi(x) - \varphi(\tau).$$

Counter-Inference Rules:

[Dual Modus Ponens] $\dfrac{\varphi, \quad \psi - \varphi}{\psi}$

[∀] $\dfrac{\psi - \varphi}{\psi - \forall x\varphi}$, where x is not free in ψ

[∃] $\dfrac{\psi - \varphi}{\exists x\psi - \varphi}$, where x is not free in φ

We use $\Gamma \dashv \varphi$, to denote that if we reject all formulas of Γ, we have to reject the formula φ. Writing $\dashv \psi$ we denote that ψ is a counter-theorem or equivalently, that ψ is rejected from the counter-axioms H^*. Thus we have satisfied the main characteristics of logical duality mentioned in [1].

In the following we will give a deictic (cf. [1]) first-order Kripke semantics for this system of counter-axioms which is correct and complete by a simple dual argumentation, cf. [3], [9], [10] and [13]. For this, consider the anti-intuitionistic language L^* without function symbols.

DEFINITION 2.1. The quadruple $\mathcal{K} := (K, \leq, D, \Vdash)$ is an *anti-intuitionistic first-order Kripke model* such that

(a) $(K, \leq)$ is a partially ordered set with $K \neq \emptyset$.

(b) D is the *domain function* which assigns to every possible world $k \in K$ a non empty set $D(k)$ the respective domain such that

$$\forall k \forall k' (k \leq k' \rightarrow D(k) \subseteq D(k')).$$

(c) Extending our language with new constant symbols for each element of $\mathcal{D} := \bigcup\{D(k) : k \in K\}$, $\Vdash$ is a binary relation on $K \times Atom(L^*_{\mathcal{D}})$ such that for a n-ary relation symbol R

$$\text{If } k \nVdash R(d_1, \ldots, d_n) \quad \text{then} \quad d_1, \ldots, d_n \in D(k) \quad \text{and}$$
$$\exists k' \geq k, \quad k' \Vdash R(d_1, \ldots, d_n) \quad \Longrightarrow \quad k \Vdash R(d_1, \ldots, d_n).$$

We will say for $k \Vdash R(d_1, ..., d_n)$ that "*k forces anti-intuitionistically* $R(d_1, ..., d_n)$". We extend the forcing relation to logically compound formulas by the following clauses, for every $k \in K$:

[∧] $k \Vdash \varphi \wedge \psi$ iff $k \Vdash \varphi$ and $k \Vdash \psi$.
[∨] $k \Vdash \varphi \vee \psi$ iff $k \Vdash \varphi$ or $k \Vdash \psi$.
[−] $k \Vdash \varphi - \psi$ iff $\exists k' \geq k$ $(k' \Vdash \varphi$ and $k' \nVdash \psi)$.
[⊤] $k \Vdash \top$.
[∃] $k \Vdash \exists x \varphi(x)$ iff $\exists k' \geq k$ $\exists d \in D(k')$, $k' \Vdash \varphi(d)$.
[∀] $k \Vdash \forall x \varphi(x)$ iff $\forall d \in D(k)$, $k \Vdash \varphi(d)$.

Remark 2.2. 1. As anti-intuitionistic negation is defined as $\neg^* \varphi := \top - \varphi$, we can show easily that

$$k \Vdash \neg^* \varphi \quad \text{iff} \quad \exists k' \geq k \;\; k' \nVdash \varphi.$$

2. Furthermore, we observe that for rejecting sentences we have the following monotonicity property - shown by induction in the complexity

$$k \nVdash \varphi \quad \Longrightarrow \quad \forall k' \geq k \;\; k' \nVdash \varphi.$$

3. With the obvious modifications, we can define easily our deictic Kripke model as a covariant functor from a poset category to the category $\boldsymbol{L^*}$**-mod**, whose objects are classical L^*-structures and morphisms are given by L^*-antimorphisms (preserving rejections in the atomic case) between the classical L^*-structures. □

In the next definition we define when sentences are valid.

Definition 2.3. A formula φ is *valid at k* in an anti-intuitionistic first-order Kripke model $\mathcal{K}$ iff $k \Vdash \varphi$. φ is *valid in $\mathcal{K}$*, denoted by $\mathcal{K} \Vdash \varphi$ iff for all $k \in K$, $k \Vdash \varphi$. For a set Γ of sentences, we define in the usual way when φ is a Kripke consequence of Γ, that is

$$\Gamma \Vdash \varphi \quad \text{iff} \quad \forall \mathcal{K} \forall k \in K (k \Vdash \gamma \text{ for all } \gamma \in \Gamma \Rightarrow k \Vdash \varphi).$$

On the other side, we introduce the following notation

$$\varphi \Vdash \Gamma \quad \text{iff} \quad \forall \mathcal{K} \forall k \in K\, (k \Vdash \varphi \Rightarrow k \Vdash \gamma \text{ for some } \gamma \in \Gamma).$$

A sentence φ is said to be *Kripke-valid* or a *tautology* iff $\emptyset \Vdash \varphi$. Contrary, we say that a formula φ is a *counter-tautology* iff there is no Kripke model $\mathcal{K}$ and no possible world $k \in K$ such that $k \Vdash \varphi$.

Using the results of [1], it is easy to see, that our anti-intuitionistic first-order logic is paraconsistent, in the sense of [2], not trivial and satisfies the law of the excluded third. The strong soundness and completeness theorems are proved by a simple dual argumentation. We will omit the details.

THEOREM 2.4 (Soundness). *If $\Gamma \dashv \varphi$ then $\varphi \Vdash \Gamma$. That is, every counter-theorem is a counter-tautology.* □

THEOREM 2.5 (Completeness). *If $\varphi \Vdash \Gamma$ then $\Gamma \dashv \varphi$.* □

3. The first-order logic with apartness

We will give in the following counter-axiomatics for our anti-intuitionistic logics *with* paraconsistent apartness relation. To our Hilbert type counter-axioms given earlier, we join the counter-axioms for our paraconsistent apartness. Some axiomatics for intuitionistic apartness relation are given for example, in [3], [11], [12] and [13], but let us observe and remark that our paraconsistent apartness is very different from Brouwer's apartness. More than this, our paraconsistent apartness is *not* clearly separated from some kind of equality, in the sense that we can think of objects being apart and not apart in a possible world. Thus, paraconsistent apartness is a kind of *quase-inequality*. It is clear therefore that paraconsistent apartness is weaker than the usual inequality relation.

In this section we will also introduce the notion of first-order Kripke model with apartness relation, and show the soundness and completeness of the anti-intuitionistic first-order calculus with paraconsistent apartness. The same restriction as in section 2 is made here, we consider languages without function symbols. Therefore, speaking of terms we are meaning variables and constant symbols.

The anti-intuitionistic theory of apartness is given by the following system of counter-axioms.

$(AP1)^{op}$ $\quad \dashv \exists x\,(x \# x)$.

$(AP2)^{op}$ $\quad \dashv \exists x \exists y\,(x \# y \,-\, y \# x)$.

$(AP3)^{op}$ $\quad \dashv \exists x \exists y \exists z\,(x \# y \,-\, (x \# z \vee y \# z))$.

$(AP4)^{op}$ $\quad$ Let φ be a $L^{*}_{\#}$-formula, then

$$\dashv \exists \vec{x} \exists \vec{y}((\varphi(\vec{x}) - \varphi(\vec{y})) - (x_1 \# y_1 \vee \cdots \vee x_n \# y_n))$$

Remark 3.6. The above counter-axiomatics is obtained by dualizing the axiomatics for identity. Therefore, also for the calculus with paraconsistent apartness the main characteristics of logical duality in the sense of [1] is satisfied. □

Before introducing our definition of first-order anti-intuitionistic Kripke model with paraconsistent apartness we will observe that the apartness relation will not be interpreted in each world as the usual inequality; this is explained by the next propostion which is the dual of a result in [3].

Proposition 3.7. *Let $\mathcal{K}$ be an anti-intuitionistic first-order Kripke model. If for all worlds $k \in K$ we have* $s = t$ *for some* $s, t \in D(k) \Rightarrow k \nVdash s \# t$, *then*

$$\mathcal{K} \nVdash \exists x \exists y (x \# y \wedge \neg^*(x \# y)).$$

Proof. We have to prove that for some $k \in K$, $s, t \in D(k)$

$$k \nVdash (s \# t \wedge \neg^*(s \# t))$$

and this is equivalent with $k \nVdash s \# t$ or $k \nVdash \neg^*(s \# t)$.
If we have $k \nVdash s \# t$ then we are done. If not, then we have $k \Vdash s \# t$ and by contraposition in the hypothesis $s \neq t$ in $D(k)$. But then we have $s \neq t$ in $D(l)$, $\forall l \geq k$. Therefore, $l \Vdash s \# t$ for $l \geq k$, and this is by remark 2.2, 1., $k \nVdash \neg^*(s \# t)$. □

We will give now the notion of first-order Kripke model for an anti-intuitionistic language with paraconsistent apartness. Then we obtain in the usual way the soundness and completeness theorems by dualizing arguments and using the notion of Kripke model with *transition functions*, cf. [3] and [13], also known as *modified* Kripke model.

Definition 3.8. The quadruple $\mathcal{K} := (K, \leq, D, \Vdash)$ is an *anti-intuitionistic first-order Kripke model with apartness* such that
(a) $(K, \leq)$ is a partially ordered set with $K \neq \emptyset$.
(b) D is the *domain* function which assigns to every possible world $k \in K$ a non empty set $D(k)$ the respective domain such that

$$\forall k \forall k' (k \leq k' \rightarrow D(k) \subseteq D(k')).$$

(c) Extending our language with new constant symbols for each element of $\mathcal{D} := \bigcup\{D(k) : k \in K\}$, $\Vdash$ is a binary relation on $K \times Atom(L^*_{\mathcal{D}})$ such that we have the following properties

(i) We have the following equivalence relation in each domain $D(k)$:

$$d \equiv_k d' \quad \Leftrightarrow \quad k \nVdash d \# d'$$

(ii) For a n-ary relation symbol R and elements $d_1, \ldots, d_n, d'_1, \ldots, d'_n \in D(k)$ we have the following

$$\text{if } k \nVdash \vec{d} \# \vec{d'} \text{ and } k \Vdash R(\vec{d}) \quad \text{then} \quad k \Vdash R(\vec{d'}),$$

where $k \nVdash \vec{d} \# \vec{d'}$ is an abbreviation for $k \nVdash d_i \# d'_i$, $\forall i = 1, \ldots, n$.

(iii) Let R be a n-ary relation symbol, then

$$\text{if } k \nVdash R(d_1, \ldots, d_n) \quad \text{then} \quad d_1, \ldots, d_n \in D(k) \quad \text{and}$$
$$\exists k' \geq k, \quad k' \Vdash R(d_1, \ldots, d_n) \quad \Longrightarrow \quad k \Vdash R(d_1, \ldots, d_n).$$

We will say for $k \Vdash R(d_1, ..., d_n)$ that "*k forces anti-intuitionistically* $R(d_1, ..., d_n)$". We extend forcing to logically compound formulas by the clauses given in Definition 2.1.

We will omit the proof of the following strong Soundness Theorem.

THEOREM 3.9 (Soundness). $\Gamma \dashv \varphi \quad \Longrightarrow \quad \varphi \Vdash \Gamma$. □

The strong completeness is proved by using a equivalence relation $\equiv_k$ in each domain $D(k)$, given by $d \equiv_k d' \iff k \nVdash d \# d'$, and the monotonicity of $\equiv_k$ (i.e., $\forall k' \geq k, \quad d \equiv_k d' \quad \Rightarrow \quad d \equiv_{k'} d'$).

Remark 3.10. Clearly, we also are able to define the Kripke model with paraconsistent apartness, as a covariant functor, as indicated in 2.2. □

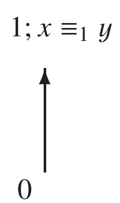
Dualizing the arguments developed in [3] we obtain the

THEOREM 3.11 (Completeness). *Anti-intuitionistic logic with paraconsistent apartness is complete with respect to modified Kripke models.* □

4. Examples of paraconsistent apartness situations

Having soundness and completeness established, we will give a few examples obtaining paraconsistencies involving our apartness relation. The following example will illustrate how we can have a paraconsistent situation in an anti-intuitionistic Kripke model.

Example 4.12. Let x, y be distinct, $D(0) = D(1) = \{x, y\}$ and consider the following anti-intuitionistic Kripke model:



Then it is easy to see that $\quad 0 \Vdash \neg^*(x \# y) \wedge x \# y$.

Firstly it is clear that $0 \Vdash x \# y$ by the hypothesis that x and y are different. On the other side, $0 \Vdash \neg^*(x \# y)$, because $1 \nVdash x \# y$. Therefore we have obtained a paraconsistent situation and by the completeness theorem, $\not\dashv \; \exists v \exists w (v \# w \wedge \neg^*(v \# w))$. Thus, our Kripke models allow contradictory situations and these contradictory situations do not lead to the trivialization of our calculus. Therefore we can speak of paraconsistency. □

The last example has shown that we can obtain structures where we have the apartness of two objects and at the same time the not-apartness of these objects. We will give now examples from philosophy and physics how these paraconsistencies can arise.

Example 4.13 (Morning and Evening Star). It is well known that the morning star (MS) that we can observe every day as well as the evening star (ES) is Venus. This leads to the proposition that MS and ES are not apart objects, because they denote the same object. Therefore, in a possible world E_2 we force that not-apartness, writing this in our mathematical language

$$E_2 \Vdash \neg^*(\mathrm{MS} \,\#\, \mathrm{ES}).$$

Let us ask now: Is that "well known" fact really well known? This is to say, every person in our world *do* know this fact? If we were a little bit realistic then we have to say that obviously not every person in our world knows this fact. Only persons who have access to a good education will know that the two planets MS and ES denote the same planet Venus. Therefore, considering another world E_1, we have in that new world the apartness of these two planets, because not every person know that MS is the same planet as ES. In our mathematical notation we are able to write this as

$$E_1 \Vdash (\mathrm{MS} \,\#\, \mathrm{ES}).$$

Formalizing our studies, consider the Kripke model $\mathcal{K}$, where the possible worlds are on the one hand the world E_1 and on the other hand the world E_2, where persons living in the first world E_1 theoretically have access—by visiting good schools or universities or studying many hours—to the second world E_2. Therefore, the accessability relation is from the first world—not knowing the not-apartness of MS and ES—to the second, now knowing that the two planets denote Venus. Our domains are $D(E_1) = D(E_2) = \{\mathrm{MS}, \mathrm{ES}\}$. The equivalence relation is self understood. Then we show the following fact:

$$E_1 \Vdash \neg^*(\mathrm{MS} \,\#\, \mathrm{ES}) \wedge (\mathrm{MS} \,\#\, \mathrm{ES}). \tag{$*$}$$

This is easy to see. Let us denote our present world as the first world E_1. Then it is clear that $E_1 \Vdash \mathrm{MS} \,\#\, \mathrm{ES}$, because not yet every person in our world has studied enough. But on the other hand, it is possible for some persons to have a good education and to make a transition from the first world to the second one. In that second world these studied persons are knowing more facts and therefore we have $E_2 \nVdash \mathrm{MS} \,\#\, \mathrm{ES}$ and this leads to $E_1 \Vdash \neg^*(\mathrm{MS} \,\#\, \mathrm{ES})$ by definition, showing $(*)$. □

The last example can be considered and justified in another way. For this we use Leibniz' principle of identity of indiscernibilities and of indiscernibility of identities. This can be considered as general criteria for identity and diversity. Let $\mathcal{P}$ be a property, then

$$x = y \iff \forall \mathcal{P}(\mathcal{P}(x) \leftrightarrow \mathcal{P}(y)).$$

In words: Two objects are identical if they are indiscernible, and vice versa. Now let us reconsider Example 4.13.

Example 4.14 (Morning and Evening Star – again). With the above notations let us consider a "temporal" property $\mathcal{T}$. This property $\mathcal{T}$ is defined as

$$\mathcal{T}(x) \quad \text{iff} \quad x \text{ happens in the morning.}$$

We know that $\mathcal{T}$(MS) and not $\mathcal{T}$(ES). Considering a new Kripke model with two worlds w_1 and w_2, and the accessibility relation from w_1 to w_2. In world w_2, we are considering all possible properties for comparing two objects *without* the temporal property $\mathcal{T}$. In the world w_1, we have all these properties including also our temporal property. Then the following paraconsistency is easy to see:

$$w_1 \Vdash \neg^*(\text{MS} \# \text{ES}) \wedge (\text{MS} \# \text{ES}). \qquad \square$$

The next example is going in the same direction. But the theme is rather banal.

Example 4.15 (Cup of Coffee). Let us consider two indistinguishable cups of coffee C_1 and C_2 on a table in a restaurant in some place in the world. What can we say about these two cups of coffee? They are *equal* and therefore *not apart*, because they are made of the same stuff, because they are of the same form, because they have the same color and they have the same properties considering the use, for example we are able to drink coffee with her help, and everything we can do with one cup we are able to do with the other. If this is so then these two cups of coffee have the same properties $\mathcal{P}$ and therefore are not apart.

But are they really not apart? Let us think about a kind of *spacial* property $\mathcal{S}$ and let us ask now: "Do the two cups C_1 and C_2 occupy the same space on the top of the table?" This is to ask, do the two cups of coffee share the property $\mathcal{S}$? Clearly, they cannot occupy the same space, if we have two cups of coffee. Therefore they do *not* share the spacial property. This kind of question has been leading to the notion of *identical* and *equal* (*indistinguishable*) objects, cf. [7]. We can say that the two cups of coffee are equal, but not identical. In our notion, we can say that the two cups of coffee are in some sense not apart and in another sense they are apart.

Now the reader is able to conclude the example in the spirit of the two examples earlier mentioned. Consider an anti-intuitionistic Kripke model with the following dates: There are two possible worlds w_1 and w_2. We are able to access from the world w_1 the world w_2. In the world w_1 we are considering all properties $\mathcal{P}$, including the spacial property $\mathcal{S}$. In the world w_2 we are considering all properties without the spacial property $\mathcal{S}$. The domains of our two possible worlds contain the same objects C_1 and C_2, our above mentioned two cups of coffee. Then we have the following paraconsistency:

$$w_1 \Vdash \neg^*(C_1 \# C_2) \wedge (C_1 \# C_2).$$

Because in world w_1 we are considering the property $\mathcal{S}$ we force the apartness of the two cups of coffee. In world w_2, accessible from w_1, forgetting about the spacial property we are not forcing the apartness of C_1 and C_2. □

If we think of the factors "time" and "space" as done in the examples 4.14 and 4.15 we can obtain many other examples for these paraconsistent happenings. We let it to the reader to develop other examples. Finally, we will explain an example from physics.

Example 4.16 (Classical and Quantum Physics). In classical and quantum physics, elementary particles having the same set of stable independent properties are indistinguishable. In classical mechanics, permutations of classical particles must be observable, and these ensembles are distinct from the permutated ones. In quantum mechanics, permutations of quantum particles do *not* provide new ensembles in the statistical point of view, and therefore we can consider them indistinguishable. See more on this subject in [4] and [5].

In our context, we will consider a classical world w_1 having two apart (because of their permutation) objects c_1 and c_2, which are indistinguishable in a quantum world w_2. With the obvious accessibility relation, we easily establish the following paraconsistent apartness situation:

$$w_1 \Vdash \neg^*(c_1 \# c_2) \wedge (c_1 \# c_2).$$ □

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ANDREAS B. M. BRUNNER
Departamento de Matemática
Universidade Federal da Bahia
Salvador, CEP 40170-110, Bahia, Brasil
andreas@im.ufba.br