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Piotr Łukowski

A DEDUCTIVE-REDUCTIVE FORM OF LOGIC: general theory and intuitionistic case

Abstract. The paper deals with reconstruction of the unique reductive counterpart of the deductive logic. The procedure results in the deductive-reductive form of logic. This extension is illustrated on the base of intuitionistic logics: Heyting's, Brouwerian and Heyting-Brouwer's ones.

1. Introduction

Usually considered logics express a deductive reasoning. No matter what is the way of their formalisation, consequence operation, consequence relation, natural deduction, and others always formalise procedures which increase the sets of accepted formulas.

However, there exists a possibility of reconstruction of the reductive counterpart of the deductive logic. This reductive part should formalise a reasoning, decreasing sets of accepted formulas.

The motivation to extend logic to its deductive-reductive form may be of various types. The most essential resides in the fact that usually, a considered logic is a part of a greater consistent whole. Moreover, this deductive-reductive whole cannot be reduced to its already existing deductive part. Further, such extended logic may constitute a base of a natural formalisation of non-monotonic reasoning. The non-monotonicity would be here obtained as a result of two monotonic procedures used alternatively: step "foreward", increasing the set of beliefs which may be realised thanks to the deductive part of logic; step "backward", decreasing the set of beliefs, made due to the reductive part of the same logic. The final result is similar to the procedures of our everyday thinking.

Let $\mathcal{L}$ be a propositional language with L, a set of all formulas. In the paper the deductive logic is represented by the well-known Tarski consequence operation $C\colon 2^L \to 2^L$, see [4]. Thanks to this operation, for any set X we can find the set of all consequences of sentences from X. In other words, C gives a set of all sentences we should accept because we have already accepted some others. The reductive counterpart of C will be E: $2^L \to 2^L$ – an elimination operation, informing us which sentences we should refuse because we have already accepted only some sentences. More precisely, let us assume that we have accepted sentences composing the set X. At the same time, we have refused all sentences from $L - X$ – the complement of X to the set of all formulas. However, there is possible to use a deduction procedure on this set. This procedure shows which sentences should be accepted as false because we have already accepted falsehood of some sentences. In such a way the set X is reduced. Moreover, let us note that as all formulas from X are the base of a deductive reasoning, the set X cannot be the base for a reductive procedure. Obviously, formulas from X cannot be reasons for removing any of them. The only base for the reductive reasoning for some set X can be the formulas from $L - X$. It coincides with the assumption that the elimination operation applied to the set X is in some sense a complement of the consequence operation employed to $L - X$.

Thus, the logic determined on language $\mathcal{L}$ in its deductive-reductive form is a triple

$$(\mathcal{L}, C, E)$$

where for any $X \subseteq L$

$$E(X) = L - C^d(L - X)$$

with $C^d\colon 2^L \to 2^L$ an operation dual to the finitary and disjunctive consequence operation C. A consequence operation C is disjunctive (see [5]), if for any finite set X and for any endomorphism e of L there exists $\alpha \in L$ such that

$$C(e\alpha) = \bigcap\{C(e\beta) : \beta \in X\}.^1$$

Let us also recall the definition of C^d formulated by Wójcicki in [5]:

$$\alpha \in C^d(X) \quad \text{iff} \quad \bigcap\{C(\beta) : \beta \in X_f\} \subseteq C(\alpha) \quad \text{for some finite} \quad X_f \subseteq X.$$

[1] For simplicity, $C(\delta)$ will be written instead of $C(\{\delta\})$.

Let us see that if at least one tautology of C is not included in X, then this tautology belongs to L − X, and $C^d(L - X) = L$. It means that $E(X) = \emptyset$. Moreover, C^d satisfies Tarski's conditions for finitary, disjunctive consequence operation, and because of this fact operation E satisfies for any $X, Y \subseteq L$ four following conditions

(E_1) $E(X) \subseteq X$

(E_2) $X \subseteq Y$ implies $E(X) \subseteq E(Y)$

(E_3) $E(X) \subseteq EE(X)$

(E_4) $E(X) = \bigcap\{E(Y) : X \subseteq Y \text{ and } Y \text{ is a cofinite set}\}$

It is clear that the condition (E_4) has an equivalent form:

(E_{41}) $\alpha \notin E(X)$ iff $\alpha \notin E(L - \{\beta_1, \ldots, \beta_n\})$ for some $\beta_1, \ldots, \beta_n \notin X$

for any $X \subseteq L$, $\alpha \in L$.

Function $E\colon 2^L \to 2^L$ satisfying (E_1)–(E_3) will be called an *elimination operation* on the language $\mathcal{L}$. An elimination operation E satisfying the condition (E_4) is called *cofinitary*. Moreover, if $E\colon 2^L \to 2^L$ is an elimination operation on $\mathcal{L}$, then function $C'\colon 2^L \to 2^L$ given by

$$C'(X) = L - E(L - X)$$

for any $X \subseteq L$; is a consequence operation on $\mathcal{L}$.

An elimination operation E is *structural*, if for any endomorphism e of the language L and for any $X \subseteq L$

(E_5) $e(L - E(X)) \subseteq L - E(L - e(L - X))$

Complexity of the condition (E_5) follows from the fact that the operation E reduces the set X. Applying some endomorphism cannot expand the set X, but must correspond to the contraction of formulas. In other words, a structural elimination operation removes some formulas with all their substitutions. For formal explanation, let us use the fact mentioned just before the condition (E_5). Since $L - E(L - X) = C'(X)$, $eC'(L - X) \subseteq C'(e(L - X))$ is an equivalent form of (E_5). Let $Y = L - X$, then $eC'(Y) \subseteq C'(eY)$. Of course, the last inclusion expresses structurality of the consequence operation C′.

The defined elimination operation is of general character and does not decide about the way it can be applied in the formalisation of the reductive reasoning. For instance we can provide arguments for replacing the definition:

$E(X) = L - C^d(L-X)$ by $E(X) = L - C^d(L-C(X))$. It is justified since, as it turns out, if even one tautology is missing in the set X, $E(X) = \emptyset$. For this reason it is better to consider E(C(X)), than E(X). But, on the other hand, this is not a sufficient precaution: as the case of intuitionism reveals, E(C(X)) would be an empty set if C(X) does not contain some formula and its negation. The situation is paralel to that of consequence operation. Here it is also better to consider C(E(X)) than C(X) because if only one countertautology occurs in X, then $C(X) = L$. While there are no countertautologies in E(X). Obviously C(E(X)) does not guarantee the full success, either, since it suffices to consider the case of intuitionism to notice that the occurrence of two formulas in E(X) of which one is the negation of the other results in $C(E(X)) = L$.

The operation C is applied to any set X (not only to E(X), i.e. X without counter-tautologies of C). Likewise operation E should be defined in such a way that it could be applied to any set X (not only to C(X)).

The above considerations confirm the general character of the elimination operation and its intended full duality to the consequence operation. Evidently, if we aim at the formalisation of some reductive reasoning we may apply this generally defined elimination operation to given sets either substituting X with its different extentions in the definition, or replacing the set $L - X$ by some disjoint with X set in the same definition. However, the general theory of elimination operation cannot provide ready-made solutions of such problems.

Similarly to C^d, we can define E^d, an operation dual to the given cofinitary and conjunctive (also see [5]) elimination operation E on $\mathcal{L}$, as follows:

$$\alpha \notin E^d(X) \text{ iff } E(L-\alpha) \subseteq \bigcup\{E(L-\beta) : \beta \notin X_{cf} \text{ for some cofinite } X_{cf} \supseteq X\}$$

where $X \subseteq L$ is any set of formulas. Obviously, E^d is a cofinitary elimination operation on $\mathcal{L}$. Moreover, for any $X \subseteq L$

$$C^{dd} = C \qquad \text{and} \qquad E^{dd} = E.$$

The diagram from Figure 1 can be obtained: with lines symbolizing mutual definabilities. Obviously, the connection between C and E^d is the following:

$$C(X) = L - E^d(L - X)$$

for any $X \subseteq L$. Not difficult examination shows that starting from the finitary consequence operation C and using the definitions given above, we can first

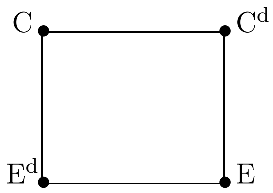

Figure 1.

build C^d, then E, E^d, and finally C' which is equivalent to C. Similarly, starting from E and passing through all operations E^d, C, and C^d we return to E. It means that for a given C (respectively, E), the reconstruction of E (respectively, C) is unique.

It is a well-known fact that every consequence operation is mutually definable by some class H of valuations h: $L \to \{0, 1\}$. In our case, i.e. when C is finitary, disjunctive and structural,

$$\alpha \in C(X) \quad \text{iff} \quad \forall h \in H(h(X) \subseteq \{1\} \text{ implies } h(\alpha) = 1);$$
$$\alpha \in C^d(X) \quad \text{iff} \quad \forall h \in H(h(X) \subseteq \{0\} \text{ implies } h(\alpha) = 0).$$

From the connections above it directly follows that similar mutual definability is characteristic for the just defined elimination operations and class H of valuations h: $L \to \{0, 1\}$ defined with respect to value v as,

$$\alpha \in E(X) \quad \text{iff} \quad \exists h \in H(h(L - X) \subseteq \{0\} \text{ and } h(\alpha) = 1);$$
$$\alpha \in E^d(X) \quad \text{iff} \quad \exists h \in H(h(L - X) \subseteq \{1\} \text{ and } h(\alpha) = 0).$$

For the matrix characterization, let Matr(C) be a class of matrices $\mathcal{M} = (\mathcal{A}, D)$ for C, with $\mathcal{A}$ an algebra similar to $\mathcal{L}$ and $D \subseteq L$. Obviously, if for $X \subseteq L$, $E(X) = L - C(L - X)$, then

$$E(L) = \bigcup\{L - TR_{\mathcal{M}} : \mathcal{M} \in Matr(C)\};$$
$$E^d(L) = \bigcup\{L - TR_{\mathcal{M}} : \mathcal{M} \in Matr(C^d)\},$$

where $TR_{\mathcal{M}}$ is the set of formulas satisfied by matrix $\mathcal{M}$ for any homomorphism $h \in Hom(\mathcal{L}, \mathcal{A})$. It turns out, however, that the duality relation between C and E operations occurs in a general case, as well. Matrix $\mathcal{M} = (\mathcal{A}, D)$ together with $Hom(\mathcal{L}, \mathcal{A})$ defines the matrix consequence operation:

$$\alpha \in C_{\mathcal{M}}(X) \quad \text{iff} \quad \forall h \in Hom(\mathcal{L}, \mathcal{A})\big(h(X) \subseteq D \text{ implies } h(\alpha) \in D\big).$$

Then,

$$C(X) = \bigcap\{C_{\mathcal{M}}(X) : \mathcal{M} \in Matr(C)\}.$$

Also, a matrix elimination operation can be defined:

$$\alpha \in \mathrm{E}_{\mathcal{M}}(\mathrm{X}) \quad \text{iff} \quad \exists \mathrm{h} \in \mathrm{Hom}(\mathcal{L}, \mathcal{A})\big(\mathrm{h}(\mathrm{L}-\mathrm{X}) \subseteq \mathrm{D} \text{ and } \mathrm{h}(\alpha) \notin \mathrm{D}\big).$$

Then, similarly:

$$\mathrm{E}(\mathrm{X}) = \bigcup\{\mathrm{E}_{\mathcal{M}}(\mathrm{X}) : \mathcal{M} \in \mathrm{Matr}(\mathrm{C})\}.$$

After remarks above, one can say that there are reconstructed two logics in deductive-reductive form: the deductive-reductive logic of truth

$$(\mathcal{L}, \mathrm{E}, \mathrm{C})$$

and the deductive-reductive logic of falsehood

$$(\mathcal{L}, \mathrm{E}^{\mathrm{d}}, \mathrm{C}^{\mathrm{d}})$$

composing the one whole. Indeed, $\mathrm{C}(\emptyset)$, $\mathrm{E}(\mathrm{L})$, $\mathrm{C}^{\mathrm{d}}(\emptyset)$ and $\mathrm{E}^{\mathrm{d}}(\mathrm{L})$ are respectively: the set of tautologies, the set of non-countertautologies, the set of countertautologies and the set of non-tautologies of the one and same logic.

For any $\mathrm{X} \subseteq \mathrm{L}$ and $\alpha \in \mathrm{L}$ the expressions $\alpha \in \mathrm{C}(\mathrm{X})$, $\alpha \in \mathrm{C}^{\mathrm{d}}(\mathrm{X})$, $\alpha \notin \mathrm{E}(\mathrm{X})$ and $\alpha \notin \mathrm{E}^{\mathrm{d}}(\mathrm{X})$ settle some relations between the formula α and the set X. These relations can be modelled by two relations of the inference $\vdash$, $\vdash^{\mathrm{d}}$ and two relations of rejection $\dashv$, $\dashv^{\mathrm{d}}$, respectively. Then we can write $\mathrm{X} \vdash \alpha$, $\mathrm{X} \vdash^{\mathrm{d}} \alpha$ and $\mathrm{X} \dashv \alpha$, $\mathrm{X} \dashv^{\mathrm{d}} \alpha$, respectively. In the two first cases X is a set of premises, α is a conclusion. In the next two cases X is a set of premises but α is a rejected formula. For $\vdash$ and $\vdash^{\mathrm{d}}$ the set of all consequences of the set X depends only on X. As it was already mentioned, such a situation may not have place in the case of elimination relations. Elimination of some formulas from the set X needs not to depend on the formulas from X. Therefore, the expression $\alpha \notin \mathrm{E}(\mathrm{X})$ will have its counterpart in the relation of the rejection $\mathrm{X} \dashv \alpha$ understood as $\mathrm{L} - \mathrm{X} \vdash^{\mathrm{d}} \alpha$. Similarly, $\alpha \notin \mathrm{E}^{\mathrm{d}}(\mathrm{X})$ has its counterpart in $\mathrm{X} \dashv^{\mathrm{d}} \alpha$, understood as $\mathrm{L} - \mathrm{X} \vdash \alpha$. Thus, for any $\mathrm{X} \subseteq \mathrm{L}$ and $\alpha \in \mathrm{L}$,

$$\mathrm{X} \dashv \alpha \quad \text{iff} \quad \mathrm{L} - \mathrm{X} \vdash^{\mathrm{d}} \alpha$$

and

$$\mathrm{X} \dashv^{\mathrm{d}} \alpha \quad \text{iff} \quad \mathrm{L} - \mathrm{X} \vdash \alpha.$$

2. Some basic notions of elimination operation

Let E be an elimination operation defined over $\mathcal{L}$. We say after [6] that if an elimination operation is cofinitary and structural, then it is called *standard.* Similarly to [6] let us introduce some additional notions.

DEFINITION 1. Let E be an elimination operation over $\mathcal{L}$. For $\mathrm{X}, \mathrm{Y} \subseteq \mathrm{L}$,

1. X and Y are equivalent under E iff $\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{Y})$;
2. X is axiomatizable under E iff X is equivalent under E to $\mathrm{L} - \mathrm{Y}$, where Y is a finite set;
3. X is sufficient under E iff $\mathrm{E}(\mathrm{X}) \neq \emptyset$;
4. X is an E-theory, if $\mathrm{X} = \mathrm{E}(\mathrm{X})$. $\mathrm{Th_E}$ denotes the set of all E-theories.

Thus, the empty set $\emptyset = \mathrm{E}(\emptyset)$ is the smallest E-theory called trivial or *insufficient*. The biggest E-theory is E(L).[2]

PROPOSITION 1. *For any elimination operation* E, $\mathrm{Th_E}$ *is an open system and* $(\mathrm{Th_E}, \subseteq)$ *is a complete lattice.*

PROOF. Obviously $\bigcup \mathbf{X} \in \mathrm{Th_E}$ for any $\mathbf{X} \subseteq \mathrm{Th_E}$ thus, the first part of the theorem is proved. For the proof of the second part it is sufficient to notice that $\sup\mathbf{X} = \bigcup \mathbf{X}$ and $\inf\mathbf{X} = \bigcup\{\mathrm{Y} \in \mathrm{Th_E} : \mathrm{Y} \subseteq \bigcap \mathbf{X}\}$, for any $\mathbf{X} \subseteq \mathrm{Th_E}$. □

DEFINITION 2. A subset $\mathbf{X} \subseteq 2^{\mathrm{L}}$ is an open base for an elimination operation E, if for any $\mathrm{X} \subseteq \mathrm{L}$, $\mathrm{E}(\mathrm{X}) = \bigcup\{\mathrm{Y} \in \mathbf{X} : \mathrm{Y} \subseteq \mathrm{X}\}$.

Obviously, $\mathrm{Th_E}$ is an open base for an elimination operation E. As in the case of a consequence operation, for any two elimination operations E_1 and E_2,

$$\mathrm{E}_1 = \mathrm{E}_2 \quad \text{iff} \quad \mathrm{Th}_{\mathrm{E}_1} = \mathrm{Th}_{\mathrm{E}_2}.$$

DEFINITION 3. Let $\mathbf{X} \subseteq 2^{\mathrm{L}}$ and $\mathrm{X} \subseteq \mathrm{L}$. $\mathrm{E}_{\mathbf{x}}(\mathrm{X}) = \bigcup\{\mathrm{Y} \in \mathbf{X} : \mathrm{Y} \subseteq \mathrm{X}\}$.

PROPOSITION 2. *For any* $\mathbf{X} \subseteq 2^{\mathrm{L}}$ *and* $\mathrm{X} \subseteq \mathrm{L}$ *the following properties hold:*

(a) $\mathrm{E}_{\mathbf{x}}$ *is an elimination operation on* $\mathcal{L}$ *(called "determined by* $\mathbf{X}$*").*
(b) $\mathrm{Th}_{\mathrm{E}_{\mathbf{x}}}$ *is the least open system including* $\mathbf{X}$.
(c) *If* $\mathbf{X}$ *is an open system, then* $\mathrm{Th}_{\mathrm{E}_{\mathbf{x}}} = \mathbf{X}$.
(d) $\mathbf{X}$ *is an open base for* $\mathrm{E}_{\mathbf{x}}$.

PROOF. (a) Very easy.

(b) By Proposition 2, $\mathrm{Th}_{\mathrm{E}_{\mathbf{x}}}$ is an open system. Moreover, $\mathrm{E}_{\mathbf{x}}(\mathrm{X}) = \mathrm{X}$ for any $\mathrm{X} \in \mathbf{X}$. Thus, we have to prove that every open system including $\mathbf{X}$

[2] If C is a consequence operation, then every set closed under C will be called a C-*theory*. $\mathrm{Th_C}$ denotes the set of all C-theories and $\mathrm{Th}^*_{\mathrm{C}}$, the set of all relatively maximal C-theories.

contains $\mathrm{Th}_{\mathrm{E}_{\mathbf{x}}}$. Let $\mathbf{Y}$ be an open system such that $\mathbf{X} \subseteq \mathbf{Y}$. If $\mathrm{X} \in \mathrm{Th}_{\mathrm{E}_{\mathbf{x}}}$, then $\mathrm{X} = \bigcup\{\mathrm{Y}{\in}\mathbf{X} : \mathrm{Y} \subseteq \mathrm{X}\}$. Because $\mathbf{Y}$ is an open system, $\bigcup \mathbf{Z} \in \mathbf{Y}$ for any $\mathbf{Z} \subseteq \mathbf{X}$. Thus, $\mathrm{X} \in \mathbf{Y}$.

(c) One inclusion follows from *b*. If $\mathrm{X} \in \mathbf{X}$, then $\mathrm{E}(\mathrm{X}) = \bigcup\{\mathrm{Y}{\in}\mathbf{X} : \mathrm{Y} \subseteq \mathrm{X}\} = \mathrm{X}$.

(d) Directly from definitions 2 and 3. □

Minimal elements of the set $\mathrm{Th}_{\mathrm{E}} - \{\emptyset\}$ will be called *minimal* E-*theories*. Thus, T is a minimal E-theory, if it is sufficient and there is no E-theory properly contained in T.

DEFINITION 4. Let E be an elimination operation on $\mathcal{L}$, $\alpha \in \mathrm{L}$ and $\mathrm{X} \subseteq \mathrm{L}$. X is an E-theory minimal relatively to α, if two following conditions are satisfied:

(a) $\alpha \in \mathrm{X}$;

(b) $\alpha \notin \mathrm{E}(\mathrm{X} - \beta)$ for any $\beta \in \mathrm{X}$.

An E-theory minimal relatively to some formula is called relatively minimal.

PROPOSITION 3. *Let* E *be an elimination operation over* $\mathcal{L}$ *and* $\mathrm{X} \subseteq \mathrm{L}$. X *is a minimal* E-*theory iff* X *is an* E-*theory which is minimal relatively to each of its formulas from* X.

PROOF. It is sufficient to notice that the right side of the theorem says that removing any formula from a theory which is minimal relatively to each of its formulas, leads to the empty set. □

DEFINITION 5. Let E be an elimination operation on $\mathcal{L}$ and $\mathrm{X} \subseteq \mathrm{L}$. X is a join irreducible E-theory, if for any $\mathbf{X} \subseteq \mathrm{Th}_{\mathrm{E}}$, $\mathrm{X} = \bigcup \mathbf{X}$ implies $\mathrm{X} \in \mathbf{X}$.

PROPOSITION 4. *Let* E *be an elimination operation on* $\mathcal{L}$ *and* $\mathrm{X} \subseteq \mathrm{L}$. X *is a relatively minimal* E-*theory iff* X *is a join irreducible* E-*theory.*

PROOF. (⇒) Let an E-theory X be minimal relatively to α. Then for any $\mathbf{X} \subseteq \mathrm{Th}_{\mathrm{E}}$, if $\bigcup \mathbf{X} \subseteq \mathrm{X}$ and $\mathrm{X} \neq \mathrm{Y}$ for any $\mathrm{Y} \in \mathbf{X}$, then $\alpha \notin \mathrm{Y}$ for any $\mathrm{Y} \in \mathbf{X}$. Thus X is join irreducible.

(⇐) Now assume that an E-theory X is not relatively minimal. Thus, for any $\alpha \in \mathrm{X}$ there exists $\beta_\alpha \in \mathrm{X}$ such that $\alpha \in \mathrm{E}(\mathrm{X} - \beta_\alpha)$. Obviously, $\mathrm{E}(\mathrm{X} - \beta_\alpha) \neq \mathrm{X}$ for any $\alpha \in \mathrm{X}$. Simultaneously, $\mathrm{X} = \bigcup\{\mathrm{E}(\mathrm{X} - \beta_\alpha) : \alpha \in \mathrm{X}\}$, thus X is not a join irreducible E-theory. □

DUAL-TO-LINDENBAUM LEMMA. *Let* E *be a cofinitary elimination operation on* $\mathcal{L}$. *For any sufficient* E-*theory* T *and for any* $\alpha \in \mathrm{T}$, *there exists an* E-*theory* T_0 *minimal relatively to* α *such that* $\mathrm{T}_0 \subseteq \mathrm{T}$.

PROOF. Assume that the assumptions of this lemma are satisfied for some formula α and for some E-theory T. Thus, $\alpha \in \mathrm{T}$. Let us express the set of all formulas L as a sequence: $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$. Now let us define the sequence of E-theories T_n as follows: if $\alpha \notin \mathrm{E}(\mathrm{T}_{n-1} - \alpha_n)$, then $\mathrm{T}_n = \mathrm{T}_{n-1}$; in other case $\mathrm{T}_n = \mathrm{E}(\mathrm{T}_{n-1} - \alpha_n)$. Notice that $\mathrm{E}(\bigcap \mathrm{T}_n) = \bigcap \mathrm{T}_n$. Indeed, assume that $\beta \notin \mathrm{E}(\bigcap \mathrm{T}_n)$, then because E is cofinitary and all E-theories T_n form a chain, there exists k such that $\beta \notin \mathrm{E}(\mathrm{T}_k) = \mathrm{T}_k \supseteq \bigcap \mathrm{T}_n$. Thus, $\bigcap \mathrm{T}_n$ is an E-theory minimal relatively to α such that $\bigcap \mathrm{T}_n \subseteq \mathrm{T}$. □

Let us notice that every open base for an elimination operation E must contain $\mathrm{Th}^*_{\mathrm{E}}$ − the set of all relatively minimal E-theories. Thus, if some elimination operation has as the least open base $\mathbf{X}$, then $\mathbf{X} = \mathrm{Th}^*_{\mathrm{E}}$. By Dual-to-Lindenbaum Lemma we obtain, moreover, the following property:

PROPOSITION 5. *The set of all relatively minimal* E*-theories is the least open base for* E.

PROOF. This theorem directly follows from the Dual-to-Lindenbaum Lemma and from the proposition 4. □

Now, let us consider the following property satisfied by certain elimination operations:

(E_6) There exists $\alpha \in \mathrm{L}$ such that $\mathrm{E}(\mathrm{L} - \alpha) = \emptyset$.

PROPOSITION 6. *If a cofinitary elimination operation* E *satisfies the condition* (E_6)*, then for every sufficient* E*-theory* T *there exists a minimal* E*-theory contained in* T.

PROOF. Let an elimination operation E satisfy all assumptions of this theorem and let $\mathrm{E}(\mathrm{X}) \neq \emptyset$ for some $\mathrm{X} \subseteq \mathrm{L}$. Moreover, let $\alpha \in \mathrm{L}$ be such formula that $\mathrm{E}(\mathrm{L} - \alpha) = \emptyset$. Of course, $\alpha \in \mathrm{X}$. Thus, by the Dual-to-Lindenbaum Lemma, there exists an E-theory T minimal relatively to α. So, $\alpha \notin \mathrm{E}(\mathrm{T} - \beta)$ for any $\beta \in \mathrm{T}$. Because $\mathrm{T} \neq \emptyset$ and $\mathrm{E}(\mathrm{T} - \beta) = \emptyset$ for any $\beta \in \mathrm{T}$, T is a minimal E-theory. □

3. Provability in the deductive-reductive form of logic

Natural differences between the deductive and reductive parts of logic should be appropriately expressed by their axiomatization. Indeed, we should be able to say what a formula has to be accepted even if nothing has been accepted so far. Similarly, we need to be able to say what a formula has to be rejected even if nothing has been rejected until now. Thus, an axiom (a rule with empty set of premises) of a deductive logic (D-axiom) will be of the following form:

$$\emptyset \vdash \alpha$$

and an axiom of a reductive logic (R-axiom) will take the following form:

$$\mathrm{L} \dashv \alpha.$$

Thus, an R-axiom says that a formula α has to be rejected even if no other formula has been rejected so far. The case of rules of inference is similar. A deduction rule is of the form $\emptyset + \{\alpha_1, \ldots, \alpha_k\} \vdash \beta$, in short

$$\{\alpha_1, \ldots, \alpha_k\} \vdash \beta,$$

the reductive rules should be of shape

$$\mathrm{L} - \{\alpha_1, \ldots, \alpha_k\} \dashv \beta.$$

Thus, as expected, we will have a separate set of axioms and a separate set of rules for each kind of reasoning. The fact that some formula α is a D-axiom (thesis) will be written: $\vdash \alpha$ which is an abbreviation of $\emptyset \vdash \alpha$. The rule stating that a formula α follows from a set of formulas Γ has usual form, viz., $\Gamma \vdash \alpha$. In the case of an elimination reasoning, a formula α is an R-axiom (antithesis) if $\mathrm{L} \dashv \alpha$, i.e., α is removed from the set of all formulas or in other words, α cannot be justified even by the set of all formulas. Similarly, an analogous rule for elimination will state that a formula α is not justified by a set of formulas Γ, $\Gamma \dashv \alpha$, i.e., α should be removed from Γ. More precisely, elimination rules will be of the form: $\Gamma - \Delta \dashv \alpha$, which means that removing all formulas of the set Δ from the set Γ removes α from Γ.

In the consequence theory it is the notion of "proof" which plays the key role. The elimination operation is closely connected with a dual notion of "disproof".

Let $\mathrm{A}_\dashv$ *be an axiom set for elimination and* $\mathbf{R}_\dashv$ *be a set of rules of elimination. A formula* α *is called disprovable from* X *by means of rules from* $\mathbf{R}_\dashv$*, if and only if there exists in* $\mathrm{L} - \mathrm{X}$ *a finite sequence of formulas* $\alpha_1, \ldots, \alpha_k$*, called a disproof of* α *from* X *by means of* $\mathbf{R}_\dashv$*, such that*

- $\alpha = \alpha_k$ *and*
- *for any* $i \in \{1, \ldots, k\}$, $\alpha_i \in \mathrm{A}_\dashv \cup (\mathrm{L} - \mathrm{X})$ *or for some* $\mathrm{Y} \subseteq \{\alpha_1, \ldots, \alpha_{k-1}\}$, $\mathrm{L} - \mathrm{Y} \dashv \alpha_i$ *is an instance of some rule from* $\mathbf{R}_\dashv$.

A formula α *is called confirmed for* X *by means of* $\mathbf{R}_\dashv$ *if there exists no disproof of* α *from* X *by means of* $\mathbf{R}_\dashv$ *in* $\mathrm{L} - \mathrm{X}$.

4. The deductive-reductive form of intuitionistic logic over a language with implication

Let $\mathcal{L}_{\mathrm{H}}$ denote the standard language

$$\mathcal{L}_{\mathrm{H}} = (\mathrm{L}_{\mathrm{H}}, \neg, \wedge, \vee, \rightarrow)$$

for the intuitionistic propositional logic. Consider the intuitionistic logic ($\mathcal{L}$,C_{H}) given by the following D-axioms (schemata $1_{\mathrm{C}_{\mathrm{H}}}$–$10_{\mathrm{C}_{\mathrm{H}}}$) and D-rule $\mathrm{MP}_{\mathrm{C}_{\mathrm{H}}}$:

$1_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash \alpha \rightarrow (\beta \rightarrow \alpha)$
$2_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
$3_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash (\alpha \wedge \beta) \rightarrow \alpha$
$4_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash (\alpha \wedge \beta) \rightarrow \beta$
$5_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$
$6_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash \alpha \rightarrow (\alpha \vee \beta)$
$7_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash \beta \rightarrow (\alpha \vee \beta)$
$8_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash (\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$
$9_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash (\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \neg\alpha)$
$10_{\mathrm{C}_{\mathrm{H}}}$	$\emptyset \vdash \neg(\alpha \rightarrow \alpha) \rightarrow \beta$
$\mathrm{MP}_{\mathrm{C}_{\mathrm{H}}}$	$\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$

Let us take an algebra $\mathcal{A} = (\mathrm{A}, \neg, \cap, \cup, \rightarrow)$ similar to $\mathcal{L}_{\mathrm{H}}$ and a non-empty set S partially ordered by $\leq$. Let every $s \in \mathrm{S}$ be associated with a subset $\mathrm{D}_s \subseteq \mathrm{A}$. It is a well-known fact that the semantics adequate for ($\mathcal{L}$,C_{H}) is the class of generalized matrices $\mathcal{M} = (\mathcal{A}, \{\mathrm{D}_s : s{\in}\mathrm{S}\})$ which are C_{H}-models (or equivalently, E_{H}-models), i.e., where for any $a, b \in \mathrm{A}$ and any $s \in \mathrm{S}$:

(i^+)	$a \in \mathrm{D}_s$	implies	for any $t \geq s$, $a \in \mathrm{D}_t$;
$(\neg^+)$	$\neg a \in \mathrm{D}_s$	iff	for any $t \geq s$, $a \notin \mathrm{D}_t$;
$(\cap^+)$	$a \cap b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ and $b \in \mathrm{D}_s$;
$(\cup^+)$	$a \cup b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ or $b \in \mathrm{D}_s$;
$(\rightarrow^+)$	$a \rightarrow b \in \mathrm{D}_s$	iff	for any $t \geq s$, $a \notin \mathrm{D}_t$ or $b \in \mathrm{D}_t$.

Thus, for the matrix consequence operation $\mathrm{C}_{\mathcal{M}}$, where

$$(\mathrm{C}_{\mathcal{M}}) \quad \alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X}) \quad \text{iff} \quad \forall \mathrm{h}{\in}\mathrm{Hom}(\mathcal{L}_{\mathrm{H}}, \mathcal{A}) \ \forall s{\in}\mathrm{S} \ (\forall\beta{\in}\mathrm{X} \ \ \mathrm{h}(\beta) \in \mathrm{D}_s \text{ implies } \mathrm{h}(\alpha) \in \mathrm{D}_s),$$

it holds the well-known completeness theorem, viz.,

$$\alpha \in \mathrm{C}_{\mathrm{H}}(\mathrm{X}) \quad \text{iff} \quad \alpha \in \mathrm{C}_{\mathcal{M}}(\mathrm{X}), \text{ for any } \mathrm{C}_{\mathrm{H}}\text{-model } \mathcal{M}, \ \mathrm{X} \subseteq \mathrm{L}_{\mathrm{H}}, \text{ and } \alpha \in \mathrm{L}_{\mathrm{H}}.$$

Following the ideas presented in the Introduction, we now consider the logic $(\mathcal{L}_{\mathrm{H}}, \mathrm{C}^{\mathrm{d}}_{\mathrm{H}})$ defined by Wójcicki. Directly from Wójcicki's definition of C^{d} it follows that the semantics adequate for $(\mathcal{L}_{\mathrm{H}}, \mathrm{C}^{\mathrm{d}}_{\mathrm{H}})$ is the class of $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-models (also called $\mathrm{E}^{\mathrm{d}}_{\mathrm{H}}$-models), i.e. all such matrices $\mathcal{M}$ that for any $a, b \in \mathrm{A}$, $s \in \mathrm{S}$:

(i^-)	$a \in \mathrm{D}_s$	implies	for any $t \geq s$, $a \in \mathrm{D}_t$;
$(\neg^-)$	$\neg a \in \mathrm{D}_s$	iff	for some $t \leq s$, $a \notin \mathrm{D}_t$;
$(\cap^-)$	$a \cap b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ or $b \in \mathrm{D}_s$;
$(\cup^-)$	$a \cup b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ and $b \in \mathrm{D}_s$;
$(\rightarrow^-)$	$a \rightarrow b \in \mathrm{D}_s$	iff	for some $t \leq s$, $a \notin \mathrm{D}_t$ and $b \in \mathrm{D}_t$;

For both kinds of models one can define the following four matrix operations:

$\alpha \in \mathrm{C}^+_{\mathcal{M}}(\mathrm{X})$ iff $\forall \mathrm{h} \in \mathrm{Hom}(\mathcal{L}_{\mathrm{H}}, \mathcal{A})\ \forall s \in \mathrm{S}$
$(\forall \beta \in \mathrm{X}\ \ \mathrm{h}(\beta) \in \mathrm{D}_s$ implies $\mathrm{h}(\alpha) \in \mathrm{D}_s)$.
$\alpha \in \mathrm{C}^-_{\mathcal{M}}(\mathrm{X})$ iff $\forall \mathrm{h} \in \mathrm{Hom}(\mathcal{L}_{\mathrm{H}}, \mathcal{A})\ \forall s \in \mathrm{S}$
$(\forall \beta \in \mathrm{X}\ \ \mathrm{h}(\beta) \notin \mathrm{D}_s$ implies $\mathrm{h}(\alpha) \notin \mathrm{D}_s)$.
$\alpha \notin \mathrm{E}^+_{\mathcal{M}}(\mathrm{X})$ iff $\forall \mathrm{h} \in \mathrm{Hom}(\mathcal{L}_{\mathrm{H}}, \mathcal{A})\ \forall s \in \mathrm{S}$
$(\forall \beta \in \mathrm{L}_{\mathrm{H}} - \mathrm{X}\ \ \mathrm{h}(\beta) \notin \mathrm{D}_s$ implies $\mathrm{h}(\alpha) \notin \mathrm{D}_s)$.
$\alpha \notin \mathrm{E}^-_{\mathcal{M}}(\mathrm{X})$ iff $\forall \mathrm{h} \in \mathrm{Hom}(\mathcal{L}_{\mathrm{H}}, \mathcal{A})\ \forall s \in \mathrm{S}$
$(\forall \beta \in \mathrm{L}_{\mathrm{H}} - \mathrm{X}\ \ \mathrm{h}(\beta) \in \mathrm{D}_s$ implies $\mathrm{h}(\alpha) \in \mathrm{D}_s)$.

Obviously, every $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model may be obtained from a C_{H}-model in which all sets of designated values become sets of nondesignated values, and vice versa − this is the semantical sense of Wójcicki's definition of $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$. Then,

for any C_{H}-model $\mathcal{M}$, $\alpha \in \mathrm{C}^+_{\mathcal{M}}(\mathrm{X})$ iff for any $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model $\mathcal{M}$, $\alpha \in \mathrm{C}^-_{\mathcal{M}}(\mathrm{X})$;
for any C_{H}-model $\mathcal{M}$, $\alpha \in \mathrm{C}^-_{\mathcal{M}}(\mathrm{X})$ iff for any $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model $\mathcal{M}$, $\alpha \in \mathrm{C}^+_{\mathcal{M}}(\mathrm{X})$;
for any C_{H}-model $\mathcal{M}$, $\alpha \notin \mathrm{E}^+_{\mathcal{M}}(\mathrm{X})$ iff for any $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model $\mathcal{M}$, $\alpha \notin \mathrm{E}^-_{\mathcal{M}}(\mathrm{X})$;
for any C_{H}-model $\mathcal{M}$, $\alpha \notin \mathrm{E}^-_{\mathcal{M}}(\mathrm{X})$ iff for any $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model $\mathcal{M}$, $\alpha \notin \mathrm{E}^+_{\mathcal{M}}(\mathrm{X})$.

If a C_{H}-model is understood as a model for the logic of truth, then a $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model should be understood as a model of the logic of falsehood. Let us emphasize that both kinds of models are generated by the same syntax: the axiom schemata $1_{\mathrm{C}_{\mathrm{H}}}$-$10_{\mathrm{C}_{\mathrm{H}}}$ and $\mathrm{MP}_{\mathrm{C}_{\mathrm{H}}}$. Indeed, it is sufficient to observe that when proving the completeness theorem for $(\mathcal{L}_{\mathrm{H}}, \mathrm{C}_{\mathrm{H}})$ one can build simultaneously a canonical C_{H}-model and a canonical $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-model. In the first case, sets of designated values are built of C_{H}-theories, while in the second case of complements of C_{H}-theories. Unfortunately, it is not possible to axiomatize $(\mathcal{L}_{\mathrm{H}}, \mathrm{C}^{\mathrm{d}}_{\mathrm{H}})$, the logic given by the class of all $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$-models in a such simple way as for the case of $(\mathcal{L}_{\mathrm{H}}, \mathrm{C}_{\mathrm{H}})$. The problem is opposite direction of the partial order relation $\leq$ in conditions: (i^-) and $(\neg^-)$, $(\rightarrow^-)$. It seems that there is no such connective, different from implication which could play the role of implication, i.e., the inference in $(\mathcal{L}_H, C^d_H)$ would be possible thanks to this additional connective. Then, the relation $\leq$ would have the same direction in the interpretation condition for the new connective as in (i^-), and so, it would be evident that C^d_H-model is a model for a deductive logic. However, it is not difficult to axiomatize the reductive logic $(\mathcal{L}_H, E^+_{\mathcal{M}})$ given by the class of all C^d_H-models. The syntax consists of the R-rule $MT_{E^d_H}$:

$$L_H - \{\alpha, \alpha \to \beta\} \dashv \beta$$

and axiom schemata 1_{C_H}-10_{C_H} with the expression "$\emptyset \vdash$ " replaced by "$L_H \dashv$ ". Then, 1_{C_H}–10_{C_H} become R-axioms.

For a simple proof of the completeness theorem, saying that

$$\alpha \notin E^d_H(X) \quad \text{iff} \quad \alpha \notin E^+_{\mathcal{M}}(X) \text{ for every } C^d_H\text{-model } \mathcal{M}$$

see [1][3]. In such a way, one logic, defined in syntactical terms as a pair consisting of a language and a consequence operator, determines two semantic objects: the deductive part of the logic of truth and the reductive part of the logic of falsehood. In spite of the lack of standard axiomatizations, let us formulate the next two completeness theorems:

$$\alpha \in C^d_H(X) \quad \text{iff} \quad \alpha \in C^+_{\mathcal{M}}(X) \text{ for every } C^d_H\text{-model } \mathcal{M}.$$
$$\alpha \notin E_H(X) \quad \text{iff} \quad \alpha \notin E^+_{\mathcal{M}}(X) \text{ for every } C_H\text{-model } \mathcal{M}.$$

It turns out that the class of all C_H-models as well as the class of all C^d_H-models define C_H, E_H, C^d_H and E^d_H. However, it is the most natural to define C_H and E_H by C_H-models, while C^d_H, E^d_H by C^d_H-models. Moreover,

$$E_H(X) = L_H - C^d_H(L_H - X) \qquad \text{and} \qquad E^d_H(X) = L_H - C_H(L_H - X)$$

and hence:

$$\begin{array}{ll} C_H(\emptyset) \subseteq E_H(L_H), & C^d_H(\emptyset) \subseteq E^d_H(L_H), \\ C_H(\emptyset) \cap E^d_H(L_H) = \emptyset, & C^d_H(\emptyset) \cap E_H(L_H) = \emptyset, \\ C_H(\emptyset) \cup E^d_H(L_H) = L_H, & C^d_H(\emptyset) \cup E_H(L_H) = L_H. \end{array}$$

Thus, there is given a syntactical characterization of two deductive-reductive forms of logics defined over the Heyting language: $(\mathcal{L}_H, C_H, E_H)$ – the intuitionistic logic of truth and $(\mathcal{L}_H, C^d_H, E^d_H)$ – the intuitionistic logic of falsehood. $C_H(\emptyset)$, $C^d_H(\emptyset)$, $E_H(L_H)$, and $E^d_H(L_H)$ are the set of tautologies, the set of countertautologies, the set of non-countertautologies and the set of non-tautologies of the same intuitionistic logic, respectively.

[3] In [1], for simplicity, the symbol "E^d" is replaced by "E".

However, only the consequence operation C_H and the elimination operation of E^d_H have natural, simple axiomatizations.

The deductive-reductive form of logic enables a better understanding of the character of the logic. As an example let us consider a complete interpretation of the negation connective. C_H- and C^d_H-models establish a distinction between the truth and the non-falsity, and between the falsehood and the non-truth of a sentence. Indeed, semantics of the deductive-reductive intuitionistic logic makes possible to fully determine the value of a sentence. The condition ($\neg^+$) determines when a sentence α is false, namely, when it is never accepted, neither now nor in future. Evidently the same condition says when α is not false; when it is possible that in future the sentence will be accepted. Hence the fact that a sentence is not accepted at present does not imply its falsehood. According to the condition ($\neg^-$), α is true when it was accepted in past, which due to the condition (i^-), means that it is also accepted at present. In other words, α is true if starting since certain point (in past) it has been accepted. It seems that this deeper, i.e. related to time comprising past, understanding of truth is closer to life. Thus we may say that α is not true if it has never happened that α was accepted. Such a simple analysis shows that neither the non-falsity considered here can be identified with the truth nor can the non-truth with the falsehood.

The first and the third condition below follow from the definition of a C^d_H-model while the second and the last condition follow from the definition of a C_H-model:

$$\begin{array}{rll} \alpha \text{ is true in } s, & \text{if} & \exists t \leq s,\ \alpha \text{ is accepted in } t \\ \alpha \text{ is not false in } s, & \text{if} & \exists t \geq s,\ \alpha \text{ is accepted in } t \\ \alpha \text{ is not true in } s, & \text{if} & \forall t \leq s,\ \alpha \text{ is not accepted in } t \\ \alpha \text{ is false in } s, & \text{if} & \forall t \geq s,\ \alpha \text{ is not accepted in } t \end{array}$$

It seems that the perspective of the past in the semantics is not artificial and cannot be defined or replaced by a perspective of the future.

5. The deductive-reductive form of intuitionistic logic over a language with coimplication

This section deals with the intuitionistic logic, in which the implication connective is replaced by coimplication. It appears that as one can easily define a deductive form of the logic of truth by implication, coimplication is a natural connective for the deductive logic of falsehood. Given the Brouwerian language

$$\mathcal{L}_{\mathrm{B}} = (\mathrm{L}_{\mathrm{B}}, \sim, \wedge, \vee, \leftharpoondown),$$

let us consider the consequence operation $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}$ given by the following D-axiom schemata $1_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}} - 10_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$ and D-rule $\mathrm{MP}_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$:

$1_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash (\alpha \leftharpoondown \beta) \leftharpoondown \alpha$
$2_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash ((\gamma \leftharpoondown \alpha) \leftharpoondown (\beta \leftharpoondown \alpha)) \leftharpoondown ((\gamma \leftharpoondown \beta) \leftharpoondown \alpha)$
$3_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash (\alpha \wedge \beta) \leftharpoondown \alpha$
$4_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash (\alpha \wedge \beta) \leftharpoondown \beta$
$5_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash ((\gamma \leftharpoondown (\alpha \wedge \beta)) \leftharpoondown (\gamma \leftharpoondown \beta)) \leftharpoondown (\gamma \leftharpoondown \alpha)$
$6_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash \alpha \leftharpoondown (\alpha \vee \beta)$
$7_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash \beta \leftharpoondown (\alpha \vee \beta)$
$8_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash (((\alpha \vee \beta) \leftharpoondown \gamma) \leftharpoondown (\beta \leftharpoondown \gamma)) \leftharpoondown (\alpha \leftharpoondown \gamma)$
$9_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash ({\sim}\beta \leftharpoondown \alpha) \leftharpoondown ({\sim}\alpha \leftharpoondown \beta)$
$10_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\emptyset \vdash \beta \leftharpoondown {\sim}(\alpha \leftharpoondown \alpha)$
$\mathrm{MP}_{\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}}$	$\{\beta, \alpha \leftharpoondown \beta\} \vdash \alpha$

A semantics for such a defined elimination operation $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}$ is the class of $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}$-models ($\mathrm{E}^{\mathrm{d}}_{\mathrm{B}}$-models), i.e. all structures $\mathcal{M}$ (introduced in the previous section) that for any $a, b \in \mathrm{A}$, $s \in \mathrm{S}$:

(i^-)	$a \in \mathrm{D}_s$	implies	for any $t \geq s$, $a \in \mathrm{D}_t$;
$(\sim^-)$	${\sim} a \in \mathrm{D}_s$	iff	for any $t \geq s$, $a \notin \mathrm{D}_t$;
$(\cap^-)$	$a \cap b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ or $b \in \mathrm{D}_s$;
$(\cup^-)$	$a \cup b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ and $b \in \mathrm{D}_s$;
$(\leftharpoondown^-)$	$a \leftharpoondown b \in \mathrm{D}_s$	iff	for any $t \geq s$, $a \in \mathrm{D}_t$ or $b \notin \mathrm{D}_t$.

C_{B}-model (E_{B}-model), a model dual in Wójcicki's sense to $\mathrm{C}^{\mathrm{d}}_{\mathrm{B}}$-model, is a structure $\mathcal{M}$ such that for any $a, b \in \mathrm{A}$, $s \in \mathrm{S}$:

(i^+)	$a \in \mathrm{D}_s$	*implies*	for any $t \geq s$, $a \in \mathrm{D}_t$;
$(\sim^+)$	${\sim} a \in \mathrm{D}_s$	iff	for some $t \leq s$, $a \notin \mathrm{D}_t$;
$(\cap^+)$	$a \cap b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ and $b \in \mathrm{D}_s$;
$(\cup^+)$	$a \cup b \in \mathrm{D}_s$	iff	$a \in \mathrm{D}_s$ or $b \in \mathrm{D}_s$;
$(\leftharpoondown^+)$	$a \leftharpoondown b \in \mathrm{D}_s$	iff	for some $t \leq s$, $a \in \mathrm{D}_t$ and $b \notin \mathrm{D}_t$.

Obviously, the class of all C_{B}-models defines a deductive logic of truth C_{B}, axiomatization of which creates difficulties analogous to those in the case of axiomatization of $\mathrm{C}^{\mathrm{d}}_{\mathrm{H}}$. Shortly speaking, there is a lack of implication such that in the definition of its interpretation the relation $\leq$ would appear with

the same direction as in (i^+). Moreover, it is evident that the conditions ($\leftharpoondown^+$) and ($\sim^+$) of the C_B-model define nothing else than, coimplication and weak negation, respectively.
As in the previous "implicational" case, also here the class of all C_B-models gives us a semantics adequate for the elimination operation E_B, where E_B is axiomatized by R-rule MT_{E_H}:

$$L_B - \{\beta, \alpha \leftharpoondown \beta\} \dashv \alpha$$

and axiom schemata obtained from $1_{C_B^d}$-$10_{C_B^d}$ by replacement of the expression "$\emptyset \vdash$ " by "$L_B \dashv$ ".

Assume that the matrix consequence and elimination operations are defined as in the previous section. Then,

$\alpha \in C_B^d(X)$ iff $\alpha \in C^+_{\mathcal{M}}(X)$ for every C_B^d-model $\mathcal{M}$;
$\alpha \notin E_B^d(X)$ iff $\alpha \notin E^+_{\mathcal{M}}(X)$ for every C_B^d-model $\mathcal{M}$;
$\alpha \in C_B(X)$ iff $\alpha \in C^+_{\mathcal{M}}(X)$ for every C_B-model $\mathcal{M}$;
$\alpha \notin E_B(X)$ iff $\alpha \notin E^+_{\mathcal{M}}(X)$ for every C_B-model $\mathcal{M}$.

Let us emphasize that simple axiomatizations of ($\mathcal{L}_B$, C_B) and ($\mathcal{L}_B$,E_B^d) are unknown, at least to our knowledge.

Similarly to the Heyting's case, there is obtained a syntactical characterization of two deductive-reductive forms for ($\mathcal{L}_B$,C_B,E_B) − the intuitionistic logic of truth and ($\mathcal{L}_B$,C_B^d,E_B^d) − the intuitionistic logic of falsehood, both defined over the Brouwerian language.

To the contrary to the intuitionistic logic defined over the Heyting's language, the truth as well as the non-truth are described by C_B^d-model using the perspective of the future. C_B-model with the interpretation of $\sim$ related to the past characterizes the falsehood and non-falsity of a sentence. A sentence α is true when it is never refused either now or in the future. Thus, α is not true when it is possible that in the future the sentence will be refused. From ($\sim^+$) it follows that α is false when it was refused in the past which, due to the condition (i^+), means that it is also refused at present. α is not false if it has never happened that α was refused. Also here, neither the non-falsity can be identified with the truth nor the non-truth with the falsehood. Thus,

α is false in s, if there exists $t \leq s$ such that α is refused in t;
α is not true in s, if there exists $t \geq s$ such that α is refused in t;
α is not false in s, if for all $t \leq s$, α is not refused in t;
α is true in s, if for all $t \geq s$, α is not refused in t.

6. The Heyting-Brouwer logic completes the puzzle

The examples of the two logics considered so far show that implication is a natural connective for the logic of deduction since it is very convenient to define two axioms of Hilbert as well as to define other connectives. The connective of coimplication has a similar role in case of reduction. It seems to be natural for the formalisation of logics of reduction. However, in none of these cases the set of connectives is sufficient for the symultaneous reconstruction of two logics in their syntactical forms: the deductive-reductive logic of truth and the deductive-reductive logic of falsehood. Fortunately, both a C_H-model with an appropriate C_B-model as well as a C_H^d-model with a corresponding C_B^d-model seem to be, respectively, two parts of the same whole. Indeed, a combination of C_H- and C_B-models will give us a C_{HB}-model (E_{HB}-model). Similarly, a combination of C_H^d- and C_B^d-models will result in a joint C_{HB}^d-model (E_{HB}^d-model).

Thus, let us take an algebra $\mathcal{A} = (A, \neg, \sim, \cap, \cup, \rightarrow, \leftharpoondown)$ similar to the language

$$\mathcal{L} = (L, \neg, \sim, \wedge, \vee, \rightarrow, \leftharpoondown)$$

and a non-empty set S partially ordered by $\leq$. As previously, let every $s \in S$ be associated with a subset $D_s \subseteq A$. A structure $\mathcal{M} = (\mathcal{A}, \{D_s : s \in S\})$ is a C_{HB}-model if for any $a, b \in A$ and for any $s \in S$,

(i^+)	$a \in D_s$	implies	for any $t \geq s$, $a \in D_t$;
($\neg^+$)	$\neg a \in D_s$	iff	for any $t \geq s$, $a \notin D_t$;
($\sim^+$)	$\sim a \in D_s$	iff	for some $t \leq s$, $a \notin D_t$;
($\cap^+$)	$a \cap b \in D_s$	iff	$a \in D_s$ and $b \in D_s$;
($\cup^+$)	$a \cup b \in D_s$	iff	$a \in D_s$ or $b \in D_s$;
($\rightarrow^+$)	$a \rightarrow b \in D_s$	iff	for any $t \geq s$, $a \notin D_t$ or $b \in D_t$;
($\leftharpoondown^+$)	$a \leftharpoondown b \in D_s$	iff	for some $t \leq s$, $a \in D_t$ and $b \notin D_t$.

A structure $\mathcal{M}$ as above is a C_{HB}^d-model if for any $a, b \in A$ and for any $s \in S$,

(i^-)	$a \in D_s$	implies	for any $t \geq s$, $a \in D_t$;
($\neg^-$)	$\neg a \in D_s$	iff	for some $t \leq s$, $a \notin D_t$;
($\sim^-$)	$\sim a \in D_s$	iff	for any $t \geq s$, $a \notin D_t$;
($\cap^-$)	$a \cap b \in D_s$	iff	$a \in D_s$ or $b \in D_s$;
($\cup^-$)	$a \cup b \in D_s$	iff	$a \in D_s$ and $b \in D_s$;
($\rightarrow^-$)	$a \rightarrow b \in D_s$	iff	for some $t \leq s$, $a \notin D_t$ and $b \in D_t$;
($\leftharpoondown^-$)	$a \leftharpoondown b \in D_s$	iff	for any $t \geq s$, $a \in D_t$ or $b \notin D_t$.

It turns out that there exists a logic defined over $\mathcal{L}$, called the Heyting-Brouwer logic (HB) and elaborated by C.Rauszer in [2], [3][4], for which the class of all C_{HB}-models is an adequate semantics. The axiomatics for C_{HB} is an axiom set for the intuitionistic logic enlarged with the following schemata:

$$\emptyset \vdash \alpha \to (\beta \vee (\alpha \leftharpoondown \beta))$$
$$\emptyset \vdash (\alpha \leftharpoondown \beta) \to \sim(\alpha \to \beta)$$
$$\emptyset \vdash ((\alpha \leftharpoondown \beta) \leftharpoondown \gamma) \to (\alpha \leftharpoondown (\beta \vee \gamma))$$
$$\emptyset \vdash \neg(\alpha \leftharpoondown \beta) \to (\alpha \to \beta)$$
$$\emptyset \vdash (\alpha \to (\beta \leftharpoondown \beta)) \to \neg\alpha$$
$$\emptyset \vdash \neg\alpha \to (\alpha \to (\beta \leftharpoondown \beta))$$
$$\emptyset \vdash ((\beta \to \beta) \leftharpoondown \alpha) \to \ \sim\alpha$$
$$\emptyset \vdash \sim\alpha \to ((\beta \to \beta) \leftharpoondown \alpha)$$

for $\alpha, \beta, \gamma \in$ L, together with the inference rules: Modus Ponens and $\alpha \vdash \neg\sim\alpha$.

Rauszer presents also HB^d – a logic dual to HB, defined by the class of all C^d_{HB}-models, see [3]. The axiomatization of C^d_{HB} is the following:

$1_{C^d_{HB}}$ $\emptyset \vdash ((\beta \leftharpoondown \gamma) \leftharpoondown (\alpha \leftharpoondown \gamma)) \leftharpoondown ((\beta \leftharpoondown \alpha) \leftharpoondown \gamma)$
$2_{C^d_{HB}}$ $\emptyset \vdash ((\gamma \leftharpoondown \alpha) \leftharpoondown (\gamma \leftharpoondown \beta)) \leftharpoondown (\beta \leftharpoondown \alpha)$
$3_{C^d_{HB}}$ $\emptyset \vdash (\alpha \wedge \beta) \leftharpoondown \alpha$
$4_{C^d_{HB}}$ $\emptyset \vdash (\alpha \wedge \beta) \leftharpoondown \beta$
$5_{C^d_{HB}}$ $\emptyset \vdash ((\gamma \leftharpoondown (\alpha \wedge \beta)) \leftharpoondown (\gamma \leftharpoondown \beta)) \leftharpoondown (\gamma \leftharpoondown \alpha)$
$6_{C^d_{HB}}$ $\emptyset \vdash \alpha \leftharpoondown (\alpha \vee \beta)$
$7_{C^d_{HB}}$ $\emptyset \vdash \beta \leftharpoondown (\alpha \vee \beta)$
$8_{C^d_{HB}}$ $\emptyset \vdash (((\alpha \vee \beta) \leftharpoondown \gamma) \leftharpoondown (\beta \leftharpoondown \gamma)) \leftharpoondown (\alpha \leftharpoondown \gamma)$
$9_{C^d_{HB}}$ $\emptyset \vdash ((\gamma \leftharpoondown (\alpha \vee \beta)) \leftharpoondown ((\gamma \leftharpoondown \beta) \leftharpoondown \alpha)$
$10_{C^d_{HB}}$ $\emptyset \vdash ((\gamma \leftharpoondown \beta) \leftharpoondown \alpha) \leftharpoondown (\gamma \leftharpoondown (\alpha \vee \beta))$
$11_{C^d_{HB}}$ $\emptyset \vdash ((\beta \to \alpha) \wedge \beta) \leftharpoondown \alpha$
$12_{C^d_{HB}}$ $\emptyset \vdash (\sim\alpha \leftharpoondown \sim\beta) \leftharpoondown (\beta \leftharpoondown \alpha)$
$13_{C^d_{HB}}$ $\emptyset \vdash ((\beta \wedge \gamma) \to \alpha) \leftharpoondown (\gamma \to (\beta \to \alpha))$
$14_{C^d_{HB}}$ $\emptyset \vdash \neg(\beta \leftharpoondown \alpha) \leftharpoondown (\beta \to \alpha)$
$15_{C^d_{HB}}$ $\emptyset \vdash (\alpha \leftharpoondown \beta) \leftharpoondown \ \sim(\alpha \to \beta)$
$16_{C^d_{HB}}$ $\emptyset \vdash \sim\alpha \leftharpoondown ((\beta \to \beta) \leftharpoondown \alpha)$

[4] In the both cited papers, the coimplication connective is named "difference".

$17_{C^d_{HB}}$ $\emptyset \vdash ((\beta \to \beta) \leftharpoondown \alpha) \leftharpoondown\ \sim \alpha$

$18_{C^d_{HB}}$ $\emptyset \vdash \neg\alpha \leftharpoondown (\alpha \to (\beta \leftharpoondown \beta))$

$19_{C^d_{HB}}$ $\emptyset \vdash (\alpha \to (\beta \leftharpoondown \beta)) \leftharpoondown \neg\alpha$

and inference rules: $\alpha \leftharpoondown \beta, \beta \vdash \alpha$ and $\sim\neg\alpha \vdash \alpha$, for any $\alpha, \beta, \gamma \in \mathrm{L}$.

Thus, the deductive-reductive forms of Heyting-Brouwer logics of truth and falsehood are, respectively, the triples:

$$(\mathcal{L}, \mathrm{C_{HB}}, \mathrm{E_{HB}}) \qquad \text{and} \qquad (\mathcal{L}, \mathrm{C^d_{HB}}, \mathrm{E^d_{HB}})$$

where the axiomatics for $\mathrm{E^d_{HB}}$ consists of two R-rules: $\mathrm{L} - \{\alpha, \alpha \to \beta\} \dashv \beta$ and $\mathrm{L} - \{\neg\sim\alpha\} \dashv \alpha$, and all axioms for $\mathrm{C_{HB}}$ with "$\emptyset \vdash$" replaced by "$\mathrm{L}\dashv$". The axiomatics for $\mathrm{E_{HB}}$ consists of two R-rules: $\mathrm{L} - \{\beta, \alpha \leftharpoondown \beta\} \dashv \alpha$ and $\mathrm{L} - \{\sim\neg\alpha\} \dashv \alpha$, and all axioms for $\mathrm{C^d_{HB}}$ with "$\emptyset \vdash$ " replaced by "$\mathrm{L} \dashv$ ".

7. Summary

The intuitionistic logic is a specially useful basis for the analysis of the reductive counterpart for the deductive logic and for the logic of falsehood. To the contrary to the classical logic, it is necessary to consider a connective of the coimplication for the reconstruction of the intuitionistic logic of falsehood. Semantical investigations show that Heyting's intuitionistic logic and Brouwerian intuitionistic logic are, respectively, some parts of the logic of truth and of the logic of falsehood of the one and the same logic. It means that the Heyting-Brouwer logic combines both logics in the one whole. The connective of implication and the connective of coimplication play the fundamental role for the inference in the logic of truth and in the logic of falsehood, respectively.

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PIOTR ŁUKOWSKI
University of Łódź
Department of Logic
Kopcińskiego 16/18
90-232 Łódź, Poland
lukowski@krysia.uni.lodz.pl