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NOTES ON THE GEOMETRY OF LOGIC AND PHILOSOPHY

Abstract. The paper is concerned with topological and geometrical characteristics of ultrafilter space which is widely employed in mathematical logic. Some philosophical applications (modal realism, theory of truth) are offered together with visulisations that reveal the beauty of logical constructions.

The ultrafilter space (US), the set of maximal filters of a given algebra $\mathfrak{A}$, plays the fundamental role of universal framework for semantical investigations. Obviously US is too weak in order to be interesting or important when considered alone. Thus routinely it is provided with additional elements. The most elementary enrichment is obtained through putting on US the Stone topology ST (whose compactness was established by Tarski). In this case a topological space $\langle \mathrm{US}, \mathrm{ST} \rangle$ is called the Stone space of the algebra $\mathfrak{A}$. Of course US might be provided with other objects. As was first observed by Lindenbaum, formalized languages can be conceived as sets with free operations determined by formation rules. Hence, if the algebra $\mathfrak{A}$ is an algebra of modal formulas, then US equipped with a binary relation constitutes a frame; if we add to the frame a valuation function we will build a canonical model of $\mathfrak{A}$. More general construction might be obtained through adding a consequence operator Cn (defined by Galois connection). Routinely a set supplied with a closure operator is regarded as a special type of topological space, namely a closure space. The closure space $\langle \mathrm{US}, \mathrm{Cn} \rangle$ constitutes so called B-natural dual space of $\mathfrak{A}$ and is fundamental for abstract logics. Logicians usually refer to US' points (or sets of its points) as possible worlds (Hughes, Cresswell), states of affairs (Hintikka), propositions (Segerberg) — depending on the field of application. This paper is concerned with philosophical applications of US. Customary, description of US is kept on the level corresponding to the properties of a given logic such like finiteness, regularity, negativity, etc. Here we are going to expand this (from philosophical point of view: cramped) field of studying; even topological separation properties, originated in geometry and with no logical use, appear in the scope of our interest. We hope that general structure of US (especially with addition of ST) will shed new light on epistemology and ontology, for both of them are rooted in questions about possible worlds (Leibniz), propositional attitudes (Frege) and other concepts that might be modeled on the ground of US. We are going to enjoy especially with metric (geometric) properties that despite of philosophical meaning will reveal us visual beauty of logical constructions.

Geometric approach to logic is present at least in some influential works on the algebraic foundations of metamathematics (Rasiowa, Sikorski [1970]; Rieger [1967]). Nevertheless these studies are confined only to the Cantor discontinuum $\mathrm{C}\,\boldsymbol{D}$. Let E be a non-empty set and D the Cartesian product U^E where U is the set consisting only of integers 0, 1. For every $a \in E$, let D_a be the set of all $u \in D$ such that $u_a = 1$. We define the class $\boldsymbol{D}_0$ of all sets D_a $(a \in E)$ and their complements. Let $\boldsymbol{D}$ be the field of subsets of D generated by $\boldsymbol{D}_0$. The Cantor discontinuum $\mathrm{C}\,\boldsymbol{D}$ is a topological space with $\boldsymbol{D}_0$ as a subbasis. Obviously, by definition, $\boldsymbol{D}$ is a basis. Two main results are as follows:

Theorem 1. *Every mapping g from $\boldsymbol{D}_0$ into a Boolean algebra $\mathfrak{B}$ can be extended to a Boolean homomorphism h of $\boldsymbol{D}$ into $\mathfrak{B}$.*

Theorem 2. *The Cantor discontinuum $\mathrm{C}\,\boldsymbol{D}$ is homeomorphic to the Stone space of the Boolean algebra $\boldsymbol{D}$.*

We would like to extend the results of Rasiowa and Sikorski through examining few more properties of the Stone space and the Cantor discontinuum. Moreover, this paper draws inspirations from topology and hence $\mathrm{C}\,\boldsymbol{D}$ plays different role here then in the two above theorems; we are concerned with the Cantor perfect set rather then the Cantor discontinuum.

Let us fix a language L (a quotient algebra modulo logical equivalence that is, as a rule, atomless). We are going to provide the set US of complete theories of the language L with topology. We associate each sentence a of the language to the set $\langle a \rangle$ of complete theories containing a. Hence $T \in \langle a \rangle$ means that $T \models a$ (read "a is a consequence of T"). Note that $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$ and therefore sets of the form $\langle a \rangle$ constitute an open base for a topology ST associated with US. The topological space $\langle \mathrm{US}, \mathrm{ST} \rangle$

(from now on abbreviated as US) is a Hausdorff space: if $T \neq T'$, then, because T and T' are complete, there exists some sentence a such that $T \models a$ and $T' \models \neg a$ and hence $\langle a \rangle$, $\langle \neg a \rangle$ are disjoint neighborhoods of T and T', respectively. The space US is also 0-dimensional, i.e., a space with a base consisting of both open and closed sets. Note that the complement of $\langle a \rangle$ is $\langle \neg a \rangle$. Compactness is the most essential logical property of US. As was proved by Gödel, for a set A of sentences to be consistent it is sufficient that every finite subset of A be consistent. On the topological ground US is, in the result, totally discontinuous and totally disconnected: all its subcontinua are single points and every connected subset either is empty or reduces to a point, respectively. Most of these facts were established by Stone's representation theorem:

THEOREM 3 (Stone's Theorem). *Every Boolean algebra $\mathfrak{B}$ is isomorphic to the clopen (both open and closed) base of a compact 0-dimensional space.*

This paper is especially concerned with geometric features of US; we would like to please our eyes. In order to be metric any topological space must have 'good' separation properties. A classical result is due to Uryson:

THEOREM 4 (Uryson's Theorem). *Every normal T_1 space with a countable base is metrizable.*

Because US is Hausdorff, i.e., T_2 space, whose base is isomorphic to the countable Boolean algebra $\mathfrak{L}$, we must ensure that US is normal. One of the peculiarities of the separation properties in compact spaces is the coincidence of axioms T_2, T_3, T_4; thus:

THEOREM 5. *Every compact Hausdorff space is normal.*

An ultrafilter of Boolean algebra $\mathfrak{A}$ is principal if it is generated by an atom of $\mathfrak{A}$.

THEOREM 6. *Principal ultrafilters of algebra $\mathfrak{A}$ coincide with isolated points of the Stone space of $\mathfrak{A}$.*

Hence US is a metrizable space without isolated points. Having these properties in a hand we can start seeking topological equivalent for US. We are especially interested with those homeomorphic spaces that are well known and *have a look*.

THEOREM 7. *Up to homeomorphism, the Cantor perfect set is the only 0-dimensional metrizable compactum, which does not have isolated points.*

The Cantor perfect set (denoted here as $\boldsymbol{C}$) is one of the most interesting examples of spaces that is often employed to create other useful mathematical objects. We construct $\boldsymbol{C}$ as a limit of an iterative process. The algorithm is as follows:

1. Divide the remaining intervals each into three equal parts.
2. Remove the open middle interval.
3. Repeat 1.

We start with the unit segment $[0,1]$, thus first we remove the interval $(\frac{1}{3},\frac{2}{3})$. This leaves a union of two closed intervals: $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. Next we split each of the remaining intervals $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$ into three smaller ones and remove the middle part $(\frac{1}{9},\frac{2}{9})$ from the first and $(\frac{7}{9},\frac{8}{9})$ from the second, respectively. We are left with the union of four intervals: $[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$. $\mathbf{C}$ is defined as the leftover collection of points. As a subspace of the real line $\mathbb{R}$, the Cantor perfect set has a number of amazing properties. We will return to it again. Now, we would like to emphasize that, by the last theorem, $\mathbf{C}$ is topologically indistinguishable from topological space US. Suppose now that S is any consistent normal propositional modal system. A canonical model for S is a triple $\langle W,R,V\rangle$. We take as members of W (typically labeled as *worlds*) sets of well formed formulas that are maximal consistent with respect to S. A valuation function V associates each formula of S with worlds in which the formula is true. Thus V generates on W the Stone topology. Hence we can conceive worlds as points of the Cantor perfect set. In the classical theory of possible worlds (abbreviated here as PWT), as is practiced by philosophers, one of these points must be designated; typically the designated world is referred to as the actual one. Lewis in *On the Plurality of Worlds* argues that all possible worlds do exist like the actual, our world. 'Actual' is indexical like 'I' or 'now': it depends for its reference on the circumstances of utterance. Thus inhabitants of other worlds may truly call their own worlds actual. Lewis, contrary to us, do not identify possible worlds with ultrafilters; he takes them as *primitives* (respectable entities in their own right) and so his argumentation is strictly philosophical. We would like to present here more formal treatement of *modal realism*.

Theorem 8. *The Cantor perfect set is homogeneous.*

Topological space X is homogeneous if for all x, $y\in X$ there exists homeomorphism $h\colon X\to X$, such that $h(x)=y$. It means that all worlds (including the actual one) are indistinguishable. Given that possible worlds constitute a homogeneous space of possibilities we can compare PWT with Everett's interpretation of quantum mechanics, well known as *many-worlds interpretation* (MWI). Everett accepts the universal validity of Schroedinger equation and consider the wave-function as a real object. MWI is a return to the classical, pre-quantum view of the universe. For example the electromagnetic fields of Maxwell or the atoms of Dalton were considered in the classical physics as true parts of the world. Therefore all possibilities involved in wave-equation (including the outcome of measure) are equally real. Following this way we might regard the *actuality* like the *quantum cut* and modal realism like MWI. Despite of counterintuitive character of MWI cosmologists and quantum field theorists like Hawking, Gell-Mann, Feynman, Weinberg regard MWI as true interpretation because it is ontology directly suggested by mathematics. Therefore, adopting this view, we may expand the classical weak ontology and regard modal realism as true theory (what is suggested by topology). According to Lewis such strong ontology would be a paradise for philosophers. There is a deeper link between quantum mechanics and PWT then the above philosophical speculations. The *Hilbert space* is a fundamental scene for description of quantum phenomena. We construct it as follows: let H be the set of all infinite sequences $(x_i)_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} x_i^2 < \infty$. For any $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ we set

$$\mathrm{p}(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

The metric space $\langle \mathrm{H}, \mathrm{p} \rangle$ is called a Hilbert space of countable weight.

THEOREM 9. *Every regular space with countable base can be embedded in the Hilbert space.*

Every metrizable space is regular and so we can regard possible worlds space as a subspace of $\langle \mathrm{H}, \mathrm{p} \rangle$. The problem of connections between the two spaces and eventual implications will be a subject of my future works. Let us come back to the Cantor perfect set. It is often presented as one of the best known and easily constructed fractals. Nowadays fractals are the most popular mathematical objects and are applied almost everywhere. According to Mandelbrot a fractal is a set whose topological dimension is lower then Hausdorff one (denoted as $\dim_{\mathrm{H}}$). Computing $\dim_{\mathrm{H}}$ is complicated even for simple sets thus we introduce self-similarity dimension. By ratio list $\boldsymbol{r}$ we mean a finite list of positive numbers $(r_1, r_2, \ldots, r_n)$. An iterated system realizing a ratio list $\boldsymbol{r}$ in a metric space S is a list $(f_1, f_2, \ldots, f_n)$, where

$f_i \colon S \to S$ is a similarity with ratio r_i. A nonempty compact set $X \subseteq S$ is an invariant set for $(f_1, f_2, \dots, f_n)$ iff $X = f_1(X) \cup f_2(X) \cup \dots \cup f_n(X)$.

THEOREM 10. *Let $f_i \colon S \to S$ be an iterated (contraction) function system, then there is a unique invariant set (attractor) if S is complete.*

It means that every iterated function system defined on the US space has the unique attractor. It is very important fact for epistemology. Scientific inquiry is usually modeled as a game between a scientist and Nature. To start the game, a set P of worlds is announced to both players. One member w of P is chosen by Nature as the actual world that is not revealed to the scientist. Nature also chooses an environment, i.e., a data-stream that provides information about the actual world. Then the scientist is (progressively) demonstrated the data in the environment and wins the game just in case his evolving conjectures stabilize to the correct answer to the question about w formulated in P. Moves made by Nature (data-stream) constitute (through Galois connection) an iterated function system that acts on the belief representation of scientist. Traditionally logicians model belief set through US space (Segerberg). Now it is very natural to define *epistemic truth* (*winning strategy*) as the only attractor of the game. Thanks to completeness of US epistemic truth must coincide with the *absolute truth* (no matter who plays the game with Nature, including God, attractor (winning strategy) is the same).

The dimension associated with ratio list $\boldsymbol{r}$ is the positive number d such that:

$$r_1^d + r_2^d + \dots + r_n^d = 1$$

Let us consider two dilations on the real line: $f_1 = \frac{x}{3}$ and $f_2 = \frac{(x+2)}{3}$. The Cantor perfect set $\mathbf{C}$ is the invariant set for the iterated function system (f_1, f_2) with ratio list $(\frac{1}{3}, \frac{1}{3})$. Thus the similarity dimension d is the solution of $2 \cdot (\frac{1}{3})^d = 1$. So $d = \frac{\log 2}{\log 3}$, what amounts to $0.6309\dots$. Generally if X is an invariant set for an iterated function system then $\dim_{\mathrm{H}}(X)$ differs from d. But because $\mathbf{C}$ satisfies *Moran's open set condition* (we omit technical details) we have $\dim_{\mathrm{H}}(\mathbf{C}) = d$. As we pointed out $\mathbf{C}$ is homeomorphic to US. Does it follow that $\dim_{\mathrm{H}} = \frac{\log 2}{\log 3}$?

THEOREM 11. *If $g \colon F \to \mathbb{R}^m$ is a bi-Lipschitz information, i.e., $c_1|x-y| \leqslant |g(x) - g(y)| \leqslant c_2|x-y|$, where $x, y \in F$, and $0 < c_1 < c_2 < \infty$, then $\dim_{\mathrm{H}}(F) = \dim_{\mathrm{H}}(g(F))$.*

In other words Hausdorff dimension is invariant under bi-Lipschitz transformation. Of course not every homeomorphism has bounded distortion, i.e., is bi-Lipschitz. Let us consider an example. We introduce now Cantor discontinuum. It plays here different role then in investigations of Rasiowa, Sikorski or Rieger and hence we define it a bit otherwise. By Cantor discontinuum (or cube) D^t of weight t we mean the product of t copies of the discrete two point space $\{0, 1\}$.

THEOREM 12. *The Cantor discontinuum D^ω of countable weight is homeomorphic to the Cantor perfect set.*

We can define many metrics on D^ω. Let m be a rational number satisfying $0 < m < 1$. A metric p_m is defined as following: if $a = sl$, $b = sl'$, where a, b are infinite strings, s is, possibly empty, finite string and the first character of l is different then the first character of l', k is the length of s then: $p_m(a, b) = m^k$.

The metric spaces constructed from D^ω using different metrics p_m are homeomorphic to each other; moreover, all of them are complete. Nevertheless, they possess different Hausdorff dimensions. The space D^ω with metric p_m has $\dim_{\mathrm{H}} = \frac{\log 2}{\log \frac{1}{m}}$. Note that under $p_m = \frac{1}{2}$ $\dim_{\mathrm{H}}(D^\omega) = 1$, while under $p_m = \frac{1}{3}$ $\dim_{\mathrm{H}} = \frac{\log 2}{\log 3}$. Thus only metric space $\langle D^\omega, p_{\frac{1}{3}} \rangle$ is bi-Lipschitz equivalent to the Cantor perfect set. So, what does it mean, in fractal horizon, that the space US is 0-dimensional?

THEOREM 13. *Let X be a separable metric space, then* $\operatorname{ind} X = \inf\{\dim_{\mathrm{H}} Y :$ *Y is homeomorphic to $X\}$, where* ind *is (small) topological dimension.*

Obviously US is separable space (it has countable base). Because $\lim_{m\to\infty} \frac{\log 2}{\log \frac{1}{m}} = 0$, the ultrafilter space is 0-dimensional. Since $\mathbf{C}$ is homeomorphic to D^ω any finite, or even countable, power of $\mathbf{C}$ is homeomorphic to $\mathbf{C}$. It allows us to visualize the space US. Lets us consider the product $\mathbf{C} \times \mathbf{C}$. First draw a square 1×1, divide it into nine squares of sides $\frac{1}{3}$. Remove five middle subsquares and repeat this process with left squares (see Figure 1.) Note that we can spread the Cantor perfect set in any dimension we want. For example take a unit cube, divide it into 27 sub-cubes, remove all central ones and repeat this process with left cubes. The first step is illustrated below (Figure 2). This way we construct three-dimensional Cantor perfect set $\mathbf{C} \times \mathbf{C} \times \mathbf{C}$ that is homeomorphic to $\mathbf{C} \times \mathbf{C}$. It is very distinctive feature of $\mathbf{C}$. Note that a unit segment $\mathbf{I}$ is, obviously, not homeomorphic to the square $\mathbf{I} \times \mathbf{I}$. Pattern arising from the above construction lies in the exponents: in dimension n we divide a cube into 3^n parts. For example the

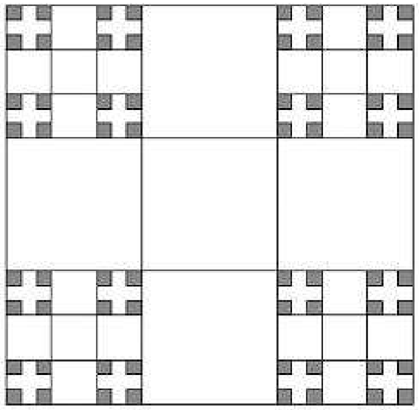

Figure 1. The Cantor perfect set

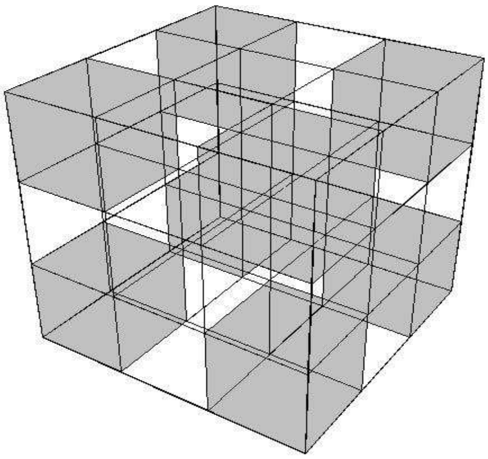

Figure 2. The Cantor perfect set

hypercube is split into 81 sub-hypercubes. After first step we leave 16 four-dimensional cubes. Although the power of **C** is attractive as mathematical object, its visual beauty is quite poor. Hence we are going to present other topological equivalents of US. Let T be a torus. Replace T by n linked (like in a chain) distinct tori $T_1, T_2, \ldots, T_n$. Repeating this process results in metrizable, 0-dimensional space that is homeomorphic to the Cantor perfect set and US. This space is known in general topology as Antoine's necklace. Hence we can imagine US (or the space of possible worlds) as is pictured out

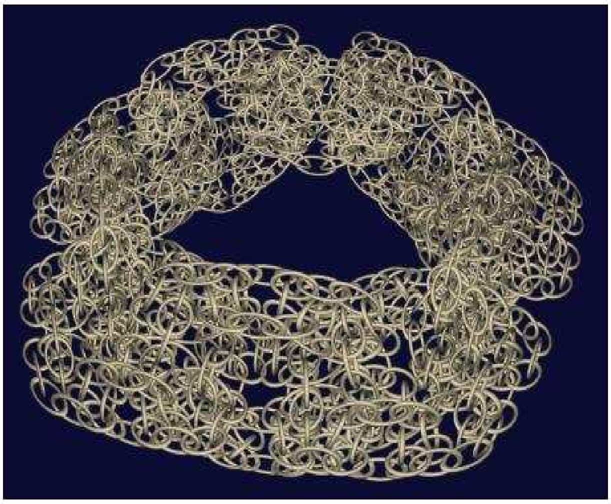

Figure 3. The space of possible worlds

on the Figure 3, where $n = 10$.* There is a peculiarity: although Antoine's necklace and **C** are homeomorphic to each other, they take different position in the $\mathbb{R}^3$ space. There is no authomorphism of $\mathbb{R}^3$ transforming the Cantor perfect set into the Antoine's necklace.

Obviously, we might continue this investigation and look for other topological equivalents — nevertheless we stop here. As we have just seen, geometry can tell us many interesting things about philosophy and logic. On the ground of philosophy we can use this results to define new definition of truth or to reconfigure our weak ontology — there was no room here to show all details. On the ground of logic we have tried to follow topological investigations started by pioneer works of Tarski and McKinsey [1944]. Nevertheless we do not present any results of logical importance; the paper is aimed to reveal the hidden beauty of well-known theorems and to embed them into the horizon of modern mathematics (such like fractal geometry). Although Hausdorff dimension does not play an important role on the scene of logic, we hope that exploration of US on the deeper level then $\dim_{\mathrm{H}}$ can bring us many interesting results about its fractal nature. But it is a goal for future works.

* We present this picture thanks to The Geometry Center, University of Minnesota. The reader can find there many fabulous mathematical objects.

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