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## THE TRUE BISIMULATIONS FOR ‘SINCE’ AND ‘UNTIL’

**Abstract.** The aim of this paper is to establish a new notion of equivalence between temporal models, so-called *S*-similarity, as the appropriate notion of bisimilarity for temporal logic with *Since* and *Until*. The main technical results of the paper provide semantical characterizations of the first-order formulas that are equivalent to a temporal formula: Theorem 3.7 concerns the equivalence of temporal and first-order formulas with respect to *pointed* temporal models, whereas Theorem 4.4 takes the level of temporal models into account.

### 1. Introduction

The present paper deals with the problem of finding a proper notion of bisimulation for the temporal logic  $\mathcal{L}_{SU}$ . Roughly speaking,  $\mathcal{L}_{SU}$  is the logic one gets by adding the two temporal operators  $\mathcal{S}$  (Since) and  $\mathcal{U}$  (Until) to the language of boolean logic, and by interpreting its formulas on Kripke-style models. Using an adaption of van Benthem’s standard translation the set of  $\mathcal{L}_{SU}$ -formulas can be mapped in a semantically well-behaved way into the set of first-order formulas over a suitable vocabulary. For a long time it had been an open problem whether this fragment has a semantical characterization in terms of bisimulations, that means, whether there exists a proper type of equivalence relations between temporal models such that a first-order formula is preserved under these relations if and only if it is equivalent to the translation of a  $\mathcal{L}_{SU}$ -formula. In [3] Kurtonina and de Rijke presented a (positive) solution to this problem. They introduced a new notion of temporal



bisimulation and proved the desired result in their Corollary 5.4. Moreover, the authors showed how to use temporal bisimulations in order to develop the basic model theory of  $\mathcal{L}_{SU}$ ; besides other results they gave a characterization of the elementary classes of  $\mathcal{L}_{SU}$ , that is, of the classes of *pointed* temporal models that are definable by means of (sets of) temporal formulas.

An alternative solution to the bisimulation problem for  $\mathcal{L}_{SU}$  is contained in my doctoral thesis [5]. In section 2.11 of this work, I introduced the notion of  $S$ -simulation, and proved this to be the right candidate for handling  $\mathcal{L}_{SU}$ .<sup>1</sup> Albeit Kurtonina and de Rijke's proof shows some resemblance with my own proof, both the proofs as well as the involved notions of equivalence, bisimilarity respectively  $S$ -similarity, were obtained independently from each other.

However, it turned out that  $S$ -similarity and bisimilarity are equivalent notions in the following sense: Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be temporal models, then two points  $a \in A$  and  $b \in B$  are related *via* a  $S$ -simulation (between  $\mathfrak{A}$  and  $\mathfrak{B}$ ) if and only if they are related *via* a bisimulation (between  $\mathfrak{A}$  and  $\mathfrak{B}$ ).<sup>2</sup> The main difference between the two notions lies in the fact that  $S$ -simulations are only defined with respect to points while bisimulations also take intervals into account. As temporal formulas are evaluated at points only this can be regarded as an advantage of  $S$ -simulations. Though Kurtonina and de Rijke concede the *prima facie* superiority of  $S$ -simulations, they nevertheless prefer their own concept; what they claim in favor of bisimulations is that *it is precisely this special two-sorted character of our notion of bisimulation that allows us to develop the model theory of Since and Until in a direct way (without making detours through richer languages).*

The correctness of this observation granted, I dare to doubt its relevance for the question under consideration. As far as I can see, there is only one step in the development of the model theory of  $\mathcal{L}_{SU}$  where it makes a difference whether to use  $S$ -simulations instead of bisimulations, namely when it comes to prove that for  $\omega$ -saturated models temporal equivalence and  $S$ -similarity coincide; for the remainder  $S$ -simulations work in the same way as bisimulations do. Contrary to Kurtonina and de Rijke's proof of their Theorem 4.3 the proof of my Lemma 3.4, where the wanted equivalence is established in the case of  $S$ -similarity, makes use of the translation of the temporal language into a first-order language.

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<sup>1</sup> To be more precise, in [5] the notion of  $S$ -simulation has been introduced for a language that only contains the forward looking operator  $U$ . However, as the temporal operators  $S$  and  $U$  are defined independently, the adaption of the proofs to the temporal case is just an easy exercise.

<sup>2</sup> This has already been observed by Kurtonina and de Rijke in their paper.



Surely, this might be seen as a kind of detour through a richer language; however, as the whole enterprise of investigating the model theory of modal and temporal logics along the line followed by [3] is based on the idea of translating modal and temporal languages into first-order languages, and, thereby, of applying techniques as well as results from the latter to the former, this is no argument against  $S$ -simulations, or in favor of bisimulations. What really matters is, contrary to an analysis presented by van Benthem and Bergstra in [1, page 822], that the power of  $\mathcal{L}_{SU}$  is strong enough to enforce the right structural relation upon  $\omega$ -saturated temporal equivalent models, and that one needs no detours through richer (possibly many-dimensional) *temporal* languages.

Therefore, and this is the message one should get from this paper,  $S$ -simulations are the true bisimulations for  $\mathcal{L}_{SU}$ .

The paper is organized as follows. In Section 2, I will recapitulate some basic notions and facts from the syntax and semantics of  $\mathcal{L}_{SU}$ , rather to fix the notation than to tell anything new. In Section 3, I will formally introduce  $S$ -simulations, and will give a semantical characterization of the first-order sentences that are equivalent to a  $\mathcal{L}_{SU}$ -formula (Theorem 3.7). The main load of the proof is carried by an application of Lemma 3.4 which states that for  $\omega$ -saturated temporal models temporal equivalence and  $S$ -similarity coincide. In accordance with common practice, Section 3 investigates temporal logic on the level of pointed models; in contrast, considering temporal models Section 4 will take a more global perspective. Theorem 4.4 presents a characterization of the set of first-order sentences that are equivalent to a  $\mathcal{L}_{SU}$ -formula with respect to temporal models.

## 2. Basic concepts

For the following fix a countable set  $\mathcal{P} := \{p_n \mid n \in \omega\}$  of propositional letters. The set  $\text{Form}_{SU}$  of temporal formulas (over  $\mathcal{P}$ ) is then defined as the smallest set  $X$  that contains the propositional letters from  $\mathcal{P}$ , is closed under  $\neg$ ,  $\vee$  and  $\wedge$ , and satisfies the following condition: if  $\varphi$  and  $\psi$  are in  $X$ , then  $\mathcal{S}(\varphi, \psi)$  and  $\mathcal{U}(\varphi, \psi)$  are in  $X$  as well.

Temporal formulas are interpreted on temporal models. A triple  $\mathfrak{A} = (A, <^{\mathfrak{A}}, V^{\mathfrak{A}})$  is called a temporal model, if  $A$  is a non-empty set,  $<^{\mathfrak{A}}$  a binary relation on  $A$ , and  $V^{\mathfrak{A}}$  a valuation function from  $\mathcal{P}$  into the power set of  $A$ . As usual, the truth of temporal formulas is defined with respect to pairs  $(\mathfrak{A}, a)$ , so-called *pointed* temporal models, consisting of a temporal model  $\mathfrak{A}$

and a distinguished element  $a \in A$ :

$$(\mathfrak{A}, a) \models p_n :\Leftrightarrow a \in V^{\mathfrak{A}}(p_n), \text{ for } n \in \omega,$$

$$(\mathfrak{A}, a) \not\models \perp,$$

$$(\mathfrak{A}, a) \models \neg\varphi :\Leftrightarrow (\mathfrak{A}, a) \not\models \varphi,$$

$$(\mathfrak{A}, a) \models \varphi \vee \psi :\Leftrightarrow (\mathfrak{A}, a) \models \varphi \text{ or } (\mathfrak{A}, a) \models \psi,$$

$$(\mathfrak{A}, a) \models \varphi \wedge \psi :\Leftrightarrow (\mathfrak{A}, a) \models \varphi \text{ and } (\mathfrak{A}, a) \models \psi,$$

$$(\mathfrak{A}, a) \models \mathcal{S}(\varphi, \psi) :\Leftrightarrow \\ \exists a'(a' <^{\mathfrak{A}} a \ \& \ (\mathfrak{A}, a') \models \varphi \ \& \ \forall a''(a' <^{\mathfrak{A}} a'' \ \& \ a'' <^{\mathfrak{A}} a \Rightarrow (\mathfrak{A}, a'') \models \psi)),$$

$$(\mathfrak{A}, a) \models \mathcal{U}(\varphi, \psi) :\Leftrightarrow \\ \exists a'(a <^{\mathfrak{A}} a' \ \& \ (\mathfrak{A}, a') \models \varphi \ \& \ \forall a''(a <^{\mathfrak{A}} a'' \ \& \ a'' <^{\mathfrak{A}} a' \Rightarrow (\mathfrak{A}, a'') \models \psi)).$$

Obviously, by a slight change of perspective a pointed temporal model  $(\mathfrak{A}, a)$  may also be regarded as a first-order model suitable for a first-order vocabulary, lets call it  $\tau$ , consisting of a countable set  $\{P_n \mid n \in \omega\}$  of predicate symbols, a binary relation symbol  $<$  and an individual constant  $c$ .<sup>3</sup> This together with the fact that the truth clauses for temporal formulas are stated in a first-order metalanguage makes it possible to define a mapping  $\text{St}$  from  $\text{Form}_{\mathcal{ST}}$  into the set of first-order sentences over  $\tau$ :

$$\text{St}(p_n) := P_n c, \text{ for } n \in \omega,$$

$$\text{St}(\neg\varphi) := \neg \text{St}(\varphi),$$

$$\text{St}(\varphi \vee \psi) := \text{St}(\varphi) \vee \text{St}(\psi),$$

$$\text{St}(\varphi \wedge \psi) := \text{St}(\varphi) \wedge \text{St}(\psi),$$

$$\text{St}(\mathcal{S}(\varphi, \psi)) := \exists x(x < c \wedge \text{St}(\varphi)[x/c] \wedge \\ \forall y(x < y \wedge y < c \rightarrow \text{St}(\psi)[y/c])),^{4,5}$$

$$\text{St}(\mathcal{U}(\varphi, \psi)) := \exists x(c < x \wedge \text{St}(\varphi)[x/c] \wedge \\ \forall y(c < y \wedge y < x \rightarrow \text{St}(\psi)[y/c])).$$

<sup>3</sup> This is possible because neglecting some harmless notational differences  $(\mathfrak{A}, a)$  might be seen as a convenient way of denoting the first-order model  $(A, <^{\mathfrak{A}}, (V^{\mathfrak{A}}(p_n))_{n \in \omega}, a)$ .

<sup>4</sup> Here and in the next clause the variables  $x$  and  $y$  are assumed to be the first two variables chosen from a list of variables that neither occur in  $\text{St}(\varphi)$  nor in  $\text{St}(\psi)$ .

<sup>5</sup> In general, for a first-order formula  $\varphi$  and individual terms  $t_1$  and  $t_2$ ,  $\varphi[t_1/t_2]$  denotes the formula one gets by replacing every occurrence of  $t_2$  in  $\varphi$  by  $t_1$ .

That this translation is a reasonable one will be obvious from the following lemma. The proof is a routine induction and can be left to the reader.

LEMMA 2.1. *For every  $\varphi \in \text{Form}_{SU}$ , temporal model  $\mathfrak{A} = (A, <^{\mathfrak{A}}, V^{\mathfrak{A}})$  and every  $a \in A$  the following equivalence holds:*

$$(\mathfrak{A}, a) \models \varphi \iff (\mathfrak{A}, a) \models \text{St}(\varphi).$$

### 3. $S$ -simulations: the local point of view

The central notion of this paper is introduced in the next definition.

DEFINITION 3.1. Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be temporal models. A relation  $Z \subseteq A \times B$  is called a  $S$ -simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ , if  $Z$  satisfies the following conditions:

- S1 For every  $a \in A$  and  $b \in B$ :  
if  $Zab$ , then  $(\mathfrak{A}, a) \models p_n \iff (\mathfrak{B}, b) \models p_n$  for every  $n \in \omega$ .
- S2a For every  $a, a_1 \in A$  and  $b \in B$ : if  $Zab$  and  $a_1 <^{\mathfrak{A}} a$ ,  
then there exists a  $b_1 \in B$  such that  $Za_1b_1$ ,  $b_1 <^{\mathfrak{B}} b$  and  
for every  $b_2 \in B$  with  $b_1 <^{\mathfrak{B}} b_2$  and  $b_2 <^{\mathfrak{B}} b$   
there is an  $a_2 \in A$  such that  $a_1 <^{\mathfrak{A}} a_2$ ,  $a_2 <^{\mathfrak{A}} a$  and  $Za_2b_2$ .
- S2b For every  $a \in A$  and  $b, b_1 \in B$ : if  $Zab$  and  $b_1 <^{\mathfrak{B}} b$ ,  
then there exist an  $a_1 \in A$  such that  $Za_1b_1$ ,  $a_1 <^{\mathfrak{A}} a$  and  
for every  $a_2 \in A$  with  $a_1 <^{\mathfrak{A}} a_2$  and  $a_2 <^{\mathfrak{A}} a$   
there is a  $b_2 \in B$  such that  $b_1 <^{\mathfrak{B}} b_2$ ,  $b_2 <^{\mathfrak{B}} b$  and  $Za_2b_2$ .
- S3a Similar to clause S2a; exchange only  $a$  and  $a_1$ , respectively  $b$  and  $b_1$ .
- S3b Similar to clause S2b; exchange only  $a$  and  $a_1$ , respectively  $b$  and  $b_1$ .

$(\mathfrak{A}, a) \sim_s (\mathfrak{B}, b)$  means that there exists a  $S$ -simulation  $Z$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  such that  $Zab$ .

LEMMA 3.2. *Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be temporal models and  $Z$  a  $S$ -simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ . For every  $a \in A$  and  $b \in B$  with  $Zab$  it holds that*

$$(\mathfrak{A}, a) \equiv_{SU} (\mathfrak{B}, b).^6$$

PROOF. By induction. □

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<sup>6</sup>  $\equiv_{SU}$  denotes the relation of elementary equivalence with respect to temporal formulas, that is, for pointed temporal models  $(\mathfrak{A}, a)$  and  $(\mathfrak{B}, b)$  we have  $(\mathfrak{A}, a) \equiv_{SU} (\mathfrak{B}, b)$  iff for all  $\varphi \in \text{Form}_{SU}$ :  $(\mathfrak{A}, a) \models \varphi \iff (\mathfrak{B}, b) \models \varphi$ .



In general, temporal equivalence is a weaker notion than  $\mathcal{S}$ -similarity; there exist temporal models  $\mathfrak{A}$ ,  $\mathfrak{B}$ , and elements  $a \in A$ ,  $b \in B$  such that  $(\mathfrak{A}, a) \equiv_{\mathcal{S}\mathcal{U}} (\mathfrak{B}, b)$  without having a  $\mathcal{S}$ -simulation between them. However, for certain classes of models these two notions coincide. Such classes are called Hennessy-Milner classes; examples are the class of finite temporal models and the class of  $\omega$ -saturated temporal models.

**DEFINITION 3.3.** Let  $\sigma$  be a first-order vocabulary. A  $\sigma$ -model  $\mathfrak{A}$  is  $\omega$ -saturated, if for every finite set  $\{c_1, \dots, c_n\}$  of new individual constants, and every set  $\Phi(x)$  of first-order formulas over  $\sigma \cup \{c_1, \dots, c_n\}$ , and every  $a_1, \dots, a_n \in A$ : if  $\Phi(x)$  is *finitely* satisfiable in  $(\mathfrak{A}, a_1, \dots, a_n)$ , then  $\Phi$  is satisfiable in  $(\mathfrak{A}, a_1, \dots, a_n)$ .

**LEMMA 3.4.** Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be  $\omega$ -saturated temporal models. Then  $\equiv_{\mathcal{S}\mathcal{U}}$  is a  $\mathcal{S}$ -simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ .

**PROOF.** We need to show that

$$\forall a \in A \forall b \in B (Zab \Leftrightarrow (\mathfrak{A}, a) \equiv_{\mathcal{S}\mathcal{U}} (\mathfrak{B}, b))$$

defines a  $\mathcal{S}$ -simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ . Clause S1 is obvious. To check S2a assume  $Zab$  and  $a_1 <^{\mathfrak{A}} a$  for  $a, a_1 \in A$  and  $b \in B$ . Let  $\Delta(x)$  be the union of the following three sets of first-order formulas:

$$\begin{aligned} & \{x < c\}, \\ & \Gamma_1(x) := \{\text{St}(\varphi)[x/c] \mid \varphi \in \text{Form}_{\mathcal{S}\mathcal{U}} \ \& \ (\mathfrak{A}, a_1) \models \varphi\}, \\ & \Gamma_2(x) := \{\forall z(x < z \wedge z < c \rightarrow \text{St}(\psi)[z/c]) \mid \\ & \quad \forall a_2(a_1 <^{\mathfrak{A}} a_2 \ \& \ a_2 <^{\mathfrak{A}} a \Rightarrow (\mathfrak{A}, a_2) \models \psi)\}. \end{aligned}$$

It is sufficient to prove that  $\Delta(x)$  is satisfiable in  $(\mathfrak{B}, b)$ , that is, that there exists a  $b_1 \in B$  such that  $(\mathfrak{B}, b) \models \Delta(x)[b_1]$ . Because  $\mathfrak{B}$  is  $\omega$ -saturated, and hence  $(\mathfrak{B}, b)$  is  $\omega$ -saturated as well, we only need to show that each *finite* subset of  $\Delta(x)$  is satisfiable in  $(\mathfrak{B}, b)$ . Moreover, as  $\Gamma_1(x)$  and  $\Gamma_2(x)$  are both closed under conjunctions, we only have to consider subsets of the form  $\{x < c, \text{St}(\varphi)[x/c], \forall z(x < z \wedge z < c \rightarrow \text{St}(\psi)[z/c])\}$ , with  $\text{St}(\varphi)[x/c] \in \Gamma_1(x)$  and  $\forall z(x < z \wedge z < c \rightarrow \text{St}(\psi)[z/c]) \in \Gamma_2(x)$ .

Assume  $\{x < c, \text{St}(\varphi)[x/c], \forall z(x < z \wedge z < c \rightarrow \text{St}(\psi)[z/c])\}$  to be fixed. Using the definition of  $\Gamma_1(x)$  and  $\Gamma_2(x)$  this yields  $(\mathfrak{A}, a) \models \mathcal{S}(\varphi, \psi)$ . By  $Zab$ , which means the same as  $(\mathfrak{A}, a) \equiv_{\mathcal{S}\mathcal{U}} (\mathfrak{B}, b)$ , we get  $(\mathfrak{B}, b) \models \mathcal{S}(\varphi, \psi)$ , hence there is a  $b' \in B$  such that

- (i)  $b' <^{\mathfrak{B}} b$ ,
- (ii)  $(\mathfrak{B}, b') \models \varphi$ , and
- (iii)  $\forall b'' \in B(b' <^{\mathfrak{B}} b'' \ \& \ b'' <^{\mathfrak{B}} b \Rightarrow (\mathfrak{B}, b'') \models \psi)$ .

Alternatively, there is a  $b' \in B$  such that

- (iv)  $b' <^{\mathfrak{B}} b$ ,
- (v)  $(\mathfrak{B}, b') \models \text{St}(\varphi)$ , and
- (vi)  $\forall b'' \in B(b' <^{\mathfrak{B}} b'' \ \& \ b'' <^{\mathfrak{B}} b \Rightarrow (\mathfrak{B}, b'') \models \text{St}(\psi))$ .

Obviously, (iv) is equivalent to

- (vii)  $(\mathfrak{B}, b) \models x < c[b']$ ,

(v) is equivalent to

- (viii)  $(\mathfrak{B}, b) \models (\text{St}(\varphi)[x/c])[b']$ ,

and (vi) to

- (ix)  $(\mathfrak{B}, b) \models \forall z(x < z \wedge z < c \rightarrow \text{St}(\psi)[z/c])[b']$ .

From (vii), (viii) and (ix) it follows that  $b'$  satisfies  $\{x < c, \text{St}(\varphi)[x/c], \forall z(x < z \wedge z < c \rightarrow \text{St}(\psi)[z/c])\}$  in  $(\mathfrak{B}, b)$ . Therefore,  $\Delta(x)$  is finitely satisfiable in  $(\mathfrak{B}, b)$ , hence, by  $\omega$ -saturation there is a  $b_1 \in B$  which satisfies  $\Delta(x)$  in  $(\mathfrak{B}, b)$ .

It remains to show that  $b_1$  has all the properties stated in S2a. Since  $x < c$  is contained in  $\Delta(x)$  we immediately get  $b_1 <^{\mathfrak{B}} b$ . That  $b_1$  satisfies  $\Gamma_1(x)$  yields  $(\mathfrak{A}, a_1) \equiv_{\text{SU}} (\mathfrak{B}, b_1)$ , hence, by the definition of  $Z$ ,  $Za_1b_1$ .

Let  $b_2 \in B$  with  $b_1 <^{\mathfrak{B}} b_2$  and  $b_2 <^{\mathfrak{B}} b$ . We must find an  $a_2 \in A$  such that  $a_1 <^{\mathfrak{A}} a_2$ ,  $a_2 <^{\mathfrak{A}} a$  and  $Za_2b_2$ , that is  $(\mathfrak{A}, a_2) \equiv_{\text{SU}} (\mathfrak{B}, b_2)$ . To show the existence of such an  $a_2$  we reason as follows: let  $\psi_1, \dots, \psi_n$  be temporal formulas such that for each  $i \leq n$ :  $(\mathfrak{B}, b_2) \models \psi_i$ . Put  $\psi := \psi_1 \wedge \dots \wedge \psi_n$ . Suppose there is no  $a' \in A$  with  $a_1 <^{\mathfrak{A}} a'$ ,  $a' <^{\mathfrak{A}} a$  and  $(\mathfrak{A}, a') \models \psi$ . Then the formula  $\chi(x) := \forall z(x < z \ \& \ z < c \rightarrow \text{St}(\neg\psi)[z/c])$  belongs to  $\Gamma_2(x)$ , hence to  $\Delta(x)$ . Therefore,  $b_1$  satisfies  $\chi(x)$  in  $(\mathfrak{B}, b)$ , but this contradicts the choice of  $b_2$  and  $\psi$ . So we have shown that there is an  $a' \in A$  such that  $a_1 <^{\mathfrak{A}} a'$ ,  $a' <^{\mathfrak{A}} a$  and  $(\mathfrak{A}, a') \models \psi$ . As the subset  $\{\psi \mid \psi \in \text{Form}_{\text{SU}} \ \& \ (\mathfrak{B}, b_2) \models \psi\}$  was chosen arbitrary, it follows that the set

$$\Phi(x) := \{\text{St}(\psi)[x/c] \mid (\mathfrak{B}, b_2) \models \psi\} \cup \{c' < x, x < c\}$$

is finitely satisfiable in  $(\mathfrak{A}, a, a_1)$ , where  $c'$  is a new individual constant. Again, by the  $\omega$ -saturation of  $\mathfrak{A}$  there is an  $a_2 \in A$  which satisfies  $\Phi(x)$



in  $(\mathfrak{A}, a, a_1)$ . It is obvious that  $a_2$  has all the desired properties:  $a_1 <^{\mathfrak{A}} a_2$ ,  $a_2 <^{\mathfrak{A}} a$  and especially  $(\mathfrak{A}, a_2) \equiv_{S\mathcal{U}} (\mathfrak{B}, b_2)$ , that is  $Za_2b_2$ . As we can repeat this argument for every  $b_2$  between  $b_1$  and  $b$  clause S2a is proved.

The other three clauses are proved similarly.  $\square$

**DEFINITION 3.5.** A first-order sentence  $\varphi$  over  $\tau$  is said to be preserved under  $S$ -simulations, if for all pointed temporal models  $(\mathfrak{A}, a)$  and  $(\mathfrak{B}, b)$  the following holds: if  $(\mathfrak{A}, a) \models \varphi$  and  $(\mathfrak{A}, a) \sim_s (\mathfrak{B}, b)$ , then  $(\mathfrak{B}, b) \models \varphi$ .

In order to streamline the proof of Theorem 3.7, I will make use of the following well-known lemma.

**LEMMA 3.6.** *Let  $\sigma$  be a first-order vocabulary and  $\Gamma$  a class of  $\sigma$ -sentences closed under (finite) disjunctions and conjunctions. Then for every  $\sigma$ -sentence  $\varphi$  the following statements are equivalent:*

1. *There exists a  $\psi \in \Gamma$  such that  $\models \varphi \leftrightarrow \psi$ .*
2. *For all  $\sigma$ -models  $\mathfrak{A}$  and  $\mathfrak{B}$ : if  $\mathfrak{A} \models \varphi$  and  $\mathfrak{A} \implies_{\Gamma} \mathfrak{B}$ , then  $\mathfrak{B} \models \varphi$ .<sup>7</sup>*

**PROOF.** For a proof see [2, section 3.2], for instance.  $\square$

**THEOREM 3.7.** *For every first-order sentence  $\varphi$  (over  $\tau$ ) the following are equivalent:*

1. *There is a temporal formula  $\psi$  such that  $\models \varphi \leftrightarrow \text{St}(\psi)$ .*
2.  *$\varphi$  is preserved under  $S$ -simulations.*

**PROOF.** The direction from 1 to 2 follows from Lemma 2.1 and Lemma 3.2. For the other direction suppose  $\varphi$  is preserved under  $S$ -simulations. Using the definition of  $\text{St}$  it is easy to check that the set of translations of temporal formulas is closed under disjunctions and conjunctions. Because of the foregoing lemma it is now sufficient to prove the second claim of this lemma with  $\Gamma := \{\text{St}(\psi) \mid \psi \in \text{Form}_{S\mathcal{U}}\}$ . Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be temporal models, and let  $a \in A$  and  $b \in B$  such that  $(\mathfrak{A}, a) \models \varphi$  and  $(\mathfrak{A}, a) \implies_{\Gamma} (\mathfrak{B}, b)$ . As  $\Gamma$  is also closed under negations this leads to  $(\mathfrak{A}, a) \equiv_{S\mathcal{U}} (\mathfrak{B}, b)$ . Choose  $\omega$ -saturated elementary extensions  $\mathfrak{A}'$  and  $\mathfrak{B}'$  of  $\mathfrak{A}$  respectively  $\mathfrak{B}$ . (A well-known result from model theory tells us that this is always possible.) By Lemma 3.4 we get:  $(\mathfrak{A}', a) \sim_s (\mathfrak{B}', b)$ . As  $\varphi$  is preserved under  $S$ -simulations, it follows that  $(\mathfrak{B}', b) \models \varphi$ , hence, by an application of Lemma 3.6, there exists a  $\text{St}(\psi) \in \Gamma$  such that  $\models \text{St}(\psi) \leftrightarrow \varphi$ , what had to be shown.  $\square$

<sup>7</sup>  $\mathfrak{A} \implies_{\Gamma} \mathfrak{B}$  means that every formula in  $\Gamma$  that is true in  $\mathfrak{A}$  is also true in  $\mathfrak{B}$ .



#### 4. $S$ -simulations: the global point of view

In the foregoing section temporal logic and its corresponding first-order fragment were studied on the level of *pointed* temporal models. This choice was made in accordance with the line of investigation usually undertaken in the literature. The present section takes a different perspective: it considers temporal formulas on the level of temporal models. Again, it is possible to define a reasonable translation from the set of temporal formulas into the set of first-order sentences (over a suitable vocabulary) which allows to consider  $\mathcal{L}_{SU}$  as a fragment of first-order logic, now on the level of temporal models: just correlate a temporal formula  $\varphi$  with the universal closure of its standard translation, that is, with the formula  $\forall x(\text{St}(\varphi)[x/c])$ . The aim of this section is to give a semantical characterization of this fragment. In Theorem 4.4 it is shown that a first-order sentence is equivalent to the universal closure of the translation of a temporal formula iff it is preserved under surjective  $S$ -simulations and disjoint unions.<sup>8</sup>

DEFINITION 4.1. Let  $Z$  be a  $S$ -simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ .  $Z$  is called a *surjective  $S$ -simulation*, if for all  $b \in B$  there is an  $a \in A$  such that  $Zab$ .

DEFINITION 4.2. Let  $\{\mathfrak{A}_i \mid i \in I\}$  be a non-empty family of temporal models, where the domains of these models are pairwise disjoint. The disjoint union of this family, abbreviated by  $\biguplus_{i \in I} \mathfrak{A}_i$ , is the following defined model  $\mathfrak{A}$ :

$$\begin{aligned} A &:= \bigcup_{i \in I} A_i, \\ V^{\mathfrak{A}}(p_n) &:= \bigcup_{i \in I} V^{\mathfrak{A}_i}(p_n), \text{ for } n \in \omega, \\ <^{\mathfrak{A}} &:= \bigcup_{i \in I} <^{\mathfrak{A}_i}. \end{aligned}$$

DEFINITION 4.3. A first-order sentence  $\varphi$  over  $\tau \setminus \{c\}$  is preserved under disjoint unions, if for every non-empty family  $\{\mathfrak{A}_i \mid i \in I\}$  of temporal models the following holds: if for every  $i \in I$   $\mathfrak{A}_i \models \varphi$ , then  $\biguplus_{i \in I} \mathfrak{A}_i \models \varphi$ .

THEOREM 4.4. For a first-order sentence  $\varphi$  over  $\tau \setminus \{c\}$  the following are equivalent:

1. There is a temporal formula  $\psi$  such that  $\models \varphi \leftrightarrow \forall x(\text{St}(\psi)[x/c])$ .
2.  $\varphi$  is preserved under disjoint unions and surjective  $S$ -simulations.

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<sup>8</sup> For a similar result in the framework of modal logic see [4].



PROOF. For the direction from 1 to 2, suppose  $\varphi$  is a first-order sentence equivalent to  $\forall x(\text{St}(\psi)[x/c])$ , where  $\psi$  is a temporal formula. We need to check that  $\forall x(\text{St}(\psi)[x/c])$  satisfies the closure conditions stated in 2. Let  $\mathfrak{A}$  be the disjoint union of the family  $\{\mathfrak{A}_i \mid i \in I\}$  and assume that for every  $i \in I$   $\mathfrak{A}_i \models \forall x(\text{St}(\psi)[x/c])$ . Let  $a \in A$ . Then there is a unique  $i \in I$  with  $a \in A_i$ . By assumption we get  $\mathfrak{A}_i \models \forall x(\text{St}(\psi)[x/c])$ , hence  $(\mathfrak{A}_i, a) \models \text{St}(\psi)$ . Moreover, it is easy to see that the identity function on  $A_i$  is a  $S$ -simulation between  $\mathfrak{A}_i$  and  $\mathfrak{A}$ . It follows that  $(\mathfrak{A}, a) \models \text{St}(\psi)$ . As  $a$  was chosen arbitrary, this yields  $\mathfrak{A} \models \forall x(\text{St}(\psi)[x/c])$ , hence  $\varphi$  is shown to be preserved under disjoint union. The other closure condition is proved similarly.

For the other direction assume that the first-order sentence  $\varphi$  is preserved under disjoint unions and surjective  $S$ -simulations. Let  $\Delta$  be the set of translations of temporal formulas. Define  $\Sigma_\varphi$  as the following set of first-order sentences:

$$\Sigma_\varphi := \{\forall x(\theta[x/c]) \mid \theta \in \Delta \ \& \ \varphi \models \forall x(\theta[x/c])\}.$$

Since  $\Delta$  is closed under finite conjunctions it is now sufficient to prove that  $\Sigma_\varphi \models \varphi$ ; an application of compactness leads to the desired result.

Let  $\mathfrak{A}'$  be a model of  $\Sigma_\varphi$ . Take  $\mathfrak{A}$  to be an  $\omega$ -saturated elementary extension of  $\mathfrak{A}'$ . For each  $a \in A$  set

$$\Delta_a := \{\neg\theta \mid \theta \in \Delta \ \& \ (\mathfrak{A}, a) \models \neg\theta\}.$$

Using the closure of  $\Delta$  under disjunctions it is easy to show that for every  $a \in A$  the set  $\Delta_a \cup \{\varphi\}$  is finitely satisfiable, and hence, by compactness, satisfiable: Assume to the contrary that there is  $a \in A$  and sentences  $\neg\theta_1, \dots, \neg\theta_n \in \Delta_a$  such that  $\{\varphi\} \cup \{\neg\theta_1, \dots, \neg\theta_n\}$  has no model. It follows that  $\varphi \models \neg(\neg\theta_1 \wedge \dots \wedge \neg\theta_n)$ . By logic we get  $\varphi \models \theta_1 \vee \dots \vee \theta_n$ , hence  $\varphi \models \forall x(\theta_1[x/c] \vee \dots \vee \theta_n[x/c])$ . As  $\Delta$  is closed under disjunctions,  $\theta_1 \vee \dots \vee \theta_n$  is in  $\Delta$ , and therefore  $\forall x(\theta_1[x/c] \vee \dots \vee \theta_n[x/c])$  in  $\Sigma_\varphi$ . But the latter implies  $\mathfrak{A} \models \forall x(\theta_1[x/c] \vee \dots \vee \theta_n[x/c])$  which contradicts the choice of the  $\theta_i$ .

Now, for each  $a \in A$  we select a model  $(\mathfrak{B}_a, b_a)$  such that  $(\mathfrak{B}_a, b_a) \models \Delta_a$  and  $(\mathfrak{B}_a, b_a) \models \varphi$ . Without any restriction we may assume the models  $\mathfrak{B}_a$  to be  $\omega$ -saturated and their domains pairwise disjoint. From  $(\mathfrak{B}_a, b_a) \models \Delta_a$  we easily get  $(\mathfrak{B}_a, b_a) \implies_\Delta (\mathfrak{A}, a)$  for every  $a \in A$ . By  $\omega$ -saturation of  $\mathfrak{B}_a$  and  $\mathfrak{A}$  and by applying Lemma 3.4 it follows that  $(\mathfrak{B}_a, b_a) \sim_s (\mathfrak{A}, a)$ , that is, there is a  $S$ -simulation  $Z_a$  between  $\mathfrak{B}_a$  and  $\mathfrak{A}$  such that  $Z_a b_a a$ . It is easy to see that  $Z := \bigcup_{a \in A} Z_a$  defines a  $S$ -simulation between  $\biguplus_{a \in A} \mathfrak{B}_a$  and  $\mathfrak{A}$ . Moreover, the relation  $Z$  is surjective.



We may now conclude as follows: By the choice of the models  $\mathfrak{B}_a$  we get  $\mathfrak{B}_a \models \varphi$  for every  $a \in A$ . As  $\varphi$  is preserved under disjoint unions it follows that  $\biguplus_{a \in A} \mathfrak{B}_a \models \varphi$ , hence, by the preservation of  $\varphi$  under surjective  $S$ -simulations:  $\mathfrak{A} \models \varphi$ , and, finally,  $\mathfrak{A}' \models \varphi$ .  $\square$

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