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RAMSEYING LIARS

Abstract. Despite the volume of discussion on the Liar Paradox recently, there is one stream of largely British thought on the matter which is hardly represented in the wider literature. This paper points out salient aspects of the history of this tradition, from its origin in forms of propositional quantification found in Ramsey, through to more precise symbolisations which have emerged more recently. But its purpose is to exposit, with respect to a number of contested cases, the ensuing results. Thus it goes on to apply the analysis to several other well known paradoxes, including one rarely discussed, which reveals more fully the consequent consistency and completeness of natural language.

Keywords: Liar, Curry and Gödel Paradoxes; propositions; epsilon calculus.

1. Kneale's account

If one formulates the Liar, and associated paradoxes, in terms of propositions it is easy to see why Kneale thought there was a trivial resolution of paradoxes in the Liar family. Kneale put the matter like this:

> [...] anyone who pronounces the sentence 'What I am now saying is false' appears to use the opening phrase to *designate* what is *expressed* by his whole utterance. If, however, he succeeds in designating any proposition by use of that phrase it should be possible in principle for him to designate the same proposition by means of a 'that' clause. ... But obviously this is impossible; for an attempt to carry out the prescription merely results in 'It is false that it is false that it is false [...],' and so on *ad infinitum*. [4, p. 242]

Kneale adds a footnote to a paper by Gilbert Ryle: in Ryle's terms there is no 'namely rider' to the phrase 'what I am saying now', in the above context. And

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there is another way of putting Kneale's point, as follows. For it might be false of a certain subject *s* that it is P, and so, in particular, that it is false might be false of the proposition that p. But for that it is false to be false of itself would be for it to be false of the proposition that it is false, i.e. something whose expression still contains a waiting pronoun. So there is no way to specify a self-referential proposition by just using a 'that' clause.

But why even talk in this way of operator constructions when there can quite clearly be self-referential sentences? That is because talk about such constructions is required to avoid the well-known paradoxes associated with self-referential sentences. Can it be said of the sentence 'The sentence at the top of the page is false', that it is false of anything—in particular that it is false of itself, if it is the sentence at the top of the page? No: for there is no place waiting in that sentence for the name of any thing it might be false of to be inserted. *That it is false* is the sort of thing which might be false of something—the sentence at the top of the page, and even that sentence when it is the sentence previously given—but that is a possibility only because 'that it is false' has a space waiting to be filled in the appropriate way. In fact, that it is false *is* false of the said sentence (no matter where it might be), because sentences are neither true nor false. What might be true or false of *s* is that it is P, but 'that *s* is P' is not a sentence; it is a noun phrase—a demonstrative referring phrase to a proposition. The use of such phrases most forcefully appears if the sentence at the top of the page is a sentence like 'The sentence at the top of the page is false, or neither true nor false'. Then—since no sentence at the top of the page is either true or false, and so any there is either false, or neither true nor false—wouldn't we, contradictorily, have to say that the sentence actually there was true, since it states the facts? No: because what is true is simply that the sentence at the top of the page is false or neither true nor false, not the sentence 'the sentence at the top of the page is false or neither true nor false'.

The identification of the referents of the '*t*'s in such identities as '*t* = "*t* is not true" ' produces confusion, if truth is a taken to be a disquotational property of sentences, because then '*t* is not true' is true iff *t* is true (on account of the identity), but also true iff *t* is not true (by disquotation). The same confusion is not obtainable if truth is taken to be a property of propositions, as above. For there is no opportunity then for paradoxical substitution of the sentential identity: one says that *t* is not true, since sentences are no longer the bearers of truth, and that simply means that '*t* is not true' is not true. In saying that *t* is not true, isn't one asserting that the sentence one then uses ('*t* is not true') is true? No: one is not saying '*p*' is true when one says it is true that p, because use is not mention, and operators are not predicates. The sentence '*t* is not true' is not said to be true when one says *that t* is not true, since one is then saying, instead, that the referent of the 'that' clause is true.

As a result, there is a very plausible case to be made for accepting such banalities as the above as a resolution of the problems in this area. Evidently there are certain mental pre-occupations which might need to be overcome, notably Quinean fears of 'intensional entities' like propositions, and Tarskian theories of truth, which involve the same fear, and try to start from properties of sentences, instead. But in addition to the above points, the latter pre-occupation ignores the difference between two senses of 'say', namely 'utter', and 'state' (c.f. [4, p. 230]); and the former ignores related aspects of 'that' clauses. We say, for instance, 'That John was already there was surprising', and so predicate something of a subject of the form 'that p', but the nominaliser 'that' is then clearly not a quotation maker, since what was surprising was not the sentence 'John was already there', but rather the fact of John's being already there.

Prior was of the opinion that the operator approach spelt the end of Tarskian semantics, if not semantics entirely (c.f. [8, Ch. 7]). With propositional operators, a truth locution is involved, but it is in the object language, which is semantically closed without contradiction. Thus the Principle of Bivalence (it is true that p or it is true that $\neg p$) becomes equivalent to the Law of the Excluded Middle ($p \vee \neg p$), and lines in a truth table such as 'If it is true that p and it is not true that q, then it is not true that $p \wedge q$' become equivalent to propositional theses like '$(p \wedge \neg q) \supset \neg(p \wedge q)$'. And that trivialises the usual soundness and completeness proofs in propositional logic.

And it is not only that which becomes elementary. For if 'r' is a referring phrase to a proposition, '§' is the nominaliser 'that', and 'T' refers to the property of being true, then we can proceed to show quite generally that there cannot be a proposition saying that it itself is not true, i.e. an r such that $r = \S\neg Tr$. For all that is needed are propositional epsilon terms. If $r = \S\neg Tr$ then Tr would be equivalent to $T\S\neg Tr$, and so to $\neg Tr$, through application of the propositional T-scheme, which equates $T\S p$ with p. But the resulting fact that $\neg(\exists r)(r = \S\neg Tr)$ still leaves the associated epsilon term '$\epsilon r(r = \S\neg Tr)$' with a referent, and so we can still speak about 'the proposition which says about itself that it is not true'. The referent of this phrase, however, is semantically arbitrary, since it is a misnomer. We know that $(\exists x)Fx \equiv F\epsilon x Fx$, [5], making $\neg(\exists x)Fx \equiv \neg F\epsilon x Fx$, so although we can refer to the lying proposition, it does not satisfy its description, and that means we cannot identify it. The proposition which says of itself that it is not true thus may be true or false depending on choice.

As we shall see, however, further use must be made of propositional epsilon terms before all doubts are expelled in this area. For Kneale's own example provides us with a case where reference is made to a proposition, but not in the form of a 'that' clause. Is it so certain then that no form of self-referential proposition can be constructed to parallel the case with self-referential sentences? Certainly there is no self-reference available with operator constructions: mereology prevents 'it is not true that p' from itself being 'p', since a whole cannot contain itself as a proper part. But if 'r' is a referring phrase to a proposition, we must allow it to be a noun phrase like 'what I am saying', as well as a 'that' clause like 'that the sentence at the top of the page is false or neither true nor false'. And isn't it evident that what Kneale was saying was that what Kneale was saying was false?

2. Some history

A little history will clear up some preliminary matters.

The study of the Liar and related paradoxes, in something like the above manner, started in the late 1950s, with a series of papers in the *Journal of Symbolic Logic* by Cohen, Goodstein and Prior. Goodstein originally showed [2, p. 418], for instance, that if A says that everything A says is false ($Sa(p)(Sap \supset \neg p)$), then something A says is true ($(\exists p)(Sap \wedge p)$), and something A says is false ($(\exists p)(Sap \wedge \neg p)$). Prior developed this form of 'protothetic', in which there can be quantification over propositions, extensively, in his posthumously published book *Objects of Thought* of 1971. In particular he proved four similar theorems: see p. 105. But a major problem with this specific kind of symbolisation, is whether the quantification is objectual or substitutional. Prior and Goodstein's symbolisation followed the quantified propositional approach deriving from Ramsey, but Ramsey's formulation was problematic, as Haack pointed out [3, p. 130]. For there is a bare 'p' in the matrix '$Sap \wedge p$', and so the 'p' in the quantifier '$\exists p$' would have to be equally not a referential phrase, but a full sentence, leading to difficulties in reading the quantifier [8, Ch. 3]. These difficulties enlarge when reading '$Sap \wedge p$' as 'A says that p, and it is true that p' since there is no explicit representation of truth, in the symbolism.

The nominaliser 'that' which turns an expressed proposition 'p' into a referring phrase 'that p' which designates the proposition is not commonly symbolised in standard logic. Kneale introduced the above symbol '§' for this purpose, and Haack, following him, used it for 'the statement that ...' [3, p. 150]. But in Haack there remained not only the problem with reading propositional quantifications, and the associated problem with an expression for truth. She also, for instance, purported to derive a paradox with her notion by first producing a sentence:

(1) $$(p)(c = \S p \supset \neg p),$$

and then letting 'c' abbreviate 'the statement made by sentence numbered (1)'. She went on to say, as a result, that 'it can be established empirically' that

$$c = \S(p)(c = \S p \supset \neg p),$$

deriving a contradiction analogous to one Tarski produced. But we can escape this paradox in Goodstein's way, by admitting that while any statement made by the sentence numbered (1) is certainly any statement made by '$(p)(c = \S p \supset \neg p)$', that does not ensure that any single, identifiable statement is made by '$(p)(c = \S p \supset \neg p)$' in this context. A sentence may be ambiguous, after all, although the form of semantic indeterminacy in paradoxical cases is not simply a double meaning.

Goodstein's original proof of non-univocality lacked Kneale's statement forming operator, and the related propositional truth predicate, so to produce a proof which is fully satisfactory we must introduce '§', where appropriate, and include the above predicate of 'that' clauses, such that $\mathrm{T}\S p$ is equivalent to p. Then we get:

> Given $\mathrm{S}ay$, where $y = \S(x)(\mathrm{S}ax \supset \neg\mathrm{T}x)$. Assume, first $(x)(\mathrm{S}ax \supset \mathrm{T}x)$, then, since $\mathrm{S}ay$ only if $\mathrm{T}y$, and $\mathrm{S}ay$, we must have $\mathrm{T}y$, and hence $(x)(\mathrm{S}ax \supset \neg\mathrm{T}x)$, but since also $(x)(\mathrm{S}ax \supset \mathrm{T}x)$ and $(\exists x)\mathrm{S}ax$, that is contradictory; hence $\neg(x)(\mathrm{S}ax \supset \mathrm{T}x)$, i.e. $(\exists x)(\mathrm{S}ax \wedge \neg\mathrm{T}x)$. But assume, second, $(x)(\mathrm{S}ax \supset \neg\mathrm{T}x)$, then $\mathrm{T}y$, but also $\mathrm{S}ay$, hence $(\exists x)(\mathrm{S}ax \wedge \mathrm{T}x)$, and a contradiction; so $\neg(x)(\mathrm{S}ax \supset \neg\mathrm{T}x)$, i.e. $(\exists x)(\mathrm{S}ax \wedge \mathrm{T}x)$.

In the modified version of Goodstein's conclusions, therefore, $(\exists x)(\mathrm{S}ax \wedge \neg\mathrm{T}x)$, $(\exists x)(\mathrm{S}ax \wedge \mathrm{T}x)$, one must first read the variables as nominal variables (unlike the 'p' above), but, second, one must be careful not to take the 'T' to be an operator; it is a predicate. And that means that there is no certainty that it can be eliminated. The replacement of '$\mathrm{T}x$' with 'x' was always criticised on this basis, in discussions of the Ramsey-Goodstein-Prior calculus, but attending to that criticism is, remarkably, just what gets us out of the semantical liar paradoxes even more clearly. Correcting the calculus, however, also means accepting that not every proposition referred to can be independently expressed. From Goodstein's conclusions it follows that in one sense in which what A says is false, it is false, i.e. $\neg\mathrm{T}\epsilon x(\mathrm{S}ax \wedge \neg\mathrm{T}x)$, and that in one sense in which what A says is true, it is true, i.e. $\mathrm{T}\epsilon x(\mathrm{S}ax \wedge \mathrm{T}x)$. So $\epsilon x(\mathrm{S}ax \wedge \neg\mathrm{T}x) \neq \epsilon x(\mathrm{S}ax \wedge \mathrm{T}x)$, and there is not just one proposition involved. But these implications are pragmatic implications, and so there need be no way to express the alternative senses just linguistically (see appendix). What a says which is false is that everything a says is false, but that is false not just because of its content, but also because a said it. More significantly, even if an identity of the form '$\epsilon x(\mathrm{S}ax \wedge \mathrm{T}x) = \S p$' is available, using further epsilon terms, it is still indeterminate what a says, i.e. there still is a choice about what '$\epsilon x\mathrm{S}ax$' refers to.

It is here that we get the needed amplification of the point which led Kneale to his resolution of the Liar Paradox. For from 'What this sentence says is not true' i.e. '$\neg T\epsilon x S a x$', where 'a' now refers to this sentence, one only gets a contradiction if some 'that' clause expression for what the sentence says is presumed, other than the tautologous one which might be granted in '$\epsilon x S a x = \S T \epsilon x S a x$'. Thus we might suppose it was automatic that what it says is that what it says is not true, i.e. $\epsilon x S a x = \S\neg T\epsilon x S a x$, from which it would follow that $T\epsilon x S a x$ iff $T\S\neg T\epsilon x S a x$, where the latter is equivalent to $\neg T\epsilon x S a x$, producing a contradiction. But if so, then the *Reductio* which was available would show that what the sentence says is not *just* that what it says is not true.

Cannot one get paradoxical self-reference in such a case as where Kneale utters (at time t) 'What Kneale states (at time t) is not true' ($\neg T\epsilon r S k r$)? Certainly then Kneale states that what he states is not true, i.e. $S k \S\neg T\epsilon r S k r$, but only supposing further that only one proposition is involved, i.e. $(\exists! r) S k r$, can one obtain a contradiction. On that assumption it follows that $(s)(S k s \supset s = \S\neg T\epsilon r S k r)$, from the uniqueness, and since $S k \S\neg T\epsilon r S k r$ entails $(\exists r) S k r$, we get $S k \epsilon r S k r$, by the epsilon definition of the quantifier. On this basis it follows that $\epsilon r S k r = \S\neg T\epsilon r S k r$, i.e. that what Kneale states is that what Kneale states is not true, and this identity then gives $T\epsilon r S k r$ iff $T\S\neg T\epsilon r S k r$, and so $T\epsilon r S k r$ iff $\neg T\epsilon r S k r$, which is the contradiction. But all that follows is that the previous assumption is false, i.e. that more than one proposition is involved. In fact there is an infinity: $\epsilon r S k r$, $\S\neg T\epsilon r S k r$, $\S\neg T\S\neg T\epsilon r S k r$, etc., as Kneale, following Ryle, anticipated.

3. Other cases

Having thus defended Kneale's specific conclusion more fully it becomes evident how it can be generalised quite readily. Turning to where there are two speakers, Jones and Nixon, say, and the first says 'What Nixon states is not true', consider first the telling case where the second speaker in fact utters nothing. We saw in the simple case where Jones states that what he is stating is false, that the crucial difference is that between '$S j \S p$' and '$\epsilon x S j x = \S p$'. Here it is even more evident why we can have the former without the latter. For the phrase 'what Nixon states' now refers to a fiction, and so the epsilon analysis simply leaves it with an indeterminate referent. We know that $S j \S\neg T\epsilon r S n r$, so we have a representation for what many have called the 'proposition' Jones expresses ($\S\neg T\epsilon r S n r$), and that necessarily contains a term for 'what Nixon states' i.e. '$\epsilon r S n r$'. But there is no way to specify the statements which either Jones or Nixon makes, because there is nothing to determine what '$\epsilon r S n r$' refers to. The distinction between propositions and statements was notably made by Strawson and Lemmon [3, Ch. 6]. The case of fictions was originally understood to provide a case where no statement is made, as with Russell's 'The King of France is bald' said at the present time. But an epsilon representation of such definite descriptions as 'The King of France' allows them to be complete individual terms, with merely an indeterminate referent in the fictional case [10].

Even if Nixon says 'What Jones states is true', there are fictions around. We have $\mathrm{S}j\S\neg\mathrm{T}\epsilon r\mathrm{S}nr$, and now also $\mathrm{S}n\S\mathrm{T}\epsilon r\mathrm{S}jr$, so if $(\exists!r)\mathrm{S}jr$ and $(\exists!r)\mathrm{S}nr$, we get that $\epsilon r\mathrm{S}jr = \S\neg\mathrm{T}\epsilon r\mathrm{S}nr$, and that $\epsilon r\mathrm{S}nr = \S\mathrm{T}\epsilon r\mathrm{S}jr$, which means, using the truth scheme, that $\mathrm{T}\epsilon r\mathrm{S}jr$ iff $\neg\mathrm{T}\epsilon r\mathrm{S}nr$, and that $\mathrm{T}\epsilon r\mathrm{S}nr$ iff $\mathrm{T}\epsilon r\mathrm{S}jr$; and those are together contradictory. It follows that at least one of the speakers does not make an identifiable statement, in which case the other is speaking about a fiction.

It is this indeterminacy of meaning which distinguishes the present approach from, for instance, the logic of ambiguity which Brown described as an equivalent alternative to standard systems of paraconsistency for handling the paradoxes [1]. One of the features of operator theories is that they do not incorporate 1-1 expression relations between sentences and propositions—so they allow for ambiguity. But if only plain ambiguity was involved then in a disambiguated language, which restored the 1-1 relationship, paradoxes could still be generated. The further, crucial feature of operator theories is that they allow for ineffability, i.e. reference to meanings which cannot be independently identified. Once the 1-1 nature of the expression relation is rejected we must accept that there may be things, as Kneale said, which can only be gestured towards, but not otherwise expressed.

This is illustrated again in other well-known paradoxes, which fall into the same category as those discussed above: The Knower Paradox, The Strengthened Liar, Curry's Paradox, and the Paradox of the Preface. For the record, solutions to The Strengthened Liar, Curry's Paradox, and the Paradox of the Knower will be given now. The Paradox of the Preface was dealt with, using Goodstein's system, in [9], and it is easily adjusted, as above.

The Strengthened Liar is sometimes said to be a difficulty for propositional solutions to the Liar: if sentences can lack meaning, then what about 'this sentence is false, or has no truth value'? Seemingly if it is false it is true and has a truth value; if it is true, it is false (since it then must have a truth value): and if it lacks a truth value then it is true (since it says so). The formulation presumes that it is sentences which are true or false, so consider instead: 'this sentence either expresses a false proposition, or does not express any proposition'. Using the modified Goodstein calculus, one starts from

$$\mathrm{S}a\S((\exists x)(\mathrm{S}ax \wedge \neg\mathrm{T}x) \vee \neg(\exists x)\mathrm{S}ax).$$

One can then obtain $(\exists x)\mathrm{S}ax$, and so, if we had $(x)(\mathrm{S}ax \supset \mathrm{T}x)$, then we would have $(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{T}x)$, because then $\mathrm{T}\S((\exists x)(\mathrm{S}ax \wedge \neg \mathrm{T}x) \vee \neg(\exists x)\mathrm{S}ax)$. Hence, absolutely, $\neg(x)(\mathrm{S}ax \supset \mathrm{T}x)$, i.e. $(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{T}x)$. But that means, also, that $(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{T}x) \vee \neg(\exists x)\mathrm{S}ax$, and so, by existential generalisation, $(\exists y)(\mathrm{S}ay \wedge \mathrm{T}y)$, where $y = \S((\exists x)(\mathrm{S}ax \wedge \neg \mathrm{T}x) \vee \neg(\exists x)\mathrm{S}ax)$. Hence a does not express a single proposition.

Curry's Paradox is significant, since Priest, for instance, admits that his dialetheism is not sufficient to resolve it—he also wants to abandon 'Absorption'. But there is no trouble with 'if what this sentence states is true then q' (for arbitrary 'q'). Certainly if we had anything like

$$p \equiv (p \supset q),$$

we could deduce the arbitrary q, by reasoning that given $\neg p$ we would have $(p \supset q)$, and hence p, so we must have p, and so $(p \supset q)$, and so q. But if instead we start from

$$\mathrm{S}a\S(\exists x)(\mathrm{S}ax \wedge (\mathrm{T}x \supset q)),$$

and suppose, for instance, $(x)(\mathrm{S}ax \supset \neg \mathrm{T}x)$, that means $\neg \mathrm{T}\S(\exists x)(\mathrm{S}ax \wedge (\mathrm{T}x \supset q))$, and so $\neg(\exists x)(\mathrm{S}ax \wedge (\mathrm{T}x \supset q))$. From this it follows, amongst other things, that $(x)(\mathrm{S}ax \supset \mathrm{T}x)$, and since we also have $(\exists x)\mathrm{S}ax$, we must say $\neg(x)(\mathrm{S}ax \supset \neg \mathrm{T}x)$, and so $(\exists x)(\mathrm{S}ax \wedge \mathrm{T}x)$. No further conclusion can be drawn, however, about whether $\mathrm{T}\S(\exists x)(\mathrm{S}ax \wedge (\mathrm{T}x \supset q))$, since there is no guarantee that a is non-ambiguous. Comparable results follow if we start from

$$\mathrm{S}a\S((\exists x)(\mathrm{S}ax \wedge \mathrm{T}x) \supset q),$$

or

$$\mathrm{S}a\S(\mathrm{T}\epsilon x\mathrm{S}ax \supset q).$$

The Paradox of the Knower was influential historically, since it motivated Montague towards his support for an operator theory of the attitudes. If the objects of attitudes like knowledge are syntactic then, by considering 'this sentence is not known to be true', for instance, we can derive a paradox, since if it is known to be true then it is true, and so not known to be true; but if it is thereby shown to be not known to be true, not only is it shown to be true (because that is what it says), also it has come to be known to be true. Hence, by *Reductio*, there cannot be a syntactic account of the objects of knowledge. Consider, however, a version which does not presume it is sentences which are known to be true, but instead the facts which they may state, viz:

$$\mathrm{S}a\S(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{K}\S\mathrm{T}x).$$

If $(x)(\mathrm{S}ax \supset \mathrm{K}\S\mathrm{T}x)$, then $\mathrm{K}\S\mathrm{T}\S(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{K}\S\mathrm{T}x)$, and so $(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{K}\S\mathrm{T}x)$. Hence $(\exists x)(\mathrm{S}ax \wedge \neg \mathrm{K}\S\mathrm{T}x)$, absolutely, and so, from the latter, we can say

$\neg K\S T\epsilon x(Sax \wedge \neg K\S Tx)$, and also $K\S T\S(\exists x)(Sax \wedge \neg K\S Tx)$. But there is no way to get from any of this that

$$\epsilon x(Sax \wedge \neg K\S Tx) = \S(\exists x)(Sax \wedge \neg K\S Tx),$$

i.e. that that thing which a says which is not known to be true is that there is something a says which is not known to be true. So when we move away from the syntactic account of the attitudes there is no trouble.

4. Priest's problem

There is another paradox which fits in here, although it is not usually considered in the same class: if the sentence ϕ is defined to be 'ϕ is (informally) unprovable', then this sentence has been said to generate 'Gödel's Paradox'. The problem it posed was made prominent in [7], although it had occupied the same writer since at least 1971. Priest said :

> We are assuming that English can be turned into a formal axiomatic system and that the truth of its Gödel sentence ϕ can be proved in English. Hence ϕ is assertible (i.e. provable) in English. Gödel's Theorem states that any such system can prove its own Gödel sentence if and only if it is inconsistent. It follows that English is inconsistent. [6, p. 130]

But that conclusion only follows if English can be turned into a formal axiomatic system, and if it is not formalistic but contentful, because it employs a provability operator, then Priest's conclusion does not follow. The expectation that any proper reasoning process should be formalistic is part of the practical grip which the pre-occupation with computers has had in recent decades. But it has a further consequence which must be remarked: that seeing English is not inconsistent is a very difficult mental challenge for people with that general formalistic expectation, despite the proof's immediacy.

There is, of course, no difficulty in finding such a sentence as Priest supposes, but, on an operator analysis, such sentences have no relevance to anything to do with truth—or therefore provability—in natural language. Used English is interpreted, and there is no comparable item leading to paradox in that area, simply because of that matter of interpretation. For what is (informally) provable is some fact, not some sentence, or formula, as with knowledge before. Forgetting the difference between facts and formulas is presumably the reason why, in common representations of the objects of formal proofs, a proposition is indicated, rather than a mentioned sentence: '$\vdash p$'. For only if that expression is used to represent an operator notion of provability, is it certainly 'p' that is involved, since "'p'"

would be the proper way of symbolising what is derived in formal proofs. By a common convention the quotation marks are omitted, but that only obscures the categorical difference between the two cases. The proved fact in question is then certainly what some sentence is standardly taken to express, i.e. what is true in its standard model. But even then, remembering the arguments against Tarski before, it is not the sentence which is what is true in that standard model: what is true is that 1 + 1 = 2, not '1 + 1 = 2', and 'that 1 + 1 = 2' is not a sentence, it is a noun phrase referring to a proposition.

The relevant locutions are not predicative locutions like ' '*p*' is derivable' but operator locutions like 'it is provable that *p*', and we have seen that it is trivial that there can be no self-reference with such locutions. For there is no '*q*' such that '*q*' = 'it is unprovable that *q*'—that is ruled out by mereology, since nothing can be a proper part of itself. In addition there can be no '*q*' such that it is necessary that

$$q \equiv \text{ it is unprovable that } q,$$

since 'it is provable that' at least obeys the rules for the modal operator in the system **T**. For that means that, with 'it is provable that' as 'L', L*q* would entail *q*, and so ¬L*q*, giving ¬L*q* absolutely, and so *q* likewise, and hence, by the Rule of Necessitation, contradictorily L*q*. Certainly we can say such things as 'that proposition where the proposition is that it itself is not provable, is provable', but there is no independent namely-rider expressible with a 'that' clause, as we saw when developing Kneale's point at the start. In addition, 'that proposition where the proposition is that it itself is not provable' cannot refer to anything properly so described, as an epsilon analysis demonstrates with complete formal accuracy. The proposition this phrase refers to is indeterminate, and so also must be its truth, and its provability.

Thus while we can construct the referring phrase 'that proposition where the proposition is that it itself is not provable', once we remember the possibility that propositional referential terms are misnomers, the conclusion in connection with this phrase is merely that no proposition with this character can exist. That is to say, although

$$(\exists s)(s = \epsilon r(r = \S\neg \mathrm{P}r)),$$

we can show that

$$\neg(\exists r)(r = \S\neg \mathrm{P}r).$$

For if the reverse of the latter were the case, then, for some '*s*', we could say that $s = \S\neg \mathrm{P}s$, and so, if P*s*, then, because T*s*, we would also have T§¬P*s*, giving, by the T-scheme, ¬P*s*, a contradiction. Hence ¬P*s*, absolutely, and that means it is provable that $\neg$Ps, i.e. P§$\neg$Ps, which gives Ps, and so another contradiction. But the conclusion to be drawn is just that

$$(\exists r)(r = \S\neg \mathrm{P}r)$$

is not the case.

The crucial difference between self-reference with sentences, and this attempted self-reference with propositions concerns the possibility of displaying the items they are respectively about. In connection with the former, the normal quotation names for sentences such as

'this sentence is not provable'

immediately identify what objects they are names of. But there is no comparable, grammatical process for automatically identifying what is referred to by 'that proposition where the proposition is that it itself is not provable'. Propositions are displayed using the demonstrative form 'that p', and not only does mereology prevent 'it is unprovable that p' from being itself the same 'p', other referring phrases besides 'that p' do not display the proposition, and so may lie i.e. be misnomers.

In short, one can have $q \equiv \neg \vdash$ 'q' (where '$\vdash$' is a predicate of mentioned sentences) but not $q \equiv \neg \vdash q$ (where '$\vdash$' is an operator on used sentences), and so there is no proof of incompleteness, following Gödel's proof, in the second case. Moreover, no induction to the first epsilon number, or anything remotely like it, is needed to previously show that the natural notion of proof is consistent. Because of the T-axiom again, there is a simple one line proof of this fact: one can no more have L$p \wedge$ L$\neg p$ than one can have $p \wedge \neg p$. Priest saw things differently:

> Thus we see that our naive notion of proof appears to outstrip the axiomatic notion of proof precisely because it can deal with semantic notions. Of course, we can formalize the semantics axiomatically but then naively we can reason about the semantics of that system. As long as a theory can not formulate its own semantics it will be Gödel incomplete, i.e. there will be sentences independent of the theory which we can establish to be true by naive semantic reasoning. [7, p. 223]

The principal corrections needed are, first, the removal of the idea that it is sentences which one can establish to be true, but then that idea's main consequences, namely that the semantics of a language cannot be expressed in the same language without contradiction, and that a consistent language therefore must be incomplete.

It seems clear, therefore, that there is no intractable difficulty with liars. It is Tarski's original T-scheme which is the trouble. For

T'p' iff p,

clearly does not hold in general. The hope was, of course, that in a perfect language such a scheme would hold, and natural language, under that expectation, was inevitably thought not to allow consistent semantic closure. But clearly it does not hold for ambiguous sentences, and many classic paradoxes, it turns out, contain a similar semantic indeterminacy, as Prior saw [8, p. 106]. Human language is not so honest, it might be said: what it states cannot always be read off its face. That point, also, provides a further explanation for why The Liar has remained unsolved for so long. For plenty of highly intelligent people have put all of their mind onto the task. But maybe they were too trusting. The problem with solving the Liar, in a large part, has been realising how thoroughgoing the lying really is.

5. Appendix

The discussion of Goodstein's ground-breaking work, at the end of section 2, pointed out that the alternative senses required by his proof(s) need not be expressible linguistically, since pragmatic implications are involved. Specifically, in the case there analysed, what a says which is true can only be given in epsilon calculus terms, since it is $\S(\mathrm{S}au \supset \neg \mathrm{T}u)$, for $u = \epsilon r \mathrm{S}a\S(\mathrm{S}ar \supset \neg \mathrm{T}r)$, and this has no equivalent in the predicate calculus.

A more elementary case will illustrate more clearly the kind of thing which is going on, say where a says that something he says is not true, i.e. $\mathrm{S}a\S(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r)$. What a says here is true, but what he says which is not true is not so readily locatable: in fact he has an instantiation of the existential remark in mind, and it is that which is not true. One needs the epsilon calculus to realise fully, for a start, that making an existential remark does involve referring, or at least alluding to an instance, since the predicate calculus hasn't got an automatic placeholder for what individual is being spoken about when someone makes such a remark. But the problem is that the existential quantifier in this case is within the scope of the intensional operator, so any instantiation which a has in mind is up to him to say, and remains a feature of the pragmatic context.

Here is the proof that the instantiation is not true. First, given

$$\mathrm{S}a\S(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r),$$

then if $(r)(\mathrm{S}ar \supset \mathrm{T}r)$ then $\mathrm{T}\S(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r)$, and so $(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r)$ absolutely, and so $(\exists r)(\mathrm{S}ar \wedge \mathrm{T}r)$, since we have the instantiation

$$\mathrm{S}a\S(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r) \wedge (\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r).$$

To find the instantiation of '$(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r)$', by contrast, we must proceed as follows. We know that $\mathrm{S}a\S(\exists r)(\mathrm{S}ar \wedge \neg \mathrm{T}r)$, so $\mathrm{S}a\S(\mathrm{S}az \wedge \neg \mathrm{T}z)$ for a certain epsilon term 'z' = 'that thing which a says which is not true', and so $(\exists r)Sa\S(Sar \wedge \neg Tr)$, by existential generalisation, and $Sa\S(Say \wedge \neg Ty)$ for another epsilon term 'y' = 'that thing about which a says that he says it and it is not true'. But $\neg(Say \wedge \neg Ty)$, i.e. if Say then Ty. For there are only two options if y is actually stated, namely $y = \S(Say \wedge \neg Ty)$, and $y = \S(Saz \wedge \neg Tz)$. In the first case we get $Ty \equiv (Say \wedge \neg Ty)$, which yields $\neg Say$, which is impossible. In the second case we get $Ty \equiv (Saz \wedge \neg Tz)$, i.e. $Ty \equiv (\exists r)(Sar \wedge \neg Tr)$, and so Ty, since $(\exists r)(Sar \wedge \neg Tr)$. Hence $Sa\S(Say \wedge \neg Ty) \wedge \neg(Say \wedge \neg Ty)$, providing the required instantiation for $(\exists r)(Sar \wedge \neg Tr)$.

In Goodstein's original case above a similar conclusion is available starting from $Sa\S(x)(Sar \supset \neg Tr)$. From this we get that $(\exists r)(Sar \wedge \neg Tr)$, and $(\exists r)(Sar \wedge Tr)$, since if $(x)(Sar \supset \neg Tr)$ then, by substitution of what was first given, $\neg T\S(x)(Sar \supset \neg Tr)$, giving $(\exists r)(Sar \wedge Tr)$, and so $(\exists r)(Sar \wedge \neg Tr)$, with the instantiation of the latter being provided by

$$Sa\S(x)(Sar \supset \neg Tr) \wedge (\exists r)(Sar \wedge Tr).$$

To get the instantiation of the former, i.e. what specific thing a says which is true, we first obtain from what was first given, $Sa\S(Sav \supset \neg Tv)$, for $v = \epsilon r \neg(x)(Sar \supset \neg Tr)$, by the epsilon definition of the universal quantifier, and then from this $(\exists r)Sa\S(Sar \supset \neg Tr)$, by existential generalisation, and so $Sa\S(Sau \supset \neg Tu)$, for $u = \epsilon r Sa\S(Sar \supset \neg Tr)$, again by the epsilon definition of the existential quantifier. But $T\S(Sau \supset \neg Tu)$, i.e. if Sau then $\neg Tu$, since if Sau, then either $u = \S(Sau \supset \neg Tu)$ or $u = \S(Sav \supset \neg Tv)$. In the first case $Tu \equiv (Sau \supset \neg Tu)$, which gives $\neg Sau$, which is impossible. In the second case $Tu \equiv (Sav \supset \neg Tv)$, i.e. $Tu \equiv (r)(Sar \supset \neg Tr)$, and so $\neg Tu$, since $(\exists r)(Sar \wedge Tr)$. It follows that $Sa\S(Sau \supset \neg Tu) \wedge T\S(Sau \supset \neg Tu)$, providing the required instantiation of $(\exists r)(Sar \wedge Tr)$.

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