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GEOMETRY AS AN EXTENSION OF THE GROUP THEORY

Abstract. Klein's Erlangen program contains the postulate to study the group of automorphisms instead of a structure itself. This postulate, taken literally, sometimes means a substantial loss of information. For example, the group of automorphisms of the field of rational numbers is trivial. However in the case of Euclidean plane geometry the situation is different. We shall prove that the plane Euclidean geometry is mutually interpretable with the elementary theory of the group of authomorphisms of its standard model. Thus both theories differ practically in the language only.

By *Euclidean plane geometry* we mean (following [4]) the elementary theory of the Cartesian plane over the real numbers. It is formalized in terms of one sort of variables called *points*, and two non-logical predicates 'B' and 'D', called respectively *betweenness* and *equidistance* relations. Its standard models are *Cartesian planes over arbitrary real closed fields*, *i.e.*, the structures

$$\mathrm{C}^2(\mathfrak{F}) = \langle F^2, \mathcal{B}_{\mathfrak{F}}, \mathcal{D}_{\mathfrak{F}} \rangle,$$

where $\mathfrak{F} = \langle F, 0, 1, +, \cdot \rangle$ is a real closed field and for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in F^2$:

$$\begin{aligned} \mathcal{B}_{\mathfrak{F}}(\boldsymbol{abc}) &\iff \exists_{x \in F} \left[0 \leqslant x \leqslant 1 \wedge \boldsymbol{b} = (1-x)\boldsymbol{a} + x\boldsymbol{c} \right], \\ \mathcal{D}_{\mathfrak{F}}(\boldsymbol{abcd}) &\iff (\boldsymbol{a} - \boldsymbol{b})^2 = (\boldsymbol{c} - \boldsymbol{d})^2. \end{aligned}$$

A structure is a model of *Euclidean plane geometry* iff it is isomorphic with $\mathrm{C}^2(\mathfrak{F})$ for some real closed field $\mathfrak{F}$ (see [4]).

Given any real closed field $\mathfrak{F} = \langle F, 0, 1, +, \cdot \rangle$, consider the class of transformations f of F^2 onto itself such that

$$f(\boldsymbol{x}) = a \cdot \boldsymbol{A} \cdot \boldsymbol{x} + \boldsymbol{a},$$

for some $\boldsymbol{a} \in F^2$ and $0 \neq a \in F$, and for some orthonormal matrix $\boldsymbol{A}$ of rank 2. The group of automorphisms of the standard model of Euclidean plane geometry is called the group of *similarities.* It contains a natural subgroup, the group of *isometries.* Their elementary theories are respectively *elementary theory of similarities* and *elementary theory of isometries.* We claim that all three theories are mutually interpretable, and therefore that Euclidean plane geometry is expressible as a theory of certain groups.

We start with the group of isometries. We restrict our attention to involutions. To simplify further considerations we denote involutions by small greek letters. Thus we have

$$\alpha\alpha = 1,$$

where 1 denotes the identity transformation, and juxtaposition — the composition of transformations.

Two involutions *commute* if their composition does not depend on its order, *i.e.*,

$$\alpha \mid \beta \Longleftrightarrow \alpha\beta = \beta\alpha.$$

We shall use the following extended denotation:

$$\alpha, \beta \mid \gamma, \delta$$

which means that both α and β commute with both γ and δ. Similarly

$$\neq(\alpha, \beta, \gamma, \delta)$$

which means that all α, β, γ and δ are mutually different.

There are two categories of involutions: *point symmetries* and *line symmetries.* Following [1] we identify point symmetries with points and line symmetries with lines. To repeat Bachmann's construction we need an elementary formula distinguishing points and lines:

$$\alpha \text{ is a } \textit{line} \qquad \text{iff} \qquad \exists_{\beta\gamma\delta} \left[\neq(\alpha, \beta, \gamma, \delta) \wedge \alpha, \beta \mid \gamma, \delta \right].$$

Thus points and lines are both definable in the theory of isometries. We shall denote points by small latin letters, and lines by capital letters. It is easy to see that a line commutes with a point iff it passes through it. Two different lines commute iff they are perpendicular, and two points commute only if they are equal.

Next we define *perpendicularity*:

$$K \perp L \iff K \mid L \ \wedge \ K \neq L,$$

parallelity:

$$K \parallel L \iff \exists_M \ K, L \perp M,$$

parallelogram:

$$\mathbb{R}(KLMN) \iff K \parallel L \ \wedge \ M \parallel N \ \wedge \ L \nparallel M,$$

vertices of parallelogram:

$$\mathbb{R}(abcd) \iff \exists_{KLMN}[\mathbb{R}(KLMN) \wedge a \mid N, K \wedge b \mid M, K \wedge c \mid M, L \wedge d \mid L, N],$$

midpoint:

$$\mathbb{M}(abc) \iff \exists_{deKL}[\mathbb{R}(adce) \ \wedge \ a, b, c \mid K \ \wedge \ d, b, e \mid L],$$

midpoint operation:

$$a \vdots b = c \iff \mathbb{M}(acb),$$

point symmetry:

$$\mathbb{S}_a(b) = c \iff \mathbb{M}(abc),$$

two *segments are perpendicular*:

$$ab \perp cd \iff \exists_{KL}[a, b \mid K \wedge c, d \mid L \ \wedge \ K \perp L],$$

three *points form equilateral triangle*:

$$ab \equiv bc \iff ac \perp b(a \vdots c).$$

Finally we define *equidistance*:

$$\mathbb{D}(abcd) \iff ab \equiv cd \iff (\mathbb{S}_{b \vdots c}(a))c \equiv cd$$

and *betweenness*:

$$\mathbb{B}(abc) \iff \exists_K \ a, b, c \mid K \ \wedge \ \exists_d[ad \perp dc \wedge ab \perp bd].$$

Thus we have constructed a model of Euclidean plane geometry in the group of isometries. In [4] a construction of real closed field is reported. It is well known how to construct the group of plane isometries starting with a real closed field. Finally we see that the elementary theory of group of plane isometries is mutually interpretable (*cf.* [2]) with elementary plane Euclidean geometry.

Theorem 1. *The plane Euclidean geometry and the theory of isometries are mutually interpretable.*

Proof. Just above is given the proof in one direction: the construction of a structure isomorphic to the original Euclidean plane in terms of the group of its isometries is given. Therefore it suffices to construct a group of isometries in a plane itself. First we define a notion of line as the bisector of non-degenerate segment:

$$\mathrm{L}\binom{a}{b} = \{p : \mathbb{D}(appb)\}.$$

The set $\mathrm{L}\binom{a}{a}$ is the whole plane, but $\mathrm{L}\binom{a}{b}$ is a line provided $a \neq b$.

Let the capital letter 'K' denote a line. Then the following notions

$$\mathbb{S}_K(a) = b \iff \mathrm{L}\binom{a}{b} = K$$

defines the symmetry with respect to the line, or shortly line symmetry. Since any isometry is a composition of two or three line symmetries, this completes the proof. □

This means that the Euclidean plane geometry may be expressed as the theory of certain groups. Moreover since any similarity which is an involution has to be an isometry, the above result may be readily repeated for the elementary theory of plane similarities. At the same time any similarity may be decomposed into isometry and polar similarity or two polar similarities. To define polar similarity we define first the following notions:

point symmetry

$$\mathbb{S}_a(b) = c \iff \mathbb{B}(abc) \wedge ab \equiv cd,$$

right angle

$$\mathbb{A}_\mathbb{R}(abc) \iff ac \equiv c\mathbb{S}_b(a),$$

perpendicularity

$$ab \perp cd \iff \exists_x [\mathbb{A}_\mathbb{R}(axc) \wedge \mathbb{A}_\mathbb{R}(axd) \wedge \mathbb{A}_\mathbb{R}(bxd) \wedge \mathbb{A}_\mathbb{R}(bxc)]$$

and *parallelity*

$$ab \parallel cd \iff \exists_{p,q}[p \neq q \wedge ab \perp pq \wedge cd \perp pq].$$

Polar similarity is determined by its center and the scale. The scale may be given by two points collinear with the center o. Thus we shall denote it by J_o^{ab}, where $\mathrm{J}_o^{ab}(a) = b$. First however it will be convenient to define a restriction of $\mathrm{J}_o^{ab'}$ to the points non-collinear with o, a, b commended by o. Moreover, we restrict ourselves to the case of negative scale, *i.e.*, we assume that $\mathbb{B}(aob)$:

$$\hat{\mathrm{J}}_o^{ab}(x) = y \iff \begin{cases} o & \text{if } x = o \\ y & \text{otherwise, where } \neg\mathbb{B}(oxa) \wedge \neg\mathbb{B}(oax) \wedge \\ & \neg\mathbb{B}(aox) \wedge \mathbb{B}(aob) \wedge \mathbb{B}(xoy) \wedge a \neq o \wedge ax \parallel by \end{cases}$$

Using a restricted polar similarity we define the similarity itself:

$$\mathrm{J}_o^{ab}(x) = y \iff \exists_{c,d}[\neg\mathbb{B}(obc) \wedge \neg\mathbb{B}(obd) \wedge \\ (y = \hat{\mathrm{J}}_o^{ac}(\hat{\mathrm{J}}_o^{cb}(x)) \vee y = \hat{\mathrm{J}}_o^{ac}(\hat{\mathrm{J}}_o^{cd}(\hat{\mathrm{J}}_o^{db}(x))))].$$

Thus elementary plane Euclidean geometry has a startling property: it is mutually interpretable with the theory of its automorphisms.

Theorem 2. *The Euclidean plane geometry is mutually interpretable with the elementary theory of similarities.*

Theorem 3. *The elementary theory of Cartesian plane over the reals is mutually interpretable with the theory of the group of automorphisms of this plane.*

References

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