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CLASSICAL ARITHMETIC IS QUITE UNNATURAL

1. Aim of the paper

It is a generally accepted idea that strict finitism is a rather marginal view within the community of philosophers of mathematics. If one therefore wants to defend such a position (as the present author does), then it is useful to search for as many different arguments as possible in support of strict finitism. Sometimes, as will be the case in this paper, the argument consists of, what one might call, a "rearrangement" of known materials. The novelty lies precisely in the rearrangement, hence on the formal-axiomatic level most of the results presented here are not new. In fact, the basic results are inspired by and based on Mycielski (1981). This does not imply of course that Mycielski would agree with my use of his results (frankly, I think he would not[1]).

The argumentative strategy of this paper is to show that classical arithmetic (I will limit myself to that theory), say in the form of Peano Arithmetic (PA), is a quite "unnatural" theory (in a sense to be specified) and, in contrast, that strict finitist arithmetics are quite natural. Obviously what needs to be specified is the meaning of "(un)natural". To do that, I want to invoke an analogy with supertasks.

In general terms a supertask is any task that consists of a countably infinite number of repeated actions to be executed in a finite time interval[2].

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[1] One of the remarks in the paper (p. 627, remark (7)) states that particular axioms are introduced to secure the potential infinity of the theory. This makes clear that Mycielski is not interested in strict finitism as such.

[2] An excellent overview of supertasks is Laraudogoitia (2002).

A typical example of a supertask is the so-called Thomson lamp. A one-minute interval is divided into a countably infinite number of subintervals — half a minute for the first interval, half of what remains for the next interval, and so forth — and in the first interval the lamp is switched on, in the second switched off, and so forth. Usually these supertasks have been devised to ask (and preferably to answer) questions about the relation between the successive stages up to the end of the one-minute interval and the state at the end of the time interval. In the case of the Thomson lamp, e.g., the question is whether the lamp is on or off at the end. There is a good argument to show that the lamp is in neither state, hence this supertask is (sometimes) used to undermine particular forms of determinism. However for the purpose of this paper it is another relation I want to focus on. Often in supertasks, even if the outcome at the end of the time-interval is determined, the outcome turns out to be *quite* different from the previous stages. A splendid illustration of how different things can be is the Ross paradox[3].

In the Ross supertask an empty box and a countably infinite number of labeled balls are given. In the first interval balls 1 up to 10 go in the box and 1 goes out, in the second interval 11 up to 20 go in and 2 comes out, In general in the nth interval, balls $10(n-1)+1$ up to $10n$ go in and n comes out. After one minute, surprisingly enough, the box has to be empty. The argument is quite simple: if there were to a ball left in the box, that ball must have a label, say k, but in the kth interval that ball was removed from the box. Formally all of this seems quite correct, but what is important for the purpose of this paper is the amazing if not staggering contrast between the final outcome — an empty box — and all the previous stages where at each stage 9 balls are added to the box.

Suppose now that we had arguments in support of the view that the stages before the final outcome are far more likely to occur "in the real world" than the final outcome itself. Or better still: suppose we had arguments to believe that the final outcome is an ideal state never to be reached. Is it not reasonable then to argue that we would do better to study the stages before the outcome rather than the outcome itself? What does it help us to make an elaborate study of an empty box to remain within the story of Ross' paradox — when the situation that we are most likely to be confronted with is a nearly uncontrollably overflowing box[4]? Little or nothing must be the

[3] Papers directly related to this paradox are: Allis and Koetsier (1991) and (1995), Holgate (1994) and Van Bendegem (1994a)

[4] A perhaps silly example. Questions about the size of the box are of no importance if we only take the final state into account, since the box in that state is empty. But at stage, say 10^{10}, there will be 9.10^{10} balls in the box. At those stages one should definitely

answer.

Here is the analogy. In terms of the numbers (or, perhaps better, the numerals) we deal with everyday, it seems obvious to me that we are always dealing with finite sets of such numbers (and a finite number of such sets). I am referring here of course to the use of actual numerals in a given notation system, e.g., in a decimal system a numeral would be 67236856. I have presented arguments in other papers to defend the thesis that the world we are in and the capacities we have to develop in order to know this world, are all perfectly finite[5]. Although many philosophers and mathematicians will accept this point — apparently that is not what they see as a problem — , they will nevertheless claim that one should make abstraction of these explicitly finite bounds, be a bit more liberal and therefore use an arithmetical theory such as PA. In short, PA is an ideal state. And this statement leads directly to the following analogy.

The basic idea or suggestion of this paper is to see PA as the final outcome of a kind of supertask[6]. Or, in other words, the question is whether it is possible to find a series of stages S_i that approach PA as closely as possible in such a way that:

(a) all stages are strictly finite, and
(b) all stages have a set of (nice) properties in common, and
(c) PA does not have these (nice) properties and therefore is too different from the previous stages to be interesting.

If such a thing is possible, then it shows that PA is quite "unnatural" and, assuming as I have done, that the stages provide us with a better description of things as they are, it is therefore more interesting to stick with these stages, and simply to forget about the final outcome, i.e. PA. In other words: one should become a strict finitist[7].

worry about the size of the box.

[5] See Van Bendegem (1994b) and (1998).

[6] The less important thing is the one-minute interval. My argumentation does not change if the one-minute interval becomes an open interval. The point is the fundamental difference between a particular stage and the final outcome.

[7] Of course for the purpose of this paper, I am talking about a countably infinite number of stages. This is only motivated by argumentative considerations. I am presenting an argument to the defender of PA and, hence, to be able to "attack" PA, at least I have to use the same language, the same concepts, and so on. A comparison: for an atheist to be able to discuss with a theologian what are, if any, God's properties, the atheist will actually have to use the word God.

The next paragraph gives a full formal description of an arbitrary stage S_i. Paragraph 3 shows how the final outcome can be reached from the previous stages. It is shown that this limit is indeed PA. It is then easily demonstrated that all the nice properties do in fact disappear. Finally in paragraph 4 some further consequences are discussed.

2. Formal presentation

The object of this part is to present a formal description of a stage S_i and discuss some of its properties. The first thing we need is a language L.

2.1. Language

(a) Vocabulary:

(a1)	constant names for the first i+1 natural numbers:	$0, s(0), s^2(0), , s^i(0)$
(a2)	a constant name for the "last" number:	$*$
(a3)	names for variables:	$x, y, z, \ldots$
(a4)	function names:	$s, +, .$
(a5)	a single predicate:	$=$
(a6)	logical constants:	$\sim, \supset, \equiv, \&, \vee, \exists, \forall$

(b) Formation rules:

(b1) the set T of terms $t, t', t'', \ldots$ is defined as usual, i.e. a term t is any expression that is either a constant name, a variable or the result of applying one of the functions to given terms.

(b2) the set of atomic formulas AF is also defined as usual, i.e., the set of all formulas of the form $t = t'$, where t and t' are terms.

(b3) the set of all formulas F is equally defined as usual.

Note concerning (b1): this definition does allow, of course, to write down expressions such as $s(s(s^i(0))$. Obviously this can be read as a constant name indicating the number $s^{i+2}(0)$. However in this paper the expression will be read as the application, two times, of the successor function to the constant name $s^i(0)$. The fact that I am rather liberal about the expressibility of the language is to go along with the infinitist as far as possible. From a strict finitist framework the length of the formulas should also be strictly finite.

2.2. Semantics

A model M_i for a stage S_i is a triple $\langle D_i, I_i, v_i \rangle$ such that:

(c1) the domain D_i is the set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{i}, \mathbf{i+1}\}$

(c2) the interpretation I_i is a function that maps:

- a constant name $s^j(0)$ onto $\mathbf{j}$, so $I(s^j(0)) = \mathbf{j}$
- * onto $\mathbf{i+1}$, so $I(^*) = \mathbf{i+1}$
- a variable x onto any element of the domain, $I(x) = \mathbf{j}$, for some $\mathbf{j}$.
- the function s corresponds to a function $\mathbf{s}$ from D_i to D_i such that:
  $$\mathbf{s}(\mathbf{j}) = \mathbf{j+1}, \text{ for all } \mathbf{j} \leq \mathbf{i} \text{ and } \mathbf{s}(\mathbf{i+1}) = \mathbf{i+1}.$$
- addition and multiplication onto the following functions:
  - a function $\oplus$ from $D_i \times D_i$ to D_i, such that

    $\mathbf{m} \oplus \mathbf{n} = \mathbf{k}$ if $\mathbf{m} \oplus \mathbf{n} \leq \mathbf{i+1}$, where $\mathbf{k}$ is the classical value
    $\mathbf{m} \oplus \mathbf{n} = \mathbf{i+1}$ for all other values,
  - a function $\oplus$ from $D_i \times D_i$ to D_i, such that

    $\mathbf{m} \oplus \mathbf{n} = \mathbf{k}$ if $\mathbf{m} \oplus \mathbf{n} \leq \mathbf{i+1}$, where $\mathbf{k}$ is the classical value
    $\mathbf{m} \oplus \mathbf{n} = \mathbf{i+1}$ for all other values.

(c3) the valuation v_i is a function that maps:

- atomic formulas to $\{0, 1\}$, such that $v_i(t = t') = 1$ iff $I(t) = I(t')$
- for all other formulas the clauses are perfectly classical.

Some comments, especially concerning (c2): the first remark concerns the successor function that has a property that the standard successor function does not have, viz. the loop at the end. That, of course, turns the model into a strict finitist model. The second remark concerns addition and multiplication. As will be shown in the next paragraph, it is not necessary once the successor function has been so restricted to introduce any additional restrictions for these functions, although semantically speaking it seems otherwise. This will turn out to be a very important element in defense of strict finitism. I will return to this problem in the last paragraph of the paper.

2.3. Axiomatics

Since we have names for all elements in the domain and the number of elements is finite, it is completely straightforward to present an axiomatisation of an arbitrary stage S_i:

(Ax0) All necessary logical axioms for first-order predicate calculus, together with the following non-logical axioms:

$(Ax1)$ $\sim (\exists x)(s(x) = 0)$

$(Ax2)$ $(\forall x)(\forall y)((x \neq^* \& y \neq^*) \supset (s(x) = s(y) \supset x = y))$

$(Ax3)$ $s(s^i(0)) =^* \& s(^*) =^*$

$(Ax4)$ $(\forall x)(x + 0 = x)$

$(Ax5)$ $(\forall x)(\forall y)(x + s(y) = s(x + y))$

$(Ax6)$ $(\forall x)(x.0 = 0)$

$(Ax7)$ $(\forall x)(\forall y)(x.s(y) = x.y + x)$

and, on the level of rules :

(R1) The usual logical rules for first-order predicate calculus, together with mathematical induction

(R2) $A(0), (\forall x)(A(x) \supset A(s(x))/(\forall x)A(x)$.

I should add here that this presentation of a strict finitist arithmetic is just one of a list of possibilities. In my (1994b), inspired by the approach of Graham Priest, I used a type of paraconsistent logic instead of classical logic. Both Vermeir (1999) and Batens (unpublished) have presented criticisms of this approach and both authors have presented alternatives[8]. Here I have tried to stay as close as possible to an axiomatic formulation of PA. As one can obviously notice, the only difference with such an axiomatisation, is the restriction in (Ax2) and the explicit addition of (Ax3). This is the idea borrowed from Mycielski and also used by Vermeir; it is a very simple, but extremely powerful idea. Next follow some theorems leading up to the "nice" properties. Although the proofs are fairly trivial, I do write them down explicitly because they have some interesting properties.

THEOREM 1. $(\forall x)(x = s(^*) \supset x = {}^*)$.

PROOF. Suppose $x = s(^*)$, use (Ax3) and conclude $x = {}^*$. ∎

THEOREM 2. $(\forall x)(x \neq {}^* \supset x \neq s(x))$.

[8] One of the paraconsistent logics used is LP (Logic of Paradox) of Graham Priest. This logic has three truth-values, viz. 0, 1, and 0,1. To be true means that 1 belongs to the truth-value. The effect is that all statements that talk about numbers larger than the limit are $\{0, 1\}$. Whereas in the presentation here, positive atomic statements turn out to be true and their negations false. So, if $^* = s^3(0)$, then $s^4(0) = s^4(0)$ and $s^4(0) = s^7(0)$ are both true, but $s^4(0) \neq s^4(0)$ and $s^4(0) \neq s^7(0)$ are both false. What remains the case however, is that all mathematical reasoning above the limit is quite uninteresting.

Proof. By induction:
Basis: $0 \neq {}^* \supset 0 \neq s(0)$. The right part is guaranteed by (Ax1), hence the result follows. Induction step: suppose $(\forall x)(x \neq {}^* \supset x \neq s(x))$ and assume that $s(x) \neq {}^*$. We have to show that $s(x) \neq s(s(x))$. Suppose not: $s(x) = s(s(x))$. Now either $x = {}^*$ or $x \neq {}^*$. The first case is excluded because it follows by (Ax3) that $s(x) = s({}^*) = {}^*$, contradicting the assumption. Thus $x \neq {}^*$. As both $x \neq {}^*$ and $s(x) \neq {}^*$, (Ax2) implies that $x = s(x)$. Contradiction. ■

Corollary 1. *$(\forall x)(x = s(x) \supset x = {}^*)$.*

Theorem 3. *$(\forall x)(x \neq 0 \ \&\ x \neq s(0) \ \&\ \dots \ \&\ x \neq s^i(0) \supset x = {}^*)$.*

Proof. By induction:
Basis: trivial.
Induction step: suppose $x \neq 0 \ \&\ x \neq s(0) \ \&\ \dots \ \&\ x \neq s^i(0) \supset x = {}^*$. Assume that we have $s(x) \neq 0 \ \&\ s(x) \neq s(0) \ \&\ \dots \ \&\ s(x) \neq s^i(0)$. We now have to show that $s(x) = {}^*$. From the formulas $s(x) \neq s^j(0)$, for j ranging from 1 to i, follows that $x \neq s^{j-1}(0)$. This takes care of all cases apart from $s^i(0)$. Now either $x = s^i(0)$ or $x \neq s^i(0)$. In the former case, it follows that $s(x) = s(s^i(0)) = {}^*$ by (Ax3). In the latter case, we use the induction step to derive $x = {}^*$, hence $s(x) = {}^*$ by the same axiom. ■

Corollary 2. *$\sim(\exists x)(A(x) \ \&\ x \neq 0 \ \&\ x \neq s(0) \ \&\ \dots \ \&\ x \neq s^i(0) \ \&\ x \neq {}^*)$. (If there was such an x, then we would have $x \neq 0 \ \&\ x \neq s(0) \ \&\ \dots \ \&\ x \neq s^i(0) \ \&\ x \neq {}^*$, contradicting the previous theorem.)*

Here is the first "nice" result.

Theorem 4. *$(\forall x)A(x)$ is equivalent to $A(0) \ \&\ A(s(0)) \ \&\ \dots A(s^i(0)) \ \&\ A({}^*)$.*

Proof. From left to right is trivial. Now suppose that $A(0) \ \&\ A(s(0)) \ \&\ \dots A(s^i(0)) \ \&\ A({}^*)$ and that $\sim (\forall x)A(x)$ or $(\exists x) \sim A(x)$. Either $(x = 0 \vee x = s(0) \vee \dots x = s^i(0) \vee x = {}^*)$ or $\sim (x = 0 \vee x = s(0) \vee \dots x = s^i(0) \vee x = {}^*)$. In the former case, each possibility leads to a contradiction and, in the latter case, we have a contradiction because of corollary 2. ■

Corollary 3. *$(\exists x)A(x)$ is equivalent to $A(0) \vee A(s(0)) \vee \dots A(s^i(0)) \vee A({}^*)$*

Although it is perhaps an obvious fact, it is interesting to point out that the reductions of the quantifiers is only possible because of the explicit axiom (Ax3) that identifies the largest number. Without (Ax3) this is not possible.

A perhaps less obvious fact to note is that mathematical induction is required as a proof tool. Without (R2) the results above could not be obtained, hence (R2) implies the reduction of the quantifier. Obviously if we have the reduction then, for any statement A(x), (R2) is derivable. For suppose that we have A(0) and $(\forall x)(A(x) \supset A(s(x))$. The second statement is equivalent to a finite conjunction of statements of the form $A(s^j(0)) \supset A(s^{j+1}(0))$ and the statement $A(s^i(0)) \supset A(^*)$. All statements that follow will be of the form $A(^*) \supset A(^*)$, but these are trivial. Therefore by a straightforward application of modus ponens, a sufficient number of times, we derive $(\forall x)A(x)$. In short, (R2) and reduction of the quantifiers imply one another.

Next come the "nice" meta-results.

THEOREM 5. *$\vdash A$ iff $\models A$. In other words, S_i is (weakly) complete.*

PROOF. Because of theorem 4 and corollary 3 the method of quantifier elimination can be used to show completeness. ∎

THEOREM 6. *S_i is consistent.*

PROOF. Any model of S_i is finite. ∎

THEOREM 7. *S_i is categorical.*

PROOF. Because each model has the same finite number of elements (due to the explicit axiom (Ax3), as mentioned before), there is a one-to-one mapping between every two models. ∎

COROLLARY 4. *Löwenheim-Skolem theorems do not apply.*

THEOREM 8. *S_i is (obviously) decidable.*

PROOF. Trivial. ∎

COROLLARY 5. *Gödel's theorems do not apply.*

As must be obvious by now, the theories at each stage S_i are really among the "nicest" theories imaginable: complete, decidable, consistent, categorical. It seems to good to be true. As mentioned before, above the limit things become quite uninteresting, but for the strict finitist that is precisely the place not to be. If in addition one tends to be rather liberal about the "true" size of * — as I have tried to show in my (1998) — then all the interesting things can be done well below the limit.

Theorem 9. *For every stage S_i, there is a later stage S_j such that the consistency of S_i can be proven in S_j.*

Proof. Although S_i is a strictly finite stage[9], one needs a later stage to be able to express all the sentences of stage S_i, (say by some form of coding), but since S_i is finite, such a stage must exist. ∎

Now that we have a description of all the stages S_i, let us turn our attention to the "ugly" limit.

3. The unnaturalness of classical arithmetic

The first question to settle is how to define the limit. There are several possibilities but, relying once more on the analogy with supertasks, one possibility seems an obvious candidate. Some supertasks, such as Ross' paradox, involves balls being moved around. It is therefore not easy to determine their final positions. A quite acceptable criterion is a (kind of) continuity principle. In terms of the movements of objects, such a principle could state that, if at a certain interval a ball reaches a specific position and in all later intervals it does not change from that position, then that will be the position of the ball in the final outcome. My proposal is to apply the same idea to the arithmetical domain, to the interpretation function and to the truth-values of arithmetical statements in order to determine the limit semantically. It is of course understood that the language of the limit is the same language as the language of the different stages, unless otherwise indicated. I formulate the "continuity" principle for arithmetic in terms of the three following rules:

Rule 1: A number n will belong to the domain D of the final outcome, if there is a stage S_i such that the number n belongs to the domain D_i of that stage and also belongs to all domains D_j of the stages S_j that follow. In addition D will contain nothing but these numbers.

Rule 2: The interpretation function I will map a term t onto an element d of the domain D, $I(t) = d$, if at a certain stage S_i, the interpretation of t is d, $I_i(t) = d$ and for all the stages S_j that follow, $I_j(t) = d$. If this is not

[9] Although we have of course the full countable language of PA at our disposal, since there are strictly finite models for a given stage, the number of sentences that are not equivalent to one another is also finite. In that sense one does not need a full coding. Or to put it otherwise: at a later stage we can always give a complete description of the model that is finite.

the case, then I does not assign anything and the term is deleted from the language (hence the "nothing but" clause in Rule 1).

Rule 3: If A is some statement and there is a stage S_i, such that $v_i(A) = 1$ (or 0), and such that for all stages S_j that follow, $v_j(A) = 1$ (or 0) – in other words, A acquires a stable truth-value —, then that will be the truth-value of A in the final outcome. For all other statements it is supposed that the final truth-value is false.

Rule 1 implies immediately that the domain D of the limit model will be the set of the standard natural numbers, $\mathbb{N}= \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}, \ldots\}$. A few things follow straightforwardly from Rule 2:

(a) It is evident that for the constant names, $I(s^i(0)) = i$.
(b) Addition and multiplication will correspond to classical addition and multiplication as this is the case at all stages. The more interesting cases are the interpretation of * and s:
(c) * has to disappear from the language. At each stage * is mapped onto a different element of the domain, hence it has no stable interpretation.
(d) It thereby follows that the interpretation of the successor function s has to be: $\mathbf{s}$ is a function from D_i to D_i such that $\mathbf{s}(\mathbf{j}) = \mathbf{j} + \mathbf{1}$, for all $\mathbf{j}$ in D.

    By the elimination of the loop, i.e., the number such that $s(\mathbf{j}) = \mathbf{j}$ at a certain stage disappears as well, thus the restriction on $\mathbf{s}$ is without meaning.

Applying rule 3 is slightly more tricky. First if we want to determine the truth-values of complex statements, we can easily reduce the question to atomic formulas, since in all stages S_i the classical rules are followed. Furthermore, since addition and multiplication are classically defined throughout all the stages S_i, we can limit ourselves to expressions of the form x = y, where x and y are names for objects in the domain. But then it is obvious that all statements of the form $\mathbf{i} = \mathbf{i}$ will turn out to be true, because such a statement is true right from the first stage (below the limit, classically so, above the limit, trivially so). Consider now a statement $\mathbf{i} = \mathbf{j}$, for $\mathbf{i}$ and $\mathbf{j}$ different numbers. Let $\mathbf{i}$ be smaller than $\mathbf{j}$. Than at all stages below $\mathbf{i}$, $\mathbf{i} = \mathbf{j}$ will turn out to be true (because it reduces to $^* = {}^*$), but after stage S_j, it will be false and remain so throughout all the remaining stages, hence it is finally false. This settles the atomic formulas. To obtain a full valuation function v, one now applies the classical rules within that stage.

(*Note:* One might wonder about the following formula $(\exists x)(x = s(x))$. At every stage S_i this formula is true, should one not therefore conclude that it is true in the limit as well? But then the limit turns out to be inconsistent. The answer is that it is not true in the limit. According to the valuation function v and the interpretation function I, to assign the formula a truth-value, we have to find out on what element of the domain D x will be mapped. Suppose it is $\mathbf{i}$, then for that element $\mathbf{i} \neq \mathbf{i} + \mathbf{1}$, hence no element of the final domain makes the formula true.)

In short, what we have here is a semantic description of classical arithmetic. Needless to say, the implication is that we loose all the "nice" properties. Out goes the categoricity, Löwenheim-Skolem's theorems apply, Gödel's theorems apply, hence consistency ceases to be a trivial matter, and the theory is undecidable. In that sense, classical arithmetic has little to do with the previous stages. Hence the argument that since we have to deal with one of the previous stages, it is a defensible attitude to ignore the limit theory. In that sense, PA is indeed "unnatural".

4. Further reflections on strict finitism and PA

4.1. Other stages and other limits

It is obvious that one needs the three rules (or similar ones) sketched above to arrive at PA in the limit. The use of a (kind of) continuity principle needs to be argued for. In Rule 3 an option is taken to declare all the undecided cases false. This is not a necessary move. In a supervaluational mood of thinking, Rule 3 could have been something like this:

(i) If A is true from a certain stage onwards, then A is true in the limit,

(ii) If A is false from a certain stage onwards, then A is false in the limit,

(iii) All other cases are undecided.

Although the limit case is a quite interesting one — the undecided cases all involve * —, it would not be PA. In short, the series of stages S_i sketched in this paper can lead to very different limits, depending on the rules that govern the transition from the elements of the series to the limit. In that sense, the purpose of this paper can be reformulated as an attempt to show that PA is a *possible* limit with the additional property that it differs maximally from the preceding stages. There is however a nice symmetry present.

Just as there is one series leading to different limits, there are different series of stages that could lead to the same limit, in our case PA. I briefly sketch three possibilities:

(a) Start at S_1 with Presburger Arithmetic (PrA). That means that we have addition but not multiplication[10]. Suppose now we have an ordering on all

[10] A slight difference that has to be noted is that in this approach we use the full

triples of natural numbers $\langle n, m, k\rangle$, such that $n.m = k$, where n, m and k are shorthand for $s^n(0), s^m(0)$ and $s^k(0)$ respectively. In the limit one will have reached PA and for all stages before that limit it will be the case that it is decidable (as we do not have full multiplication, but just a finite set of particular instances), but in the limit this property will have disappeared. Actually there is a body of research on extensions of PrA, initiated by A. L. Semenov[11], to investigate at what point decidability breaks down before one reaches PA. A typical example is the addition of a predicate P to PrA that expresses the property "is a prime".

(b) Start at S_1 with the theory Q or Robinson Arithmetic (RA)[12]. RA consists of the axioms of PA, thus in (Ax2) the restriction to * can be dropped and (Ax3) is replaced by:

$$(Ax3^Q) \qquad\qquad (\forall x)(x \neq 0 \supset (\exists y)(x = s(y))).$$

The originality of RA is that the rule of mathematical induction is missing. (*Note:* sometimes the label Q is reserved for the theory without $(Ax3^Q)$; here I will stick to RA). If we set up a listing of all formulas of the language, $A_1, A_2, , A_n, \ldots$, then at each stage we can add the induction rule for that formula. In the limit all formulas have been treated and so we end up in PA.

(c) A perhaps rather exotic example is to start at S_1 with the domain D_1 consisting only of primes and including $\mathbf{0}$ and $\mathbf{1}$, thus $D_1 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{7}, \ldots, p_n, \ldots\}$. If we define the successor function as defined over the primes, thus:

$I(0) = \mathbf{0}$,

$I(s(0)) = \mathbf{1}$,

$I(s^n(0)) = p_n$, where p_n is the n-th prime (taking $p_2 = 2$),

then addition and multiplication are defined although these will differ from their "natural" interpretation. Take $s^2(0) + s^3(0)$. This will equal $s^5(0)$, but $I(s^2(0)) = \mathbf{2}, I(s^3(0)) = \mathbf{3}$, but $I(s^5(0)) = \mathbf{7}$, thus $\mathbf{2} \oplus \mathbf{3} = \mathbf{7}$. At the next stage, the domain can be extended with all numbers consisting of the product of two primes (implying of course a suitable reinterpretation of the successor function). Bizarre though this structure may seem, it will still be the case that from a certain stage onwards, mathematical statements

language of PA. Thus right from the start we have the multiplication function (or, better, what will turn out to be multiplication in the limit), but, axiomatically, say at stage S_2, we only state that, e.g., $s(0).s(0) = s(0)$.

[11] See Michaux (2000) for an overview of these results.

[12] See Nelson (1986) for a full treatment.

will have stable truth-values. Since at stage S_2, the domain will contain $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \ldots$, we will have that $\mathbf{2} + \mathbf{2}$ does equal $\mathbf{4}$ from that stage onwards (and not $\mathbf{5}$ if we are in S_1).

I want to emphasize once more that this presentation does not lead to new results (not at first sight apparently). Its focus is on a different way of looking at a theory, PA, that we seem to know so well and that appears to be so "natural".

4.2. Recovering PA within strict finitism.

Usually strict finitist arithmetic is seen as a severe reduction of PA, hence it is seen as a very weak and poor theory. The straightforward counterargument to this observation is the claim that:

$$\text{if } \vdash_{\mathrm{PA}} A, \text{ then } \vdash_{\mathrm{S_i}} A.$$

It is easy enough to see why: the only difference between the axiomatization of a stage S_i and PA are the conditions that have been added to (Ax2). This comes down to a (kind of) relativization[13] of PA. Hence the result that says that all stages are actually *extensions* of PA. However, one might observe that, although the result looks impressive, all it amounts to is the following:

(a) A universally true statement in PA will obviously remain true if you only consider a finite, initial fragment.

(b) An existential statement in PA will remain true if the number making the statement true is below the limit and above the limit it will be trivially true because the conditions are not satisfied.

(c) Both universal and existential false statements in PA can turn out to be true. This is indeed not truly interesting.

There is however another way of looking at things. Suppose we are at a certain stage S_i. Consider the set of possible proofs we can write down. Some of these proofs will make use of (Ax2) and/or (Ax3). All other proofs that do not involve these axioms are entirely classically acceptable. Of course, this idea introduces the delicate notion of "making use of". I will not go into this matter here, but assume that we can identify and ignore such trivial

[13] I add the expression "kind of", because it does not correspond to the classical definition of a relativization, where $(\forall x)A(x)$ is replaced by $(\forall x)(C \supset A(x))$, where C stands for the conditions, and $(\exists x)A(x)$ is replaced by $(\exists x)(C\&A(x))$. In this particular case, $(\exists x)A(x)$ is replaced by $(\exists x)(C \supset A(x))$.

cases where, e.g., in a proof where there is a transition from A to B, one can always change it into a transition from A to A & (Ax3) and then to B, so (Ax3) has been used. This is of the greatest importance, of course, when mathematical induction is used. At stage S_i we have two formulations, the classical one and the strict finitist one. In some cases, the explicit version can be replaced by the classical one, in some cases not. When things become interesting, to show that a step in a proof is essential becomes itself subject to proof. What follows is therefore intended as a first survey.

There are two possibilities:

(a) Proofs involving (Ax3) and possibly (Ax2):

These proofs are to be considered exclusively strict finitist. Because quantifier elimination is possible, this means that all proofs are or can be reduced to proofs by cases. It is in this sense, of course, that strict finitism is trivial. Take a famous open problem, say Goldbach's conjecture. Of course, at stage S_{1000} (and for a few stages more as well) the conjecture will be true. But that is (somewhat) misleading. Note that in this case too the strict finitist will not to be satisfied. For the question that remains is: is there an alternative proof that proves the same statement, but does not involve (Ax3)? Which brings us to case (b).

(b) Proofs involving (Ax2), but not (Ax3):

Consider any classical PA-proof involving the unconditional axiom (Ax2). If in the proof every occurrence of that axiom is replaced by the conditional axiom (Ax2), then we will have a strict finitist proof of the same theorem. However, there is a very neat way, I believe, to retrieve the classical proof from the strict finitist. I call this the "bookkeeping" approach. Say we have a proof where on line (j) we have:

(j) $$(\forall x)(\forall y)((x \neq {}^* \& y \neq {}^*) \supset (s(x) = s(y) \supset x = y)).$$

A different way of writing down this line is as follows:

(j) $$(\forall x)(\forall y)(s(x) = s(y) \supset x = y) \qquad \{x \neq {}^*, y \neq {}^*\}.$$

This rewriting system[14] has the effect that the proof itself will correspond to the classical proof and the conditions to be satisfied will be

[14] In Vermeir (1999) a similar idea is used, though in his case the object is to deal with inconsistencies. Vermeir's idea itself goes back to Batens' adaptive logic. See, e.g., Batens (2002) for a general introduction.

gathered together on the right. A different way of formulating this idea is that the classical mathematician is a strict finitist who (temporarily) ignores the limit and acts *as if* there is no limit[15]. If one looks at a standard proof for the theorem:

$$(\forall x)(\forall y)(\forall z)(x + z = y + z \supset x = y),$$

then it is easy to see that the corresponding strict finitist proof will be for the theorem:

$$(\forall x)(\forall y)(\forall z)((x + z \neq {}^{*} \;\&\; y + z \neq {}^{*}) \supset (x + z = y + z \supset x = y)).$$

In the proof itself, (Ax2) is used once, hence if the conditions $x + z \neq {}^{*}$ and $y + z \neq {}^{*}$ are put aside, what one is left with is the classical proof itself.

To end this paragraph, an important element to add is this: quite a large number of fundamental theorems in PA do not involve (Ax2). Some examples:

(i) $(\forall x)(\forall y)(\forall z)(x = y \supset x + z = y + z)$

(ii) $(\forall x)(\forall y)(\forall z)(x = y \supset x.z = y.z)$

(iii) $(\forall x)(\forall y)(\forall z)(x + y = y + x)$

(iv) $(\forall x)(\forall y)(\forall z)((x + y) + z = x + (y + z))$

(v) $(\forall x)(\forall y)(\forall z)((x.y).z = x.(y.z))$

(vi) $(\forall x)(x + 0 = x)$

(vii) $(\forall x)(x.0 = 0)$

Are these not the properties of addition and multiplication we really worry about? In that sense, becoming a strict finitist does not seem all that disastrous from the point of view of mathematical productivity. Taken all

[15] I must emphasize the "as if". Another way of looking at this situation, is this: given an axiomatic theory T that models some part of the world, it is always possible to try to find out what happens if this or that axiom is deleted or changed. In some cases this will still lead to useful models, but in some cases not. If I give a (as good as) complete description of our solar system and then I ignore the Sun, I should not be surprised that my model ceases to be interesting. Likewise, ignoring the limit in the arithmetical case, has given rise to all kinds of explorations, very satisfying and challenging on their own, but philosophically dangerous. Before you know, people actually believe that infinity exists.

together, in practice hardly anything changes, but philosophically we end up at the other end of the spectrum.

At the same time some care has to be taken. If one would restrict oneself to such universal statements, then one will end up with a theory that will be very close to so-called PA^-. This theory counts the statements (i) up to (vii) among its axioms. All other axioms are also universal statements, but induction axioms are lacking. (For a full treatment, see Kaye (1991), especially chapter 2). On the one hand, the resulting theory turns out to be quite powerful, meaning that a set of important classical theorems can be shown to hold in this theory, but, on the other hand, some trivial results do not hold, e.g.,

$$(\forall x)(\exists y)(2.y = x \vee 2.y + 1 = x).$$

(The result is shown by the classical method of constructing a counter-model, in this case the model is the ring $\mathbb{Z}[X]$ of polynomials in one variable X with coefficients from $\mathbb{Z}$, the set of integers.) This is, of course, a rather surprising result and serves as a warning: it is not because *most* results match, that therefore all results match. Being able to talk (strict finitistically) about commutativity of addition and multiplication is fine, but not being able to make the distinction between even and odd is precisely that: odd.

4.3. Yessenin-Volpin, Isles and Wittgenstein

I have already mentioned the work of Mycielski, Vermeir and Batens. To complete this picture, I should also mention the work of A. S. Yessenin-Volpin and David Isles[16]. Both have worked with the notion of a natural number notation system (NNNS). My claim is not that a NNNS corresponds to a stage S_i in the construction outlined here, but there are definitely similarities (although I have to add that I do not share Volpin's basic research program which was to find a finitary (in a rather unusual meaning of the term) proof for the consistency of ZF). One of the similarities is that there are different NNNSs. In their view there could be a NNNS that is closed under the function s, but not under the function $s \circ s$. One of the dissimilarities is that a NNNS comes in two parts, so to speak: a realized or actual part (this would correspond to the domain of a stage S_i) and a future part (this future part has a more tree-like structure to indicate that different actualisations are possible). In the construction here all stages are strictly speaking actual.

[16] An excellent overview of their work is to be found in Epstein & Carnielli (2000), pp. 260-270.

There is however one other similarity that is worth mentioning. The idea of a “bookkeeping” device also occurs in the work of Isles, with a different purpose however. A simple example will clarify the matter. Take the following PA-theorem:

$$(\forall x)(\exists y)(\exists z)(z = s(y) \& y = s(x)).$$

In strict finitist terms it is important to know how many times the successor function is applied. At first glance, one is tempted to say, looking at the formula, that s is only used in a one-step application. Isles’ idea is to construct, as the proof goes along, the necessary requirements a model (in his terms, a substitution graph) must satisfy to make the formula true. It is easy to see that the partial model:

$$\underset{x}{\circ} \xrightarrow{\quad \mathrm{s} \quad} \underset{y}{\circ} \xrightarrow{\quad \mathrm{s} \quad} \underset{z}{\circ}$$

will do the job[17]. The model reveals that a double application of s is required. Note at the same time that if we wanted to construct a model specifically for this formula, then a domain with three elements is sufficient, which is an important consideration for a strict finitist.

The idea to construct a model as the proof goes along suggests an alternative way to bring back together syntax and semantics. Above all it also suggests that the couple $\langle P, M\rangle$, where P is a proof and M a (partial) model for that proof, actually forms the basis of mathematical thinking. I will not explore this suggestion any further in this paper, but I will end with a philosopher that is usually accused of not understanding mathematics in a proper way. I am referring to Ludwig Wittgenstein (of course?) and his *Remarks on the Foundations of Mathematics*. Without claiming that he was a strict finitist (which he probably was not), if one thinks about the structure of the stages S_i in a temporal setting, does not the following paragraph become quite meaningful:

> “However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics” (excerpt from V–9, p. 267).

Although he talks about an irrational number, nothing should prevent us of paraphrasing the quote as follows:

[17] I must emphasize that this is my reading of the work of Isles. From his own writings, it is rather clear that his research problem is to capture as well as possible what Yessenin-Volpin might have had in mind, not to develop a “bookkeeping device”.

> "However queer it sounds, to move from one stage to the next stage of a mathematical theory is a further expansion of mathematics."

If we find ourselves at a certain stage S_i and we perform a calculation that goes over the limit of that stage and we insist on having an answer, then we move to another stage S_j that corresponds to an expansion. I have added the phrase that "we insist on having an answer" to highlight another Wittgensteinian idea, viz., that mathematics is a purposeful enterprise.

In conclusion (and to be clear about my purposes), all the considerations and reflections presented in this paper show, besides my explicit thesis that PA is an "unnatural" theory, the richness of strict finitist theories in contrast to "popular" opinion that claims that such theories are trivial.

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