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PARAINCONSISTENCY OF CREDIBILITY-BASED BELIEF STATES*

Abstract. In our approach credibility of information plays an important role in modeling of both belief state and belief change [4]. It turns out that the credibility-based consequence operators used to define the notion of belief state tolerate inconsistency under some conditions.

Keywords: parainconsistency, credibility of information, belief state.

1. Introduction

In [4] we proposed a general model of belief state and belief change, where both rationality of an agent as well as credibility of information were taken into account. In the present paper we develop some of the ideas, concerning the concept of belief state, presented there and discuss the problem of parainconsistency of such belief states.

It is hardly an exaggeration to say that our notion of belief state is credibility-based, while rationality, if understood along the standard lines, plays an auxiliary role only. It is a simple consequence of the fact that the consequence operators used to define a belief state are "credibility-oriented". Moreover, these operators tolerate inconsistency of sets of formulae, which is another interesting feature.

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Information and beliefs, viewed as sufficiently credible information, are represented by formulae of the classical propositional logic (PC). Starting with credibility of information, expressed in the form of pairs of real numbers attached to formulae, we arrive at credibility-based preference relations on formulae. Any such preference relation induces a consequence operator in the Tarski sense. Next, information is classified into three classes: highly credible information, lowly credible information, and the rest. Finally, a notion of belief state is defined by means of our consequence operators and the concept of high and low credibility. Belief states are pairs of sets of formulae that are closed under the consequence operators mentioned above and consist of sufficiently credible information. The first element of a belief state defined in such a way is called a belief set. It contains stronger beliefs and is a subset of the second element called a plausibility set. The plausibility set may contain weaker beliefs as well.

When information comes from several sources, it can happen that there is much evidence both for a considered fact as well as for its negation. As a consequence, both a piece of information and its negation may be highly (or at least not-lowly) credible to a given agent. On the other hand, contradictory statements like $p \wedge \neg p$ are not believed in general. Along the standard lines [17], a set of formulae X of a given language is parainconsistent (or, in other words, paraconsistent) if there is a formula α such that α and $\neg\alpha$ can be derived from it by means of a considered consequence operation but the set of all consequences of X is a proper subset of the set of all formulae. In our case we are faced with a credibility-based form of parainconsistency of an agent's state of beliefs. In the paper we consider this form of parainconsistency and investigate whether (and under what conditions if answered positively) it may be the case that (i) both a formula and its negation are highly credible; (ii) a formula is highly credible and its negation is not lowly credible; and (iii) neither a formula nor its negation are lowly credible. Similarly, we answer the question whether and under what conditions it may be the case that (i) both a formula and its negation belong to the same belief set; (ii) a formula belongs to a belief set, while its negation is a member of the corresponding plausibility set; (iii) both a formula and its negation are members of the same plausibility set. Finally, we obtain necessary conditions for belief and plausibility sets to be parainconsistent in our sense.

Section 2 contains preliminaries. Among others, we define a consequence operator which is based on a preference relation and which plays a fundamental role in defining of the notion of belief state. In Section 3 we present some notions concerning credibility of information, recall their properties given in [4], and also give new results useful or simply interesting from our standpoint. In particular, we recall such notions as pre-degree of credibility, credibility mapping, degree of credibility, credibility-based preference relation, and low and high credibility. In Section 4 we define a notion of belief state based on credibiity of information and investigate the question of parainconsistency. Final remarks are given in Section 5.

2. Preliminaries

Information and, in particular, beliefs are represented by formulae of the classical propositional logic (PC). Propositional letters are denoted by p, q with sub/superscripts if needed, propositional connectives by $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$, formulae, formed along the standard lines, by α, β, γ with sub/superscripts if needed, the set of all formulae by FOR, and the power set of a set X by $\mathcal{P}(X)$.

An operator $C\colon \mathcal{P}(FOR) \rightarrow \mathcal{P}(FOR)$ is a *consequence operator* in the Tarski sense if for any $X, Y \subseteq FOR$, C is *increasing* (i.e., $X \subseteq C(X)$), *monotonic* ($C(X) \subseteq C(Y)$ if $X \subseteq Y$), *idempotent* ($C(C(X)) \subseteq C(X)$), and *compact* ($C(X) = \bigcup\{C(X') \mid X' \subseteq X \wedge \mathrm{card}(X') < \aleph_0\}$). Cn denotes the classical consequence operator. Whenever convenient, $\alpha \in Cn(X)$ and $X \vdash \alpha$ are used interchangeably, while $TAUT$ stands for the set $Cn(\emptyset)$ of all PC-tautologies. *co-*$TAUT$ denotes the set of all PC-countertautologies, that is:

$$co\text{-}TAUT = \{\alpha \in FOR \mid \neg\alpha \in TAUT\}.$$

A set $X \subseteq FOR$ is C-*closed* if $C(X) = X$. It is *absolutely* (resp., *traditionally*) C-*consistent* iff $C(X) \neq FOR$ (resp., $\neg(\exists\alpha)(\alpha, \neg\alpha \in C(X))$); otherwise X is *absolutely* (*traditionally*) C-*inconsistent*.[1] Along the standard lines [17], sets of formulae traditionally C-inconsistent and absolutely C-consistent are called C-*parainconsistent* (or *paraconsistent* when looking from another perspective). An intersection $C_1 \cap C_2$ of two consequence operators C_1 and C_2, defined as

$$(C_1 \cap C_2)(X) = C_1(X) \cap C_2(X),$$

is a consequence operator.

A relation $\preceq\, \subseteq X^2$ is a *pre-ordering* on X, while $(X, \preceq)$ is a *pre-ordered* set if $\preceq$ is reflexive and transitive. The corresponding relations $\prec$ (*strict*

[1] In general, absolute and traditional C-consistency are different notions. However, they coincide for $C = Cn$. Clearly, traditional C-consistency implies the absolute one.

pre-ordering) and $\approx$ on X are defined as usual, i.e., for any $x, y \in X$:

$$x \prec y \quad \text{iff} \quad x \preceq y \wedge y \not\preceq x\,.$$
$$x \approx y \quad \text{iff} \quad x \preceq y \wedge y \preceq x\,.$$

Pre-orderings may be viewed as *preference* relations, where $x \preceq y$ (resp., $x \prec y$, $x \approx y$) is read as 'y is *preferred* to x' ('y is *strictly preferred* to x', 'x and y are equally preferred'). Along the standard lines if $\preceq$ is also antisymmetric, it is called *partial ordering* and $(X, \preceq)$ is *partially ordered*. Partial orderings which are also connected (i.e., any two elements are comparable) are called *total orderings*. Note that $\approx$ is the equality relation for partial orderings. An element $x \in X$ is the *greatest* (resp., *least*) in a partially ordered set $(X, \preceq)$ iff $(\forall y \in X) y \preceq x$ (resp., $x \preceq y$).

Let $D = \{(x, y) \in [0, 1]^2 \mid x \leqslant y\}$, where $[0, 1]$ is the unit interval and $\leqslant$ is the natural total ordering of real numbers. Some pre-orderings on D are defined below.

$$(x_1, y_1) \preceq_1 (x_2, y_2) \quad \text{iff} \quad x_1 \leqslant x_2 \wedge y_1 \leqslant y_2\,.$$
$$(x_1, y_1) \preceq_2 (x_2, y_2) \quad \text{iff} \quad x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \leqslant y_2)\,.$$
$$(x_1, y_1) \preceq_3 (x_2, y_2) \quad \text{iff} \quad y_1 < y_2 \vee (y_1 = y_2 \wedge x_1 \leqslant x_2)\,.$$
$$(x_1, y_1) \preceq_4 (x_2, y_2) \quad \text{iff} \quad y_1 \leqslant x_2 \vee (x_1 = x_2 \wedge y_1 = y_2)\,.$$

Relation $\preceq_2$ (resp., $\preceq_3$) is a total ordering known as the lexicographic (anti-lexicographic) ordering. Relations $\preceq_1$ and $\preceq_4$ are partial orderings. Let us note that $(1, 1)$ (resp., $(0, 0)$) is the greatest (least) element in $(D, \preceq_i)$ for $i = 1, \ldots, 4$.

Proposition 2.1. *For any $(x_j, y_j) \in D$ ($j = 1, 2$) and $i = 1, \ldots, 4$, we have that:*

$$(x_1, y_1) \prec_1 (x_2, y_2) \quad \textit{iff} \quad (x_1 < x_2 \wedge y_1 \leqslant y_2) \vee (x_1 \leqslant x_2 \wedge y_1 < y_2)\,.$$
$$(x_1, y_1) \prec_2 (x_2, y_2) \quad \textit{iff} \quad x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 < y_2)\,.$$
$$(x_1, y_1) \prec_3 (x_2, y_2) \quad \textit{iff} \quad y_1 < y_2 \vee (y_1 = y_2 \wedge x_1 < x_2)\,.$$
$$(x_1, y_1) \prec_4 (x_2, y_2) \quad \textit{iff} \quad x_1 < y_1 = x_2 \vee y_1 < x_2 \vee y_1 = x_2 < y_2\,.$$
$$\textit{If } (x_1, y_1) \neq (0, 0) \quad \textit{then } (0, 0) \prec_i (x_1, y_1)\,.$$

For any pre-ordering $\preceq \subseteq X^2$, we define two operators $\nabla_{\preceq}, \triangle_{\preceq} : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

$$\nabla_{\preceq} Y = \{x \in X \mid (\exists y \in Y)\, y \preceq x\}\,.$$
$$\triangle_{\preceq} Y = \{x \in X \mid (\exists y \in Y)\, x \preceq y\}\,.$$

For simplicity, $\nabla_{\preceq} x$ stands for $\nabla_{\preceq}\{x\}$, and similarly for $\triangle_{\preceq}$. Some basic properties of $\nabla_{\preceq}$ and $\triangle_{\preceq}$ are given below. Easy proofs are left as exercises.

PROPOSITION 2.2. *Let O stand for $\nabla_{\preceq}$ or $\triangle_{\preceq}$. For any $X_1, X_2 \subseteq X$, we have that:*

$$\begin{aligned} O(X_1) = \emptyset \quad &\textit{iff} \quad X_1 = \emptyset\,. \\ \textit{If } X_1 \subseteq X_2 \textit{ then } &O(X_1) \subseteq O(X_2)\,. \\ X_1 &\subseteq O(X_1)\,. \\ O(O(X_1)) &\subseteq O(X_1)\,. \\ O(X_1) &= \bigcup\{O(x) \mid x \in X_1\}\,. \\ O(X_1 \cup X_2) &= O(X_1) \cup O(X_2)\,. \\ O(X_1 \cap X_2) &\subseteq O(X_1) \cap O(X_2)\,. \end{aligned}$$

COROLLARY 2.3. *If $X = FOR$, then $\nabla_{\preceq}$ and $\triangle_{\preceq}$ are consequence operators.*

3. Credibility of information

In [4] we defined a *pre-degree of credibility* of information as a quantity (e.g., a number, a pair of numbers, an interval) describing initially how credible a considered piece of information is in a given situation. Such a quantity may be obtained as the result of some measurement, calculation or logical inference. In the next step pre-degrees are refined to satisfy some rationality postulates. In the present paper pre-degrees of credibility are elements of D, i.e. pairs of real numbers (x, y) such that $0 \leqslant x \leqslant y \leqslant 1$. This is a generalization of the following situation.

Example 3.1. In a group of k $(0 < k)$ experts or sources of information, m experts state that α holds, while n experts state that $\neg\alpha$ holds in the sense that it is not the case that α holds. Assume that experts have consistent opinions, i.e., no one states that both α and its negation hold simultaneously. However, some experts may be indecisive, i.e., $m + n \leqslant k$. Credibility of α is initially assessed by two fractions $\frac{m}{k}$ and $\frac{n}{k}$ of positive and negative answers, respectively. Thus, $(\frac{m}{k}, 1 - \frac{n}{k})$ may be taken as the pre-degree of credibility of α. Similarly, the pre-degree of credibility of $\neg\alpha$ is $(\frac{n}{k}, 1 - \frac{m}{k})$.

From a rational point of view, degrees of credibility of information should satisfy some postulates. For instance, it is reasonable to require for logically equivalent formulae to be given the same degree of credibility. Therefore pre-degrees of credibility may need an appropriate adjustment. Let us consider the ordered sets $(D, \preceq_i)$ $(i = 1, \ldots, 4)$ defined in Section 2. Every mapping

$cr_i \colon FOR \to D$ satisfying the following postulates for any formulae α, β is called a *credibility mapping* and its values are referred to as *credibility degrees*:

(CR1) If $\alpha \vdash \beta$, then $cr_i(\alpha) \preceq_i cr_i(\beta)$.

(CR2) If $\vdash \alpha$, then $cr_i(\alpha) = (1,1)$.

(CR3) If $\vdash \neg\alpha$, then $cr_i(\alpha) = (0,0)$.

The constraints imposed on credibility by (CR1)–(CR3) are natural. Tautologies are credible to the highest degree as opposite to countertautologies. Logically equivalent pieces of information are equally credible by (CR1). Starting with pre-degrees, one can arrive at credibility degrees [4]. Henceforth we will consider such credibility mappings cr_i that $cr_i(\alpha) = (x, y)$ iff $cr_i(\neg\alpha) = (1-y, 1-x)$.

Every cr_i induces a preference (pre-ordering) relation on FOR, $\preceq_{cr_i}$, as follows:

$$\alpha \preceq_{cr_i} \beta \quad \text{iff} \quad cr_i(\alpha) \preceq_i cr_i(\beta)\,.$$

Relations $\approx_{cr_i}$ and $\prec_{cr_i}$ are defined along the standard lines. Henceforth ∇_i will stand for $\nabla_{\preceq_{cr_i}}$ for the sake of simplicity, and analogously for $\triangle_{\preceq_{cr_i}}$.

PROPOSITION 3.2. *For any $X \subseteq FOR$ and $\alpha, \beta \in FOR$, we have that:*

$$\begin{aligned}
\alpha \approx_{cr_i} \beta \quad & \textit{iff} \quad cr_i(\alpha) = cr_i(\beta)\,. \\
\alpha \prec_{cr_i} \beta \quad & \textit{iff} \quad cr_i(\alpha) \prec_i cr_i(\beta)\,. \\
\textit{If } \vdash \alpha \leftrightarrow \beta, \quad & \textit{then } \alpha \approx_{cr_i} \beta\,. \\
\alpha \preceq_{cr_i} \beta \quad & \textit{iff} \quad \neg\beta \preceq_{cr_i} \neg\alpha, \ \textit{for } i = 1, 4\,. \\
\alpha \preceq_{cr_2} \beta \quad & \textit{iff} \quad \neg\beta \preceq_{cr_3} \neg\alpha\,. \\
\textit{If } \vdash \alpha, \quad & \textit{then } (\forall\beta)(\beta \preceq_{cr_i} \alpha \ \wedge \ \beta \approx_{cr_i} (\alpha \wedge \beta))\,. \\
\textit{If } \vdash \neg\alpha, \quad & \textit{then } (\forall\beta)(\alpha \preceq_{cr_i} \beta \ \wedge \ \beta \approx_{cr_i} (\alpha \vee \beta))\,. \\
\textit{If } \alpha \vdash \beta, \quad & \textit{then } \forall\gamma)\alpha \wedge \gamma \preceq_{cr_i} \beta \wedge \gamma\,. \\
\textit{If } X \neq \emptyset, \quad & \textit{then } TAUT \subseteq \nabla_i X \wedge co\text{-}TAUT \subseteq \triangle_i X\,. \\
\nabla_i co\text{-}TAUT = {} & FOR = \triangle_i TAUT\,. \\
\textit{If } X \neq \emptyset, \quad & \textit{then } \triangle_i \nabla_i X = FOR = \nabla_i \triangle_i X\,. \\
\nabla_i X = FOR \quad & \textit{iff} \quad co\text{-}TAUT \cap \nabla_i X \neq \emptyset\,.
\end{aligned}$$

PROOF. We only prove the last property. Assume that there is a countertautology $\alpha \in \nabla_i X$. Hence $cr_i(\alpha) = (0,0)$ and there is $\beta \in X$ such that $\beta \preceq_{cr_i} \alpha$. Thus $cr_i(\beta) = (0,0)$ and $(\forall\gamma)\beta \preceq_{cr_i} \gamma$. In other words, $FOR \subseteq \nabla_i X$. The remaining part is obvious. □

It is easy to see that operators ∇_i tolerate ∇_i-inconsistency in the sense that there exist ∇_i-parainconsistent sets of formulae.

Example 3.3. Let $X = \{q_1, q_2\}$, $cr_i(q_1) = (0.5, 0.6)$, $cr_i(q_2) = (0.4, 0.4)$, and $cr_i(p) = (0.5, 0.6)$ $(i = 1, \ldots, 4)$. Hence $cr_i(\neg p) = (0.4, 0.5)$. Clearly, $\nabla_i X \neq FOR$. On the other hand, $p, \neg p \in \nabla_i X$.

Given credibility of information, one may have to decide whether a particular piece of information is highly or lowly credible. The concepts of high and low credibility are subjective and change with agents, situation or domain of application.[2] In our model we distinguish three classes of information: highly credible information, lowly credible information, and the rest. The rules for decision making are simple: a piece of information is highly (resp., lowly) credible if its degree of credibility is higher (lower) than some threshold.[3] A formula is highly (resp., lowly) credible in case the information represented by the formula is highly (lowly) credible.

Let $l_i(\alpha)$ (resp., $h_i(\alpha)$) denote that α is *lowly (highly) credible* relative to the ordering $\preceq_i$ $(i = 1, \ldots, 4)$ and k_i stand for h_i or l_i. Given a pair of threshold values $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, low and high credibility of a formula may be defined as follows:

$$l_i(\alpha) \quad \text{iff} \quad cr_i(\alpha) \prec_i (t_1, t_2) \text{ and } h_i(\alpha) \quad \text{iff} \quad (t_1, t_2) \preceq_i cr_i(\alpha)\,.$$

Some basic properties of the notions just introduced are given below.

Proposition 3.4. *For any formulae α, β, we have that:*

$$\begin{aligned}
\textit{If } \alpha \preceq_{cr_i} \beta \wedge l_i(\beta), &\quad \textit{then} \quad l_i(\alpha)\,.\\
\textit{If } \alpha \vdash \beta \wedge l_i(\beta), &\quad \textit{then} \quad l_i(\alpha)\,.\\
\textit{If } \alpha \preceq_{cr_i} \beta \wedge h_i(\alpha), &\quad \textit{then} \quad h_i(\beta)\,.\\
\textit{If } \alpha \vdash \beta \wedge h_i(\alpha), &\quad \textit{then} \quad h_i(\beta)\,.\\
\textit{If } \alpha \approx_{cr_i} \beta, &\quad \textit{then} \quad (k_i(\alpha) \quad \textit{iff} \quad k_i(\beta))\,.\\
\textit{If } \vdash \alpha \leftrightarrow \beta, &\quad \textit{then} \quad (k_i(\alpha) \quad \textit{iff} \quad k_i(\beta))\,.\\
\textit{If } l_i(\alpha) \wedge h_i(\beta), &\quad \textit{then} \quad \alpha \prec_{cr_i} \beta\,.\\
\textit{If } i = 2, 3, &\quad \textit{then} \quad h_i(\alpha) \quad \textit{iff} \quad \neg l_i(\alpha)\,.
\end{aligned}$$

The proof is easy and hence omitted.

[2] For instance, information about a new physical phenomenon may be highly credible for ordinary people and of low credibility for physicists working in that area.

[3] First, note that our present approach is quantitative. In [4] a qualitative version is considered as well. Next, several interesting and non-trivial questions arise, e.g., how to obtain the threshold which is the most relevant in a given situation.

There arises a problem: Whether and under what conditions is there a formula α such that

(1) $h_i(\alpha)$ and $h_i(\neg\alpha)$;

(2) $h_i(\alpha)$ and $\neg l_i(\neg\alpha)$;[4]

(3) neither $l_i(\alpha)$ nor $l_i(\neg\alpha)$?

If (1) (resp., (2), and (3)) holds, then the set $A_i = \{(x, y) \in D \mid (t_1, t_2) \preceq_i (x, y) \wedge (t_1, t_2) \preceq_i (1 - y, 1 - x)\}$ (resp., $B_i = \{(x, y) \in D \mid (t_1, t_2) \preceq_i (x, y) \wedge (1 - y, 1 - x) \not\preceq_i (t_1, t_2)\}$, and $C_i = \{(x, y) \in D \mid (x, y) \not\preceq_i (t_1, t_2) \wedge (1 - y, 1 - x) \not\preceq_i (t_1, t_2)\}$) is non-empty. The converse, however, is not true in general since the co-domain $cr_i^{\rightarrow}(FOR)$ is a proper subset of $[0, 1]$. The following equivalences hold.

PROPOSITION 3.5.

$$\begin{aligned}
A_1 \neq \emptyset \quad &\text{iff} \quad B_1 \neq \emptyset \quad \text{iff} \quad t_2 \leqslant 1 - t_1 \,. \\
C_1 \neq \emptyset \quad &\text{iff} \quad t_1 < 0.5 \vee t_2 < 1 \vee t_2 \leqslant 1 - t_1 \,. \\
A_2 = B_2 = C_2 \neq \emptyset \quad &\text{iff} \quad t_1 < 0.5 \vee t_2 \leqslant 1 - t_1 \,. \\
A_3 = B_3 = C_3 \neq \emptyset \quad &\text{iff} \quad t_2 < 1 \vee t_2 \leqslant 1 - t_1 \,. \\
A_4 \neq \emptyset \quad &\text{iff} \quad t_2 \leqslant 0.5 \vee t_2 = 1 - t_1 \,. \\
B_4 \neq \emptyset \quad &\text{iff} \quad t_1 < 0.5 \vee t_1 = t_2 \leqslant 0.5 \,. \\
C_4 \neq \emptyset \quad &\text{iff} \quad t_1 < 1 \vee t_1 = t_2 \leqslant 0.5 \,.
\end{aligned}$$

PROOF. Leaving the remaining cases as exercises, we only show that $B_1 \neq \emptyset$ iff $t_2 \leqslant 1 - t_1$. Let us note that $(t_1, t_2) \preceq_1 (x, y)$ and $(1-y, 1-x) \not\preceq_1 (t_1, t_2)$ iff $t_1 \leqslant x$, $t_2 \leqslant y$, and neither $(1 - y < t_1 \wedge 1 - x \leqslant t_2)$ nor $(1 - y \leqslant t_1 \wedge 1 - x < t_2)$. After necessary transformations we obtain that $(t_1, t_2) \preceq_1 (x, y)$ and $(1 - y, 1 - x) \not\preceq_1 (t_1, t_2)$ iff $t_1 \leqslant x < 1 - t_2 \wedge t_2 \leqslant y$ or $t_1 \leqslant x \wedge t_2 \leqslant y < 1 - t_1$ or $t_1 \leqslant x = 1 - t_2 \wedge t_2 \leqslant y = 1 - t_1$. There are such $(x, y) \in D$ iff $t_2 \leqslant 1 - t_1$. □

Now we can give a partial answer[5] to the problem formulated above.

[4] The case that there is α such that $h_i(\neg\alpha)$ and $\neg l_i(\alpha)$ can be easily reduced to that one.

[5] The conditions below are necessary but not sufficient.

Corollary 3.6. *Let m_i stand for h_i or $\neg l_i$.*

$$\begin{aligned}
&\text{If } (\exists\alpha)(h_1(\alpha) \wedge m_1(\neg\alpha)), && \text{then} \quad t_2 \leqslant 1 - t_1\,.\\
&\text{If } (\exists\alpha)(\neg l_1(\alpha) \wedge \neg l_1(\neg\alpha)), && \text{then} \quad t_1 < 0.5 \vee t_2 < 1 \vee t_2 \leqslant 1 - t_1\,.\\
&\text{If } (\exists\alpha)(h_2(\alpha) \wedge m_2(\neg\alpha)), && \text{then} \quad t_1 < 0.5 \vee t_2 \leqslant 1 - t_1\,.\\
&\text{If } (\exists\alpha)(\neg l_2(\alpha) \wedge \neg l_2(\neg\alpha)), && \text{then} \quad t_1 < 0.5 \vee t_2 \leqslant 1 - t_1\,.\\
&\text{If } (\exists\alpha)(h_3(\alpha) \wedge m_3(\neg\alpha)), && \text{then} \quad t_2 < 1 \vee t_2 \leqslant 1 - t_1\,.\\
&\text{If } (\exists\alpha)(\neg l_3(\alpha) \wedge \neg l_3(\neg\alpha)), && \text{then} \quad t_2 < 1 \vee t_2 \leqslant 1 - t_1\,.\\
&\text{If } (\exists\alpha)(h_4(\alpha) \wedge h_4(\neg\alpha)), && \text{then} \quad t_2 \leqslant 0.5 \vee t_2 = 1 - t_1\,.\\
&\text{If } (\exists\alpha)(h_4(\alpha) \wedge \neg l_4(\neg\alpha)), && \text{then} \quad t_1 < 0.5 \vee t_1 = t_2 \leqslant 0.5\,.\\
&\text{If } (\exists\alpha)(\neg l_4(\alpha) \wedge \neg l_4(\neg\alpha)), && \text{then} \quad t_1 < 1 \vee t_1 = t_2 \leqslant 0.5\,.
\end{aligned}$$

For any set of formulae X, let $l_i(X) = \{\alpha \in X \mid l_i(\alpha)\}$ (the *set of lowly credible members* of X) and $h_i(X) = \{\alpha \in X \mid h_i(\alpha)\}$ (the *set of highly credible members* of X). Note that $X - l_i(X) = X - l_i(FOR)$. Below we relate the notions of low and high credibility to Cn, ∇_i, and $\triangle_i$. As earlier k_i stands for h_i or l_i.

Proposition 3.7. *For any sets of formulae X, Y, we have that:*

$$\begin{aligned}
k_i k_i(X) &= k_i(X)\,.\\
k_i(X \cap Y) &= k_i(X) \cap k_i(Y), \text{ and similarly for } \cup\,.\\
\triangle_i l_i(FOR) = l_i(FOR) \wedge \nabla_i h_i(FOR) &= h_i(FOR)\,.\\
l_i \triangle_i l_i(X) = \triangle_i l_i(X) &\subseteq l_i \triangle_i(X)\,.\\
h_i \nabla_i h_i(X) = \nabla_i h_i(X) &\subseteq h_i \nabla_i(X)\,.\\
\nabla_i(X - l_i(FOR)) &\subseteq \nabla_i X - l_i(FOR)\,.\\
h_i(Cn(h_i(X))) &\subseteq h_i(Cn(X))\,.\\
(Cn \cap \nabla_i)(X - l_i(FOR)) &\subseteq (Cn \cap \nabla_i)(X) - l_i(FOR)\,.\\
(Cn \cap \nabla_i) h_i(X) &\subseteq h_i(Cn \cap \nabla_i)(X)\,.\\
h_i(FOR) = FOR \quad \text{iff} \quad l_i(FOR) &= \emptyset \quad \text{iff} \quad t_1 = t_2 = 0\,.
\end{aligned}$$

Proof. We only prove the last property.[6] First, let us note that $h_i(FOR) \cap l_i(FOR) = \emptyset$. Hence if $h_i(FOR) = FOR$, then $l_i(FOR) = \emptyset$. Now suppose that $l_i(FOR) = \emptyset$, i.e., $(\forall\alpha)\, cr_i(\alpha) \not\preceq_i (t_1, t_2)$. In particular if $\alpha \in co\text{-}TAUT$ and hence $cr_i(\alpha) = (0, 0)$, then $(*)\ (0, 0) \not\preceq_i (t_1, t_2)$. Let $i = 2, 3$. Since $\preceq_i$ is total, $(t_1, t_2) \preceq_i (0, 0)$. Hence $t_1 = t_2 = 0$. If $i = 1$, then $(*)$ is equivalent

[6] A proof of the remaining cases can be found in [4].

to the disjunction: $t_1 < 0$ or $t_2 < 0$ or $t_1 = t_2 = 0$. Hence $t_1 = t_2 = 0$. If $i = 4$, then $(*)$ is equivalent to the disjunction: $t_1 = t_2 = 0$ or $t_1 < 0$. Hence $t_1 = t_2 = 0$ as earlier. Now if $t_1 = t_2 = 0$, then $h_i(\textit{co-TAUT}) = \textit{co-TAUT}$. Hence $h_i(FOR) = FOR$. □

4. Belief states

Many various models for representing states of beliefs are known in the areas of logic, philosophy, and artificial intelligence, depending on the perspective taken by the author or the intended area of application. For lack of space, instead of giving an exhaustive survey of the proposals, let us mention belief sets, belief bases, hypertheories, and pairs of belief sets.

As before it is assumed that beliefs are represented by formulae of a given language. Along the standard lines, *belief sets* are C-closed sets of formulae, where C is a logical consequence operator. In the AGM theory [1], it is assumed that the underlying logic contains the classical one. In modal logics approaches [10, 11], C is the consequence operator of an epistemic/doxastic modal logic. In nonmonotonic approaches, C may be the consequence operator of PC like in the autoepistemic logic [14, 15], the consequence operator of the classical first-order logic like in the default logic [18] or the consequence operator of a modal logic like in nonmonotonic modal logics [13, 20]. In nonmonotonic approaches however, being C-closed is a necessary but not sufficient condition for a set of formulae to represent beliefs. Indeed, belief sets based on a set of premises I called extensions (or expansions) of I are defined as C-closed sets of formulae satisfying some groundedness and stability conditions. Whenever direct or indirect practical applications are considered, belief states are represented by finite sets of formulae called *belief bases* [3, 6, 8, 12, 16]. In this case there arises a problem of independence of the results of operations on belief bases from the syntax. In [21] belief states of a rational agent are represented by *hypertheories* to grasp how doxastic dispositions of a rational agent change. In [5] a nonmonotonic framework for representing belief states in the presence of incomplete knowledge is proposed. Belief states are represented by *AE2* extensions which are pairs of sets of formulae, where the first (resp., second) set consists of all beliefs (disbeliefs)[7] of a rational agent in a given situation. The first set is Cn-closed, while the second one is closed under a rejection consequence Cn'. Both have to satisfy some groundedness and stability conditions in the spirit of autoepistemic logic.

[7] In [5] the terms 'accepted' and 'rejected information' were used.

In the present approach the main stress is laid on credibility of information. From this standpoint beliefs may be stronger or weaker. In low-risk situations we usually use beliefs of both types, while in risky cases we rather trust our stronger beliefs only. It can also be the case that normally we use stronger beliefs, while in problematic situations we also use weaker beliefs to enhance our reasoning capabilities or decision making. Keeping with the common terminology [19, 22], the stronger beliefs are simply called beliefs, while the weaker ones are referred to as plausible information. Belief states are represented by pairs of sets of formulae, where the first (resp., second) set called a *belief set* (*plausibility set*) consists of all believed (plausible) information.[8] In our approach two aspects are combined: *rationality* of an agent and *credibility* of information. In [4] we consider three kinds of consequence operators: (1) Cn; (2) ∇_i; and (3) $Cn \cap \nabla_i$ ($i = 1, \dots, 4$). The first operator yields purely rational consequences of premises, operators of the second kind yield consequences which are at least as credible as some premise, while operators of the last kind take into account both rationality and credibility. As traditional and absolute Cn-consistency coincide, there are no Cn-parainconsistent sets of formulae. For Cn-inconsistent sets of formulae, $Cn \cap \nabla_i = \nabla_i$. In the present paper we will only take into account operators of the second kind. Given a set of premises I and $i = 1, \dots, 4$, the belief state based on I is a pair $B_i(I) = (BS_i(I), PS_i(I))$, where $PS_i(I)$ (the plausibility set) and $BS_i(I) \subseteq PS_i(I)$ (the belief set) are defined as follows:

$$BS_i(I) = \nabla_i h_i(I) \text{ and } PS(I) = \nabla_i(I - l_i(FOR))$$

Informally speaking, beliefs have to be at least as credible as some highly credible premises and plausible information has to be at least as credible as some non-lowly credible premises. In this way, beliefs are highly credible and plausible information is not lowly credible. Given a belief state $B_i(I)$ as above, we will say that the plausibility set $PS_i(I)$ is *corresponding* for $BS_i(I)$, and similarly for the belief set. Let S_i stand for $BS_i(I)$ or $PS_i(I)$.

PROPOSITION 4.1. *For any formulae α, β, we have that:*

1. *If $\alpha \vdash \beta \wedge \alpha \in S_i$, then $\beta \in S_i$.*
2. *If $\vdash \alpha \leftrightarrow \beta$, then $\alpha \in S_i$ iff $\beta \in S_i$.*
3. *$S_i = FOR$ iff $S_i \cap co\text{-}TAUT \neq \emptyset$ iff*
   $t_1 = t_2 = 0$ and $(\exists \beta \in I)\, cr_i(\beta) = (0, 0)$.

[8] In [4] we also consider the case that belief/plausibility sets are not closed under a logical consequence operator.

PROOF. We only prove that $BS_i(I) = FOR$ iff $t_1 = t_2 = 0$ and $(\exists\beta \in I)\, cr_i(\beta) = (0,0)$, for a given cr_i and a threshold (t_1, t_2). The equivalence

$$BS_i(I) = FOR \quad \text{iff} \quad BS_i(I) \cap co\text{-}TAUT \neq \emptyset$$

follows from Proposition 3.2. Thus if $BS_i(I) = FOR$, then $(\exists\alpha)\alpha \wedge \neg\alpha \in BS_i(I)$. Clearly, $cr_i(\alpha \wedge \neg\alpha) = (0,0)$. Hence there is $\beta \in I$ such that $h_i(\beta)$ and $\beta \preceq_{cr_i} \alpha \wedge \neg\alpha$, i.e., $(t_1, t_2) \preceq_i cr_i(\beta) = (0,0)$. Thus $t_1 = t_2 = 0$ and there is $\beta \in I$ such that $cr_i(\beta) = (0,0)$. If $t_1 = t_2 = 0$ and $(\exists\beta \in I)\, cr_i(\beta) = (0,0)$, then $h_i(FOR) = FOR$ by Proposition 3.7 and $(\exists\alpha)\alpha \wedge \neg\alpha \in \nabla_i I$. Finally $BS_i(I) = \nabla_i h_i(I) = \nabla_i I = FOR$. □

Roughly speaking, property 1 (resp., 2) states that belief and plausibility sets are closed under direct derivation (equivalence) in PC. According to 3, belief and plausibility sets based on a set of premises I are either both absolutely ∇_i-consistent or both inconsistent in this sense. Moreover, they are inconsistent iff the threshold is $(0,0)$ and so is the degree of credibility of some premise.

Below we formulate the necessary and sufficient conditions for (1) both a formula and its negation to be in a belief set $BS_i(I)$; (2) a formula to be in $BS_i(I)$ and its negation to be in the corresponding plausibility set $PS_i(I)$; (3) both a formula and its negation to be in $PS_i(I)$. In other words, the conditions are necessary and sufficient for $BS_i(I)$ (resp., $PS_i(I)$) to be traditionally ∇_i-inconsistent in the case (1) (resp., (3)). Clearly, (1) and (2) are special cases of (3) since $BS_i(I) \subseteq PS_i(I)$. Needless to say, necessary (resp., sufficient) conditions for ∇_i-inconsistency of $PS_i(I)$ ($BS_i(I)$) remain necessary (sufficient) for $BS_i(I)$ ($PS_i(I)$) as well.

PROPOSITION 4.2. *For any formula α, we have that:*[9]

1. $\alpha, \neg\alpha \in BS_i(I)$ *iff* $(\exists\beta, \gamma \in I)(h_i(\beta) \wedge h_i(\gamma)$ *and* $\beta \preceq_{cr_i} \alpha$ *and* $\gamma \preceq_{cr_i} \neg\alpha)$ *iff* $h_i(I) \cap \triangle_i\alpha \neq \emptyset$ *and* $h_i(I) \cap \triangle_i\neg\alpha \neq \emptyset$.
2. $\alpha \in BS_i(I)$ *and* $\neg\alpha \in PS_i(I)$ *iff* $(\exists\beta, \gamma \in I)(h_i(\beta)$ *and* $\wedge\neg l_i(\gamma)$ *and* $\beta \preceq_{cr_i} \alpha$ *and* $\gamma \preceq_{cr_i} \neg\alpha)$ *iff* $h_i(I) \cap \triangle_i\alpha \neq \emptyset$ *and* $(I \cap \triangle_i\neg\alpha) - l_i(FOR) \neq \emptyset$.
3. $\alpha, \neg\alpha \in PS_i(I)$ *iff* $(\exists\beta, \gamma \in I)(\neg l_i(\beta)$ *and* $\neg l_i(\gamma)$ *and* $\beta \preceq_{cr_i} \alpha$ *and* $\gamma \preceq_{cr_i} \neg\alpha)$ *iff* $(I \cap \triangle_i\alpha) - l_i(FOR) \neq \emptyset$ *and* $(I \cap \triangle_i\neg\alpha) - l_i(FOR) \neq \emptyset$.

The proof is left as an exercise.

[9] Since $(\exists\alpha)(\alpha \in PS_i(I) \wedge \neg\alpha \in BS_i(I))$ is equivalent to $(\exists\alpha)(\alpha \in BS_i(I) \wedge \neg\alpha \in PS_i(I))$, the former case need not to be considered separately.

Despite of the elegant form of the above conditions and their sufficiency, what we would rather like to have is a characterization in terms of the threshold values t_1 and t_2. It is interesting that the obtained conditions are the same as in Corollary 3.6. They are necessary but not sufficient, unfortunately. As earlier S_i stands for $BS_i(I)$ or $PS_i(I)$.

PROPOSITION 4.3. 1. *If* $(\exists\alpha)(\alpha \in BS_1(I) \wedge \neg\alpha \in S_1)$, *then* $t_2 \leqslant 1 - t_1$.

2. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_1(I))$, *then* $t_1 < 0.5 \vee t_2 < 1 \vee t_2 \leqslant 1 - t_1$.
3. *If* $(\exists\alpha)(\alpha \in BS_2(I) \wedge \neg\alpha \in S_2)$, *then* $t_1 < 0.5 \vee t_2 \leqslant 1 - t_1$.
4. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_2(I))$, *then* $t_1 < 0.5 \vee t_2 \leqslant 1 - t_1$.
5. *If* $(\exists\alpha)(\alpha \in BS_3(I) \wedge \neg\alpha \in S_3)$, *then* $t_2 < 1 \vee t_2 \leqslant 1 - t_1$.
6. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_3(I))$, *then* $t_2 < 1 \vee t_2 \leqslant 1 - t_1$.
7. *If* $(\exists\alpha)(\alpha, \neg\alpha \in BS_4(I))$, *then* $t_2 \leqslant 0.5 \vee t_2 = 1 - t_1$.
8. *If* $(\exists\alpha)(\alpha \in BS_4(I) \wedge \neg\alpha \in PS_4(I))$, *then* $t_1 < 0.5 \vee t_1 = t_2 \leqslant 0.5$.
9. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_4(I))$, *then* $t_1 < 1 \vee t_1 = t_2 \leqslant 0.5$.

PROOF. Since the proof is technical and rather long, we show its idea only. Let us consider, e.g., the third condition for $S_2 = BS_2(I)$. Assume there is a formula α such that $\alpha, \neg\alpha \in BS_2(I)$. By Proposition 4.2, there are $\beta_1, \beta_2 \in I$ such that $h_2(\beta_j)$ $(j = 1, 2)$, $\beta_1 \preceq_{cr_2} \alpha$, and $\beta_2 \preceq_{cr_2} \neg\alpha$. Let $cr_2(\alpha) = (x, y)$ (hence $cr_2(\neg\alpha) = (1 - y, 1 - x)$) and $cr_2(\beta_j) = (x_j, y_j)$. One can check that there are $(x, y), (x_j, y_j) \in D$ $(j = 1, 2)$ such that $(t_1, t_2) \preceq_2 (x_j, y_j)$, $(x_1, y_1) \preceq_2 (x, y)$, and $(x_2, y_2) \preceq_2 (1 - y, 1 - x)$ iff $t_1 < 0.5$ or $t_2 \leqslant 1 - t_1$. Hence our condition follows. □

From Propositions 4.1 and 4.3 we obtain the necessary conditions for $BS_i(I)$ (or equivalently $PS_i(I)$) to be absolutely ∇_i-consistent and a formula α to exist, where (1) $\alpha, \neg\alpha \in BS_i(I)$; (2) $\alpha \in BS_i(I)$ and $\neg\alpha \in PS_i(I)$; (3) $\alpha, \neg\alpha \in PS_i(I)$. In other words, the conditions are necessary for ∇_i-parainconsistency of $BS_i(I)$ (resp., $PS_i(I)$) in the case (1) (resp., (3)). For simplicity, let Ψ_i be $(\forall\beta \in I)(0, 0) \prec_i cr_i(\beta)$ $(i = 1, \ldots, 4)$ and Φ_j $(j = 1, \ldots, 9)$ be the necessary conditions from Proposition 4.3, respectively.[10]

COROLLARY 4.4. *If* $BS_i(I) \neq FOR$ $(i = 1, \ldots, 4)$, *then the following conditions hold:*

1. *If* $(\exists\alpha)(\alpha \in BS_1(I) \wedge \neg\alpha \in S_1)$, *then*
   $0 = t_1 < t_2 \vee\ t_2 \leqslant 1 - t_1 < 1 \vee (\Phi_1 \wedge \Psi_1)$.

[10] That is, $\Phi_1 = t_2 \leqslant 1 - t_1$, $\Phi_2 = (t_1 < 0.5 \vee t_2 < 1 \vee t_2 \leqslant 1 - t_1)$, etc.

2. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_1(I))$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 < 0.5 \vee 0 < t_1 \leqslant t_2 < 1 \vee (\Phi_2 \wedge \Psi_1)$.

3. *If* $(\exists\alpha)(\alpha \in BS_2(I) \wedge \neg\alpha \in S_2)$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 < 0.5 \vee t_2 \leqslant 1 - t_1 < 1 \vee (\Phi_3 \wedge \Psi_2)$.

4. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_2(I))$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 < 0.5 \vee t_2 \leqslant 1 - t_1 < 1 \vee (\Phi_4 \wedge \Psi_2)$.

5. *If* $(\exists\alpha)(\alpha \in BS_3(I) \wedge \neg\alpha \in S_3)$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 \leqslant t_2 < 1 \vee t_2 \leqslant 1 - t_1 < 1 \vee (\Phi_5 \wedge \Psi_3)$.

6. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_3(I))$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 \leqslant t_2 < 1 \vee t_2 \leqslant 1 - t_1 < 1 \vee (\Phi_6 \wedge \Psi_3)$.

7. *If* $(\exists\alpha)(\alpha, \neg\alpha \in BS_4(I))$, *then*
   $0 = t_1 < t_2 \leqslant 0.5 \vee (t_1 = 0 \wedge t_2 = 1) \vee 0 < t_1 \leqslant t_2 \leqslant 0.5 \vee$
   $t_2 = 1 - t_1 < 1 \vee (\Phi_7 \wedge \Psi_4)$.

8. *If* $(\exists\alpha)(\alpha \in BS_4(I) \wedge \neg\alpha \in PS_4(I))$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 < 0.5 \vee 0 < t_1 = t_2 \leqslant 0.5 \vee (\Phi_8 \wedge \Psi_4)$.

9. *If* $(\exists\alpha)(\alpha, \neg\alpha \in PS_4(I))$, *then*
   $0 = t_1 < t_2 \vee 0 < t_1 < 1 \vee 0 < t_1 = t_2 \leqslant 0.5 \vee (\Phi_9 \wedge \Psi_4)$.

Except for the conditions of the form $\Phi_j \wedge \Psi_i$ not discussed here for the sake of simplicity, the remaining ones may be classified into several groups. (a) The conditions, where $t_1 = 0$ (i.e., $(t_1 = 0 \wedge t_2 = 1)$, $0 = t_1 < t_2$, and $0 = t_1 < t_2 \leqslant 0.5$) seem to be unintuitive from the common-sense standpoint. (b) According to the conditions $0 < t_1 \leqslant t_2 \leqslant 0.5$ and $0 < t_1 = t_2 \leqslant 0.5$ (cases 7–9), the threshold values should be relatively low which is not very promising. In the most optimistic case $t_1 = t_2 = 0.5$. (c) The condition $0 < t_1 < 0.5$ (cases 2–4, and 8) requires that t_1 be fairly low but, on the other hand, t_2 may be arbitrary (i.e., $t_1 \leqslant t_2 \leqslant 1$). ($d$) The conditions $t_2 \leqslant 1 - t_1 < 1$ and $t_2 = 1 - t_1 < 1$ (cases 1 and 3–7) form yet another class. In this case $t_1 = 0.5 - a$ and $t_2 \leqslant 0.5 + a$ for some $0 \leqslant a < 0.5$. Hence the distance between t_1 and t_2 should not be greater than $2a$. Intuitively, the situation is realistic, especially for small a, and it should be no particular problems with finding parainconsistent belief and plausibility sets. (e) The last class considered here consists of the conditions $0 < t_1 \leqslant t_2 < 1$ and $0 < t_1 \leqslant t_2 \leqslant 1$ (cases 2, 5, 6, and 9). Here it should be easy to find parainconsistent belief states since the terms imposed on t_1 and t_2 are indeed minimal.

5. Summary

The aim of the paper was to show that the situation, where a belief state is parainconsistent in the sense that its belief and/or plausibility sets are C-parainconsistent, is natural if the consequence operator C takes into account credibility of information like the operators ∇_i $(i = 1, \ldots, 4)$ do. Indeed, the necessary conditions for belief and plausibility sets to be ∇_i-parainconsistent obtained above can be satisfied in a fairly large number of cases.

Since ∇_i tolerates inconsistency (in the sense that there exist ∇_i-parainconsistent sets of formulae) and for any formulae α, β, we have that $cr_i(\alpha \to (\beta \to \alpha)) = (1, 1)$, operator ∇_i is not closed under the *adjunction rule*:

$$\frac{\alpha,\ \beta}{\alpha \wedge \beta}$$

It is easy to give an informal example justifying this statement, however, systematic considerations are postponed to a separate paper.

Example 5.1. Let p_i $(i = 1, 2)$ represent the information that 'Skier S_i wins the competition.' Suppose each p_i is highly credible. On the other hand, $p_1 \wedge p_2$ is almost incredible.

Another problem worthy of consideration is the problem of change of (possibly parainconsistent) belief states and the threshold values.

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