Logic and **L**ogical **P**hilosophy
Volume 11 (2003), 5–37

Diderik Batens

CRITERIA CAUSING INCONSISTENCIES. GENERAL GLUTS AS OPPOSED TO NEGATION GLUTS*

Abstract. This paper studies the question: How should one handle inconsistencies that derive from the inadequacy of the criteria by which one approaches the world. I compare several approaches. The adaptive logics defined from **CLuN** appear to be superior to the others in this respect. They *isolate* inconsistencies rather than spreading them, and at the same time allow for genuine deductive steps from inconsistent and mutually inconsistent premises.

Yet, the systems based on **CLuN** seem to introduce an asymmetry between negated and non-negated formulas, and this seems hard to justify. To clarify and understand the source of the asymmetry, the epistemological presuppositions of **CLuN**, viz. inadequate criteria, are investigated. This leads to a new type of paraconsistent logic that involves gluts with respect to all logical symbols. The larger part of the paper is devoted to this logic, to the adaptive logics defined from it, and to the properties of these systems.

While the resulting logics are sensible and display interesting features, the search for variants of the justification leads to an unexpected justification for **CLuN**.

1. The Problem

The first papers on adaptive logics concerned inconsistency-adaptive logics defined from the logic (now called) **CLuN** (see especially [3] and [2]). This was largely accidental. In earlier work on paraconsistent logic, especially [1],

---

* The research for this paper was supported by the Fund for Scientific Research – Flanders, and indirectly by the Flemish Minister responsible for Science and Technology (contract BIL98/37 and BIL01/80). I am indebted to Dagmar Provijn and to Lieven Haesaert for comments on a former draft.

CLuN had turned out the basic system[1] of a (very large) family of paraconsistent subsystems of **CL** (Classical Logic)—many of these subsystems have been studied in the literature. So it was only natural that the first attempts to construct an inconsistency-adaptive logic started from **CLuN**.

Adaptive logics defined from **CLuN** turned out to have the property to maximally isolate inconsistencies: from the inconsistency of a formula A need not follow the inconsistency of any subformula or superformula of A.[2] For an inconsistency-adaptive logic, this clearly is an advantage. Inconsistency-adaptive logics interpret a set of premises 'as consistently as possible'. Apart from the rules of their (paraconsistent) lower limit logic, they validate all *applications* of further **CL**-correct rules, provided these applications do not lead to triviality in view of the disjunctions of abnormalities that are derivable by the lower limit logic. That **CLuN** isolates inconsistencies warrants that more applications of **CL**-correct rules are validated by the adaptive logics defined from it—the point is further discussed in Section 3. In this respect, the original inconsistency-adaptive logics appeared to be the most attractive ones.

In [24], Graham Priest presented the inconsistency-adaptive logic $\mathbf{LP}^m$ defined from his **LP**. Priest chose the latter system because it is his preferred paraconsistent logic. As **LP** spreads inconsistencies, so do the inconsistency-adaptive logics defined from it (or from **CLuNs**, of which **LP** is the implicationless fragment). I shall discuss this point in Section 3.

In [20] and [21], Joke Meheus objected against inconsistency-adaptive logics defined from **CLuN**. Her main argument was that they treat Modus Ponens and Modus Tollens in an asymmetric way, and that this is unrealistic with respect to the aim of explicating the intuitive reasoning of a scientist who tries to surmount the inconsistency of some theory. I shall not deal with this objection here—I did so in [7]—but rather discuss the deeper asymmetry underlying **CLuN** that causes the asymmetry between Modus Ponens and Modus Tollens.

This deeper asymmetry concerns the fact that, according to **CLuN**, formulas of the form $\sim A$, and no other formulas, may be true independently of the truth of their subformulas. I shall consider a justification for this behaviour of formulas of the form $\sim A$, and next argue that *this* justification

[1] According to **CL** (Classical Logic), negation is characterized by the consistency requirement (if $v_M(A) = 1$ then $v_M(\sim A) = 0$) and by the completeness requirement (if $v_M(A) = 0$ then $v_M(\sim A) = 1$). **CLuN** is obtained by dropping the former and keeping the latter, and by giving all other logical symbols the same meaning as in **CL**.

[2] The only exception is where the inconsistent formula implies itself a contradiction. For example, $((r \wedge p) \wedge \sim p) \wedge \sim((r \wedge p) \wedge \sim p)$ entails $p \wedge \sim p$.

cannot be restricted to such formulas. The point will be spelled out in Section 7. The main part of the paper is devoted to the study of the system that is the direct outcome of this line of justification. The system will be called **G** because it leads to gluts with respect to all logical symbols.

The reader deserves to be warned that this paper should not be read as a defense of **G** as the correct paraconsistent logic. First, **G** is much more 'para' than any popular paraconsistent logic. Next, as I argued in several papers, the true logic is a chimera because a logic is an instrument that is more or less suited to a specific purpose or application context.

In my view, the main application contexts for inconsistency-adaptive logics are those in which an unexpected inconsistency surfaces. A theory T was meant as consistent but turns out to be inconsistent. In trying to replace T by a consistent improvement T', we have to reason from T in order to locate its inconsistencies. The latter may then be removed in view of specific experiments or observations, or in view of a conceptual analysis if T is a mathematical theory.

The aim of inconsistency-adaptive logics is merely to see what T comes to, in its full richness, except for the pernicious consequences of its inconsistency. This interpretation of T, which is neither offered by **CL** nor by any (monotonic) paraconsistent logic, should enable one to start the search for a consistent improvement T', either by devising and performing experiments or by disentangling concepts. An inconsistency-adaptive logic should not spell out the consistent improvements of T. It should not display all consequences of all possible consistent improvements of T. It should locate the problems (the inconsistencies) and separate them from the other consequences of the premises (or theory). As, step by step, inconsistencies are resolved,[3] the inconsistency-adaptive logic should depict the new situation, locate the remaining inconsistencies, if any, and provide a guide to locate the experimental or conceptual means that enable one to surmount them. Our judgement of **G** should depend on the merits of the inconsistency-adaptive logics that are obtained from it.

2. The CLuN-Based Systems

Syntactically, **CLuN** consists of full positive predicative logic together with the axiom $A \vee \sim A$. Neither Replacement of Equivalents nor Replacement of Identicals is valid in this extremely poor paraconsistent logic.

---

[3] This should be done in a tentative or defeasible way. Logic should not provide the possible resolutions, but should handle attempted resolutions. A suitable logic is spelled out in [12].

Let $\mathcal{L}$ be the usual predicative language schema, with $\mathcal{S}$, $\mathcal{C}$, $\mathcal{V}$, $\mathcal{P}^r$, $\mathcal{F}$ and $\mathcal{W}$ respectively the sets of sentential letters, individual constants, individual variables, predicate letters of rank r, formulas, and wffs (closed formulas).[4] To handle the quantifiers in a simple way in the semantics, $\mathcal{L}$ is extended to the pseudo-language schema $\mathcal{L}^+$ by adding a set $\mathcal{O}$ of pseudo-constants, requiring that any element of D is named by at least one member of $\mathcal{C} \cup \mathcal{O}$.[5] Let $\mathcal{F}^+$ and $\mathcal{W}^+$ denote respectively the set of formulas and wffs of $\mathcal{L}^+$ and let ${}^{\sim}\mathcal{W}^+$ be the set of wffs of the form $\sim A$.

A model $M = \langle D, v\rangle$, in which D is a set and v an assignment function, is an interpretation of $\mathcal{W}^+$, and hence of $\mathcal{W}$, which is what we are interested in. The assignment function v is defined by:

C1.1 $v : \mathcal{S} \longrightarrow \{0,1\}$
C1.2 $v : \mathcal{C} \cup \mathcal{O} \longrightarrow D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C1.3 $v : \mathcal{P}^r \longrightarrow \wp(D^r)$ (the power set of the r-th Cartesian product of D)
C1.4 $v : {}^{\sim}\mathcal{W}^+ \longrightarrow \{0,1\}$

The valuation function $v_M : \mathcal{W}^+ \longrightarrow \{0,1\}$ determined by M is defined as follows:

C2.1 if $A \in \mathcal{S}$, $v_M(A) = v(A)$; $v_M(\bot) = 0$
C2.2 $v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C2.3 $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C2.4 $v_M({\sim}A) = 1$ iff $v_M(A) = 0$ or $v({\sim}A) = 1$
C2.5 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C2.6 $v_M(A \wedge B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
C2.7 $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$
C2.8 $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$
C2.9 $v_M((\forall\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$
C2.10 $v_M((\exists\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$

A is true in M (M verifies A) iff $v_M(A) = 1$; $\Gamma \vDash A$ iff all models of Γ verify A; A is valid iff it is verified by all models.[6]

---

[4] I shall suppose that "$\bot$" belongs to $\mathcal{L}$ and is characterized syntactically by $\bot \supset A$. Classical negation can be defined by $\neg A =_{df} A \supset \bot$.

[5] It is called a set of pseudo-constants because the cardinality of $\mathcal{O}$ should be at least that of the largest domain considered—to handle all possible domains, a different $\mathcal{O}$ may be chosen for each model.

[6] The present semantics does not exclude the model that is trivial with respect to $\bot$-free formulas of $\mathcal{L}$, but no model verifies $\bot$. This is quite all right. We are interested in semantic consequence, and the model that verifies $\bot$ cannot possibly make any difference in this respect.

The two best studied adaptive logics defined from **CLuN** are **ACLuN1** and **ACLuN2**—see especially [4]. In preparation of the logic **G**, I present them in a way that slightly departs from other papers. By an abnormality I shall mean a contradiction, $\sim A \wedge A$, or, if it is an open formula, its existential closure, which will be denoted by $\exists(\sim A \wedge A)$. In the sequel, $!\sim A$ and $\exists!\sim A$ will abbreviate abnormalities.[7] The set of abnormalities will be called Ω. Where M is a **CLuN**-model, let $Ab(M)$, the abnormal part of M, be $\{A \in \Omega \mid v_M(A) = 1\}$. Remark that $Ab(M)$ contains $(\exists x)!\sim Px$ iff it contains $(\exists y)!\sim Py$ (and vice versa) and that it contains $(\exists x)!\sim Px$ if it contains $!\sim Pa$ (but not vice versa).

In the sequel I shall have to refer to *disjunctions* of abnormalities. I shall write $Dab(\Delta)$ to denote the disjunction of the finite $\Delta \subset \Omega$. $Dab(\Delta)$ will be called a *minimal* Dab-consequence of Γ iff $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ whereas $\Gamma \nvdash_{\mathbf{CLuN}} Dab(\Theta)$ for all $\Theta \subset \Delta$. Similarly, some Dab-formulas are *minimal* at a stage of a proof.

All theorems mentioned in the present paper are either proved in [4] or their proof is obvious in view of the proof of the corresponding theorem in that paper.

Theorem 1. *$\vdash_{\mathbf{CL}} A$ iff $\vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ for some (possibly empty) Δ. (Theorem Adjustment Theorem)*

This theorem provides the basis for the dynamic proof theory. If $A_1, \ldots, A_n \vdash_{\mathbf{CL}} B$, then, by Theorem 1, there is a Δ such that $\vdash_{\mathbf{CLuN}} ((A_1 \wedge \ldots \wedge A_n) \supset B) \vee Dab(\Delta)$. This **CLuN**-theorem may be interpreted as: B is derivable from $A_1, \ldots, A_n$ on the condition that all members of Δ behave normally. So this provides us with a conditional rule if Δ is not empty, and with an unconditional rule otherwise.

It is useful to illustrate this by listing some **CLuN**-theorems that correspond to popular **CL**-rules. The last example concerns an application of Replacement of Identicals within the scope of a negation.

MT	$((\sim B \wedge (A \supset B)) \supset \sim A) \vee !\sim B$
DS	$((\sim B \wedge (A \vee B)) \supset A) \vee !\sim B$
ND	$(\sim(A \vee B) \supset \sim A) \vee !\sim(A \vee B)$
RI	$((\sim Pa \wedge a = b) \supset \sim Pb) \vee !\sim Pa$

Let us now turn to the adaptive logics. The dynamic proofs are characterized by three (generic) rules of inference and a marking definition. The rules of inference are common to both adaptive logics :

[7] In earlier papers on inconsistency-adaptive logics, an (open or closed) formula A is called an abnormality iff it is true together with its negation.

PREM At any stage of a proof one may add a line consisting of (i) an appropriate line number, (ii) a premise, (iii) a dash, (iv) 'PREM', and (v) '$\emptyset$'.

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line that has B as its second element and $\Delta_1 \cup \ldots \cup \Delta_n$ as its fifth element.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee Dab(\Delta_0)$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line that has B as its second element and $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$ as its fifth element.

ACLuN1 is defined from **CLuN** by the Reliability strategy, which is the most obvious strategy from a proof theoretic point of view. In preparation of its marking definition we define, with respect to a proof from a set of premises Γ, $U_s(\Gamma)$ as the set of all abnormalities that are disjuncts of the minimal *Dab*-formulas that occur in the proof at stage s. If, for example, p, $\sim p \vee q$, and $\sim q$ have been unconditionally derived (that is, derived on a line the fifth element of which is empty) in a proof from Γ, then one may unconditionally derive

$$!\sim p \vee !\sim q \tag{1}$$

If (1) is a minimal *Dab*-formula at stage s, then both $!\sim p$ and $!\sim q$ are members of $U_s(\Gamma)$. If, at a later stage s', q is unconditionally derived, then one may unconditionally derive

$$!\sim q$$

So (1) is not a minimal *Dab*-formula at stage s', whence $!\sim p$ is not a member of $U_{s'}(\Gamma)$ (unless it is a disjunct of another minimal *Dab*-formula at stage s'). The marking definition reads:

Definition 1. *Marking for Reliability:* Line i is marked at stage s of a proof from Γ iff a member of its fifth element is a member of $U_s(\Gamma)$.

We say that A is derived at stage s of a proof from Γ iff A occurs as the second element of a non-marked line of the proof. A is *finally derived* on line i in a proof from Γ iff A is derived on line i at some stage s of the proof *and* any extension of the proof in which line i is marked may be further extended in such a way that line i is not marked. $\Gamma \vdash_{\mathbf{ACLuN1}} A$ iff A is *finally derivable* from Γ.

Let $U(\Gamma)$ denote the set of the disjuncts of the minimal *Dab*-consequences that are **CLuN**-derivable from Γ.

THEOREM 2. *$\Gamma \vdash_{\mathbf{ACLuN1}} A$ iff there is a finite set of abnormalities Δ such that $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. (Monotonic Characterization)*

THEOREM 3. *If $\Gamma \vdash_{\mathbf{ACLuN1}} A$ then A can be finally derived in any* **ACLuN1***-proof from Γ. (Proof Invariance)*

Semantically, **ACLuN1** is characterized by:

DEFINITION 2. A **CLuN**-model M of Γ is an **ACLuN1**-model (a reliable model) of Γ iff $Ab(M) \subseteq U(\Gamma)$.

DEFINITION 3. $\Gamma \vDash_{\mathbf{ACLuN1}} A$ iff A is verified by all **ACLuN1**-models of Γ.

THEOREM 4. *$\Gamma \vdash_{\mathbf{ACLuN1}} A$ iff $\Gamma \vDash_{\mathbf{ACLuN1}} A$. (Soundness and Completeness)*

ACLuN2 is defined from **CLuN** by the Minimal Abnormality strategy. This strategy is most straightforward from a semantic point of view.

DEFINITION 4. A **CLuN**-model M of Γ is an **ACLuN2**-model (a minimally abnormal model) of Γ iff there is no **CLuN**-model M' of Γ such that $Ab(M') \subset Ab(M)$.

DEFINITION 5. $\Gamma \vDash_{\mathbf{ACLuN2}} A$ iff A is verified by all **ACLuN2**-models of Γ.

THEOREM 5. *If $\Gamma \vDash_{\mathbf{ACLuN1}} A$, then $\Gamma \vDash_{\mathbf{ACLuN2}} A$.*

The converse of this theorem fails. As an example, consider the set of premises $\{p, {\sim}p \vee q, {\sim}q, q \vee s, {\sim}p \vee s\}$. Remark that $U(\Gamma) = \{!{\sim}p, !{\sim}q\}$. So the **ACLuN1**-models of Γ are those **CLuN**-models M of Γ such that $Ab(M)$ comprises either $!{\sim}p$ or $!{\sim}q$ or both. The **ACLuN2**-models of Γ are those that verify either $!{\sim}p$ or $!{\sim}q$, but *not* both. Hence, some **ACLuN1**-models of Γ falsify s, whereas all **ACLuN2**-models of Γ verify s.

The marking definition for **ACLuN2**-proofs is somewhat tiresome. I shall first characterize the **ACLuN2**-models of Γ in terms of the set of all minimal *Dab*-consequences of Γ. Let $\Phi^{\circ}(\Gamma)$ be the set of all sets that contain one disjunct out of each minimal *Dab*-consequence of Γ. Let $\Phi^{\star}(\Gamma)$ contain, for any $\varphi \in \Phi^{\circ}(\Gamma)$, the set $Cn_{\mathbf{CLuN}}(\varphi) \cap \Omega$. Finally let $\Phi(\Gamma)$ contain those members of $\Phi^{\star}(\Gamma)$ that are not proper supersets of other members of $\Phi^{\star}(\Gamma)$. A **CLuN**-model M of Γ is an **ACLuN2**-models of Γ iff $Ab(M) \in \Phi(\Gamma)$.

With respect to a stage s of a proof from Γ, we define $\Phi_s(\Gamma)$ in the same way as $\Phi(\Gamma)$, but now in terms of the set of minimal *Dab*-formulas that occur in the proof. The integrity criterion is defined as follows in terms of $\Phi_s(\Gamma)$:

Definition 6. Where A is the second element of line j, line j *fulfills the integrity criterion* at stage s iff (i) the intersection of some member of $\Phi_s(\Gamma)$ and of the fifth element of line j is empty, and (ii) for each $\varphi \in \Phi_s(\Gamma)$ there is a line k such that the intersection of φ and of the fifth element of line k is empty and A is the second element of line k.

The proof theory of **ACLuN2** consists of the rules PREM, RU, and RC together with a specific marking definition:

Definition 7. *Marking for Minimal Abnormality:* Line i is marked at stage s of a proof from Γ iff it does not fulfill the integrity criterion.

That, at stage s of a proof from Γ, A is finally derived on line i of the proof is defined as for **ACLuN1**. Here, however, we need to take into account that even if i is marked in an *infinite* extension of the proof, then this proof may be further extended (that is: finitely many lines may be inserted in the infinite proof) in such a way that line i is not marked in the extension—see [4, p. 466] for an example. $\Gamma \vdash_{\mathbf{ACLuN2}} A$ (A is finally derivable from Γ) is defined as for **ACLuN1**.

Here too, $\Gamma \vdash_{\mathbf{ACLuN2}} A$ has a Monotonic Characterization, and Proof Invariance, Soundness and Completeness are provable.

If Γ has **CL**-models, then no disjunction of abnormalities is **CLuN**-derivable from it, and hence $Cn_{\mathbf{ACLuN1}}(\Gamma) = Cn_{\mathbf{ACLuN2}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$. If Γ has no **CL**-models (and hence is inconsistent), we have $Cn_{\mathbf{CLuN}}(\Gamma) \subseteq Cn_{\mathbf{ACLuN1}}(\Gamma) \subseteq Cn_{\mathbf{CL}}(\Gamma)$ (and similarly for **ACLuN2**). It can be shown that, except for border cases, the subset-relations are proper.

Theorem 6. *If a* **CLuN**-*model* M *of* Γ *is not an* **ACLuN1**-*model of* Γ, *then there is an* **ACLuN1**-*model* M' *of* Γ *such that* $Ab(M') \subset Ab(M)$. *(Strong Reassurance)*

Corollary 1. *If* Γ *has* **CLuN**-*models, then it has* **ACLuN1**-*models. (Reassurance)*

The equivalents of the theorem and corollary hold also for **ACLuN2**.

3. Spreading Abnormalities

Some other adaptive logics are defined from richer paraconsistent logics. The most popular such logic one is no doubt **CLuNs** (known under many different names). **CLuNs** is obtained by extending **CLuN** with Double Negation, De Morgan properties, and all similar negation-reducing equivalences. By dropping the implication and equivalence from **CLuNs** one obtains Priest's **LP** (in which a non-detachable implication is defined by $\sim A \vee B$).

As is explained in Section 1, such logics spread inconsistencies, and hence the adaptive logics defined from them validate less applications of conditional rules. Here is a simple but instructive example of a premise set:

$$p,\ s \wedge q,\ \sim(p \wedge q),\ \sim p \vee r\,.$$

According to **ACLuN1** (or **ACLuN2**), the only minimal *Dab*-consequence of the premises is $!\sim(p \wedge q)$—this formula, and no other one, behaves abnormally. Hence, r is finally derivable from the premises, along with the **CLuN**-derivable p, q and $\sim(p \wedge q)$. The situation is radically different for adaptive logics based on **CLuNs** and **LP**. According to these, p, q, $\sim p \vee \sim q$ are all unconditionally derivable, and hence so is $!\sim p \vee !\sim q$. It follows that r is *not* finally derivable from the premises.

In general, and as noted already in [3], adaptive logics defined from **CLuN** have the following advantage. Where (locally) no abnormality is involved, they deliver all **CL**-consequences (such as applications of Double Negation, De Morgan, etc.). Thus, $\sim\sim p$, $\sim\sim r$, $\sim(p \wedge \sim r)$, ... are derivable from the above premise set. So by not spreading abnormalities, for example by not making $!p \vee !q$ a consequence of the above premise set, they are able to deliver other **CL**-consequences (such as r in the example). The matter is discussed at length in [7].

The force of the above argument depends obviously on the intended application contexts of inconsistency-adaptive logics. For example, in [24] Graham Priest stresses the Classical Recapture: in as far as classical reasoning is applied to consistent theories, it is recognized as correct by the inconsistency-adaptive logic.[8]

Actually, this property is shared by any sensible inconsistency-adaptive logic, independently of its lower limit logic.[9] A different, and in my view more important, application context concerns theories that were intended as consistent but turn out to be inconsistent—see Section 1. Let us for a moment return to the above premise set, now taking into account that the premises were intended as consistent—that this is unrealistic for the simple premise set is irrelevant for my argument. We discovered a problem with

[8] For Priest applications of Disjunctive Syllogism (and similar rules) are not correct for logical reasons alone, but partly by virtue of the supposition that the described domain is consistent.

[9] Incidentally, the property fails for the original formulation of $\mathbf{LP}^m$; see [25] for a correction and [5] for some further discussion.

$p \wedge q$, and we want to resolve this. What the consistent replacement for the premises will be, we cannot predict; it might affect the truth value of p as well as that of q. But, as the premises were intended to be consistent, the fourth premise is not taken serious if its truth is taken as possibly deriving from the truth of $\sim p$.

If newly gained information enables one to remove the inconsistency in the above premises, and it turns out that $\sim p$, and hence $\sim(p \wedge q)$ is rejected, for example replaced by $\sim(p \wedge \sim q)$, then r will obviously be derivable from the thus modified premises. This is so independently of the inconsistency-adaptive logic that was actually used to handle the original theory. The difference between inconsistency-adaptive logics defined from **CLuN** and those defined from lower limit logics that spread inconsistencies, is that only the former point to *the* problem, rather than to a set of problems.[10]

In specific circumstances, the inconsistency-spreading logics may be preferable. Suppose, for example, that it is only possible to devise tests for primitive statements (those that correspond to formulas in which no logical symbol occurs, except possibly for identity). Under this supposition, the only conclusive tests to resolve the inconsistency in the above premise set is the test $?(p, \sim p)$ and the test $?(q, \sim q)$. Needless to say, the supposition is unrealistic—or rather it will be justified in few circumstances only.

4. Consistent Chunks

Several approaches to handling inconsistency proceed by dividing an inconsistent set of premises into consistent chunks, sometimes maximal ones, and defining consequence relations from the inconsistent set in terms of the **CL**-consequences of these chunks. Jaskowski's system **D2** proceeded from this idea, but it is a monotonic paraconsistent logic without any adaptive properties.[11] More closely related to adaptive logics are the Rescher–Manor consequence relations: the flat ones (Free Consequence, Strong Consequence, Argued Consequence, C-Based Consequence, and Weak Consequence), and a set of prioritized consequence relations. I refer to [27], [28], [29], and especially [30]. Interesting surveys of the whole family are presented in [15] and [16]. A related approach is presented in [31].

In [8], it was shown that all flat Rescher–Manor consequence relations

[10] If A and $\sim A$ are derivable from the premises, then, according to the logics that spread inconsistencies, $!\sim(A \wedge B)$ is unconditionally derivable for *any* B derivable from the premises; similarly for many other inconsistencies.

[11] See [22] for adaptive logics that are defined from **D2**.

are characterized by an adaptive logic defined from **CLuN**. There it is proved that, for example, where all negations in members of Γ are classical (that is, are formalized as "¬") and $\Gamma^G = \{\sim\neg A \mid A \in \Gamma\}$, $\Gamma \vdash_{Free} A$ iff $\Gamma^G \vdash_{\mathbf{ACLuN1}} A$. This provides all those consequence relations with a dynamic proof theory—before, they were just abstract definitions in terms of possibly undecidable sets. In [11], an interesting strengthening of the flat Rescher–Manor consequence relations is presented and applied to discussions.

Suitable application contexts for chunking logics are those in which the premises derive from distinct sources. For example, the Free consequences of some set of premises are the **CL**-consequences of those premises that are not contradicted by any other subset of the premises. This (very cautious) approach relies on the idea that sources that contradict some other (set of) sources are not considered as reliable. If the information does not derive from different sources, the suitability of the Rescher–Manor consequence relations is doubtful. For example, if some theory is inconsistent, it is usually impossible to consider its "axioms" as deriving from different sources. Take Frege's set theory as an example. If we consider the whole theory as deriving from a single source (and take the conjunction of the axioms as *the* single axiom), then the whole theory reduces to the empty set on all Rescher–Manor consequence relations. If Frege's axioms are considered as deriving from different sources, then the abstraction axiom is simply removed from them (it leads to an inconsistency by logic alone). Both approaches are obviously inadequate if the aim is to study the theory in order to find a consistent replacement for the abstraction axiom. One wants to keep at least *some* of its consequences.

That mutually inconsistent premises do not deliver any common consequences according to any Rescher–Manor consequence relation, seems especially problematic for empirical theories. In [10] two such problems are spelled out in the context of explanation from inconsistent theories.

If two theories are mutually inconsistent and there are good reasons to forge a consistent single theory from consequences of both of them, then again the Rescher–Manor consequence relations seem problematic. A historical example is discussed in [19]: the way in which Clausius forged a consistent theory from, on the one hand, Sadi Carnot's thermodynamics and, on the other hand, Joule's principle and experimental findings. Here we have two clearly distinct sources. But we do not want to choose between them; Clausius's problem was to modify Carnot's theory in such a way that its coherence was preserved but that it did not contradict Joule's findings and principle.

Many of these problems can be resolved by applying the Rescher–Manor consequence relations to **CL**-*consequences* of the statements made by the different 'sources'. There are, however, two problems with proceeding thus. First, if some 'source' is internally inconsistent, we still do not want to kick it out (as all Rescher–Manor consequence relations do). Next, precisely because these consequence relations are so strongly dependent on the formulation of the premises,[12] it is extremely easy to offer a rational reconstruction of a historical case in terms of this procedure, but it is nearly impossible to demonstrate that the reconstruction is not *ad hoc*.

A fair summary of the situation seems to be that the Rescher–Manor consequence relations are sensible adaptive logics (in the broad sense) but that their application context is rather restricted. More particularly, they seem unfit for the analysis of theories that were intended as consistent but turn out to be inconsistent—and remember that this analysis should be useful for deciding which further specific information might enable one to resolve the inconsistency.

5. Ambiguities in the Non-Logical Symbols

An important contribution to the adaptive logic programme was offered by Guido Vanackere in [32] and [33]—see [9] for a slightly different lower limit logic. The idea is that inconsistencies may be caused by ambiguities in the non-logical symbols rather than by non-standard meanings of the logical symbols. Unfortunately, in its present guise this approach has the disadvantage to spread abnormalities. This disadvantage might be overcome by allowing for ambiguities in complex formulas that do not reduce to ambiguities in primitive formulas. For the time being, however, I need not further consider the approach in the present context.

6. The Original Justification of CLuN

Before considering the objection to **CLuN**, it is worthwhile to briefly summarize its original justification. **CLuN** relies on a clear and simple idea, both from a semantic and from a proof theoretic point of view. The justification for the propositional fragment was first spelled out in [1]. This justification derives from the way in which **CLuN** is obtained, viz. by dropping the consistency requirement (if $v_M(A) = 1$, then $v_M({\sim}A) = 0$) from

[12] The set $\Gamma \cup \{p, q\}$ does not in general have the same consequences as the set $\Gamma \cup \{p \wedge q\}$.

CL.[13] That a **CL**-rule is valid, respectively invalid, in **CLuN** is easily justified in view of this simple idea: the meaning of negation is reduced to the completeness requirement (if $v_M(A) = 0$, then $v_M({\sim}A) = 1$) and the meaning of all other logical symbols is exactly as in **CL**. Below, I merely consider some examples.

Let us start with Disjunctive syllogism. If ${\sim}A$ and A are true, then ${\sim}A$ and $A \vee B$ are true even if B is false. Hence, B is not a semantic consequence of ${\sim}A$ and $A \vee B$ because some models are inconsistent. Modus Tollens too is invalid: in models that verify A, B and ${\sim}B$, and falsify ${\sim}A$, both $A \supset B$ and ${\sim}B$ are true whereas ${\sim}A$ is false. Unlike Modus Tollens, Modus Ponens is valid—the truth conditions for $A \supset B$ are distinct from those for ${\sim}A \vee B$; if A is false, ${\sim}A$ is true, but not conversely. As a last example, consider Double Negation, both directions of which are invalid because of the absence of the consistency requirement. Some models verify A and ${\sim}A$ but falsify ${\sim}{\sim}A$, other models verify ${\sim}{\sim}A$ and ${\sim}A$ but falsify A.

The transition to the predicative level is straightforward. If both ${\sim}(\forall x)A(x)$ and $(\forall x)A(x)$ are true, $(\exists x){\sim}A(x)$ may very well be false. Conversely, the truth of $(\exists x){\sim}A(x)$ does not rule out the truth of $(\forall x)A(x)$, and hence cannot warrant the truth of ${\sim}(\forall x)A(x)$. Also Replacement of Identicals is not generally valid. If $a = b$ and Pa are true, then $v(b) = v(a)$ and $v(a) \in v(P)$ and hence Pb. However, the truth of $a = b$ and ${\sim}Pa$ does not exclude that $v(a) \in v(P)$; if this is the case, Pb is true and ${\sim}Pb$ may be false.

It is easy enough to make Replacement of Identicals hold generally. This, however, would undermine the systematic character of **CLuN**. As appears from the previous two paragraphs, the idea underlying **CLuN** is clear and simple from a semantic point of view. From a proof theoretic point of view, **CLuN** consists of full positive **CL** (or of full **CL**, as classical negation may be defined) extended with $A \vee {\sim}A$.[14] Making Replacement of Identicals hold generally would require an exception with respect to both the semantics and the proof theory. A further argument derives from the obvious similarity between identity and equivalence. Some models verify A, B and ${\sim}A$ and falsify ${\sim}B$ (even if A and B are valid formulas); hence they verify $A \equiv B$ and falsify ${\sim}A \equiv {\sim}B$.[15] It follows that Replacement of Equivalents does

[13] One should not confuse this justification with the structure of the models—this will become even more obvious in Section 12.

[14] There is nothing puzzling about this. I take "${\sim}$" to be the standard negation. This negation is classical in **CL** and paraconsistent in **CLuN**. However, if one defines $\neg A =_{df} A \supset \bot$, then "$\neg$" behaves in **CLuN** as "${\sim}$" behaves in **CL**.

[15] This holds even if $A \equiv B$ is a theorem. Remark, however, that Replacement of

not generally hold in **CLuN**. In view of this, it seems hard to justify that Replacement of Identicals would generally hold.

Some people object to the absence of Modus Tollens (in the presence of Modus Ponens), or to the absence of Contraposition. Apart from the above justification, it is worthwhile to recall that Modus Tollens and Contraposition fail for many familiar implications, for example counterfactual implications (see, e.g., [17]), default rules (see, e.g., [18]), probabilistic implications, etc. An excellent example (borrowed from Guido Vanackere) for default rules is that although it is true that (typical) humans are non-logicians, it is false that (typical) logicians are non-human.

7. Invoking Criteria

Let us now consider the asymmetry between Modus Ponens on the one hand and Modus Tollens and Disjunctive Syllogism on the other hand—see Section 1. The asymmetry between Modus Ponens and Modus Tollens derives from a more deeply rooted asymmetry. **CLuN** does not spread inconsistencies because of the special way in which negative formulas are handled. Unlike all other formulas, formulas of the form $\sim A$ may be true independently of the truth value of their subformulas—technically, this is realized by clauses C1.4 and C2.4 of the semantics. As a result some **CLuN**-models verify, for example, $\sim(p \wedge q))$, but falsify both $\sim p$ and $\sim q$.[16]

Clause C1.4 of the assignment 'drops in' certain true formulas of the form $\sim A$. As their truth is not a result of the truth values of their subformulas, I shall say that these formulas hang from a skyhook. The truth of skyhook formulas has effects for more complex formulas. Thus, if p and hence $p \vee q$ is true, $\sim(p \vee q)$ hangs from a skyhook, and $\sim\sim(p \vee q)$ does not hang from a skyhook, then $\sim\sim(p \vee q)$ is *false* whereas $\sim\sim\sim(p \vee q)$ and $p \wedge \sim(p \vee q)$ are true.

In the subsequent paragraphs, I consider one possible interpretation that justifies skyhook formulas. I shall show that a sensible and systematic application of this interpretation leads to a paraconsistent logic—I shall call it **G**—that is very different from **CLuN**. In subsequent paragraphs I articulate **G** and two inconsistency-adaptive logics defined from it.

Identicals proceeds in terms of logically contingent identities.

[16] **CLuN** was devised for reasons that have nothing to do with isolating inconsistencies—see the previous section—but it turned out that it isolates inconsistencies.

Our knowledge may be taken to derive from a multiplicity of criteria that apply to statements of different complexity.[17] The combination of criteria need not lead to consistent results. Thus, one criterion may deliver p whereas another delivers $\sim(p \vee q)$.

Restricting our attention to the simplest possible case, let each application of a criterion be a test of the type $?(A, \sim A)$—a positive outcome will be described as A and a negative one as $\sim A$. It is possible that the test $?(p, \sim p)$ delivers p whereas the test for $?(p \vee q, \sim(p \vee q))$ delivers $\sim(p \vee q)$. These outcomes obviously reveal that the criteria are problematic. However, it is obvious enough from the history of the sciences that many problematic criteria could only be replaced by the introduction of new concepts (and of connected criteria).

So, as a first approach, a paraconsistent model may be seen as a model of the results of a set of criteria. The criteria may cause the model to be inconsistent (even if 'the world is consistent'). Needless to say, the outcome of the criteria does not in itself constitute a model. In the example considered, the outcome leaves the truth value of $p \vee q$ undetermined. In order to fix that value, one has to rely on the intended meaning of the logical symbols. Suppose then that we intend to use disjunction in such a way that $p \vee q$ is true whenever p is true, and that we intend to use negation in such a way that $\sim(p \vee q)$ is false whenever $p \vee q$ is true. If we moreover intend 'true' and 'false' to exclude each other, our intentions are clearly overruled by the outcome of the criteria. This seems a good reason to see the criteria as problematic.[18]

We may proceed in several ways to handle cases in which criteria lead to jointly inconsistent outcomes. The classical logician will have to consider the situation as hopeless. As, on the classical logician's approach, the resulting knowledge about the domain is trivial, one cannot possibly rely on it to arrive at a consistent replacement—one would have to start from scratch.

In situations of the considered kind, adaptive logics offer a sensible way out: they interpret the outcome of applications of the criteria as normally as possible. It has been argued at length, in [3] and in many other papers, that

[17] The nature of the criteria may vary according to the domain and according to the way in which the domain is approached. The criteria may be observational. They may also contain observational as well as theoretical aspects. In mathematical cases, they may refer to provability and disprovability from a given set of axioms by means of a set of given rules. And so on.

[18] Given that we have no a priori warrant that our criteria are unproblematic, no form of decent knowledge would be possible if the result of applying of our criteria could not reveal their problematic character.

a monotonic paraconsistent logic delivers too weak a set of consequences for such situations.

In the sequel I shall on use a classical semantic metalanguage. This means that I shall use 'true' and 'false' in such a way that they exclude each other.[19]

How then may an inconsistency-adaptive logic handle the outcomes of criteria? One possibility is to follow the Rescher–Manor approach and to consider the outcomes of the criteria as the premises (the criteria thus being the sources).[20]

A second possibility consists in applying all *analysing* rules of **CL** to separate formulas, and to close the result under the *constructive* rules for the binary connectives.[21] Thus, in the above example, $p \vee q$ will be derived from p and $\sim p$ will be derived from $\sim(p \vee q)$. This is the approach followed by $\mathbf{LP}^m$ (see [24]).[22] As we have seen in Section 3, such adaptive logics spread inconsistencies; the same holds for the specific approach followed by **AN** from [21].

CLuN and the adaptive logics that have **CLuN** as their lower limit logic proceed in a different way. The truth values of complex formulas are determined by the truth values of their subformulas, just as in the **CL**-semantics. The only difference with the **CL**-semantics is that the skyhook formulas interfere whenever they are met. This procedure may be justified in terms of criteria. Given the intended meaning of the logical symbols, the combination of a criterion for p and of a criterion for q constitutes a criterion for $p \wedge q$; that the outcome of both former criteria is positive counts as a positive outcome of the criterion for $p \wedge q$. A different combination of criteria for p and for q constitutes a criterion for $p \vee q$; that the outcome of the criterion for p is positive counts as a positive outcome of the criterion for $p \vee q$. And so on.

There is absolutely no problem with this approach in itself. If a semantics allows for models that verify both A and $\sim A$ for some but not all A, then negation cannot possibly be a binary truth-function. If the semantics allows for models that verify both A and $\sim A$ for complex A and inconsistencies should not be spread, then some formulas of the form $\sim A$ should be allowed to hang from a skyhook.

---

[19] Dialetheists and many relevantists will object for philosophical reasons that need not be considered here (and with which I disagree).

[20] This answers some objections from Section 4, but the others remain.

[21] See [21] for the distinction between analysing and constructive rules.

[22] **CLuNs** (see [13]) proceeds somewhat differently in that its implication is detachable.

However, if the skyhooks have to be justified in terms of criteria, then it is problematic that the *only* formulas hanging from skyhooks have the form $\sim A$. Consider again the simple example used throughout this section. The truth of $\sim(p \vee q)$ derived from the negative outcome of the test for $?(p \vee q, \sim(p \vee q))$. Suppose that the outcome of this test had been positive. Then $p \vee q$ would have been considered as true. However, nothing prevents that, in this situation, other criteria give us $\sim p$ as well as $\sim q$ whereas no criterion gives us either p or q. The upshot seems to be that we need models that verify $p \vee q$ but falsify both p and q, and no **CLuN**-model does so.

An interpretation of paraconsistency in terms of criteria is implicitly used in the previous paragraph. This interpretation leads to a logic that allows for gluts with respect to all logical symbols. Even if A is false, $A \wedge B$ may be true; even if A and B are false, $A \vee B$ may be true, etc. This suggests that the interpretation leads to one of the many possible logics discussed in [6], viz. gluts with respect to all logical symbols (and no gaps). However, the present interpretation requires something more. If any formula may be hanging from a skyhook, then so may *primitive* formulas. This requires some special attention.

Let us first consider sentential letters. As these are not in any way composed, it does not make any difference whether they are true in the regular way or because they are hanging from a skyhook. So sentential letters do not require special care. That a formula of the form $\alpha = \beta$ hangs from a skyhook means that $\alpha = \beta$ is true whereas $v(\alpha) \neq v(\beta)$. This is quite all right. If there are gluts with respect to all the other logical symbols, identity should not form an exception. This leaves us with primitive predicative formulas: even if $v(a) \notin v(P)$, Pa may be verified by the model. One of the effects of this is that, if $v(a) = v(b)$, it is still possible that the model verifies Pa but falsifies Pb.[23]

In the preceding discussion, I implicitly presupposed that there is at most one criterion for each (primitive or complex) formula. No change is required if this presupposition is removed. That several tests for $?(A, \sim A)$ have different outcomes simply causes a negation-glut. Incidentally, I did not exclude tests for $?(\sim A, \sim\sim A)$, and so on.

A final remark concerns gaps. I presupposed that each criterion leads to a negative or positive outcome. In order to have gaps, for example both A and $\sim A$ false, we need to presuppose that all criteria for A fail (and that all

[23] That some criterion provides us with Pa might be interpreted as a reason to take $v(a) \in v(P)$. I have investigated this approach, and it appears to undermine both the coherence and the justification of the resulting logic—however, see Section 12.

criteria for $\sim A$ fail or have a negative outcome). Even if this were the case, we may still presuppose that either A or $\sim A$ is true. The reasons for doing so are not ontological, but derive from the intended meaning of negation. That our criteria may cause inconsistencies, does not force us to give up this intended meaning in as far as it is compatible with the possibly inconsistent outcomes of criteria.[24] The **G**-models will be presented in the next section. It is useful to remark that all **CLuN**-models are **G**-models, but not the other way around.

8. The Paraconsistent Logic G

There are several ways to obtain the effect that all formulas may be verified directly in view of the assignment. I apply the simplest one that came to my mind. A model $M = \langle D, v\rangle$, in which D is a set and v an assignment function, is an interpretation of $\mathcal{W}^+$, and hence of $\mathcal{W}$. The assignment function v is defined by:

C1.1 $v : \mathcal{W}^+ \longrightarrow \{0, 1\}$
C1.2 $v : \mathcal{C} \cup \mathcal{O} \longrightarrow D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C1.3 $v : \mathcal{P}^r \longrightarrow \wp(D^r)$ (the power set of the r-th Cartesian product of D)

The valuation function $v_M : \mathcal{W}^+ \longrightarrow \{0, 1\}$ determined by M is defined as follows:

C2.1 where $A \in \mathcal{S}$, $v_M(A) = v(A)$; $v_M(\bot) = 0$
C2.2 $v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$ or $v(\pi^r\alpha_1 \ldots \alpha_r) = 1$
C2.3 $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$ or $v(\alpha = \beta) = 1$
C2.4 $v_M(\sim A) = 1$ iff $v_M(A) = 0$ or $v(\sim A) = 1$
C2.5 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$ or $v(A \supset B) = 1$
C2.6 $v_M(A \wedge B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$ or $v(A \wedge B) = 1$
C2.7 $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$ or $v(A \vee B) = 1$
C2.8 $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$ or $v(A \equiv B) = 1$
C2.9 $v_M((\forall\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$ or $v((\forall\alpha)A(\alpha)) = 1$
C2.10 $v_M((\exists\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$ or $v((\exists\alpha)A(\alpha)) = 1$

[24] This presupposition may obviously be given up and in some cases there are good reasons to do so—see, for example, [6] and [14]. However, the matter need not concern us here. Our present problem is paraconsistency, not paracompleteness.

Truth in a model, semantic consequence and validity are defined as usual. Incidentally, C2.8 does not lead to the same values as

$$v_M(A \equiv B) = 1 \text{ iff } v_M(A \supset B) = v_M(B \supset A) = 1 \text{ or } v(A \equiv B) = 1\,.$$

Indeed, nothing excludes that $v_M(A) = 1$, $v_M(B) = 0$, $v_M(A \equiv B) = v(A \equiv B) = 0$, but $v(A \supset B) = v_M(A \supset B) = v_M(B \supset A) = 1$.

G is a very poor logic. $\Gamma \cup \{A\} \vDash A$ holds in it, as well as many constructive rules of **CL**, for example $A \vDash B \supset A$, $A \vDash A \vee B$, $A, B \vDash A \wedge B$, $A, B \vDash A \equiv B$, and $A(\beta) \vDash (\exists\alpha)A(\alpha)$. Moreover, $A \vee {\sim}A$ is valid as are most formulas obtained by the Deduction Theorem from **G**-correct inferences, for example $A \supset (B \supset A)$, $A \supset (A \vee B)$, etc. However, **G** falsifies other constructive rules, such as ${\sim}A \vDash A \supset B$, and all analyzing rules of **CL**: Modus Ponens, Modus Tollens, Disjunctive Syllogism, Simplification, etc.

An axiomatization of the **G**-semantics is rather useless with respect to the inconsistency-adaptive logics I want to define from it. Indeed, I am first and foremost interested in the proof theory of these adaptive logics, and this requires that one is able to derive abnormalities as well as 'disjunctions' of abnormalities from a set of premises. The language $\mathcal{L}$ does not allow one to do so. Let me consider an example. Suppose that $p \supset q$, p, and ${\sim}q$ are derivable from the premises. It follows that either ${\sim}q$ or $p \supset q$ is abnormal (either ${\sim}q$ is true together with q, or $p \supset q$ is true whereas p is true and q is false). These two possibilities cannot be distinguished within the language $\mathcal{L}$ because the falsehood of q cannot be expressed in it (some models verify even ${\sim}q$, $q \supset \bot$ and q).

To resolve this inconvenience, I extend the language $\mathcal{L}$ to $\mathcal{L}^\dagger$ by adding the logical symbols of **CL** with their usual meaning—I shall write $\neg$, $\sqcap$, $\sqcup$, $\Rightarrow$, $\Leftrightarrow$, $(\sqcap\alpha)$, $(\sqcup\alpha)$ and $\approx$ to denote negation, conjunction, disjunction, implication, equivalence, the universal quantifier, the existential quantifier, and identity respectively. In the present paper, premises will always be closed formulas of $\mathcal{L}$; the classical connectives will merely function as a technical means to express abnormalities.

The resulting system will still be called **G**. The semantics is obtained by adding clauses for the classical connectives. These are identical to the clauses for the corresponding logical symbols of $\mathcal{L}$, except that the disjunct referring to the assignment is dropped. Thus, the clause for identity reads:

C2.11 $v_M(\alpha \approx \beta) = 1$ iff $v(\alpha) = v(\beta)$

The axiomatization of (the full system) **G** is obtained by extending an axiomatization for **CL** with the following axioms:

$\neg A \Rightarrow {\sim} A$
$(A \Rightarrow B) \Rightarrow (A \supset B)$
$(A \sqcap B) \Rightarrow (A \wedge B)$
$(A \sqcup B) \Rightarrow (A \vee B)$
$(A \Leftrightarrow B) \Rightarrow (A \equiv B)$
$a \approx b \Rightarrow a = b$
$(\sqcap\alpha)A(\alpha) \Rightarrow (\forall\alpha)A(\alpha)$
$(\sqcup\alpha)A(\alpha) \Rightarrow (\exists\alpha)A(\alpha)$

The obvious Soundness and Completeness proofs are left to the reader.

9. Abnormalities

A complex formula will be said to be normal iff it classically implies the corresponding **CL**-formula; otherwise it will be said to be abnormal. Thus $A \vee B$ is normal iff it is either false, or else true together with $A \sqcup B$, that is iff $(A \vee B) \Rightarrow (A \sqcup B)$ is true; $A \vee B$ is abnormal iff it is true together with $\neg A \sqcap \neg B$, that is iff $(A \vee B) \sqcap \neg A \sqcap \neg B$ is true.

One cannot express in $\mathcal{L}^\dagger$ that a primitive predicative formula $\pi\alpha_1 \dots \alpha_r$ is normal, which is the case iff $\langle v(\alpha_1) \dots v(\alpha_r)\rangle \in v(\pi)$. However, there is a way around this. Consider a model M that verifies Pa. Whether this is caused by $v(a) \in v(P)$ or by $v(Pa) = 1$ has no effect on superformulas of Pa, such as $Pa \vee Qc$. Nor does it have any effect on universal or existential generalizations of Pa. For example, as M verifies Pa, it verifies $(\exists x)Px$ as well as $(\sqcup x)Px$. However, there is a relevant difference between $v(a) \in v(P)$ and $v(Pa) = 1$: only the first warrants that M verifies $P\alpha$ whenever $v(\alpha) = v(a)$. The upshot is that Pa is normal if and only if $Pa \Leftrightarrow (\sqcap x)(x \approx a \Rightarrow Px)$ is true. In general, $\pi\alpha_1 \dots \alpha_r$ is normal iff it is classically equivalent to

$$((\sqcap x)(x \approx \alpha_1 \Rightarrow \pi x\alpha_2 \dots \alpha_r) \sqcap \dots \sqcap (\sqcap x)(x \approx \alpha_r \Rightarrow \pi\alpha_2 \dots \alpha_{r-1}x))$$

We shall have to compare sets of *abnormalities*, and an abnormality is not simply an abnormal formula. It is indeed essential to distinguish between, on the one hand, the fact that a formula hangs from a skyhook and, on the other hand, the trouble that is caused by this. Thus, a model in which $v(A \wedge B) = 1$, $v_M(A) = 0$ and $v_M(B) = 0$ is more abnormal than one in which $v(A \wedge B) = 1$, $v_M(A) = 0$ and $v_M(B) = 1$. The same holds for primitive predicative expressions formed by a predicate with rank greater than one. Thus, a model M in which $v_M(Pab) = 1$ is abnormal if $v(c) = v(a)$ and $v_M(Pcb) = 0$, but it is more abnormal if moreover $v(d) = v(b)$ and $v_M(Pad) = 0$.

If open formulas behave abnormally, we need to prefix the (open) abnormality with a *classical* existential quantifier over the variables that occur free in the formula. To easily handle such cases, let $(\sqcup)C$ abbreviate the result of prefixing C with a classical existential quantifier over each variable free in C. Remark that, in Section 2, the abbreviation $\exists C$ had exactly the same meaning—the existential quantifier is classical in **CLuN**.

For the sake of precision, I now list all types of abnormalities. It is useful to introduce abbreviations for the abnormalities; I list the abbreviations on the left hand side:

$$\begin{array}{rl}
!_1\pi\alpha_1\ldots\alpha_r\text{:} & (\sqcup)(\pi\alpha_1\ldots\alpha_r \sqcap (\sqcup x)(x \approx \alpha_1 \sqcap \neg\pi x\alpha_2\ldots\alpha_r)) \\
!_2\pi\alpha_1\ldots\alpha_r\text{:} & (\sqcup)(\pi\alpha_1\ldots\alpha_r \sqcap (\sqcup x)(x \approx \alpha_2 \sqcap \neg\pi\alpha_1 x\alpha_3\ldots\alpha_r)) \\
\ldots & \ldots \\
!_r\pi\alpha_1\ldots\alpha_r\text{:} & (\sqcup)(\pi\alpha_1\ldots\alpha_r \sqcap (\sqcup x)(x \approx \alpha_r \sqcap \neg\pi\alpha_1\ldots\alpha_{r-1}x)) \\
!{\sim}A\text{:} & (\sqcup)({\sim}A \sqcap A) \\
!(A \supset B)\text{:} & (\sqcup)((A \supset B) \sqcap A \sqcap \neg B) \\
!_l(A \wedge B)\text{:} & (\sqcup)((A \wedge B) \sqcap \neg A) \\
!_r(A \wedge B)\text{:} & (\sqcup)((A \wedge B) \sqcap \neg B) \\
!(A \vee B)\text{:} & (\sqcup)((A \vee B) \sqcap \neg A \sqcap \neg B) \\
!_l(A \equiv B)\text{:} & (\sqcup)((A \equiv B) \sqcap A \sqcap \neg B) \\
!_r(A \equiv B)\text{:} & (\sqcup)((A \equiv B) \sqcap B \sqcap \neg A) \\
!a = b\text{:} & (\sqcup)(a = b \sqcap \neg a \approx b) \\
!_\beta(\forall\alpha)A(\alpha)\text{:} & (\sqcup)((\forall\alpha)A(\alpha) \sqcap \neg A(\beta)) \quad (\beta \in \mathcal{C}) \\
!_x(\forall\alpha)A(\alpha)\text{:} & (\sqcup)((\forall\alpha)A(\alpha) \sqcap (\sqcup\alpha)\neg A(\alpha)) \\
!(\exists\alpha)A(\alpha)\text{:} & (\sqcup)((\exists\alpha)A(\alpha) \sqcap \neg(\sqcup\alpha)A(\alpha))
\end{array}$$

Let $¡_1\pi\alpha_1\ldots\alpha_r$ denote $\neg!_1\pi\alpha_1\ldots\alpha_r$, etc.—this will be handy to indicate, for example, that a conditional step in a dynamic proof presupposes that a formula behaves normally in a specific sense.

The abnormal part of a **G**-model M, $Ab(M)$, is the set of all abnormalities — see the above list — that are verified by M. Remark that $(\sqcup x)A(x) \in Ab(M)$ iff $(\sqcup y)A(y) \in Ab(M)$, etc. Remark also that $(\sqcup x)A(x) \in Ab(M)$ if $A(a) \in Ab(M)$, but not vice versa. This also explains the difference between $!_\beta(\forall\alpha)A(\alpha)$, which denotes a different formula for different $\beta \in \mathcal{C}$, and $!_x(\forall\alpha)A(\alpha)$ which (with some notational abuse) denotes an infinite set of equivalent formulas. Again, $!_a(\forall\alpha)A(\alpha)$ entails $!_x(\forall\alpha)A(\alpha)$ but not vice versa.

Where we need disjunctions of abnormalities, the disjunction obviously has to be classical. Also, $Dab(\Delta)$ denotes the classical disjunction of the finite set of abnormalities Δ.

It is useful to have a closer look at the way in which abnormalities interfere with some familiar inferences. $A \supset B$ is **G**-equivalent to

$$(A \Rightarrow B) \sqcup ((A \supset B) \sqcap (A \sqcap \neg B))$$

in other words,

$$(A \Rightarrow B) \sqcup !(A \supset B)$$

This has an immediate effect whenever $A \supset B$ is a premise. For example, consider what becomes of Modus Ponens:

$$A, A \supset B \vdash_{\mathbf{G}} B \sqcup !(A \supset B)$$

So from A and $A \supset B$ **G** does not enable one to derive B, but only "B *or* $A \supset B$ is abnormal" (in which the "or" is classical). Even if $A \supset B$ is not a premise but a subformula of the conclusion, a valid **CL**-inference may become invalid. Here is an example:

$$A, {\sim}B \vdash_{\mathbf{G}} {\sim}(A \supset B) \sqcup !{\sim}B \sqcup !(A \supset B)$$

Suppose that A and ${\sim}B$ are true. If ${\sim}B$ is abnormal, then B is true, and hence $A \supset B$ is true; whence ${\sim}(A \supset B)$ need not be true. If ${\sim}B$ is normal (and hence $\neg B$ is true), $A \supset B$ may be abnormal (true together with A and $\neg B$); whence ${\sim}(A \supset B)$ need not be true.

In some cases there are even more formulas that need to behave normally:

$${\sim}A \wedge {\sim}B \vdash_{\mathbf{G}} {\sim}(A \vee B) \sqcup !_1({\sim}A \wedge {\sim}B) \sqcup !_2({\sim}A \wedge {\sim}B) \sqcup !(A \vee B) \sqcup !{\sim}A \sqcup !{\sim}B$$

Compare this to:

$${\sim}(A \vee B) \vdash_{\mathbf{G}} ({\sim}A \wedge {\sim}B) \sqcup !{\sim}(A \vee B)$$

Here is an example with an abnormal open formula:

$$(\forall x)Px \vdash_{\mathbf{G}} {\sim}(\exists x){\sim}Px \sqcup !_x(\forall x)Px \sqcup (\sqcup x)!{\sim}Px \sqcup !(\exists x){\sim}Px$$

Indeed, if a model verifies $(\forall x)Px$, it may still falsify ${\sim}(\exists x){\sim}Px$ in three cases. The first is that $(\forall x)Px$ is abnormal (the model falsifies $(\sqcap x)Px$). Next, even if the model verifies $(\sqcap x)Px$, it may also verify $(\sqcup x){\sim}Px$, in which case it verifies $(\sqcup x)({\sim}Px \sqcap Px)$. Finally, even if the model verifies $(\sqcap x)Px$ and falsifies $(\sqcup x){\sim}Px$, it may still verify $(\exists x){\sim}Px$. Here is a more complex example:

$$(\forall x)(Px \supset Qx), (\exists x){\sim}Qx \;\vdash_{\mathbf{G}}$$
$$(\exists x){\sim}Px \sqcup !_x(\forall x)(Px \supset Qx) \sqcup !(\exists x){\sim}Qx \sqcup (\sqcup x)!(Px \supset Qx) \sqcup (\sqcup x)!{\sim}Qx$$

Theorem 7. $\vdash_{\mathbf{CL}} A$ *iff* $\vdash_{\mathbf{G}} A \sqcup Dab(\Delta)$ *for some (possibly empty)* Δ. *(Theorem Adjustment Theorem)*

As is the case for other adaptive logics, this theorem provides the basis for the dynamic proof theory. If $A_1, \ldots, A_n \vdash_{\mathbf{CL}} B$, then $\vdash_{\mathbf{CL}} (A_1 \sqcap \ldots \sqcap A_n) \Rightarrow B$ and hence, by Theorem 7, there is a Δ such that $\vdash_{\mathbf{G}} ((A_1 \sqcap \ldots \sqcap A_n) \Rightarrow B) \sqcup Dab(\Delta)$. The latter will be interpreted as: B is derivable from $A_1, \ldots, A_n$ on the condition that all members of Δ behave normally. Here are some examples of **G**-theorems that correspond to popular **CL**-rules—RI corresponds to a specific application only:

DN	$({\sim}{\sim}A \Rightarrow A) \sqcup !{\sim}A$
MP	$((A \sqcap (A \supset B)) \Rightarrow B) \sqcup !(A \supset B)$
MT	$(({\sim}B \sqcap (A \supset B)) \Rightarrow {\sim}A) \sqcup (!{\sim}B \sqcup !(A \supset B))$
DS	$(({\sim}B \sqcap (A \vee B)) \Rightarrow A) \sqcup (!{\sim}B \sqcup !(A \vee B))$
SIM	$((A \wedge B) \Rightarrow A) \sqcup !_l(A \wedge B)$
ND	$({\sim}(A \vee B) \Rightarrow {\sim}A) \sqcup !{\sim}(A \vee B)$
UI	$((\forall\alpha)A(\alpha) \Rightarrow A(\beta)) \sqcup !_\beta(\forall\alpha)A(\alpha)$
RI	$((Pac \sqcap a = b) \Rightarrow Pbc) \sqcup !a = b \sqcup !_1 Pac$

Here are some examples of unconditional rules:

IRR	$A \Rightarrow (B \supset A)$
ADJ	$A, B \Rightarrow (A \wedge B)$
ADD	$A \Rightarrow (A \vee B)$

10. The Adaptive Logics AG1 and AG2

Applying the Reliability strategy and the Minimal Abnormality strategy, we obtain respectively the adaptive logics **AG1** and **AG2**. *Dab*-formulas are now *classical* disjunctions of abnormalities. The premise rule is as in Section 2 and the two other rules of inference require cosmetic changes only:

RU If $A_1, \ldots, A_n \vdash_{\mathbf{G}} B$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line that has B as its second element and $\Delta_1 \cup \ldots \cup \Delta_n$ as its fifth element.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{G}} B \sqcup Dab(\Delta_0)$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line that has B as its second element and $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$ as its fifth element.

Let $U(\Gamma)$ and $U_s(\Gamma)$ be defined as before, but now with respect to minimal classical disjunctions of (the present) abnormalities.[25] The Marking definition for **AG1** is identical to that for **ACLuN1**. For the **AG2**-proof theory, we define $\Phi(\Gamma)$ and $\Phi_s(\Gamma)$ as for **ACLuN2**. The integrity criterion is as in Section 2 and the Marking definition is identical to that for **ACLuN2**. The definitions of "derivability at a stage" and of all other proof theoretic technicalities remain unchanged.

Semantically, the systems are characterized by:

DEFINITION 8. A **G**-model M of Γ is an **AG1**-model (a reliable model) of Γ iff $Ab(M) \subseteq U(\Gamma)$.

DEFINITION 9. $\Gamma \vDash_{\mathbf{AG1}} A$ iff A is verified by all **AG1**-models of Γ.

DEFINITION 10. A **G**-model M of Γ is an **AG2**-model (a minimally abnormal model) of Γ iff there is no **G**-model M' of Γ such that $Ab(M') \subset Ab(M)$.

DEFINITION 11. $\Gamma \vDash_{\mathbf{AG2}} A$ iff A is verified by all **AG2**-models of Γ.

The derivability relation of both adaptive logics has an obvious Monotonic Characterization, *e.g.*, for **AG1**:

THEOREM 8. *$\Gamma \vdash_{\mathbf{AG1}} A$ iff there is a finite set of abnormalities Δ such that $\Gamma \vdash_{\mathbf{G}} A \sqcup Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. (Monotonic Characterization)*

Moreover, the proofs of Soundness and Completeness, Proof Invariance, Strong Reassurance (and Reassurance) are straightforward. As expected, we also have:

THEOREM 9. *If $\Gamma \vDash_{\mathbf{AG1}} A$, then $\Gamma \vDash_{\mathbf{AG2}} A$.*

The following instructive example illustrates that the converse of this theorem fails: $\{p, p \supset q, {\sim}q, q \vee s, {\sim}(p \supset q) \vee s\}$. Remark that $U(\Gamma) = \{!{\sim}q, !(p \supset q)\}$. So the **AG1**-models of Γ are those **G**-models M of Γ such that $Ab(M)$ comprises either $!{\sim}q$, or $!(p \supset q)$, or both. Of these, only those of the first and second kind are **AG2**-models of Γ. Hence, s is only an **AG2**-consequence of Γ.

If Γ has **CL**-models, $Cn_{\mathbf{AG1}}(\Gamma) = Cn_{\mathbf{AG2}}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$. If Γ has no **CL**-models (and hence is (**CL**-)inconsistent), we have $Cn_{\mathbf{G}}(\Gamma) \subseteq Cn_{\mathbf{AG1}}(\Gamma) \subseteq Cn_{\mathbf{CL}}(\Gamma)$ (and similarly for **AG2**). Except for border cases, these subset-relations are proper. So I have established that **AG1** and **AG2** are decent adaptive logics.

[25] An example: $!(p \supset q) \sqcup !{\sim}q$ is unconditionally derivable from p, $p \supset q$, and ${\sim}q$.

11. Comparison with the CLuN-based Systems

It is worth discussing the way in which the present logics differ from **ACLuN1** and **ACLuN2**. The most striking difference is that, in the systems defined from **G**, a single abnormality—a one-member disjunction of abnormalities—can only be derived from a set of premises Γ iff (i) it has the form $!{\sim}A$ and (ii) $A, {\sim}A \in \Gamma$. Where one 'half' of an inconsistency is **CLuN**-derivable from a set of premises, for example, by Modus Ponens, only a classical disjunction of several abnormalities will be **G**-derivable. I present some examples in the top half of Table 1. In the bottom half of that table, I list two examples where even **CLuN** delivers a disjunction—$!_e({\sim}A \wedge {\sim}B)$ abbreviates $!_l({\sim}A \wedge {\sim}B) \sqcup !_r({\sim}A \wedge {\sim}B)$.[26]

premises	**CLuN**	**G**
$A, {\sim}A$	$!{\sim}A$	$!{\sim}A$
$A, A \supset B, {\sim}B$	$!{\sim}A$	$!{\sim}A \sqcup !(A \supset B)$
$A \wedge B, {\sim}A$	$!{\sim}A$	$!{\sim}A \sqcup !_l(A \wedge B)$
$(\forall x)Px, {\sim}Pa$	$!{\sim}Pa$	$!{\sim}Pa \sqcup !(\forall x)Px$
$Pa, a = b, {\sim}Pb$	$!{\sim}Pb$	$!{\sim}Pb \sqcup !a = b$
$A \vee B, {\sim}A, {\sim}B$	$!{\sim}A \vee !{\sim}B$	$!{\sim}A \sqcup !{\sim}B \sqcup !(A \vee B)$
$A \vee B, {\sim}A \wedge {\sim}B$	$!{\sim}A \vee !{\sim}B$	$!{\sim}A \sqcup !{\sim}B \sqcup !(A \vee B) \sqcup !_e({\sim}A \wedge {\sim}B)$

Table 1. Strongest Derivable Disjunctions of Abnormalities

The situation has rather dramatic consequences. Consider, for example, the premises $p \wedge q, {\sim}p$. The only derivable disjunction of abnormalities is $!{\sim}p \sqcup !_l(p \wedge q)$. According to both **AG1** and **AG2**, q is finally derivable from these premises, whereas p and hence $!{\sim}p$ are not.[27] According to **ACLuN1** and **ACLuN2**, q, p, and ${\sim}p$, and hence $!{\sim}p$, are *unconditionally* derivable from the premises.

Let us extend these premises to $p \wedge q, {\sim}p, {\sim}p \supset r, q \supset s, p \supset t$. The only derivable disjunction of abnormalities is still $!{\sim}p \sqcup !_l(p \wedge q)$. So, from these premises, q, r, and s are finally derivable, but p and t are not—as t is only derivable on the condition $\{¡(p \supset t), ¡_l(p \wedge q)\}$, and hence, the line at which it is derived will be marked on both strategies. Compare this to: according

[26] To interpret the table, recall that, in **CLuN**, "$\wedge$" and "$\vee$" have the same force as "$\sqcap$" and "$\sqcup$" respectively.

[27] Incidentally, that q is finally derivable depends on the fact that we distinguished between $!_l(p \wedge q)$ and $!_r(p \wedge q)$; if we would take $p \wedge q$ to be abnormal as soon as one of its disjuncts is false, q would not be derivable.

to **ACLuN1** and **ACLuN2**, p, $\sim p$, q, r, s, and t are all unconditionally derivable from the premises.

Finally let us modify the premises to $p \wedge q, \sim p, p \vee r, \sim q \vee s, \sim p \vee t$. Again, the only derivable disjunction of abnormalities is $!\sim p \sqcup !_l(p \wedge q)$. So q and s are finally derivable whereas p, r, and t are not.[28] **ACLuN1** and **ACLuN2** deliver p, q and s as final consequences.

All these examples illustrate two characteristics of the adaptive logics defined from **G**. First, the set of consequences is reduced in comparison to systems defined from **CLuN**. Many formulas that are unconditionally derivable from a set of premises by the **CLuN**-systems become conditionally derivable only or even non-derivable. Some formulas that are conditionally derivable on the **CLuN**-systems become non-derivable. Next, the same holds for abnormalities. In general, we obtain longer disjunctions of abnormalities than on the **CLuN**-systems, and it is exceptional that all the disjuncts are inconsistencies.

In this sense, the systems defined from **G** isolate abnormalities as strongly as the **CLuN**-systems. One should not misunderstand this. One half of $!\sim p$ is non-derivable in the examples from the previous paragraphs, but a disjunction of abnormalities that has $!\sim p$ as a disjunct is unconditionally derivable. The effect is that the deductive force of p is lost in those examples (as p is not finally derivable) whereas p is nevertheless seen as suspect (for example in that $!\sim p \in U(\Gamma)$).

It is instructive to reconsider the example from Section 3 in connection with the isolation of abnormalities:

$$p,\ s \wedge q,\ \sim(p \wedge q),\ \sim p \vee r$$

The only derivable disjunction of abnormalities is $!\sim(p \wedge q) \sqcup !_r(s \wedge q)$. Hence p and r are still finally derivable, as desired. However, q is not finally derivable by the present logics. So if $q \supset t$ were added to the premises, t would not be finally derivable. Remark the asymmetry between p and q. The formula p derives from a direct criterion, whereas q is known in a more indirect way: a deduction from the result of a criterion for $s \wedge q$. So it is sensible that q is considered as more problematic than p.

The fact that inconsistency-adaptive logics defined from **G** deliver less consequences than those defined from **CLuN** is not necessarily a disadvantage. Let us reconsider the last example (with $q \supset t$ a premise) from the

[28] As $\sim p$ is unconditionally derivable, so is $\sim p \vee t$, which incidentally (and somewhat uselessly) has the force of $\sim p \sqcup t$.

viewpoint of the intended application context of inconsistency-adaptive logics. Two comments are at hand.

First, the systems defined from **CLuN** point to $!{\sim}(p \wedge q)$ as the problem and hence advise us to test $?(p \wedge q, {\sim}(p \wedge q))$. The systems defined from **G** point to $!{\sim}(p \wedge q) \sqcup !_r(s \wedge q)$ as the problem and hence advise us to test $?(p \wedge q, {\sim}(p \wedge q))$ as well as $?(s \wedge q, \neg q)$.

This deserves a small digression. The test for $?(p \wedge q, {\sim}(p \wedge q))$ is clear enough. Depending on the available criteria, this might be a direct test, or an indirect one, built up from a test of $?(p, {\sim}p)$ and a test of $?(q, {\sim}q)$. But what is a test for $?(s \wedge q, \neg q)$? As s is unproblematic (and useless with respect to the derivability of t), we are really interested in a test for $?(q, \neg q)$. However, we should be serious at this point. We cannot presuppose that the tests always deliver consistent answers. In other words, we can at best perform a test for $?(q, {\sim}q)$. Moreover, as the original tests led to an inconsistent set of premises, we can only hope to resolve the inconsistency by means of an improved set of tests. *If* the outcome of the new tests are consistent, an answer to $?(q, {\sim}q)$ is sufficient to resolve our problem. Indeed, if the outcome is q, we can derive t and know that the problem with the test for $?(p \wedge q, {\sim}(p \wedge q))$ reduces to a test for $?(p, {\sim}p)$. If the outcome is ${\sim}q$, we retain s and consider $q \supset t$ as true in virtue of ${\sim}q$, and hence as inconclusive with respect to t.[29]

The second comment concerns a rather different consideration. In some cases, the choice between different consistent alternatives depends on their consequences. Thus, in the absence of conclusive empirical results, one might prefer the theory that has the greater explanatory power. In view of this, it is useful that a dynamic proof reveals the consequences of the 'halves' of inconsistencies. In dynamic proofs of the **CLuN**-systems, these consequences may be traced by spelling out the path of derived formulas. The fact that the **G**-systems deliver less final consequences need not be a serious drawback in this respect. The consequences of 'halves' of abnormalities may still be traced.[30] Only this time they will be revealed by *marked* lines of the proof. As an illustration, consider again the same example. We have seen that any line on which q is derived will be marked (as soon as $!{\sim}(p \wedge q) \sqcup !_r(s \wedge q)$ is derived). Nevertheless, nothing prevents one to derive further consequences of q, most importantly t. It will be derived on a line that has the condi-

[29] If a different test for $?(q, {\sim}q)$ delivers q, or if a test for $?(s \wedge q, {\sim}(s \wedge q))$ delivers $s \wedge q$, we know that the tests are still problematic and hence have to revise them. This is a nuisance, but an unavoidable one. However, if the tests deliver consistent results, then the test for $?(q, {\sim}q)$ leads to a consistent improvement of the theory.

[30] This is, of course, not typical for the **G**-systems but holds for all adaptive logics.

tion $¡_r(s \wedge q)$ and hence will be marked. But this is not important. In the presence of the premise $s \wedge q$, $¡_r(s \wedge q)$ entails that q is true, and hence the (marked) lines with this condition adequately depict the consequences of the *modified* premises that make q true. By a hardly more complex reasoning, (marked) lines with the condition $¡{\sim}(p \wedge q)$ contain consequences of the *modified* premises that make ${\sim}q$ true—because p is unconditionally derivable. If ${\sim}q \supset u$ or even $q \vee u$ were added to the premises, then u would be derivable on such a line.

This too deserves a small digression. Some people, among them Joke Meheus, have argued that one should look for an adaptive logic that delivers *all* consequences of *all* possible (but sensible) consistent selections that may be obtained from an inconsistent theory. No such system is possible because different adaptive logics define different sets of abnormalities, as is obvious from the logics discussed in this paper. However, one may still compare the result of different adaptive logics, and compare the different 'normal selections' to which each of them leads.

12. Skyhook Formulas Plus Choices

In the previous sections, paraconsistent logics were justified in terms of the skyhook idea, viz. that formulas of any complexity are provided by criteria. Let us now reconsider more systematically the way in which models may be obtained from a set of skyhook formulas.

Needless to say, a set of skyhook formulas by itself does not constitute a model. The reason is that the skyhook formulas by themselves do not lead to any decisions on the meaning of the logical symbols; this is handled by clauses. Some clauses, for example "if $M \models A$ and $M \models B$, then $M \models A \wedge B$", introduce a logical symbol and hence extend the valuation to superformulas. Other clauses, for example "if $M \models A \wedge B$, then $M \models A$ and $M \models B$", eliminate a logical symbol and hence extend the valuation to subformulas. Each clause provides an *indirect criterion*, relying on the intended interpretation of the logical symbols—the outcomes A and B, provide an indirect criterion for $A \wedge B$. So, while skyhook formulas derive from the direct criteria, other formulas may derive from the indirect criteria.

How do the clauses that determine **CL**-models relate to **G**-models and to **CLuN**-models? Let us start with **G**. Clauses that introduce a single logical symbol and have a positive consequent—one that does not contain "$\not\models$"—hold unrestrictedly in **G**-models. An example is: "if $M \models B$, then $M \models A \supset B$". Clauses that eliminate a single logical symbol and have a positive antecedent function as defaults: if their antecedent is true, M

will agree with their consequent unless this is contradicted by a skyhook formula or by one of the aforementioned introduction clauses. Examples are "if $M \models \sim A$, then $M \not\models A$", "if $M \models A \supset B$, then $M \not\models A$ or $M \models B$". Other clauses that introduce or eliminate a single symbol are contrapositives of the discussed ones and hence need not further be considered.

It is interesting to compare the situation with **CLuN**. Here all aforementioned clauses hold unconditionally, with the exception of "if $M \models A$, then $M \not\models \sim A$", which functions as a default. Remark first that this plot is formally coherent: a set of skyhook formulas may be inconsistent, in that both A and $\sim A$ belong to it, but cannot itself display any other logical abnormalities—*any* set of formulas has a **CLuN**-model. It follows at once that an approach in terms of skyhook formulas does not force one to chose for the plot behind **G**. Moreover, the plot behind **CLuN** is attractive, even from a philosophical point of view.

If the outcome of a criterion is $A \wedge B$, this provides a reason to believe A as well as B; if the outcome is $A \vee B$, it provides a reason to believe A or B. What if the outcome of a criterion is $\sim(A \wedge B)$? This provides a reason to believe that $A \wedge B$ is false. If indeed $A \wedge B$ is false, we have a reason to believe that either $\sim A$ or $\sim B$ is true. However, the reason to believe that $A \wedge B$ is false may be overruled by the outcome of a different criterion. If $A \wedge B$ is the outcome of a different criterion, the previous reasoning is blocked. The reason to believe that $A \wedge B$ is false is overruled, and hence we have no reason for believing that either $\sim A$ or $\sim B$ is true.[31]

The plot that leads from skyhook formulas to **CLuN** is simpler and in a sense more attractive than the one that leads to **G**. The only derived criterion that may be overruled is the one that leads from a reason to consider A (respectively $\sim A$) as true to a reason to consider $\sim A$ (respectively A) as false.[32] Even a derived criterion for Pa comes to a criterion for $v(a) \in v(P)$.

What was going on in this section? First and foremost, I was not trying to justify an adaptive logic, but a type of models. From the epistemological side, I supposed that our theories rely upon certain criteria (for possibly

[31] If both A and B are outcomes of criteria, we have a reason to believe that $A \wedge B$ is true. Remark that, even on the plot behind **G**, this forces one to consider $A \wedge B$ as true. As we cannot avoid inconsistency on the skyhook approach, the upshot is that we have to let the reason to consider $A \wedge B$ as true prevail over the reason to consider it as false. The only alternative is to put both reasons on a par. This, however, leads to spreading inconsistency.

[32] The situation for material implication is very different. A reason to believe $A \supset B$, provides a reason to believe that A is false or that B is true. If we moreover have a reason to believe that A is true, then we have a reason to believe that B is true.

complex formulas). Where the direct criteria leave the truth-value of certain formulas undetermined, it has to be supplied by the intended meaning of the logical symbols. Remark that this not only settles the truth-value of subformulas and superformulas, but even of formulas that are unrelated to the skyhook formulas: even if A is not a subformula of any skyhook formula, if we take the model M to falsify A, then M verifies $\sim A$ (and vice versa). The difference between the logics turns out to depend on the choice of the clauses that are considered as defaults only. Both **G** and **CLuN** are fine in this respect, but rely on different presuppositions.

It turned out, somewhat unexpectedly, that all sorts of formulas may be hanging from skyhooks in the **CLuN**-models. This is not clear at once because only negations lead to abnormalities. However, as we have seen, this effect results from a very sensible plot. The skyhook approach need not lead to gluts for all logical symbols. And the effects of the skyhooks need not be explicit in the *characterization* of the valuation function.

13. In Conclusion

The somewhat odd line of exposition of the present paper corresponds to the way in which the results were obtained. The justification of a paraconsistent logic requires that one has independent reasons for believing certain complex formulas. I described the matter in terms of skyhook formulas. Trying to avoid an apparent asymmetry of **CLuN**-models (only negations seem to be hanging from skyhooks), we arrived at the logic **G**. Its models allow for gluts with respect to all logical symbols. To define adaptive logics from **G** turned out to be straightforward.

Next, we found an asymmetry in the **G**-models: clauses that introduce logical symbols (in the sense that a complex formula is verified) hold unrestrictedly, whereas clauses that eliminate logical symbols function only as defaults. Trying to overcome this asymmetry led to the idea to make positive conclusions of clauses hold unconditionally, and to let only conclusions of the form $M \not\models A$ be subject to being overruled. Unexpectedly, this plot resulted in the **CLuN**-models.

Presumably, other plots may be formulated, and they may lead to different logics. A systematic research in this direction is wholly beyond the confines of the present paper. Nevertheless, it is clear that the skyhook approach cannot justify logics that are not paraconsistent—nothing prevents different criteria from leading to the outcomes A and $\sim A$ respectively.

An interesting feature of the skyhook approach is that it may be applied directly to a set of premises to arrive at adaptive models of the premises.

Given a set of premises (a 'theory') Γ, the criteria underlying a model of Γ provide a reason to believe Γ. In the intended application contexts of *adaptive* logics, we consider 'the world' to be as normally as possible with respect to Γ. And indeed, it is easily seen that applying the reasoning from the previous section to a set of skyhook formulas Γ leads to the adaptive models of Γ, and not to all its lower limit models. Which adaptive models will be obtained will depend on the chosen plot (determining the lower limit logic) and on the chosen adaptive strategy. Remark that such an approach directly connects the adaptive models to the dynamic proof theory.[33]

References

[1] Diderik Batens, 'Paraconsistent extensional propositional logics', *Logique et Analyse*, 90–91:195–234, 1980.

[2] Diderik Batens, 'Dialectical dynamics within formal logics', *Logique et Analyse*, 114:161–173, 1986.

[3] Diderik Batens, 'Dynamic dialectical logics', in Priest et al. [26], pp. 187–217.

[4] Diderik Batens, 'Inconsistency-adaptive logics', in Orłowska [23], pp. 445–472.

[5] Diderik Batens, 'Linguistic and ontological measures for comparing the inconsistent parts of models', *Logique et Analyse*, 165–166:5–33, 1999. Appeared 2002.

[6] Diderik Batens, 'Zero logic adding up to classical logic', *Logical Studies*, 2:15, 1999, (Electronic Journal: `http://www.logic.ru/LogStud/02/LS2.html`).

[7] Diderik Batens, 'Rich inconsistency-adaptive logics. The clash between heuristic efficiency and realistic reconstruction', in François Beets and Éric Gillet, (eds.), *Logique en perspective. Mélanges offerts à Paul Gochet*, pp. 513–543. Éditions OUSIA, Brussels, 2000.

[8] Diderik Batens, 'Towards the unification of inconsistency handling mechanisms', *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.

[9] Diderik Batens, 'On some remarkable relations between paraconsistent logics, modal logics, and ambiguity logics', in Walter A. Carnielli, Marcelo E. Coniglio, and Itala M. Loffredo D'Ottaviano, (eds.), *Paraconsistency. The Logical Way to the Inconsistent*, pp. 275–293. Marcel Dekker, New York / Basel, 2002.

[33] Most unpublished papers in the reference section (and many others) are available from the internet address `http://logica.rug.ac.be/centrum/writings/`.

[10] Diderik Batens, 'The theory of the process of explanation generalized to include the inconsistent case', *Synthese*, in print.

[11] Diderik Batens, 'A strengthening of the Rescher–Manor consequence relations', *Logique et Analyse*, To appear.

[12] Diderik Batens, 'Narrowing down suspicion in inconsistent premise sets', To appear in the *Festschrift* for Jerzy Perzanowski.

[13] Diderik Batens and Kristof De Clercq, 'A rich paraconsistent extension of full positive logic', To appear.

[14] Diderik Batens and Guido Vanackere, 'Incompleteness-adaptive logics', To appear.

[15] Salem Benferhat, Didier Dubois, and Henri Prade, 'Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 1: The flat case', *Studia Logica*, 58:17–45, 1997.

[16] Salem Benferhat, Didier Dubois, and Henri Prade, 'Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case', in Orłowska [23], pp. 473–511.

[17] David Lewis, *Counterfactuals*, Harvard University Press, Cambridge, Mass., 1973.

[18] Witold Łukaszewicz, *Non-Monotonic Reasoning. Formalization of Commonsense Reasoning*, Ellis Horwood, New York, 1990.

[19] Joke Meheus, 'Adaptive logic in scientific discovery: the case of Clausius', *Logique et Analyse*, 143–144:359–389, 1993. Appeared 1996.

[20] Joke Meheus, *Wetenschappelijke ontdekking en creativiteit. Een poging tot theorievorming op basis van een conceptuele, methodologische en logische studie*, PhD thesis, Universiteit Gent (Belgium), 1997. Unpublished PhD thesis.

[21] Joke Meheus, 'An extremely rich paraconsistent logic and the adaptive logic based on it', in Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, (eds.), *Frontiers of Paraconsistent Logic*, pp. 189–201, Research Studies Press, Baldock, UK, 2000.

[22] Joke Meheus, 'An adaptive logic based on Jaśkowski's **D2**', To appear.

[23] Ewa Orłowska, (ed.), *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*, Physica Verlag (Springer), Heidelberg, New York, 1999.

[24] Graham Priest, 'Minimally inconsistent **LP**', *Studia Logica*, 50:321–331, 1991.

[25] Graham Priest, 'Paraconsistent logic', in D. Gabbay and F. Guenthner, (eds.), *Handbook of Philosophical Logic.*, volume 2. Kluwer, Dordrecht, 2 edition, in print.

[26] Graham Priest, Richard Routley, and Jean Norman, (eds.), *Paraconsistent Logic. Essays on the Inconsistent*, Philosophia Verlag, München, 1989.

[27] Nicholas Rescher, *Hypothetical Reasoning*, North-Holland, Amsterdam, 1964.

[28] Nicholas Rescher, *The Coherence Theory of Truth*, Clarendon, Oxford, 1973.

[29] Nicholas Rescher, *Plausible Reasoning. An Introduction to the Theory and Practice of Plausibilistic Inference*, Van Gorcum, Assen/Amsterdam, 1976.

[30] Nicholas Rescher and Ruth Manor, 'On inference from inconsistent premises', *Theory and Decision*, 1:179–217, 1970.

[31] Peter K. Schotch and Raymond E. Jennings, 'On detonating', in Priest et al. [26], pp. 306–327.

[32] Guido Vanackere, 'Ambiguity-adaptive logic', *Logique et Analyse*, 159:261–280, 1997. Appeared 1999.

[33] Guido Vanackere, 'Minimizing ambiguity and paraconsistency', *Logique et Analyse*, 165–166:139–160, 1999. Appeared 2002.

Diderik Batens
Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
Diderik.Batens@UGent.be