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DERIVABILITY OF RULES FROM RULE COMPLEXES*

Abstract. In the paper we focus upon the problem of derivability of rules from rule complexes. The notion of a rule complex is the main mathematical notion of *generalized game theory* (GGT for short). Derivability of rules, as defined here, comprises the concept of an extension from a default theory [9] as well as the classical notion of derivability of rules in logic. The idea of localness of reasoning, reasoning with a limited access to rules, is realized by the concept of relative derivability. Starting with derivability of rules, we next touch upon the questions of the activation of rules and (in)consistency of rule complexes.

Keywords: rule complex, derivability of rules, activation of rules, (in)consistency of a rule complex, generalized game theory.

1. Introduction

Nowadays one can observe a growing interest of social and political scientists, and even economists to study various kinds of social interactions that

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essentially depart from the classical economic game paradigm. Some important aspects of the mentioned interactions are: (1) participating actors (or in other words, players, agents) may not be ordinary rule followers but can innovate, modify, fabricate or even refuse to follow rules of the game; (2) actors' social roles and, in particular, values and norms are important factors having impact on the behaviour of actors in interaction situations; (3) information may be not only incomplete but also vague, and this fuzziness is an additional source of uncertainty.

Tom R. Burns'[1] idea to build a theory of such social game-like interactions has been realized in the form of *generalized game theory* (GGT in short). GGT, being still in progress, extends and reconceptualizes the classical game theory by von Neumann and Morgenstern [10]. In GGT all "rules of the game", information as well as actors' values, norms, beliefs, and knowledge that are expressible in a considered language are formalized in the form of rules. The key mathematical concept of GGT is *rule complex*, a notion coined by the author of the present paper [2, 6, 7]. A prototype of a rule complex is an algorithm containing procedures. Interactions considered in GGT, social actors, and their systems are uniformly represented by rule complexes.

In economic game theory by von Neumann and Morgenstern [10], a game may be specified by a collection of pre-determined rules, a set of players, and possible moves and strategies for each and every player. Players have to follow the "rules of the game" and are not allowed to change them unless stated otherwise. Nevertheless, they may choose a strategy and decide which of the possible moves to take. The classical approach was intended for the purpose of formalization of economic behaviour of fully rational agents. Why should we expect then that all kinds of game-like social interactions can be captured by the framework proposed by von Neumann and Morgenstern? Clearly we should not.

In GGT we generalize von Neumann and Morgenstern's classical work (cf. [2, 3, 4, 5, 6]) in that the rules of a game may be imprecise and tolerating exceptions; games may be underspecified or specified in a vague way; actors may not know strategies for themselves and/or for other participants but, nevertheless, they may modify or change the rules, construct or plan actions, fabricate rules. In GGT actors are not merely rule followers or pure rationalists maximizing a value. They are social beings trying to realize their relationships and cultural forms, and hence they engage in processes of restructurization of games. With every actor we associate social roles played

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by the actor, his/her values, norms, actions and action modalities, judgement rules and algorithms, beliefs, and knowledge. All these constituents are represented as appropriate rules or rule complexes. Also actors are represented as rule complexes, having as their parts, among other things, the just mentioned components. Actors apply their rule complexes in situations of action or interaction to achieve private or group objectives, to plan and implement necessary activities, and to solve problems. Needless to say, systems of actors may also be viewed as rule complexes. On the other hand, a social interaction, and a game in particular, may be given the form of a rule complex as well. Such a rule complex specifies more or less precisely who the actors are, what their roles, rights, and obligations are, what the interaction is, what action opportunities, resources, goals, procedures, and payoffs of the game are, etc. Examples of games well-known from the literature can be rewritten to the form of appropriate rule complexes without special difficulties. To the contrary, it is quite easy to find a game-like social interaction (e.g., "school") for which it would be very difficult or impossible at all to find an adequate representation in normal form, in the form of a tree or by means of the characteristic function. Thus, rule complexes seem to be a flexible and powerful tool to represent social actors and to represent and analyze social interactions. Nevertheless, one has to emphasize that we do not pretend to be able to formalize every aspect of social interaction or to find a formula describing the whole complexity of social actors and their behaviour.

In the present paper we focus upon the problem of derivability of rules from rule complexes. According to [8], the first formal definition of derivability of inference rules had been formulated by K. Ajdukiewicz in 1928 [1], before the general notion of an inference rule was studied in a systematic way. In Hilbert-style logical systems, derivability of rules corresponds to provability of theorems. In formalisms where exceptions to rules are allowed (for instance, in Reiter's default logic [9]), the concept of an *extension* generalizes that of the *set of theorems.* Members of such an extension do not have the force of theorems. They may be seen as tentative candidates for theorems only. Taking into account logical aspects of our approach, it belongs to the family of rule-based formalisms, where formulas play an auxiliary role only. The concept of derivability of rules from a rule complex, studied in the present paper, extends the classical logical notion of derivability of rules in the way as to take into account possible exceptions to rules. At the same time it comprises the notion of "provability" of candidates for theorems which are elements of possible extensions. The idea of localness of reasoning, understood as reasoning with a limited access to rules, is realized by means of the relative derivability of rules. Starting with derivability of rules, we can discuss the activation of rules and define (in)consistency of rule complexes. Activation of a rule is a key concept needed to speak of the application of rule complexes, viz., only activated rules can possibly be applied. The notion of (in)consistency of a rule complex is important to study the problem of (in)compatibility of rule complexes. One can say that two or more rule complexes are compatible if the rule complex obtained as the result of their composition is consistent.

In the present definition of derivability of rules from a rule complex, two important aspects have been taken into account, viz. exceptions to rules and localness of reasoning. Some preliminary attempts to incorporate judgements of similarity have been made but, generally, vagueness of reasoning has not been incorporated yet. Reasoning under vague information in the context of rule complexes and GGT will be discussed elsewhere.

For any set X, we denote the cardinality of X by $\#X$ and the power set of X by $\wp(X)$. The set of natural numbers (with 0) will be denoted by $\mathbf{N}$. $x_0 \in x_1 \in \cdots \in x_n \in x_{n+1}$ is an abbreviation of $x_0 \in x_1 \wedge x_1 \in x_2 \wedge \cdots \wedge x_n \in x_{n+1}$.

In the paper we present revised and generalized versions of the notions of a rule (Sect. 2), a rule complex (Sect. 3), and related concepts: a complex base, a rule base, a generalized element, and a subcomplex (Sect. 4).[2] Next, we define derivability of rules step by step (Sect. 5). In Sect. 6 we give some preliminary remarks on the activation of rules and the application of rule complexes. In Sect. 7 we address the problem of (in)consistency of a rule complex. A brief summary is given in Sect. 8.

2. Rules

A rule and a rule complex are key mathematical concepts underlying generalized game theory (GGT). Rules are major components of games and interactions. Instructions of procedures and algorithms may be seen as rules. Values and norms as well as beliefs and knowledge of social actors may also be represented in the form of rules. One can mention action rules specifying pre- and post-conditions of various actions and interactions, highly context-independent rules of logical inference, generative rules and specific situational rules, control rules and, in particular, judgement rules, strict rules and rules

[2] In [2, 3, 5, 6] we use earlier versions of the mentioned notions.

with exceptions, precise and vague rules, and last but not least meta-rules of various kinds.

In this section we introduce a notion of rule and propose a few alternative forms of representation of rules. Such questions as transformation of rules and, in particular, composition and decomposition will be discussed somewhere else.

Assume a language L is given, where $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are propositional connectives of negation, disjunction, conjunction, implication, and equivalence, respectively. At the present stage we do not specify the language totally nor insist on a classical understanding of the connectives above. The parentheses $(,)$ are auxiliary symbols. Next, $\mathrm{F}(L)$ denotes the set of all formulas of L formed according to some formation rules. We use lowercase Greek letters, with subscripts whenever needed, to denote formulas. Thus if $\alpha, \beta \in \mathrm{F}(L)$, then at least $\neg\alpha$, $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta) \in \mathrm{F}(L)$. As usual, parentheses will be omitted whenever possible. Given a set of formulas $X \subseteq \mathrm{F}(L)$, we denote by $\neg X$ the image of X given by the operation $\neg$, that is,

$$\neg X \stackrel{\text{def}}{=} \{\neg\alpha \mid \alpha \in X\}. \tag{1}$$

From formulas and finite sets of them we form rules, usually denoted by r with subscripts if needed. We write $\mathrm{rul}(x)$ to say that x is a rule. By a *rule* r over L we mean a triple

$$\text{(r-1)} \quad r = (X, Y, \alpha) \tag{2}$$

where X, Y are finite sets of formulas of L called *premises* and *justifications* of r, respectively, and α is a formula of L called the *conclusion* of r. Premises and justifications form *pre-conditions* of r, while the conclusion is the *post-condition*. Pre-conditions have to be declarative statements, thus no order (e.g., *Do* β!) may be a pre-condition. On the other hand, post-conditions may be declarative or imperative statements but not interrogative statements. The set of all rules over L is denoted by $\mathrm{R}(L)$. The reference to L will be omitted if no confusion will result. The informal meaning of r is that if all elements of X hold and all elements of Y possibly hold, then α is concluded. The definition of a rule taken above differs from the definition used in our earlier papers [2, 3, 5, 6]. From the present perspective rules in the old sense are sets of rules in the new sense. Thus, rules of logical inference as, for instance, *modus ponens* MP: $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$ where α, β are arbitrary formulas of L, are not longer rules but sets of rules. Without going into details, the motivation for such a change of the definition of a rule came from the field of knowledge discovery in information systems. However, it can be useful to recall a traditional notion of an inference rule, e.g. that one described in [8].

Definition 2.1. By a *rule* of inference* over L we mean a non-empty binary relation $r \subseteq \wp(\mathrm{F}(L)) \times \mathrm{F}(L)$ such that for any pairs $(X_i, \alpha_i) \in r$ $(i = 1, 2)$, $\#X_1 = \#X_2 < \aleph_0$.

Rules without premises and justifications are called *axiomatic*. For every formula α, there is a unique axiomatic rule corresponding to α, viz. $\alpha^{\mathrm{rul}} = (\emptyset, \emptyset, \alpha)$. Where X is a set of formulas,

$$X^{\mathrm{rul}} \stackrel{\mathrm{def}}{=} \{\alpha^{\mathrm{rul}} \mid \alpha \in X\}. \tag{3}$$

In particular, $\emptyset^{\mathrm{rul}} = \emptyset$. The set of all axiomatic rules over L is denoted by $\mathrm{AXR}(L)$. Thus, $\mathrm{AXR}(L) \subseteq \mathrm{R}(L)$. In our approach two kinds of pre-conditions are distinguished: premises and justifications. Premises of a rule r are stronger than justifications; they have to hold if r is to be applied. On the other hand, it suffices for the sake of application of r that justifications possibly hold. That is, a justification β may actually not hold but it suffices that we do not know for sure that it does not hold. We use the name 'justification' for historical reasons mainly. The term is adopted from the formalism introduced by Reiter and widely known as *default logic* [9]. Indeed, our rules make possible to reason "by default" and to deal with "exceptions to rules". Such features of rules may be particularly useful when formalizing commonsense reasoning. Needless to say, such a form of reasoning is common in social life, and hence in social actions and interactions. Of course, not all rules admit of exceptions and the set of justifications is empty in many cases. Taking into account rules which may have exceptions usually adds to the complexity of the formalism. On the other hand, a form of incompleteness of information may be grasped.

Consider a rule $r = (X, Y, \alpha)$, where $X = \{\alpha_0, \ldots, \alpha_m\}$ and $Y = \{\alpha_{m+1}, \ldots, \alpha_n\}$. The rule r may also be written in the following forms whenever convenient:

$$\begin{array}{ll}
\text{(r-2)} & (\alpha_0, \ldots, \alpha_m : \alpha_{m+1}, \ldots, \alpha_n, \alpha) \\
\text{(r-3)} & \dfrac{\alpha_0, \ldots, \alpha_m : \alpha_{m+1}, \ldots, \alpha_n}{\alpha} \\
\text{(r-4)} & \alpha_0 \wedge \cdots \wedge \alpha_m \wedge b\alpha_{m+1} \wedge \cdots \wedge b\alpha_n \Rightarrow \alpha \\
\text{(r-5)} & \alpha_0, \ldots, \alpha_m, b\alpha_{m+1}, \ldots, b\alpha_n \Rightarrow \alpha
\end{array} \tag{4}$$

In the cases (r-4) and (r-5), 'b' is used to distinguish justifications from premises and $\Rightarrow$ is to separate the pre-conditions from the post-condition of

r. If $r = (X, \emptyset, \alpha)$, i.e., $Y = \emptyset$, then r is a rule without exceptions and may also be written as follows:

$$(\alpha_0, \ldots, \alpha_m, \alpha)$$

$$\frac{\alpha_0, \ldots, \alpha_m}{\alpha}$$

$$\alpha_0 \wedge \cdots \wedge \alpha_m \Rightarrow \alpha$$

$$\alpha_0, \ldots, \alpha_m \Rightarrow \alpha \tag{5}$$

If $r = (\emptyset, Y, \alpha)$, i.e., $X = \emptyset$, then r may also take the following forms:

$$(: \alpha_{m+1}, \ldots, \alpha_n, \alpha)$$

$$\frac{: \alpha_{m+1}, \ldots, \alpha_n}{\alpha}$$

$$b\alpha_{m+1} \wedge \cdots \wedge b\alpha_n \Rightarrow \alpha$$

$$b\alpha_{m+1}, \ldots, b\alpha_n \Rightarrow \alpha \tag{6}$$

Finally, if $r = (\emptyset, \emptyset, \alpha)$, i.e., both X and Y are empty, then r may also be written as follows:

$$\alpha^{\mathrm{rul}} \text{ or } \frac{}{\alpha} \text{ or } \Rightarrow \alpha. \tag{7}$$

Consider two rules r_1 and r_2 over L. If the application of our rule-based formalism to GGT is to be taken seriously, we should be able, for instance, to write formally that: *If both r_1 and r_2 are applicable in a situation s, then apply r_1 first.* Let r be the name of the just written rule. As a matter of fact, r is a *meta-rule* relative to r_1 and r_2. We may assume that the language L is a multi-level language where various levels are not separated. In this case r, when written formally, would be a rule over L. A well-known drawback of such an approach is the possibility of circularities. To avoid such problems, we could consider, e.g., a hierarchy of languages instead of one language. In this paper we choose the first possibility and shall treat meta-rules relative to rules (or rule complexes introduced in the next section) over L as rules over L as well.

3. Rule Complexes

In GGT social actors and their interactions are represented by means of systems of rules called rule complexes. Consider the actor case. We organize the totality of rules associated with an actor i in a situation s into a rule complex, i's *actor complex* in s, written $\mathrm{ACTOR}_{i,s}$. The part of $\mathrm{ACTOR}_{i,s}$

which is the most interesting from our perspective concerns i's social roles in s. Such roles can be, for instance, family roles like father, mother, son, dauther, etc., roles played at the work place like supervisor, subordinate, teacher, student, physician, politician, businessman or businesswoman, etc., customer, church member, party member, society member, friend, and so on. The totality of rules associated with i's roles in s are arranged in i's *role complex* in s, $\mathrm{ROLE}_{i,s}$, which is a rule complex being a subcomplex of $\mathrm{ACTOR}_{i,s}$. The notion of a subcomplex will be defined later on. For the time being, a subcomplex of a rule complex C may be understood as a rule complex which is a part of C. Roughly speaking, the rule complex $\mathrm{ROLE}_{i,s}$ is obtained from $\mathrm{ACTOR}_{i,s}$ by neglecting all these rules of $\mathrm{ACTOR}_{i,s}$ that are irrelevant for the notion 'social role'. With every role j played by i in s one can associate a corresponding rule complex $\mathrm{ROLE}^j_{i,s}$, specifying and describing this role. This rule complex is a subcomplex of $\mathrm{ROLE}_{i,s}$.

To play their social roles, actors are equipped with systems of norms and values, telling them what is good, bad, worth being strived for, what ought to be done, and what is forbidden. In the case of actor i in situation s, these systems of norms and values are represented as i's *value complex* in s, $\mathrm{VALUE}_{i,s}$. Norms and values relevant for a particular role j in s form an appropriate rule complex, $\mathrm{VALUE}^j_{i,s}$, being a subcomplex of $\mathrm{VALUE}_{i,s}$ and $\mathrm{ROLE}^j_{i,s}$, simultaneously. On the other hand, both $\mathrm{VALUE}_{i,s}$ and $\mathrm{ROLE}^j_{i,s}$ are subcomplexes of $\mathrm{ROLE}_{i,s}$.

Actors also have beliefs and knowledge about themselves, other actors involved in the interaction situation, and about the situation. The actor i's beliefs and knowledge in s form a model of the actor i, other actors, and the situation s. The model is represented by a rule complex $\mathrm{MODEL}_{i,s}$ called simply i's *model* in s. This rule complex is a subcomplex of $\mathrm{ROLE}_{i,s}$.

Actors are provided with repertoires of actions and activities but can also construct and plan appropriate actions. Modes of acting and interacting are determined by procedures called *action modalities*. There are several major action modalities distinguished by social scientists: instrumental rationality (modality of consequentialism), normatively oriented action, procedural modality, ritual and communication, and play [3]. Repertoires of actions associated with i in s are composed into i's *action complex* in s, $\mathrm{ACT}_{i,s}$. Moreover, the collection of actions associated with a particular role j in s is arranged into a rule complex $\mathrm{ACT}^j_{i,s}$. The last rule complex is a subcomplex of $\mathrm{ACT}_{i,s}$ as well as $\mathrm{ROLE}^j_{i,s}$. Action modalities of the actor i in s are represented in the form of a rule complex called i's *action modality complex* in

s, $\mathrm{ACTMOD}_{i,s}$. Both $\mathrm{ACT}_{i,s}$ as well as $\mathrm{ACTMOD}_{i,s}$ are subcomplexes of $\mathrm{ROLE}_{i,s}$ like i's value complex and model.

Apart from rules and their systems mentioned above, actor complexes also contain various rules and rule complexes concerning control over the whole actor complex, where 'control' is understood in a broad sense. Control procedures available to the actor i in situation s are represented by a rule complex called i's *control complex* in s, $\mathrm{CTRL}_{i,s}$, being a subcomplex of $\mathrm{ACTOR}_{i,s}$. The rule complex $\mathrm{CTRL}_{i,s}$ comprises, among other things, judgemental and reasoning procedures that play an important role in almost every social action and interaction. Human actors are able to reason in a rational (or logical) way, draw conclusions, derive new rules from the existing ones, etc. Collective social actors can "reason" indirectly via their human members. All rules of logical inference (including meta-rules of derivability describing how to derive rules from a given rule complex), available to the actor i in s, are arranged in a rule complex $\mathrm{LOGIC}_{i,s}$ called i's *logic* in s and being a subcomplex of $\mathrm{CTRL}_{i,s}$.

What are rule complexes then? Informally speaking, rule complexes are particular sets formed of rules and/or the empty set. Usually we shall denote them by C, D, E with subscripts whenever needed. The expression $\mathrm{cpl}(x)$ means that x is a rule complex.

Definition 3.2. The class of *rule complexes* over the language L, written $\mathrm{CPL}(L)$, is the least class of sets containing all sets of rules and closed under the following formation rules:

(rc-1) If $\mathcal{C}$ is a family of rule complexes over L, then $\bigcup\mathcal{C}$ is a rule complex over L.

(rc-2) If C is a rule complex over L, then the power set of C, $\wp(C)$, is a rule complex over L.

(rc-3) If $C \subseteq D$ and D is a rule complex over L, then C is a rule complex over L.

We shall omit the reference to L if no confusion will result. The following two properties may be directly obtained from the definition.

Proposition 3.3. (a) *For any non-empty family of rule complexes* $\mathcal{C}$, $\bigcap\mathcal{C}$ *is a rule complex.* (b) *Where* C *is a rule complex and* X *is a set,* $C - X$ *is a rule complex.*

Proof. Let us note that for any rule complex $C \in \mathcal{C}$, $\bigcap\mathcal{C} \subseteq C$. Similarly, for any rule complex C, $C - X$ is a subset of C. □

Example 3.4. The sets $C_1 = \{r_1, r_2\}$, $C_2 = \{r_2, C_1\}$, and $C_3 = \{r_1, \{r_1\}, \{\{r_1\}\}, \ldots\}$ are rule complexes.

Example 3.5. Algorithms as collections of instructions and/or procedures may be seen as rule complexes. As a matter of fact, an algorithm with embedded procedures was a prototype of the notion of a rule complex.

Let us consider the following statements:
(rc-4) Every set of rules is a rule complex.
(rc-5) Every set of rule complexes is a rule complex.
(rc-6) $\mathrm{cpl}(C)$ iff for each $x \in C$, $\mathrm{rul}(x)$ or $\mathrm{cpl}(x)$.

THEOREM 3.6. (a) *Conditions* (rc-3) *and* (rc-5) *imply* (rc-2).
(b) *Conditions* (rc-1)–(rc-3) *imply* (rc-5).
(c) *Conditions* (rc-1) *and* (rc-4)–(rc-6) *imply* (rc-3).
(d) *Conditions* (rc-1)–(rc-4) *imply* (rc-6).

PROOF. For (a) consider a rule complex C. For each $D \subseteq C$, $\mathrm{cpl}(D)$ by (rc-3). Hence $\mathrm{cpl}(\wp(C))$ by (rc-5). For (b) assume X is a set of rule complexes. By (rc-1), $\mathrm{cpl}(\bigcup X)$. Hence $\mathrm{cpl}(\wp(\bigcup X))$ by (rc-2). Since $X \subseteq \wp(\bigcup X)$, $\mathrm{cpl}(X)$ by (rc-3). For (c) assume $C \subseteq D$ where $\mathrm{cpl}(D)$. Let x be any element of C. By assumption, $x \in D$. Hence by (rc-6), $\mathrm{rul}(x)$ or $\mathrm{cpl}(x)$. Again by (rc-6), $\mathrm{cpl}(C)$. For (d) consider the class $\mathcal{A}$ of all sets consisting of rules and/or rule complexes. Let C be a member of $\mathcal{A}$. Then $X = \{x \in C \mid \mathrm{rul}(x)\}$ and $Y = \{x \in C \mid \mathrm{cpl}(x)\}$ are rule complexes by (rc-4) and (rc-5), respectively. Since $C = X \cup Y$, $\mathrm{cpl}(C)$ by (rc-1). For the left-to-right part of (rc-6) we prove that every rule complex is a member of $\mathcal{A}$. To this end we show that $\mathcal{A}$ is closed under (rc-1)–(rc-3). Assume $C = \bigcup \mathcal{C}$ where $\mathcal{C}$ is a family of sets of $\mathcal{A}$, and consider $x \in C$. There is $D \in \mathcal{C}$ such that $x \in D$. By assumption, $\mathrm{rul}(x)$ or $\mathrm{cpl}(x)$ as required. Now assume that $C \subseteq D$, where D is a member of $\mathcal{A}$. For each $x \in C$, $x \in D$ as well. By assumption, $\mathrm{rul}(x)$ or $\mathrm{cpl}(x)$, i.e., C belongs to $\mathcal{A}$. Finally, assume $C = \wp(D)$ where D is in $\mathcal{A}$. Consider $x \in C$. By assumption $x \subseteq D$ and for each $y \in x$, $\mathrm{rul}(y)$ or $\mathrm{cpl}(y)$. By the right-to-left part of (rc-6), $\mathrm{cpl}(x)$. Hence C belongs to $\mathcal{A}$ as needed. Since $\mathcal{A}$ contains all sets of rules and is closed under (rc-1)–(rc-3), it contains the class of all rule complexes. $\square$

COROLLARY 3.7. *For any rule complex C, there are sets C_1 and C_2 such that $C = C_1 \cup C_2$ and one of the following cases holds:*
(a) $C_1 = C_2 = \emptyset$.
(b) $C_1 = \emptyset$ *and C_2 is a non-empty set of rules.*

(c) $C_1 = \emptyset$ *and* C_2 *is a non-empty set of rule complexes.*
(d) C_1 *is a non-empty set of rules and* C_2 *is a non-empty set of rule complexes.*

All rule complexes over L do not form a set, i.e., $\mathrm{CPL}(L)$ is a proper class. Suppose to the contrary that $\mathrm{CPL}(L)$ is a set. By part (b) of Theorem 3.6, $\mathrm{CPL}(L)$ must be a rule complex consisting of all rule complexes. Since each subset of $\mathrm{CPL}(L)$ is a rule complex by (rc-3), $\wp(\mathrm{CPL}(L)) \subseteq \mathrm{CPL}(L)$. Then $\#\wp(\mathrm{CPL}(L)) \leq \#\mathrm{CPL}(L)$ contrary to Cantor's Theorem.

4. Complex Bases, Rule Bases, and Subcomplexes

In this section we define a few auxiliary concepts related to the notion of a rule complex like a complex part, a rule part, a complex base, and a rule base. Next, we show how the notions of membership and a subset may be generalized in the case of rule complexes.

The *complex part* of a rule complex C, $\mathrm{cp}(C)$, is the set of all elements of C that are complexes.

$$\mathrm{cp}(C) \stackrel{\mathrm{def}}{=} \{x \in C \mid \mathrm{cpl}(x)\}. \tag{8}$$

Similarly, the *rule part* of C, $\mathrm{rp}(C)$, consists of all elements of C being rules.

$$\mathrm{rp}(C) \stackrel{\mathrm{def}}{=} \{x \in C \mid \mathrm{rul}(x)\}. \tag{9}$$

Example 4.8. Consider the rule complexes from Example 3.4. Their complex parts are: $\mathrm{cp}(C_1) = \emptyset$, $\mathrm{cp}(C_2) = \{C_1\}$, and $\mathrm{cp}(C_3) = C_3 - \{r_1\}$. On the other hand, their rule parts are: $\mathrm{rp}(C_1) = C_1$, $\mathrm{rp}(C_2) = \{r_2\}$, $\mathrm{rp}(C_3) = \{r_1\}$.

Several properties of operations cp and rp are given below. For simplicity, let $\tau \in \{\mathrm{cp}, \mathrm{rp}\}$.

Proposition 4.9. *For any rule complexes* C, D, *we have that:*

(a) $\mathrm{rp}(C) \cup \mathrm{cp}(C) = C$ *and* $\mathrm{rp}(C) \cap \mathrm{cp}(C) = \emptyset$.
(b) *If* $C \subseteq D$, *then* $\tau(C) \subseteq \tau(D)$.
(c) *If* $\mathrm{rp}(C) = \emptyset$, *then* $\tau(\bigcup C) = \bigcup\{\tau(D) \mid D \in C\}$.
(d) *If* $\mathrm{rp}(C) = \emptyset$ *and* $C \neq \emptyset$, *then* $\tau(\bigcap C) = \bigcap\{\tau(D) \mid D \in C\}$.
(e) $\tau(C - D) = \tau(C) - \tau(D)$.
(f) $\mathrm{rp}(\wp(C)) = \emptyset$ *and* $\mathrm{cp}(\wp(C)) = \wp(C)$.

PROOF. We only prove (c) for $\tau = \mathrm{cp}$, leaving the remaining cases as exercises. To this end, assume that $\mathrm{rp}(C) = \emptyset$. For any x, $x \in \mathrm{cp}(\bigcup C)$ iff $\mathrm{cpl}(x)$ and $\exists D \in C.x \in D$ iff $\exists D \in C.(\mathrm{cpl}(x) \wedge x \in D)$ iff $\exists D \in C.x \in \mathrm{cp}(D)$ iff $x \in \bigcup\{\mathrm{cp}(D) \mid D \in C\}$. □

We generalize the above two notions and define the complex and rule bases of C as the sets of all rule complexes and rules, respectively, constituting (or, in other words, occurring in) C. Precisely, the *complex base* of C, $\mathrm{cb}(C)$, is defined as follows:

$$\text{(10)} \quad \mathrm{cb}(C) \stackrel{\mathrm{def}}{=} \mathrm{cp}(C) \cup \{D \mid \exists n \in \mathbf{N}.\exists D_0, \ldots, D_n.(\forall i = 0, \ldots, n.\mathrm{cpl}(D_i) \wedge D \in D_0 \in \cdots \in D_n \in C)\}.$$

Now the *rule base* of C, $\mathrm{rb}(C)$, is defined as

$$\text{(11)} \qquad \mathrm{rb}(C) \stackrel{\mathrm{def}}{=} \mathrm{rp}(C \cup \bigcup \mathrm{cb}(C)).$$

Example 4.10. Again consider the rule complexes from Example 3.4. Their complex bases are: $\mathrm{cb}(C_1) = \emptyset$, $\mathrm{cb}(C_2) = \{C_1\}$, and $\mathrm{cb}(C_3) = C_3 - \{r_1\}$; and their rule bases are: $\mathrm{rb}(C_1) = \mathrm{rb}(C_2) = \{r_1, r_2\}$ and $\mathrm{rb}(C_3) = \{r_1\}$.

Complex and rule bases of rule complexes have interesting properties some of which are stated in the following two propositions.

PROPOSITION 4.11. *For any rule complexes C, D, and E, we have that:*

(a) $\mathrm{cb}(C) = \emptyset$ iff $\mathrm{cp}(C) = \emptyset$.
(b) *If* $C \subseteq D$, *then* $\mathrm{cb}(C) \subseteq \mathrm{cb}(D)$.
(c) *If* $C \in D$, *then* $\mathrm{cb}(C) \cup \{C\} \subseteq \mathrm{cb}(D)$.
(d) $\mathrm{cb}(C) = \mathrm{cp}(C) \cup \bigcup\{\mathrm{cb}(D) \mid D \in C\}$.
(e) *If* $C \in D \in \mathrm{cb}(E)$, *then* $C \in \mathrm{cb}(E)$.
(f) *If* $C \in \mathrm{cb}(D)$ *and* $D \in \mathrm{cb}(E)$, *then* $C \in \mathrm{cb}(E)$.
(g) $\mathrm{cb}(\wp(C)) = \wp(C) \cup \mathrm{cb}(C)$.
(h) *If* $\mathrm{rp}(C) = \emptyset$, *then* $\mathrm{cb}(\bigcup C) = \bigcup\{\mathrm{cb}(D) \mid D \in C\}$.

PROOF. (a) easily follows from the definition. For (b) assume $C \subseteq D$. Consider any $E \in \mathrm{cb}(C)$. By definition, $E \in \mathrm{cp}(C)$ or there are $n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_n$ such that $E \in x_0 \in \cdots \in x_n \in C$. In the former case $E \in \mathrm{cp}(D)$ by assumption and Proposition 4.9, and hence

$E \in \mathrm{cb}(D)$ by definition. In the latter case notice that $x_n \in D$ by assumption. Hence $E \in x_0 \in \cdots \in x_n \in D$, and finally $E \in \mathrm{cb}(D)$ by definition. For (c) assume $C \in D$. Obviously $C \in \mathrm{cb}(D)$. Now consider $E \in \mathrm{cb}(C)$. As in (b), $E \in \mathrm{cp}(C)$ or there are $n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_n$ such that $E \in x_0 \in \cdots \in x_n \in C$. In the former case $E \in C \in D$ which entails $E \in \mathrm{cb}(D)$ by definition. In the latter case notice that $E \in x_0 \in \cdots \in x_n \in C \in D$ and apply the definition of cb. For (d) assume $E \in \mathrm{cb}(C)$ first. By definition, $E \in \mathrm{cp}(C)$ or there are $n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_n$ such that $E \in x_0 \in \cdots \in x_n \in C$. In the former case we are done. In the latter one notice that $E \in \mathrm{cb}(x_n)$ and $x_n \in C$ as required. To prove the remaining part, it suffices to assume that there is (i) $D \in C$ such that $E \in \mathrm{cb}(D)$. Indeed, $\mathrm{cp}(C) \subseteq \mathrm{cb}(C)$ by definition. $E \in \mathrm{cb}(D)$ implies that $E \in \mathrm{cp}(D)$ or there are $n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_n$ such that $E \in x_0 \in \cdots \in x_n \in D$. In the former case $E \in \mathrm{cb}(C)$ by (i) and definition. In the latter one, $E \in x_0 \in \cdots \in x_n \in D \in C$ and $E \in \mathrm{cb}(C)$ by definition. For (e) assume $C \in D \in \mathrm{cb}(E)$. By definition, $D \in \mathrm{cp}(E)$ or there are $n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_n$ such that $D \in x_0 \in \cdots \in x_n \in E$. In the former case $C \in \mathrm{cb}(E)$ by assumption and definition. In the latter one, $C \in D \in x_0 \in \cdots \in x_n \in E$ and $C \in \mathrm{cb}(E)$ by definition. Actually (e) is a particular case of (f). Now for (f) assume (ii) $C \in \mathrm{cb}(D)$ and (iii) $D \in \mathrm{cb}(E)$. Hence the following cases hold: (iv) $C \in D \in E$ or (v) $C \in D$ and there are $n \in \mathbf{N}$ and rule complexes $y_0, \ldots, y_n$ such that $D \in y_0 \in \cdots \in y_n \in E$ or (vi) there are $m \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_m$ such that $C \in x_0 \in \cdots \in x_m \in D$ and $D \in E$ or (vii) there are $m, n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_m$, $y_0, \ldots, y_n$ such that $C \in x_0 \in \cdots \in x_m \in D$ and $D \in y_0 \in \cdots \in y_n \in E$. In each case $C \in \mathrm{cb}(E)$ is easily obtained by definition. For (g) notice first that (viii) $\mathrm{cp}(C) = C$ implies $\mathrm{cb}(C) = C \cup \bigcup\{\mathrm{cb}(D) \mid D \in C\}$ by (d). Clearly, $\bigcup\{\mathrm{cb}(D) \mid D \in \wp(C)\} = \bigcup\{\mathrm{cb}(D) \mid D \subseteq C\} = \mathrm{cb}(C)$ by (b). Finally, $\mathrm{cb}(\wp(C)) = \wp(C) \cup \mathrm{cb}(C)$. For (h) assume that $\mathrm{rp}(C) = \emptyset$. ($\subseteq$) Suppose that for some rule complex x, $x \in \mathrm{cb}(\bigcup C)$. Then (ix) $x \in \mathrm{cp}(\bigcup C)$ or (x) there are $n \in \mathbf{N}$ and rule complexes $x_0, \ldots, x_n$ such that $x \in x_0 \in \cdots \in x_n \in \bigcup C$. In the former case there is $D \in C$ such that $x \in \mathrm{cp}(D)$ by assumption and Proposition 4.9. Then $x \in \mathrm{cb}(D)$ easily. In summary, $x \in \bigcup\{\mathrm{cb}(D) \mid D \in C\}$. In the case (x), there is a rule complex $D \in C$ such that $x \in x_0 \in \cdots \in x_n \in D$. That is, for some $D \in C$, $x \in \mathrm{cb}(D)$ by definition. Again $x \in \bigcup\{\mathrm{cb}(D) \mid D \in C\}$. ($\supseteq$) Now suppose that $x \in \bigcup\{\mathrm{cb}(D) \mid D \in C\}$. Hence there is $D \in C$ such that $x \in \mathrm{cb}(D)$. Since $D \in C$, $D \subseteq \bigcup C$. Thus by (b), $x \in \mathrm{cb}(\bigcup C)$ as required. □

PROPOSITION 4.12. *For any rule complexes C, D, and E, we have that:*

(a) $\mathrm{rb}(C) \cap \mathrm{cb}(C) = \emptyset$.

(b) $\mathrm{rb}(C) = \bigcup\{\mathrm{rp}(D) \mid D = C \vee D \in \mathrm{cb}(C)\}$.

(c) $\mathrm{rb}(C) = \emptyset$ iff $\mathrm{rp}(C) = \emptyset$ and $\forall D \in \mathrm{cb}(C).\ \mathrm{rp}(D) = \emptyset$.

(d) *If* $C \subseteq D$, *then* $\mathrm{rb}(C) \subseteq \mathrm{rb}(D)$.

(e) *If* $C \in D$, *then* $\mathrm{rb}(C) \subseteq \mathrm{rb}(D)$.

(f) *If* $C \in \mathrm{cb}(D)$, *then* $\mathrm{rb}(C) \subseteq \mathrm{rb}(D)$.

(g) *If* $\mathrm{rp}(C) = \emptyset$, *then* $\mathrm{rb}(\bigcup C) = \mathrm{rb}(C)$.

(h) $\mathrm{rb}(\wp(C)) = \mathrm{rb}(C)$.

The proof is left as an exercise.

Now we can generalize the notions of an element and a subset to the case of rule complexes. x is a *generalized element* (or simply *g-element*) of C, $x \in_{\mathrm{g}} C$, in case x is an element of the complex base or the rule base of C. That is,

$$x \in_{\mathrm{g}} C \overset{\mathrm{def}}{\leftrightarrow} x \in \mathrm{cb}(C) \cup \mathrm{rb}(C). \tag{12}$$

Two different rule complexes may have the same g-elements.

Example 4.13. *Consider* $C = \{r_0, C_0\}$ *and* $D = \{C_0\}$ *where* $C_0 = \{r_0, r_1\}$. *Rule complexes C and D are different but have the same g-elements.*

The notion of a subcomplex, introduced below in a semi-formal way, is of great importance in GGT. In the preceding section we briefly described fundamental ideas of our representation of social actors.[3] In GGT a social actor i in an interaction situation s is represented by i's actor complex in s $\mathrm{ACTOR}_{i,s}$. Parts of this rule complex, being of particular interest, are i's role and control complexes in s, $\mathrm{ROLE}_{i,s}$ and $\mathrm{CTRL}_{i,s}$, respectively. Important parts of $\mathrm{ROLE}_{i,s}$ are: (1) i's value complex $\mathrm{VALUE}_{i,s}$, (2) i's model of him/herself, other actors, and the situation, written $\mathrm{MODEL}_{i,s}$, (3) i's action complex $\mathrm{ACT}_{i,s}$, and (4) i's action modality complex $\mathrm{ACTMOD}_{i,s}$; all related to s. These parts are examples of subcomplexes. More formally, a rule complex C is a *subcomplex* of a rule complex D, $C \sqsubseteq D$, if $C = D$ or C is obtained from D by deleting some occurrences of g-elements of D and/or by removing redundant parentheses. The notion of redundancy deserves a detailed elaboration. For lack of space however, we only give an

[3] Also social interactions may be similarly modelled by means of rule complexes.

example of such a redundancy of parentheses. Consider rule complexes C and $D = \underbrace{\{\ldots\{}_{n} C \underbrace{\}\ldots\}}_{n}$ where $n > 0$. As far as the application of rule complexes is concerned, these parentheses are redundant since the application of D intuitively resolves itself into the application of C. A simple operation of removing of parentheses, κ, may be defined as follows. For any rule complex C,

$$\kappa(C) \stackrel{\text{def}}{=} \begin{cases} D & \text{if } C = \{D\} \text{ and } \mathrm{cpl}(D) \\ C & \text{otherwise.} \end{cases} \tag{13}$$

Applying κ a sufficient number of times, we eventually remove all parentheses redundant in the above sense.

One can see that every subset of a rule complex is a subcomplex of it, but not vice versa.

Example 4.14. *Let r_i ($i = 0, \ldots, 3$) be different rules. Consider a rule complex $D = \{r_0, r_1, C_0, C_1\}$, where $C_0 = \{r_2, r_3, C_2\}$, $C_1 = \{r_0, C_0\}$, and $C_2 = \{r_1\}$. Let $C_3 = \{r_2, C_2\}$, $C_4 = \{r_0, C_5\}$, and $C_5 = \{r_2, r_3\}$. The rule complex $C = \{r_1, C_3, C_4\}$ is a subcomplex of D, in symbols $C \sqsubseteq D$. However, C is not a subset of D.*

Let us note a useful observation that if C is a subcomplex of D, then C is also a subcomplex of $D \cup E$, for any rule complex E.

5. Derivability of Rules

Derivation of (possibly new) rules from given rules is a form of reasoning. Actually, it is the main form of reasoning in the case of our formalism since formulas play an auxiliary role only.

The notion of derivability of inference rules was formally defined by K. Ajdukiewicz in 1928 [1]. It was before the notion of inference rule was studied systematically. We recall Ajdukiewicz's notion using the terminology proposed in [8]. First, we say that a set X of formulas of L is *closed* under a set of inference rules* R (cf. Definition 2.1), written $\mathrm{cl}_R(X)$, in case

$$\forall r \in R.\forall Y \subseteq \mathrm{F}(L).\forall \alpha \in \mathrm{F}(L).(((Y, \alpha) \in r \wedge Y \subseteq X) \to \alpha \in X). \tag{14}$$

Consider a Hilbert-style logical system (R, A), where R is a set of inference rules* and A is a set of formulas of L. With (R, A) we can associate a unique consequence operator $C_{R,A} : \wp(\mathrm{F}(L)) \mapsto \wp(\mathrm{F}(L))$ such that for any set of formulas X,

$$C_{R,A}(X) \stackrel{\text{def}}{=} \bigcap\{Y \subseteq \mathrm{F}(L) \mid A \cup X \subseteq Y \wedge \mathrm{cl}_R(Y)\}. \tag{15}$$

Now we can formulate the definition of Ajdukiewicz's concept of derivability of rules.

DEFINITION 5.15. A rule* r is *derivable* in the system (R, A) iff

$$\forall X \subseteq \mathrm{F}(L). C_{\{r\},\emptyset}(X) \subseteq C_{R,A}(X). \tag{16}$$

One can prove that condition (16) is equivalent to the following one:

$$\forall X \subseteq \mathrm{F}(L). \forall \alpha \in \mathrm{F}(L).((X, \alpha) \in r \to \alpha \in C_{R,A}(X)). \tag{17}$$

Along the standard lines, the latter formula is just taken as the condition defining that r is derivable in (R, A).

Thus roughly speaking, derivability of rules corresponds to provability of theorems in Hilbert-style logical systems. In our framework however, exceptions to rules are admitted like in Reiter's default logic [9]. In default logic the notion of the set of theorems of a theory is generalized to the notion of an extension. Since a rule can block the application of another one, there can be no, one or more than one extension of a given theory. Elements of an extension may be seen as tentative candidates for theorems.

The notion of derivability of rules from a rule complex, proposed in this paper, is introduced in a few steps. First, we define derivability of axiomatic rules from a set of rules. As axiomatic rules represent formulas (viz., conclusions of the rules) in our formalization, the possible sets of derived axiomatic rules are counterparts of extensions in default logic. In the second step we generalize, in some sense, the classical concept of derivability of rules described above[4] as to obtain the notion of derivability of arbitrary rules from a set of rules. According to the definition formulated in the next step, derivability of rules from an arbitrary rule complex C is understood as derivability from the rule base of C, $\mathrm{rb}(C)$. Such a view is not oversimplified if it is assumed that all rules of C are equally accessible. In practice an unrestricted access to rules is rare since most actors are "local" reasoners. Therefore, we finally define the notion of relative derivability of rules.

We start with derivability of axiomatic rules from a set of rules. This notion is related to that of an extension in Reiter's default logic [9].

[4] Recall that our notion of a rule differs from the usual one.

Definition 5.16. Given a set of rules D, we define a sequence of operators Δ^n_D ($n \in \mathbf{N}$) as follows. For any set of rules C, let

(i) $\Delta^0_D(C) = C \cap \mathrm{AXR}(L)$;

(ii) $(X, Y, \alpha) \in \Delta^{n+1}_D(C)$ iff $X = Y = \emptyset \wedge \exists T, Z \subseteq \mathrm{F}(L).((T, Z, \alpha) \in C \wedge T^{\mathrm{rul}} \subseteq \Delta^n_D(C) \wedge (\neg Z)^{\mathrm{rul}} \cap D = \emptyset)$.

D is called a *possible set of axiomatic rules derived* from C, $D \in \mathrm{PSARD}(C)$, if

(iii) $D = \bigcup_{n\in\mathbf{N}} \Delta^n_D(C)$.

It is easy to see that for any $n \in \mathbf{N}$, $\Delta^n_D(C)$ (and hence $\bigcup_{n\in\mathbf{N}} \Delta^n_D(C)$) consists of axiomatic rules only. Moreover, $\mathrm{PSARD}(\emptyset) = \{\emptyset\}$. A few other basic properties are given below.

Proposition 5.17. (a) *For each $D \in \mathrm{PSARD}(C)$, $C \cap \mathrm{AXR}(L) \subseteq D$.*

(b) *For any sets of rules C, D and $m < n$, $\Delta^m_D(C) \subseteq \Delta^n_D(C)$.*

(c) *If $C \subseteq \mathrm{AXR}(L)$, then $\mathrm{PSARD}(C) = \{C\}$.*

Proof. If $\mathrm{PSARD}(C) = \emptyset$, then (a) is obvious. In the remaining case it suffices to notice that for each $n \in \mathbf{N}$ and a set of axiomatic rules D, $\Delta^0_D(C) \subseteq \Delta^n_D(C)$, and to apply Definition 5.16. For (b) we first prove by induction on $n \in \mathbf{N}$ that $(*)\Delta^n_D(C) \subseteq \Delta^{n+1}_D(C)$. For $n = 0$ apply simply the definition. Now assume inductively that $(*)$ holds for some $n \in \mathbf{N}$. Suppose that $(X, Y, \alpha) \in \Delta^{n+1}_D(C)$. By definition, $X = Y = \emptyset$ and there exist finite sets of formulas T, Z such that $(T, Z, \alpha) \in C$, $T^{\mathrm{rul}} \subseteq \Delta^n_D(C)$, and $(\neg Z)^{\mathrm{rul}} \cap D = \emptyset$. By the inductive assumption, $T^{\mathrm{rul}} \subseteq \Delta^{n+1}_D(C)$. Hence $(X, Y, \alpha) \in \Delta^{n+2}_D(C)$ by definition, and we are done by the principle of induction. Let $k = n - m$. Thus we prove by induction on $k > 0$ that $(**)\Delta^m_D(C) \subseteq \Delta^{m+k}_D(C)$. The case $k = 1$ has been already proved. Assume inductively that $(**)$ holds for all $k \leq l$. Then $\Delta^m_D(C) \subseteq \Delta^{m+l}_D(C)$ and $\Delta^{m+l}_D(C) \subseteq \Delta^{m+l+1}_D(C)$ by the inductive assumption. Hence $\Delta^m_D(C) \subseteq \Delta^{m+l+1}_D(C)$ as required, and we are done by the principle of induction. For (c) assume $C \subseteq \mathrm{AXR}(L)$. Let D be any set of formulas. First notice that $\Delta^0_D(C) = C$. Hence $C \subseteq \bigcup_{n\in\mathbf{N}} \Delta^n_D(C)$. Consider an arbitrary $n > 0$ and a rule (X, Y, α). Suppose that $(X, Y, \alpha) \in \Delta^n_D(C)$. By definition, $X = Y = \emptyset$ and there exist sets T, Z such that $(T, Z, \alpha) \in C$, $T^{\mathrm{rul}} \subseteq \Delta^{n-1}_D(C)$, and $(\neg Z)^{\mathrm{rul}} \cap D = \emptyset$. By assumption, $T = Z = \emptyset$. Clearly, $(X, Y, \alpha) \in C$. Thus $\bigcup_{n\in\mathbf{N}} \Delta^n_D(C) = C$ for any set of rules D. Hence $\mathrm{PSARD}(C) = \{C\}$. □

One can see that $C \cap \mathrm{AXR}(L) \neq \emptyset$ implies $\emptyset \notin \mathrm{PSARD}(C)$.

Example 5.18. Derivability of axiomatic rules is non-monotonic, viz., it can be that $C_1 \subseteq C_2$ but their correponding possible sets, D_1 and D_2, of axiomatic rules derived from C_1 and C_2 are not comparable. Suppose that $\beta \neq \neg\alpha$ and consider $C_1 = \{(\emptyset, \{\alpha\}, \beta)\}$ and $C_2 = C_1 \cup \{\neg\alpha^{\mathrm{rul}}\}$. $\mathrm{PSARD}(C_1) = \{\{\beta^{\mathrm{rul}}\}\}$ and $\mathrm{PSARD}(C_2) = \{\{\neg\alpha^{\mathrm{rul}}\}\}$.

Example 5.19. There can be more than one possible set of axiomatic rules derived from a set of rules. Let $C = \{(\emptyset, \{\alpha\}, \alpha), (\emptyset, \{\neg\alpha\}, \neg\alpha), (\{\alpha\}, \emptyset, \neg\neg\alpha)\}$. $\mathrm{PSARD}(C) = \{D_1, D_2\}$ where $D_1 = \{\alpha, \neg\neg\alpha\}^{\mathrm{rul}}$ and $D_2 = \{\neg\alpha^{\mathrm{rul}}\}$.

Example 5.20. There can be no possible set of axiomatic rules derived from a set of rules. Let $C = \{(\emptyset, \{\alpha\}, \neg\alpha)\}$. Suppose $D \in \mathrm{PSARD}(C)$. Then $(\neg\alpha)^{\mathrm{rul}} \in D$ iff $(\neg\alpha)^{\mathrm{rul}} \notin D$. Thus $\mathrm{PSARD}(C) = \emptyset$.

Now we can formulate the definition of derivability of arbitrary rules from a set of rules. In that definition we try to generalize the classical idea of derivability to the case of rules with exceptions.

DEFINITION 5.21. A rule (X, Y, α) is *derivable from a set of rules* C, $(X, Y, \alpha) \in \mathrm{Der}(C)$, if there are a $\subseteq$-maximal set of rules $D \subseteq C$ such that $\mathrm{PSARD}(D \cup X^{\mathrm{rul}}) \neq \emptyset$ and a set of rules $E \in \mathrm{PSARD}(D \cup X^{\mathrm{rul}})$ such that $(\neg Y)^{\mathrm{rul}} \cap E = \emptyset$ and $\alpha^{\mathrm{rul}} \in E$.

At first sight, one could wonder why one should look for a set of rules $D \subseteq C$ instead of taking simply C in the definition above. It is mainly for pragmatic reasons. Observe that adding new axiomatic rules X^{rul} to the whole set of given rules C may block derivability for good, i.e., it can easily be that $\mathrm{PSARD}(C \cup X^{\mathrm{rul}}) = \emptyset$. Moreover, the question arises why one should require the set of rules E to be disjoint with the set of all axiomatic rules obtained from negated justifications, $(\neg Y)^{\mathrm{rul}}$. The motivation is related to the informal reading of the rule (X, Y, α). Namely, to conclude α, all premises of X have to hold and it has to be the case that all justifications of Y possibly hold, i.e., no negated justification of Y may hold.

Example 5.22. Axiomatic rules β^{rul} and γ^{rul} are derivable from $C = \{(\emptyset, \{\alpha\}, \neg\alpha), \beta^{\mathrm{rul}}, (\{\beta\}, \emptyset, \gamma)\}$ in spite of the fact that $\mathrm{PSARD}(C) = \emptyset$. Indeed, $\mathrm{PSARD}(C) = \emptyset$ because of the rule $(\emptyset, \{\alpha\}, \neg\alpha)$. Thus in accordance with Definition 5.21, we consider the set $D = C - \{(\emptyset, \{\alpha\}, \neg\alpha)\} = \{\beta^{\mathrm{rul}}, (\{\beta\}, \emptyset, \gamma)\}$. It is easy to see that $\mathrm{PSARD}(D) = \{E\}$ where $E = \{\beta, \gamma\}^{\mathrm{rul}}$. Thus $\beta^{\mathrm{rul}}, \gamma^{\mathrm{rul}} \in \mathrm{Der}(C)$.

Example 5.23. Let us observe that

$$\mathrm{Der}(\emptyset) = \{(X, Y, \alpha) \in \mathrm{R}(L) \mid \alpha \in X \wedge X \cap \neg Y = \emptyset\}.$$

Indeed, for every rule (X, Y, α) as above, $\mathrm{PSARD}(X^{\mathrm{rul}}) = \{X^{\mathrm{rul}}\}$, $\alpha^{\mathrm{rul}} \in X^{\mathrm{rul}}$, and $X^{\mathrm{rul}} \cap (\neg Y)^{\mathrm{rul}} = \emptyset$. One can also see that for C from Example 5.20, $\mathrm{Der}(C) = \mathrm{Der}(\emptyset)$ though $\mathrm{PSARD}(C) = \emptyset$. Finally, let us note that $C \not\subseteq \mathrm{Der}(C)$.

Now we can define an unrestricted form of derivability of rules from an arbitrary rule complex. We assume implicitly that all rules of a considered rule complex are equally and unrestrictedly accessible. Under such an assumption, deriving rules from a rule complex C may actually be the same as deriving rules from the rule base of C.

Definition 5.24. A rule r is *derivable from a rule complex* C, $r \in \mathrm{Der}(C)$, iff r is derivable from the rule base of C, i.e.,

$$\mathrm{Der}(C) \stackrel{\mathrm{def}}{=} \mathrm{Der}(\mathrm{rb}(C)).$$

Clearly, $\mathrm{rb}(C) = \mathrm{rb}(D)$ implies $\mathrm{Der}(C) = \mathrm{Der}(D)$. One can see that all axiomatic rules, being g-elements of C, are derivable from C. Observe also that α^{rul} is not derivable from C unless α is the conclusion of a rule $r \in_{\mathrm{g}} C$.

Example 5.25. Let $\alpha_i \neq \neg\alpha_2$ for $i = 0, \ldots, 4$. Consider the following rules: $r_0 = (\{\alpha_0, \alpha_1\}, \{\alpha_2\}, \alpha_3)$, $r_1 = (\{\alpha_0\}, \emptyset, \alpha_4)$, $r_2 = (\{\alpha_1, \alpha_4\}, \emptyset, \alpha_3)$, $r_3 = (\emptyset, \{\alpha_2\}, \neg\alpha_2)$, and the rule complex $C = \{r_1, \{r_2, r_3\}\}$. In this case $\mathrm{rb}(C) = \{r_1, r_2, r_3\}$. Notice that $\mathrm{PSARD}(\mathrm{rb}(C) \cup \{\alpha_0, \alpha_1\}^{\mathrm{rul}}) = \emptyset$ because of the rule r_3. According to the definition, we look for a $\subseteq$-maximal subset D of $\mathrm{rb}(C)$ such that $\mathrm{PSARD}(D \cup \{\alpha_0, \alpha_1\}^{\mathrm{rul}}) \neq \emptyset$. Let $D = \mathrm{rb}(C) - \{r_3\} = \{r_1, r_2\}$. Then $\mathrm{PSARD}(D \cup \{\alpha_0, \alpha_1\}^{\mathrm{rul}}) = \{E\}$ where $E = \{\alpha_0, \alpha_1, \alpha_3, \alpha_4\}^{\mathrm{rul}}$. Clearly, $(\neg\alpha_2)^{\mathrm{rul}} \notin E$ and $\alpha_3^{\mathrm{rul}} \in E$. Hence $r_0 \in \mathrm{Der}(C)$.

According to Definition 5.24, all rules of a rule complex are accessible in the process of deriving rules. However, it is more realistic to assume that only some rules of a given rule complex may be used. Such a form of local reasoning can be modelled by means of relative derivability.

Definition 5.26. Given a rule complex C and a non-empty family $\mathcal{X}$ of subcomplexes of C. A rule r is *derivable* from C *relative to* $\mathcal{X}$, $r \in \mathrm{Der}(C|\mathcal{X})$, iff there is $D \in \mathcal{X}$ such that $r \in \mathrm{Der}(D)$, i.e.,

$$\mathrm{Der}(C|\mathcal{X}) = \bigcup\{\mathrm{Der}(D) \mid D \in \mathcal{X}\}.$$

In this case we are allowed to use only rules occurring in those subcomplexes of C that are members of the family $\mathcal{X}$. Nevertheless, if $\mathcal{X} = \{C\}$, then $\mathrm{Der}(C|\mathcal{X}) = \mathrm{Der}(C)$.

Example 5.27. Consider again Example 5.25. Let $\mathcal{X} = \{\{r_1, \{r_2\}\}\}$ and $\mathcal{Y} = \{\{r_1\}, \{r_2\}\}$. Observe that $r_0 \in \mathrm{Der}(C|\mathcal{X}) - \mathrm{Der}(C|\mathcal{Y})$. Indeed, $\mathrm{Der}(C|\mathcal{X}) = \mathrm{Der}(C)$ and $\mathrm{Der}(C|\mathcal{Y}) = \mathrm{Der}(\{r_1\}) \cup \mathrm{Der}(\{r_2\})$.

6. Some Remarks on the Activation of Rules

The activation of a rule is a notion relating derivability of rules to such an important issue as the application of rule complexes. In this section we only give some preliminary remarks on the problem of activation of rules, postponing a more systematic study to a separate paper.

As expected, the application of rule complexes resolves itself into the application of rules. A necessary but usually insufficient condition for a rule r to be applied in a situation s is that r is activated in s. Generally, only rules that are activated in s may possibly be applied in s. The activation of a rule may be defined with help of the notion of derivability. Informally speaking, a rule r is *activated* in a situation s if each premise of r holds in s and it is possible for each justification of r to hold in s. Thus, the question of activation of a rule in a given situation may be reduced to checking whether or not some formulas hold or possibly hold in the situation.

Whether or not a formula α holds in s is not a simple matter. For instance, we can say that α *holds* in s in an actor i's opinion, written $(s, i) \models \alpha$, in case there is a formula β similar to α to some sufficient extent (determined by s and i) and such that the corresponding axiomatic rule β^{rul} is derived from i's model in s, $\mathrm{MODEL}_{i,s}$, under some conditions (again determined by s and i) and given some rules of logical inference, represented in the form of a rule complex $\mathrm{LOGIC}_{i,s}$. Suppose that $\mathrm{sim}_{i,s}(\alpha, \beta)$ means that β is sufficiently similar to α with respect to i and s. Along the standard lines, we may assume that the similarity relation $\mathrm{sim}_{i,s}$ is reflexive and symmetric. Next, let $\mathcal{X}$ be a non-empty family of subcomplexes of $\mathrm{MODEL}_{i,s}$, where the elements of $\mathcal{X}$ consist of those rules of $\mathrm{MODEL}_{i,s}$ that are accessible to i in the process of derivation of rules in the situation s. Thus, the above definition of the fact that α holds in s in i's opinion may be formulated as follows:

$$(18)\quad (s, i) \models \alpha \overset{\mathrm{def}}{\leftrightarrow} \exists \beta.(\mathrm{sim}_{i,s}(\alpha, \beta) \wedge \beta^{\mathrm{rul}} \in \mathrm{Der}(\mathrm{MODEL}_{i,s} \cup \mathrm{LOGIC}_{i,s} \mid \mathcal{X})).$$

We stop at this point and postpone further investigations to a separate paper.

7. (In)consistency of a Rule Complex

The notion of derivability of rules is a good starting point to discuss (in)compatibility of rule complexes. Generally, two or more rule complexes (e.g., social roles) are *compatible* if the rule complex, obtained as the result of their composition under some specified conditions, is consistent; otherwise the rule complexes are *incompatible*. Clearly, to obtain a working version of the definition, ready to be applied in GGT, we should modify the above definition by relating it to a particular actor and a situation and by specifying what we mean by composition and (in)consistency of rule complexes. In this section we only focus upon a few theoretical notions of (in)consistency of rule complexes.

Like in the case of activation of rules, (in)consistency of a rule complex may be defined with help of derivability of rules. Below we distinguish four forms of inconsistency of a rule complex: (1) xs-inconsistency ('xs' for 'extra strong'), (2) s-inconsistency ('s' for 'strong'), (3) inconsistency, and (4) relative inconsistency. There are also four corresponding forms of consistency: (1) xw-consistency ('xw' for 'extra weak'), (2) w-consistency ('w' for 'weak'), (3) consistency, and (4) relative consistency, respectively.

The first of the mentioned forms of inconsistency of a rule complex C is called extra strong because it arises in every possible set of axiomatic rules derived from a $\subseteq$-maximal subset D of the rule base of C. Inconsistency of such a sort cannot be avoided by separating elements of D or rejecting of some of them.

Definition 7.28. A rule complex C is called *xs-inconsistent* if there exists a $\subseteq$-maximal set of rules $D \subseteq \mathrm{rb}(C)$ such that $\mathrm{PSARD}(D) \neq \emptyset$ and for every set $E \in \mathrm{PSARD}(D)$ of axiomatic rules derived from D, there is a formula α such that $\alpha^{\mathrm{rul}}, \neg\alpha^{\mathrm{rul}} \in E$; otherwise C is *xw-consistent*.

Example 7.29. To illustrate the above notion it suffices to modify slightly Example 5.19. Let

$$C = \{(\emptyset, \{\alpha\}, \alpha), (\emptyset, \{\alpha\}, \beta), (\emptyset, \{\neg\alpha\}, \neg\alpha), (\emptyset, \{\neg\alpha\}, \gamma),$$
$$(\{\alpha\}, \emptyset, \neg\neg\alpha), \neg\beta^{\mathrm{rul}}, \neg\gamma^{\mathrm{rul}}\}.$$

$\mathrm{PSARD}(\mathrm{rb}(C)) = \{E_1, E_2\}$ where $E_1 = \{\alpha, \neg\neg\alpha, \beta, \neg\beta, \neg\gamma\}^{\mathrm{rul}}$ and $E_2 = \{\neg\alpha, \neg\beta, \gamma, \neg\gamma\}^{\mathrm{rul}}$. The rule complex C is xs-inconsistent.

Weaker, yet still strong is the second form of inconsistency, where contradictory axiomatic rules like α^{rul} and $\neg\alpha^{\mathrm{rul}}$ obtain in at least one of the possible sets of axiomatic rules derived from the set $D \subseteq \mathrm{rb}(C)$ as mentioned above. In the optimistic case this form of inconsistency can be avoided by rejecting of "infected" elements of $\mathrm{PSARD}(D)$.

DEFINITION 7.30. A rule complex C is called *s-inconsistent* if there exist a $\subseteq$-maximal set of rules $D \subseteq \mathrm{rb}(C)$ such that $\mathrm{PSARD}(D) \neq \emptyset$, a set of axiomatic rules $E \in \mathrm{PSARD}(D)$, and a formula α that $\alpha^{\mathrm{rul}}, \neg\alpha^{\mathrm{rul}} \in E$; otherwise C is *w-consistent*.

Example 7.31. Suppose that $\alpha \neq \neg\beta$ and $\beta \neq \neg\alpha$. Let

$$C = \{\{(\emptyset, \{\alpha\}, \alpha), (\{\alpha\}, \emptyset, \neg\neg\alpha), (\emptyset, \{\alpha\}, \beta), (\emptyset, \{\gamma\}, \neg\gamma)\},\\ (\emptyset, \{\neg\alpha\}, \neg\alpha), \neg\beta^{\mathrm{rul}}\}.$$

In this case $\mathrm{PSARD}(\mathrm{rb}(C)) = \emptyset$ because of the rule $(\emptyset, \{\gamma\}, \neg\gamma)$. We consider the $\subseteq$-maximal subset D of $\mathrm{rb}(C)$ such that $\mathrm{PSARD}(D)$ is non-empty. Thus $D = \{(\emptyset, \{\alpha\}, \alpha), (\{\alpha\}, \emptyset, \neg\neg\alpha), (\emptyset, \{\alpha\}, \beta), (\emptyset, \{\neg\alpha\}, \neg\alpha), \neg\beta^{\mathrm{rul}}\}$. In this case $\mathrm{PSARD}(D) = \{E_1, E_2\}$ where $E_1 = \{\alpha, \neg\neg\alpha, \beta, \neg\beta\}^{\mathrm{rul}}$ and $E_2 = \{\neg\alpha, \neg\beta\}^{\mathrm{rul}}$. In summary, the rule complex C is both s-inconsistent and xw-consistent.

Inconsistency of the third kind, called simply *inconsistency* is weaker. In the optimistic case one can omit the problem by separating sets in $\mathrm{PSARD}(D)$, where D is as earlier.

DEFINITION 7.32. A rule complex C is called *inconsistent* if there is a formula α such that $\alpha^{\mathrm{rul}}, \neg\alpha^{\mathrm{rul}} \in \mathrm{Der}(C)$; otherwise C is *consistent*.

It is easy to show that this kind of inconsistency may be characterized by the condition below.

PROPOSITION 7.33. *A rule complex C is inconsistent iff there are a $\subseteq$-maximal set of rules $D \subseteq \mathrm{rb}(C)$ such that $\mathrm{PSARD}(D) \neq \emptyset$, sets of axiomatic rules $E_1, E_2 \in \mathrm{PSARD}(D)$, and a formula α that $\alpha^{\mathrm{rul}} \in E_1$ and $(\neg\alpha)^{\mathrm{rul}} \in E_2$.*

Example 7.34. Consider any rule complex D such that $\mathrm{rb}(D) = C$, where C is the rule complex from Example 5.19. The rule complex D is inconsistent since $\alpha^{\mathrm{rul}}, \neg\alpha^{\mathrm{rul}} \in \mathrm{Der}(D)$. On the other hand, D is w-consistent.

In monotonic logics, every set of formulas containing an inconsistent set of formulas is inconsistent as well. Conversely, every subset of a consistent set of formulas is consistent. In our framework this is not true in general.

Example 7.35. *Consider the rule complexes C, D from Example 7.34 and let $E = D \cup \{\neg\alpha^{\mathrm{rul}}\}$. The rule complex E is consistent since* $\mathrm{PSARD}(\mathrm{rb}(E)) = \{\{\neg\alpha^{\mathrm{rul}}\}\}$. *In summary, $D \subseteq E$ (and hence $D \sqsubseteq E$), D is inconsistent, while E is consistent.*

Interdependencies among the above three forms of (in)consistency are clear.

PROPOSITION 7.36. *Let C be any rule complex. (a) If C is xs-inconsistent, then C is s-inconsistent. On the other hand, if C is w-consistent, then it is xw-consistent. (b) If C is s-inconsistent, then C is inconsistent. To the contrary, if C is consistent, then it is w-consistent.* ∎

Starting with relative derivability we obtain the corresponding form of (in)consistency.

DEFINITION 7.37. A rule complex C is *inconsistent relative to* a non-empty family $\mathcal{X}$ of subcomplexes of C if there is a formula α such that $\alpha^{\mathrm{rul}}, \neg\alpha^{\mathrm{rul}} \in \mathrm{Der}(C|\mathcal{X})$; otherwise C is *consistent relative to* $\mathcal{X}$.

Notice that inconsistency as described in Definition 7.32 is a particular case of relative inconsistency, where the rule complex C is taken as the only member of the family $\mathcal{X}$. Indeed, C is inconsistent iff C is inconsistent relative to $\{C\}$, and analogously for consistency.

Example 7.38. Let D and E be as in Example 7.35. Recall that E is consistent. On the other hand, E is inconsistent relative to $\{D\}$.

It is also easy to find an example of a rule complex C and a non-empty family $\mathcal{X}$ of subcomplexes of C such that C is inconsistent and at the same time consistent relative to $\mathcal{X}$.

8. Summary

The aim of the paper was to define and study an appropriate notion of derivability of rules from a rule complex. To this end, we first introduced updated and improved versions of fundamental notions of our theory of rule complexes. Starting with the concept of derivability of rules we were able to obtain the corresponding notions of (in)consistency of a rule complex. We also formulated some ideas on the activation of rules and the application of rule complexes. An important direction for future research on derivability of rules from a rule complex seems to be vagueness of information and, in particular, reasoning about similarity.

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