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A DEDUCTIVE-REDUCTIVE FORM OF LOGIC: intuitionistic S4 modalities

Abstract. The paper is a continuation of *A deductive-reductive form of logic: general theory and intuitionistic case* ([1]) and considers the problem of definability of modal operators on the intuitionistic base. Contrary to the classical case, it seems that the fact whether the connective is Heyting's or Brouwerian is essential for the intuitionistic logic. The connective of possibility has the classical interpretation, i.e. $w \models \Diamond\alpha$ iff $\exists t(w\mathrm{R}t$ and $t \models \alpha)$ (with R, a relation of accesibility), if it is defined on the base of the logic with Brouwerian connective of coimplication.

The logic determined over the language $\mathcal{L}$ with L, the set of all formulas, in its complete, i.e. deductive-reductive form is a triple

$$(\mathcal{L}, \mathrm{E}, \mathrm{C})$$

where for any $\mathrm{X} \subseteq \mathrm{L}$

$$\mathrm{E}(\mathrm{X}) = \mathrm{L} - \mathrm{C}^{\mathrm{d}}(\mathrm{L} - \mathrm{X})$$

with $\mathrm{C}^{\mathrm{d}} \colon 2^{\mathrm{L}} \to 2^{\mathrm{L}}$ an operation dual to the finitary, structural and either disjunctive or conjunctive consequence operation C, formulated by Wójcicki in [4]:

$$\alpha \in \mathrm{C}^{\mathrm{d}}(\mathrm{X}) \quad \text{iff} \quad \bigcap\{\mathrm{C}(\beta) : \beta \in \mathrm{X_f}\} \subseteq \mathrm{C}(\alpha) \text{ for some finite } \mathrm{X_f} \subseteq \mathrm{X}.$$

1. Intuitionistic logic with identity on the language with implication

Let $\mathrm{C}_{\mathrm{H}\equiv}$ be an intuitionistic propositional logic with identity defined on the language

$$\mathcal{L}_{\mathrm{H}\equiv} = (\mathrm{L}_{\mathrm{H}\equiv}, \neg, \wedge, \vee, \to, \leftrightarrow, \equiv)$$

by the standard axiom set for intuitionistic propositional logic, Modus Ponens and

$1_{\mathrm{C}_{\mathrm{H}\equiv}}$ $\quad \emptyset \vdash \alpha \equiv \alpha$
$2_{\mathrm{C}_{\mathrm{H}\equiv}}$ $\quad \emptyset \vdash (\alpha \equiv \beta) \to (\neg\alpha \equiv \neg\beta)$
$3_{\mathrm{C}_{\mathrm{H}\equiv}}$ $\quad \emptyset \vdash ((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \to ((\alpha \S \gamma) \equiv (\beta \S \delta)), \quad \S \in \{\wedge, \vee, \to, \leftrightarrow, \equiv\}$
$4_{\mathrm{C}_{\mathrm{H}\equiv}}$ $\quad \emptyset \vdash (\alpha \equiv \beta) \to (\alpha \to \beta)$

where, $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$.

For the Kripke style semantics let us take $\mathcal{A} = (\mathrm{A}, \neg, \cap, \cup, \to, \equiv)$ similar to $\mathcal{L}_{\mathrm{H}\equiv}$, and for every $\mathrm{D} \subseteq \mathrm{A}$, $\simeq$ a congruence of the matrix $(\mathcal{A}, \mathrm{D})$. Let us present a class

$$\mathcal{M} = \{(\mathcal{A}/\!\simeq, \mathrm{D}/\!\simeq) : \mathrm{D} \subseteq \mathrm{A},\ \simeq \text{ is a congruence of } (\mathcal{A}, \mathrm{D})\}$$

as

$$\mathcal{M} = \{(\mathcal{A}_s, \mathrm{D}_s) : s \in \mathrm{S}\}$$

with S a set of indices of elements of $\mathcal{M}$ partially ordered by $\leq$ such that for any $s_1, s_2 \in$S and $\mathrm{D}_s i = \mathrm{D}_i/\!\simeq_i$, for $i \in \{1, 2\}$: $s_1 \leq s_2$ iff $\mathrm{D}_1 \subseteq \mathrm{D}_2$.

A semantics adequate for $\mathrm{C}_{\mathrm{H}\equiv}$ is a class of $\mathrm{C}_{\mathrm{H}\equiv}$-models, i.e. such $\mathcal{M} = (\mathcal{A}_s, \{\mathrm{D}_s : s \in \mathrm{S}\})$ that for any $a, b \in$A, $s \in$S:

(i^+)	$[a]_s \in \mathrm{D}_s$	implies	$\forall t \geq s \quad [a]_t \in \mathrm{D}_t$
$(\neg^+)$	$\neg_s [a]_s \in \mathrm{D}_s$	iff	$\forall t \geq s \quad [a]_t \notin \mathrm{D}_t$
$(\cap^+)$	$[a]_s \cap_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ and $[b]_s \in \mathrm{D}_s$
$(\cup^+)$	$[a]_s \cup_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ or $[b]_s \in \mathrm{D}_s$
$(\to^+)$	$[a]_s \to_s [b]_s \in \mathrm{D}_s$	iff	$\forall t \geq s \quad ([a]_t \notin \mathrm{D}_t$ or $[b]_t \in \mathrm{D}_t)$
$(\equiv^+)$	$[a]_s \equiv_s [b]_s \in \mathrm{D}_s$	iff	$\forall t \geq s \quad [a]_t = [b]_t$.

If

$$(\mathrm{C}_{\mathcal{M}}) \quad \alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X}) \quad \text{iff} \quad \forall \mathrm{h} \in \mathrm{Hom}(\mathcal{L}_{\mathrm{H}\equiv}, \mathcal{A})\ \forall s \in \mathrm{S}\ \forall \beta \in \mathrm{X}\ \ \mathrm{h}(\beta) \in \mathrm{D}_s \text{ implies } \mathrm{h}(\alpha) \in \mathrm{D}_s$$

then

$$\alpha \in \mathrm{C}_{\mathrm{H}\equiv}(\mathrm{X}) \quad \text{iff} \quad \alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X}) \text{ for any } \mathrm{C}_{\mathrm{H}\equiv}\text{-model } \mathcal{M}$$

$\mathrm{E}_{\mathrm{H}\equiv}$, a reductive counterpart of $\mathrm{C}_{\mathrm{H}\equiv}$ is semantically defined as follows

$$\alpha \notin \mathrm{E}_{\mathrm{H}\equiv}(\mathrm{X}) \quad \text{iff} \quad \alpha \notin \mathrm{E}_{\mathcal{M}}(\mathrm{X}) \text{ for any } \mathrm{C}_{\mathrm{H}\equiv}\text{-model } \mathcal{M}$$

with

$$(\mathrm{E}_{\mathcal{M}}) \quad \alpha \notin \mathrm{E}_{\mathcal{M}}(\mathrm{X}) \quad \text{iff} \quad \forall \mathrm{h} \in \mathrm{Hom}(\mathcal{L}_{\mathrm{H}\equiv}, \mathcal{A}) \ \forall s \in \mathrm{S}$$
$$\forall \beta \in \mathrm{L}_{\mathrm{H}\equiv} - \mathrm{X} \ \ \mathrm{h}(\beta) \notin \mathrm{D}_s \text{ implies } \mathrm{h}(\alpha) \notin \mathrm{D}_s.$$

Obviously, the axiomatization of $\mathrm{E}_{\mathrm{H}\equiv}$ is not standard, see [1]. Thus, the deductive-reductive Heyting's intuitionistic propositional logic of truth with identity is a triple:

$$(\mathcal{L}_{\mathrm{H}\equiv}, \mathrm{C}_{\mathrm{H}\equiv}, \mathrm{E}_{\mathrm{H}\equiv}).$$

A structure $\mathcal{M} = (\mathcal{A}_s, \{\mathrm{D}_s : s \in \mathrm{S}\})$ is a $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}$-model, if for any $a, b \in$A, $s \in$S:

(i^-)	$[a]_s \in \mathrm{D}_s$	implies	$\forall t \geq s \ \ [a]_t \in \mathrm{D}_t$
$(\neg^-)$	$\neg_s[a]_s \in \mathrm{D}_s$	iff	$\exists t \leq s \ \ [a]_t \notin \mathrm{D}_t$
$(\cap^-)$	$[a]_s \cap_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ or $[b]_s \in \mathrm{D}_s$
$(\cup^-)$	$[a]_s \cup_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ and $[b]_s \in \mathrm{D}_s$
$(\to^-)$	$[a]_s \to_s [b]_s \in \mathrm{D}_s$	iff	$\exists t \leq s \ \ ([a]_t \notin \mathrm{D}_t$ and $[b]_t \in \mathrm{D}_t)$
$(\equiv^-)$	$[a]_s \equiv_s [b]_s \in \mathrm{D}_s$	iff	$\exists t \leq s \ \ [a]_t \neq [b]_t$.

Using the definitions $(\mathrm{C}_{\mathcal{M}})$ and $(\mathrm{E}_{\mathcal{M}})$ one can semantically present:

$$\alpha \in \mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}(\mathrm{X}) \quad \text{iff} \quad \alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X}) \text{ for any } \mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}\text{-model } \mathcal{M},$$
$$\alpha \notin \mathrm{E}^{\mathrm{d}}_{\mathrm{H}\equiv}(\mathrm{X}) \quad \text{iff} \quad \alpha \notin \mathrm{E}_{\mathcal{M}}(\mathrm{X}) \text{ for any } \mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}\text{-model } \mathcal{M}.$$

The axiomatization of both, $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}$ and $\mathrm{E}_{\mathrm{H}\equiv}$, brings about the same problems, however the syntax for $\mathrm{E}^{\mathrm{d}}_{\mathrm{H}\equiv}$ can be formulated: it suffices to replace all occurrences of "$\emptyset \vdash$" by "$\mathrm{L}_{\mathrm{H}\equiv} \dashv$" in all axioms for $\mathrm{C}_{\mathrm{H}\equiv}$. The only R-rule is $\mathrm{L}_{\mathrm{H}\equiv} - \{\alpha, \alpha \to \beta\} \dashv \beta$.

Thus, the class of all $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}$-models defines deductive-reductive intuitionistic propositional logic of falsehood with identity:

$$(\mathcal{L}_{\mathrm{H}\equiv}, \mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}, \mathrm{E}^{\mathrm{d}}_{\mathrm{H}\equiv}).$$

1.1. Intuitionistic S4 modal systems on the language with implication

As it was shown by R. Suszko (e.g. [3]) the classical S4 system is some Boolean theory of SCI (Sentential Calculus with Identity). Similar connection between S4-necessity and appropriately strengthened identity holds also on the base of intuitionistic logic.

Let us consider C_{H}, an axiomatic extension of $\mathrm{C}_{\mathrm{H}\equiv}$ by:

$$\{\alpha \equiv \beta : \ \alpha \leftrightarrow \beta \in \mathrm{C}_{\mathrm{H}\equiv}(\emptyset)\}.$$

Translations formulated by R.Suszko:

$$\alpha \equiv \beta = \Box(\alpha \leftrightarrow \beta) \qquad \text{and} \qquad \Box\alpha = \alpha \equiv 1$$

establish an equivalence between $\mathrm{C_H}(\emptyset)$ and $\mathrm{S4_{DH\Box}}$, a deductive intuitionistic Heyting modal system of kind S4 defined on the language

$$\mathcal{L}_{\mathrm{H}\Box} = (\mathrm{L}_{\mathrm{H}\Box}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \Box)$$

by intuitionistic propositional axioms, Modus Ponens and

$$\begin{aligned}
&\emptyset \vdash \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)\\
&\emptyset \vdash \Box\alpha \rightarrow \alpha\\
&\emptyset \vdash \Box\alpha \rightarrow \Box\Box\alpha\\
&\emptyset \vdash \alpha \;/\; \emptyset \vdash \Box\alpha.
\end{aligned}$$

The well known semantics for $\mathrm{S4_{DH\Box}}$ is a class of $\mathrm{S4_{H\Box}}$-models, i.e. these structures $\langle \mathrm{W}, \leq, \mathrm{R}, \models\rangle$ with reflexive and transitive relations $\leq$ and R on W, for which

(i^+)	$w \models p$	implies	$\forall v \geq w \quad v \models p$
$(\neg^+)$	$w \models \neg\alpha$	iff	$\forall v \geq w \quad v \not\models \alpha$
$(\cap^+)$	$w \models \alpha \wedge \beta$	iff	$w \models \alpha$ *and* $w \models \beta$
$(\cup^+)$	$w \models \alpha \vee \beta$	iff	$w \models \alpha$ or $w \models \beta$
$(\rightarrow^+)$	$w \models \alpha \rightarrow \beta$	iff	$\forall v \geq w \quad (v \not\models \alpha$ or $v \models \beta)$
$(\Box^+)$	$w \models \Box\alpha$	iff	$\forall v \quad (w\mathrm{R}v$ implies $v \models \alpha)$

for any $\alpha, \beta \in \mathrm{L_{H\Box}}$, $w \in$W.

Deductively:

- a formula α is $\mathrm{S4_{H\Box}}$-satisfiable, if there is $\mathcal{M} = \langle \mathrm{W}, \leq, \mathrm{R}, \models\rangle$, a $\mathrm{S4_{H\Box}}$-model, and a world $w \in$W, such that $w \models \alpha$.
- a formula α is valid in a $\mathrm{S4_{H\Box}}$-model $\mathcal{M} = \langle \mathrm{W}, \leq, \mathrm{R}, \models\rangle$, if for any world $w \in$W, $w \models \alpha$. Then, usually we write $\mathcal{M} \models \alpha$.
- a formula α is $\mathrm{S4_{H\Box}}$-valid, if for any $\mathrm{S4_{H\Box}}$-model $\mathcal{M}$, $\mathcal{M} \models \alpha$.

Naturally,

$$\alpha \in \mathrm{S4_{DH\Box}} \quad \text{iff} \quad \alpha \text{ is } \mathrm{S4_{H\Box}}\text{-valid.}$$

Reductively:

- a formula α is $\mathrm{S4_{H\Box}}$-falsified, if there is $\mathcal{M} = \langle \mathrm{W}, \leq, \mathrm{R}, \models\rangle$, a $\mathrm{S4_{H\Box}}$-model, and a world $w \in$W, such that $w \not\models \alpha$.

- a formula α is false in a $\mathrm{S4_{H\Box}}$-model $\mathcal{M} = \langle \mathrm{W}, \leq, \mathrm{R}, \models \rangle$, if for any world $w \in \mathrm{W}$, $w \not\models \alpha$.
- a formula α is $\mathrm{S4_{H\Box}}$-false, if α is false in every $\mathrm{S4_{H\Box}}$-model.

Then,

$$\alpha \notin \mathrm{S4_{RH\Box}} \quad \text{iff} \quad \alpha \text{ is } \mathrm{S4_{H\Box}}\text{-false}.$$

Thus, the intuitionistic Heyting modal system of kind S4 in deductive-reductive form is the following triple:

$$(\mathcal{L}_{\mathrm{H\Box}}, \mathrm{S4_{DH\Box}}, \mathrm{S4_{RH\Box}})$$

Let us notice that the axiomatization of $\mathrm{S4_{EH\Box}}$ is not standard. It means that neither is the syntax for the deductive part of

$$(\mathcal{L}_{\mathrm{H\Box}}, \mathrm{S4^d_{DH\Box}}, \mathrm{S4^d_{RH\Box}})$$

i.e., $\mathrm{S4^d_{DH\Box}} = \mathrm{L_{H\Box}} - \mathrm{S4_{RH\Box}}$, the modal system given in the above deductive sense by the class of $\mathrm{S4^d_{H\Box}}$-models. $\langle \mathrm{W}, \leq, \mathrm{R}, \models \rangle$ with reflexive and transitive relations $\leq$ and R on W is a $\mathrm{S4^d_{H\Box}}$-model, if for any $\alpha, \beta \in \mathrm{L_{H\Box}}$, $w \in \mathrm{W}$,

(i^-)	$w \models p$	implies	$\forall v \geq w \quad v \models p$
$(\neg^-)$	$w \models \neg\alpha$	iff	$\exists v \leq w \quad v \not\models \alpha$
$(\cap^-)$	$w \models \alpha \wedge \beta$	iff	$w \models \alpha$ or $w \models \beta$
$(\cup^-)$	$w \models \alpha \vee \beta$	iff	$w \models \alpha$ and $w \models \beta$
$(\rightarrow^-)$	$w \models \alpha \rightarrow \beta$	iff	$\exists v \leq w \quad (v \not\models \alpha$ and $v \models \beta)$
$(\Box^-)$	$w \models \Box\alpha$	iff	$\exists v \ (w\mathrm{R}v$ and $v \models \alpha)$.

However, the axiomatization of $\mathrm{S4^d_{RH\Box}} = \mathrm{L_{H\Box}} - \mathrm{S4_{DH\Box}}$, i.e. of the modal system defined in the reductive sense by the class of $\mathrm{S4^d_{H\Box}}$-models causes no problem. To this aim, it is sufficient to take all axioms for $\mathrm{S4_{DH\Box}}$ in reductive sense, i.e. with "$\emptyset \vdash$" replaced by "$\mathrm{L_{H\Box}} \dashv$", and instead of Modus Ponens and the rule of generalization, their reductive counterparts: $\mathrm{L_{H\Box}} - \{\alpha, \alpha \rightarrow \beta\} \dashv \beta$ and $\mathrm{L_{H\Box}} \dashv \alpha \ / \ \mathrm{L_{H\Box}} \dashv \Box\alpha$.

2. Intuitionistic logic with non-identity on the language with coimplication

On the language with coimplication and non-identity

$$\mathcal{L}_{\mathrm{B}\not\equiv} = (\mathrm{L_{B\not\equiv}}, \sim, \wedge, \vee, \leftharpoondown, \rightleftharpoons, \not\equiv)$$

let us consider $\mathrm{C^d_{B\not\equiv}}$ a consequence operation given by

$$\emptyset \vdash (\alpha \leftharpoondown \beta) \leftharpoondown \alpha$$

$\emptyset \vdash ((\gamma \leftharpoondown \alpha) \leftharpoondown (\beta \leftharpoondown \alpha)) \leftharpoondown ((\gamma \leftharpoondown \beta) \leftharpoondown \alpha)$
$\emptyset \vdash (\alpha \wedge \beta) \leftharpoondown \alpha$
$\emptyset \vdash (\alpha \wedge \beta) \leftharpoondown \beta$
$\emptyset \vdash ((\gamma \leftharpoondown (\alpha \wedge \beta)) \leftharpoondown (\gamma \leftharpoondown \beta)) \leftharpoondown (\gamma \leftharpoondown \alpha)$
$\emptyset \vdash \alpha \leftharpoondown (\alpha \vee \beta)$
$\emptyset \vdash \beta \leftharpoondown (\alpha \vee \beta)$
$\emptyset \vdash (((\alpha \vee \beta) \leftharpoondown \gamma) \leftharpoondown (\beta \leftharpoondown \gamma)) \leftharpoondown (\alpha \leftharpoondown \gamma)$
$\emptyset \vdash (\sim \beta \leftharpoondown \alpha) \leftharpoondown (\sim \alpha \leftharpoondown \beta)$
$\emptyset \vdash \beta \leftharpoondown \sim(\alpha \leftharpoondown \alpha)$
$\emptyset \vdash \alpha \not\equiv \alpha$
$\emptyset \vdash (\sim \alpha \not\equiv \sim \beta) \leftharpoondown (\alpha \not\equiv \beta)$
$\emptyset \vdash ((\alpha \mathbin{\S} \gamma) \not\equiv (\beta \mathbin{\S} \delta)) \leftharpoondown ((\alpha \not\equiv \beta) \vee (\gamma \not\equiv \delta)), \quad \S \in \{\wedge, \vee, \leftharpoondown, \rightleftharpoons, \not\equiv\}$
$\emptyset \vdash (\alpha \leftharpoondown \beta) \leftharpoondown (\alpha \not\equiv \beta)$
$\{\beta, \alpha \leftharpoondown \beta\} \vdash \alpha$

with $\alpha \rightleftharpoons \beta = (\alpha \leftharpoondown \beta) \wedge (\beta \leftharpoondown \alpha)$.

As in the previous case, let $\mathcal{A} = (\mathrm{A}, \sim, \cap, \cup, \leftharpoondown, \not\equiv)$ be an algebra similar to $\mathcal{L}_{\mathrm{B}\not\equiv}$, and for every $\mathrm{D} \subseteq \mathrm{A}$, $\simeq$ a congruence of the matrix $(\mathcal{A}, \mathrm{D})$. Similarly, a class

$$\mathcal{M} = \{(\mathcal{A}/\simeq, \mathrm{D}/\simeq) : \mathrm{D} \subseteq \mathrm{A}, \ \simeq \text{ is a congruence of } (\mathcal{A}, \mathrm{D})\}$$

will be rewritten as

$$\mathcal{M} = \{(\mathcal{A}_s, \mathrm{D}_s) : \ \ s \in \mathrm{S}\}$$

with S a set of indices of elements of $\mathcal{M}$ partially ordered by $\leq$ such that for any $s_1, s_2 \in$S and $\mathrm{D}_s i = \mathrm{D}_i/\simeq_i$, dla $i \in \{1, 2\}$: $s_1 \leq s_2$ iff $\mathrm{D}_1 \subseteq \mathrm{D}_2$.

A structure $\mathcal{M}$ will be:

a $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$-model, if for any $a, b \in \mathrm{A}$, $s \in \mathrm{S}$

(i^-)	$[a]_s \in \mathrm{D}_s$	implies	$\forall t \geq s \ \ [a]_t \in \mathrm{D}_t$
$(\sim^-)$	$\sim_s [a]_s \in \mathrm{D}_s$	iff	$\forall t \geq s \ \ [a]_t \notin \mathrm{D}_t$
$(\cap^-)$	$[a]_s \cap_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ or $[b]_s \in \mathrm{D}_s$
$(\cup^-)$	$[a]_s \cup_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ and $[b]_s \in \mathrm{D}_s$
$(\leftharpoondown^-)$	$[a]_s \leftharpoondown_s [b]_s \in \mathrm{D}_s$	iff	$\forall t \geq s \ \ ([a]_t \in \mathrm{D}_t$ or $[b]_t \notin \mathrm{D}_t)$
$(\not\equiv^-)$	$[a]_s \not\equiv_s [b]_s \in \mathrm{D}_s$	iff	$\forall t \geq s \ \ [a]_t = [b]_t$

a $\mathrm{C}_{\mathrm{B}\not\equiv}$-model, if for any $a, b \in \mathrm{A}$, $s \in \mathrm{S}$

(i^+)	$[a]_s \in \mathrm{D}_s$	implies	$\forall t \geq s \quad [a]_t \in \mathrm{D}_t$
$(\sim^+)$	$\sim_s[a]_s \in \mathrm{D}_s$	iff	$\exists t \leq s \quad [a]_t \notin \mathrm{D}_t$
$(\cap^+)$	$[a]_s \cap_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ and $[b]_s \in \mathrm{D}_s$
$(\cup^+)$	$[a]_s \cup_s [b]_s \in \mathrm{D}_s$	iff	$[a]_s \in \mathrm{D}_s$ or $[b]_s \in \mathrm{D}_s$
$(\leftharpoondown^+)$	$[a]_s \leftharpoondown_s [b]_s \in \mathrm{D}_s$	iff	$\exists t \leq s \quad ([a]_t \in \mathrm{D}_t$ and $[b]_t \notin \mathrm{D}_t)$
$(\not\equiv^+)$	$[a]_s \not\equiv_s [b]_s \in \mathrm{D}_s$	iff	$\exists t \leq s \quad [a]_t \neq [b]_t$

Every class of the above models defines a deductive-reductive Brouwerian intuitionistic propositional logic of truth with non-identity

$$(\mathcal{L}_{\mathrm{B}\not\equiv}, \mathrm{C}_{\mathrm{B}\not\equiv}, \mathrm{E}_{\mathrm{B}\not\equiv})$$

or the analogous logic of falsehood,

$$\mathcal{L}_{\mathrm{B}\not\equiv}, \mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}, \mathrm{E}^{\mathrm{d}}_{\mathrm{B}\not\equiv})$$

which are defined semantically, as in the Heyting's case

$\alpha \in \mathrm{C}_{\mathrm{B}\not\equiv}(\mathrm{X})$	iff	$\alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X})$ for any $\mathrm{C}_{\mathrm{B}\not\equiv}$-model $\mathcal{M}$
$\alpha \notin \mathrm{E}_{\mathrm{B}\not\equiv}(\mathrm{X})$	iff	$\alpha \notin \mathrm{E}_{\mathcal{M}}(\mathrm{X})$ for any $\mathrm{C}_{\mathrm{B}\not\equiv}$-model $\mathcal{M}$
$\alpha \in \mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}(\mathrm{X})$	iff	$\alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X})$ for any $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$-model $\mathcal{M}$
$\alpha \notin \mathrm{E}^{\mathrm{d}}_{\mathrm{B}\not\equiv}(\mathrm{X})$	iff	$\alpha \notin \mathrm{E}_{\mathcal{M}}(\mathrm{X})$ for any $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$-model $\mathcal{M}$

As in the Heyting's case, the standard axiomatization of $\mathrm{C}_{\mathrm{B}\not\equiv}$ and of $\mathrm{E}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$ also evokes some serious difficulties. However, one can put an axiomatization for $\mathrm{E}_{\mathrm{B}\not\equiv}$ taking all axioms for $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$ in the reductive sense, i.e. with "$\emptyset \vdash$ " replaced by "$\mathrm{L}_{\mathrm{B}\not\equiv} \dashv$ ". The only R-rule is $\mathrm{L}_{\mathrm{B}\not\equiv} - \{\beta, \alpha \leftharpoondown \beta\} \dashv \alpha$.

2.1. Intuitionistic S4 modal systems on the language with coimplication

In order to obtain modal system analogous to the previous one, let us extend $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$ to $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}$ by the following axiom set

$$\{\alpha \not\equiv \beta : \ \alpha \rightleftharpoons \beta \in \mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}(\emptyset)\}.$$

Using the translations

$$\alpha \not\equiv \beta = \Diamond(\alpha \rightleftharpoons \beta) \qquad \text{and} \qquad \Diamond\alpha = \alpha \not\equiv 0$$

one can prove an equivalence between $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}(\emptyset)$ and $\mathrm{S4}^{\mathrm{d}}_{\mathrm{DB}\Diamond}$, a deductive intuitionistic Brouwerian modal system of kind S4 defined on the language

$$\mathcal{L}_{\mathrm{B}\Diamond} = (\mathrm{L}_{\mathrm{B}\Diamond}, \sim, \wedge, \vee, \leftharpoondown, \rightleftharpoons, \Diamond)$$

by first ten axioms and one rule of $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$, and

$$\emptyset \vdash (\Diamond\alpha \leftharpoondown \Diamond\beta) \leftharpoondown \Diamond(\alpha \leftharpoondown \beta)$$
$$\emptyset \vdash \alpha \leftharpoondown \Diamond\alpha$$
$$\emptyset \vdash \Diamond\Diamond\alpha \leftharpoondown \Diamond\alpha$$
$$\emptyset \vdash \alpha \,/\, \emptyset \vdash \Diamond\alpha$$

The semantics for $\mathrm{S4}^{\mathrm{d}}_{\mathrm{DB}\Diamond}$ is a class of $\mathrm{S4}^{\mathrm{d}}_{\mathrm{B}\Diamond}$-models, i.e. these structures $\langle \mathrm{W}, \leq, \mathrm{R}, \models \rangle$ with reflexive and transitive relations $\leq$ and R on W, for which

(i^-)	$w \models p$	implies	$\forall v \geq w \quad v \models p$
$(\sim^-)$	$w \models \sim \alpha$	iff	$\forall v \geq w \quad v \not\models \alpha$
$(\cap^-)$	$w \models \alpha \wedge \beta$	iff	$w \models \alpha$ or $w \models \beta$
$(\cup^-)$	$w \models \alpha \vee \beta$	iff	$w \models \alpha$ and $w \models \beta$
$(\leftharpoondown^-)$	$w \models \alpha \leftharpoondown \beta$	iff	$\forall v \geq w \quad (v \models \alpha$ or $v \not\models \beta)$
$(\Diamond^-)$	$w \models \Diamond\alpha$	iff	$\forall v \ (w\mathrm{R}v$ implies $v \models \alpha)$

for any $\alpha, \beta \in \mathrm{L}_{\mathrm{B}\Diamond}$, $w \in$W.

Using notions of satisfaction, non-satisfaction, validity and falsity given for the Heyting's case one can express the following two completeness theorems:

$$\alpha \in \mathrm{S4}^{\mathrm{d}}_{\mathrm{DB}\Diamond} \quad \text{iff} \quad \alpha \text{ is } \mathrm{S4}^{\mathrm{d}}_{\mathrm{B}\Diamond}\text{-valid}$$
$$\alpha \notin \mathrm{S4}^{\mathrm{d}}_{\mathrm{RB}\Diamond} \quad \text{iff} \quad \alpha \text{ is } \mathrm{S4}^{\mathrm{d}}_{\mathrm{B}\Diamond}\text{-false}$$

which semantically define

$$(\mathcal{L}_{\mathrm{B}\Diamond}, \mathrm{S4}^{\mathrm{d}}_{\mathrm{DB}\Diamond}, \mathrm{S4}^{\mathrm{d}}_{\mathrm{RB}\Diamond}),$$

in deductive-reductive form the intuitionistic Brouwerian modal S4 system of falsehood.

Similarly, the intuitionistic Brouwerian modal system S4 of truth in deductive-reductive form

$$(\mathcal{L}_{\mathrm{B}\Diamond}, \mathrm{S4}_{\mathrm{DB}\Diamond}, \mathrm{S4}_{\mathrm{RB}\Diamond})$$

is defined by the class of all $\mathrm{S4}_{\mathrm{B}\Diamond}$-models, i.e. the structures $\langle \mathrm{W}, \leq, \mathrm{R}, \models \rangle$ such that for any $\alpha, \beta \in \mathrm{L}_{\mathrm{B}\Diamond}$, $w \in$W:

(i^+)	$w \models p$	implies	$\forall v \geq w \quad v \models p$
$(\sim^+)$	$w \models \sim \alpha$	iff	$\exists v \leq w \quad v \not\models \alpha$
$(\cap^+)$	$w \models \alpha \wedge \beta$	iff	$w \models \alpha$ and $w \models \beta$
$(\cup^+)$	$w \models \alpha \vee \beta$	iff	$w \models \alpha$ or $w \models \beta$
$(\leftharpoondown^+)$	$w \models \alpha \leftharpoondown \beta$	iff	$\exists v \leq w \quad (v \models \alpha$ and $v \not\models \beta)$
$(\Diamond^+)$	$w \models \Diamond\alpha$	iff	$\exists v \ (w\mathrm{R}v$ and $v \models \alpha)$.

As in other cases, there is probably no standard axiomatization for $\mathrm{S4}_{\mathrm{DB}\Diamond}$ nor $\mathrm{S4}^{\mathrm{d}}_{\mathrm{RB}\Diamond}$. The syntax for $\mathrm{S4}_{\mathrm{RB}\Diamond} = \mathrm{L}_{\mathrm{B}\Diamond} - \mathrm{S4}^{\mathrm{d}}_{\mathrm{DB}\Diamond}$, the modal system reductively defined by the class of $\mathrm{S4}^{\mathrm{d}}_{\mathrm{B}\Diamond}$-models, consists of all axioms for $\mathrm{S4}^{\mathrm{d}}_{\mathrm{DB}\Diamond}$ taken in reductive position. The only R-rules are: $\mathrm{L}_{\mathrm{B}\not\equiv} - \{\beta, \alpha \leftharpoondown \beta\} \dashv \alpha$ and $\mathrm{L}_{\mathrm{B}\Diamond} \dashv \alpha \;/\; \mathrm{L}_{\mathrm{B}\Diamond} \dashv \Diamond\alpha$.

3. Intuitionistic modalities defined on the Heyting-Brouwer language

[1] deals with two logics: Heyting intuitionistic logic of truth and of falsehood, and Brouwerian intuitionistic logic of truth and of falsehood; both given in the deductive-reductive forms. Problems with the complete axiomatization of those logics are similar to ours. Indeed, a standard axiomatization of the deductive part of the Heyting intuitionistic logic of falsehood given by the class of all $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}$-models, and a standard axiomatization of the reductive part of the Heyting intuitionistic logic of truth given by the class of all $\mathrm{C}_{\mathrm{H}\equiv}$-models would be possible if we had defined some additional connective, for example coimplication, interpreted on the future in the deductive case (for $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}$) and on the past in the reductive one (for $\mathrm{E}_{\mathrm{H}\equiv}$). Also in the Brouwerian case, there is a need of a connective interpreted on the future for $\mathrm{C}_{\mathrm{B}\not\equiv}$ and on the past for $\mathrm{E}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$.

However, it is possible to define a model for the intuitionistic logic of truth with Heyting and Brouwerian connectives together with identity and non-identity, $\mathrm{C}_{\mathrm{HB}\equiv\not\equiv}$-model, whose appropriate reducts would be a $\mathrm{C}_{\mathrm{H}\equiv}$-model and a $\mathrm{C}_{\mathrm{B}\not\equiv}$-model, respectively. Similarly, one can consider a model for intuitionistic logic of falsehood with Heyting and Brouwerian connectives together with identity and non-identity, a $\mathrm{C}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}$-model, which appropriate reducts would be a $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}\equiv}$-model and a $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}\not\equiv}$-model, respectively. Of course, both logics are defined on the extended language:

$$\mathcal{L}_{\mathrm{HB}\equiv\not\equiv} = (\mathrm{L}_{\mathrm{HB}\equiv\not\equiv}, \neg, \sim, \wedge, \vee, \rightarrow, \leftrightarrow, \leftharpoondown, \rightleftharpoons, \equiv, \not\equiv).$$

The class of all $\mathrm{C}_{\mathrm{HB}\equiv\not\equiv}$-models defines the Heyting-Brouwer logic of truth with identity and non-identity ($\mathcal{L}_{\mathrm{HB}\equiv\not\equiv}$, $\mathrm{C}_{\mathrm{HB}\equiv\not\equiv}$, $\mathrm{E}_{\mathrm{HB}\equiv\not\equiv}$), while all $\mathrm{C}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}$-models give the same logic but for the falsehood ($\mathcal{L}_{\mathrm{HB}\equiv\not\equiv}$, $\mathrm{C}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}$, $\mathrm{E}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}$). The syntax of the Heyting-Brouwer logic (without identity and non-identity) of truth as well as of falsehood is presented by C. Rauszer in [2]. For the syntactical characterisation of the Heyting-Brouwer logic with identity and non-identity it suffices to extend the Rauszer's axiomatization

- for the logic of truth by:

$$\alpha \equiv \alpha$$
$$(\alpha \equiv \beta) \to (\neg\alpha \equiv \neg\beta)$$
$$(\alpha \equiv \beta) \to ({\sim}\,\alpha \equiv {\sim}\,\beta)$$
$$((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \to ((\alpha \,\S\, \gamma) \equiv (\beta \,\S\, \delta)), \qquad \S \in \{\wedge, \vee, \to, \leftrightarrow, \leftharpoondown, \rightleftharpoons, \equiv, \not\equiv\}$$
$$(\alpha \equiv \beta) \to (\alpha \to \beta)$$
$$\neg(\alpha \not\equiv \alpha)$$
$$(\neg\alpha \not\equiv \neg\beta) \to (\alpha \not\equiv \beta)$$
$$({\sim}\,\alpha \not\equiv {\sim}\,\beta) \to (\alpha \not\equiv \beta)$$
$$((\alpha \,\S\, \gamma) \not\equiv (\beta \,\S\, \delta)) \to ((\alpha \not\equiv \beta) \vee (\gamma \not\equiv \delta)), \qquad \S \in \{\wedge, \vee, \to, \leftrightarrow, \leftharpoondown, \rightleftharpoons, \equiv, \not\equiv\}$$
$$(\alpha \leftharpoondown \beta) \to (\alpha \not\equiv \beta)$$

- for the logic of falsehood by:

$${\sim}(\alpha \equiv \alpha)$$
$$(\alpha \equiv \beta) \leftharpoondown (\neg\alpha \equiv \neg\beta)$$
$$(\alpha \equiv \beta) \leftharpoondown ({\sim}\,\alpha \equiv {\sim}\,\beta)$$
$$((\alpha \equiv \beta) \wedge (\gamma \equiv \delta)) \leftharpoondown ((\alpha \,\S\, \gamma) \equiv (\beta \,\S\, \delta)), \qquad \S \in \{\wedge, \vee, \to, \leftrightarrow, \leftharpoondown, \rightleftharpoons, \equiv, \not\equiv\}$$
$$(\alpha \equiv \beta) \leftharpoondown (\alpha \to \beta)$$
$$\alpha \not\equiv \alpha$$
$$(\neg\alpha \not\equiv \neg\beta) \leftharpoondown (\alpha \not\equiv \beta)$$
$$({\sim}\,\alpha \not\equiv {\sim}\,\beta) \leftharpoondown (\alpha \not\equiv \beta)$$
$$((\alpha \,\S\, \gamma) \not\equiv (\beta \,\S\, \delta)) \leftharpoondown ((\alpha \not\equiv \beta) \vee (\gamma \not\equiv \delta)), \qquad \S \in \{\wedge, \vee, \to, \leftrightarrow, \leftharpoondown, \rightleftharpoons, \equiv, \not\equiv\}$$
$$(\alpha \leftharpoondown \beta) \leftharpoondown (\alpha \not\equiv \beta)$$

Every formula above should be taken either in deductive (with $\emptyset \vdash$) or in reductive (with $\mathrm{L}_{\mathrm{HB}\equiv\not\equiv} \dashv$) position, depending on the part of logic defined: deductive (consequence operation) or reductive (elimination operation), respectively.

Let us consider an extension of $\mathrm{C}_{\mathrm{HB}\equiv\not\equiv}$ as well $\mathrm{E}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}$ by the same set of formulas[1]

$$\{\alpha \equiv \beta : \ \alpha \leftrightarrow \beta \in \mathrm{C}_{\mathrm{HB}\equiv\not\equiv}(\emptyset)\} \cup \{\neg(\alpha \not\equiv \beta) : \ \alpha \leftrightarrow \beta \in \mathrm{C}_{\mathrm{HB}\equiv\not\equiv}(\emptyset)\}$$

[1] Of course, an axiomatic extension of an elimination operation is de facto reduction, because the set of rejected formulas is enlarged.

and an extension of $\mathrm{C}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}$ and $\mathrm{E}_{\mathrm{HB}\equiv\not\equiv}$ also by the same set

$$\{\alpha \not\equiv \beta : \ \alpha \rightleftharpoons \beta \in \mathrm{C}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}(\emptyset)\} \cup \{\sim(\alpha \equiv \beta) : \ \alpha \rightleftharpoons \beta \in \mathrm{C}^{\mathrm{d}}_{\mathrm{HB}\equiv\not\equiv}(\emptyset)\}.$$

These extensions give C- and E-theories being, in the sense of the appropriate already quotated translations, deductive and reductive Heyting-Brouwer modal systems of kind S4.

For the logic of truth a syntax of

$$(\mathcal{L}_{\mathrm{HB}\Box\Diamond}, \mathrm{S4}_{\mathrm{DHB}\Box\Diamond}, \mathrm{S4}_{\mathrm{RHB}\Box\Diamond})$$

consists of the axioms and rules of Heyting-Brouwer logic extended by the following

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$$
$$\Box\alpha \to \alpha$$
$$\Box\alpha \to \Box\Box\alpha$$
$$(\Diamond\alpha \leftharpoondown \Diamond\beta) \to \Diamond(\alpha \leftharpoondown \beta)$$
$$\alpha \to \Diamond\alpha$$
$$\Diamond\Diamond\alpha \to \Diamond\alpha$$
$$\emptyset \vdash \alpha \ / \ \emptyset \vdash \Box\alpha \quad \text{and} \quad \mathrm{L}_{\mathrm{HB}\Box\Diamond} \dashv \alpha \ / \ \mathrm{L}_{\mathrm{HB}\Box\Diamond} \dashv \Diamond\alpha$$

A semantics for this logic is a class of HB-models, i.e. all structures $\langle \mathrm{W}, \leq, \mathrm{R}_1, \mathrm{R}_2, \models\rangle$ with reflexive and transitive relations $\leq$, R_1, R_2 on W, for which

(i^+)	$w \models p$	implies	$\forall v \geq w \ \ v \models p$
$(\neg^+)$	$w \models \neg\alpha$	iff	$\forall v \geq w \ \ v \not\models \alpha$
$(\sim^+)$	$w \models \sim \alpha$	iff	$\exists v \leq w \ \ v \not\models \alpha$
$(\cap^+)$	$w \models \alpha \wedge \beta$	iff	$w \models \alpha$ and $w \models \beta$
$(\cup^+)$	$w \models \alpha \vee \beta$	iff	$w \models \alpha$ or $w \models \beta$
$(\to^+)$	$w \models \alpha \to \beta$	iff	$\forall v \geq w \ \ (v \not\models \alpha$ or $v \models \beta)$
$(\leftharpoondown^+)$	$w \models \alpha \leftharpoondown \beta$	iff	$\exists v \leq w \ \ (v \models \alpha$ and $v \not\models \beta)$
$(\Box^+)$	$w \models \Box\alpha$	iff	$\forall t \ \ (w\mathrm{R}_1 t$ implies $t \models \alpha)$
$(\Diamond^+)$	$w \models \Diamond\alpha$	iff	$\exists t \ \ (w\mathrm{R}_2 t$ and $t \models \alpha)$

for any $\alpha, \beta \in \mathrm{L}_{\mathrm{HB}\Box\Diamond}$, $w \in$W.

The proof of completeness theorem is presented in [2] and is based on the fact that the complement of every prime C_{HB}-theory is a prime $\mathrm{C}^{\mathrm{d}}_{\mathrm{HB}}$-theory, and the complement of every prime $\mathrm{C}^{\mathrm{d}}_{\mathrm{HB}}$-theory is a prime C_{HB}-theory.

It can be easily seen that the possibility does not depend on the necessity nor the necessity depends on the possibility. However, since R_1 and R_2 are reflexive, a formula $\Box\alpha \to \Diamond\alpha$ is a tautology of the system.

In HB logic interpretations of implication and coimplication use the same relation $\leq$ but in opposite directions. It is possible to extend our HB-modal system to such wherein interpretations of necessity and possibility have the same relation R but in opposite directions. Indeed, an easy verification shows that formulas $\alpha \to \Box\Diamond\alpha$ and $\Diamond\Box\alpha \to \alpha$ are tautologies of the system if and only if in every HB-model for any $w, t \in$W, ($w\mathrm{R}_1 t$ implies $t\mathrm{R}_2 w$) and ($w\mathrm{R}_2 t$ implies $t\mathrm{R}_1 w$), respectively. It means that the HB-modal system extended by these two formulas contains modalities interpreted, as the remaining connectives, depending on whether the character is Heyting's or Brouwerian. Either the necessity is interpreted by the future and then the possibility is interpreted by the notion of the past or the possibility is interpreted by the future and the necessity is interpreted by the notion of the past.

Let us assume that both formulas are axioms of the system, and the necessity is interpreted by the future, and the possibility by the past. Then, due to relations between identity and necessity and between non-identity and possibility, one can say that some sentence is necessary not because its negation is not possible, but because today and thus always in the future the sentence is equal to the logical truth. Similarly, some sentence is possible not because its negation is not necessary but because always, i.e. in the past and today, the sentence was different from the logical falsehood.

In the end, let us notice that one can define even on the pure, i.e. without $\alpha \to \Box\Diamond\alpha$ and $\Diamond\Box\alpha \to \alpha$, HB-modal system other "secondary" modalities.

- Future possibilities: strong $\Diamond^{11}_{\Box} = \neg\Box\sim$ and weak $\Diamond^{12}_{\Box} = \neg\Box\neg$, both informing that something is possible in the future.
- Past necessities: strong $\Box^{11}_{\Diamond} = \sim\Diamond\sim$, and weak $\Box^{12}_{\Diamond} = \sim\Diamond\neg$, saying that something was necessary in the past.
- Another past possibilities: strong $\Diamond^{21}_{\Box} = \sim\Box\sim$ and weak $\Diamond^{22}_{\Box} = \sim\Box\neg$.
- Another future necessities: strong $\Box^{21}_{\Diamond} = \neg\Diamond\sim$, and weak $\Box^{22}_{\Diamond} = \neg\Diamond\neg$.

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