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THE PROBLEM OF FORMALIZATION OF SOME NONSTANDARD SEMANTICS*

1. I shall propose the formalization of some nonstandard semantics in sequential form.

On semantical level the problem of paraconsistency has the following aspects:

First of all, what does it mean that "contradictions entails everything"? What kind of entailment is involved here? Below we are going to show that under the different presuppositions different notions of entailment relation can be formalized in the same system.

Secondly, what is our interpretation of contradiction. This question is related to the treatment of negation.

The notion of impossible worlds and its analogous are not admitted in semantics as considered to be less clear. Instead of this predicates of truth and falsity are considered to be partially defined.

We develop the idea of the symmetry of concepts of truth and falsity (and this is very important). Falsity is considered to be an *independent* notion and not as absence or negation of the truth.

2. Let W be a nonempty set of possible worlds and $\varphi\colon \mathrm{Var} \to \mathcal{P}(W) \times \mathcal{P}(W)$, where Var is the set of all propositional variables and $\mathcal{P}(W)$ is the power set of W, i.e. φ is a function ascribing a par of sets $\langle H_1, H_2\rangle$ to propositional variables where $H_1 \subseteq W$ and $H_2 \subseteq W$.

* Presented by title.

If $\varphi(p) = \langle H_1, H_2 \rangle$, then we put $\varphi_{\mathrm{t}}(p) := H_1$ and $\varphi_{\mathrm{f}}(p) := H_2$. H_1 is the set of worlds in which p *holds* (the domain of a sentence) and H_2 is the set of worlds in which p *does not hold* (the anti-domain of a sentence).

The relation between the sets $\varphi_{\mathrm{t}}(A)$ and $\varphi_{\mathrm{f}}(A)$ may be either satisfy or not satisfy the following conditions:

$$(*) \qquad \varphi_{\mathrm{t}}(A) \cap \varphi_{\mathrm{f}}(A) = \emptyset\,,$$

$$(**) \qquad \varphi_{\mathrm{t}}(A) \cup \varphi_{\mathrm{f}}(A) = W\,.$$

For the different cases we get the following semantics:

$(*)$	$(**)$	*semantics*
accepting	accepting	standard
accepting	rejecting	with truth value gaps
rejecting	accepting	with glut evaluations
rejecting	rejecting	relevant

For any formula A we shall mark the complement of the set $\varphi_{\mathrm{f}}(A)$ by $\varphi_{-\mathrm{f}}(A)$, i.e., we put $\varphi_{-\mathrm{f}}(A) := W \setminus \varphi_{\mathrm{f}}(A)$.

Remark 1. Of course, in standard semantics we have $\varphi_{-\mathrm{f}}(A) = \varphi_{\mathrm{t}}(A)$; in semantics with truth value gaps (resp. with glut evaluations) we have $\varphi_{\mathrm{t}}(A) \subseteq \varphi_{-\mathrm{f}}(A)$ (resp. $\varphi_{-\mathrm{f}}(A) \subseteq \varphi_{\mathrm{t}}(A)$).

3. With our approach, it is possible to introduce not one, but a whole class of different relations of a type of logical entailment. We define these relations of logical entailment in the terms of relations between domains and anti-domains of formulas.

DEFINITION. Let s be a semantics from the table, $n \in \{1, \ldots, 9\}$, and A, B be any formulas. Then:

A logically entails B in s and in the type n iff

for any model $\langle W, \varphi \rangle$ for s, A and B satisfy below condition n.

1. $\varphi_{\mathrm{t}}(A) \cap \varphi_{-\mathrm{f}}(A) \subseteq \varphi_{\mathrm{t}}(B)$
2. $\varphi_{\mathrm{t}}(A) \cap \varphi_{-\mathrm{f}}(A) \subseteq \varphi_{-\mathrm{f}}(B)$
3. $\varphi_{\mathrm{t}}(A) \cap \varphi_{-\mathrm{f}}(A) \subseteq \varphi_{\mathrm{t}}(B) \cup \varphi_{-\mathrm{f}}(B)$
4. $\varphi_{\mathrm{t}}(A) \subseteq \varphi_{\mathrm{t}}(B)$
5. $\varphi_{\mathrm{t}}(A) \subseteq \varphi_{-\mathrm{f}}(B)$

6. $\varphi_{\mathrm{t}}(A) \subseteq \varphi_{\mathrm{t}}(B) \cup \varphi_{-\mathrm{f}}(B)$
7. $\varphi_{-\mathrm{f}}(A) \subseteq \varphi_{\mathrm{t}}(B)$
8. $\varphi_{-\mathrm{f}}(A) \subseteq \varphi_{-\mathrm{f}}(B)$
9. $\varphi_{-\mathrm{f}}(A) \subseteq \varphi_{\mathrm{t}}(B) \cup \varphi_{-\mathrm{f}}(B)$

Remark 2. (i) In standard semantics we have only one type of logical entailment (4); because every condition from 1 to 9 is equivalent to condition 4.

(ii) In semantics with truth value gaps we have only four types of logical entailment (4, 5, 7, 8); condition 1 is equivalent to condition 4, conditions 2, 3 and 6 are equivalent to 5, and condition 9 is equivalent to 8.

(iii) In semantics with glut evaluations we have only four types of logical entailment (4, 5, 7, 8); conditions 1, 3 and 9 are equivalent to 7, condition 2 is equivalent to 8, and condition 6 is equivalent to 4.

The other relations relations between domains and anti-domains of formulas may be reduced to the conjunction of some of these nine relations.

4. Now, let us consider the problem of the formalization of logics (resp. relations of logical entailment $\rightarrow$), describe above. Let us formulate some well known logical systems in sequential form. The following figures of conclusion are common for these systems:

$$\frac{\Gamma \rightarrow \Theta, A \qquad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \,\&\, B} \qquad \frac{\Gamma \rightarrow \Theta, \neg A, \neg B}{\Gamma \rightarrow \Theta, \neg(A \,\&\, B)}$$

$$\frac{A, B, \Gamma \rightarrow \Theta}{A \,\&\, B, \Gamma \rightarrow \Theta} \qquad \frac{\neg A, \Gamma \rightarrow \Theta \qquad \neg B, \Gamma \rightarrow \Theta}{\neg(A \,\&\, B), \Gamma \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow \Theta, A, B}{\Gamma \rightarrow \Theta, A \vee B} \qquad \frac{\Gamma \rightarrow \Theta, \neg A \qquad \Gamma \rightarrow \Theta, \neg B}{\Gamma \rightarrow \Theta, \neg(A \vee B)}$$

$$\frac{A, \Gamma \rightarrow \Theta \qquad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta} \qquad \frac{\neg A, \neg B, \Gamma \rightarrow \Theta}{\neg(A \vee B), \Gamma \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow \Theta, \neg A \qquad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \supset B} \qquad \frac{\Gamma \rightarrow \Theta, A \qquad \Gamma \rightarrow \Theta, \neg B}{\Gamma \rightarrow \Theta, \neg(A \supset B)}$$

$$\frac{\neg A, \Gamma \rightarrow \Theta \qquad B, \Gamma \rightarrow \Theta}{A \supset B, \Gamma \rightarrow \Theta} \qquad \frac{A, \neg B, \Gamma \rightarrow \Theta}{\neg(A \supset B), \Gamma \rightarrow \Theta}$$

$$\frac{\Gamma \rightarrow \Theta, A}{\Gamma \rightarrow \Theta, \neg\neg A} \qquad \frac{A, \Gamma \rightarrow \Theta}{\neg\neg A, \Gamma \rightarrow \Theta}$$

Structural rules are usual. Observe that cut elimination theorem holds.

5. Basic sequences are different in these different systems. Let us consider the following four types:

$$A, \Gamma \rightarrow \Theta, A$$
$$A, \neg A, \Gamma \rightarrow \Theta$$
$$\Gamma \rightarrow \Theta, B, \neg B$$
$$A, \neg A, \Gamma \rightarrow \Theta, B, \neg B$$

If sequences only of type (1) are considered as basic, we get de Morgan's logic (M); if sequences of types (1) and (2) — then we get the system of Hao Wang (WH) in sequential form (axiomatic construction in which is given by Allan Rose); if we have the sequences of types (1) and (3) — then we get the logical system dual to Hao Wang's logic (DWN); with sequences of types (1) and (4) we have a variant of Łukasiewicz's logic; finally, with sequences of types (1), (2) and (3) we have the classical system (C).

The following theorems hold:

Theorem 1. *In the semantics with truth value gaps the relation of logical entailment:*

(a) *of type 7 is empty,*

(b) *of type 4 is formalized by* WH,

(c) *of type 8 is formalized by* DWH,

(d) *of type 5 is formalized by* C.

Theorem 2. *In the semantics with glut evaluations the relation of logical entailment:*

(a) *of type 5 is empty,*

(b) *of type 4 is formalized by* DWH,

(c) *of type 8 is formalized by* WH,

(d) *of type 7 is formalized by* C.

Theorem 3. *In the relevant semantics the relation of logical entailment:*

(a) *of types 5 and 7 are empty,*

(b) *of types 4 and 8 are formalized by* M,

(c) *of types 1 and 2 are formalized by* WH,

(d) *of types 6 and 9 are formalized by* DWH.

The proofs of theorems 1–3 are given in the book [2], chapter 5 (cf. also [3]).

References

[1] Smirnova, E. D., "Semantics with thruth value gaps, glut evaluations, and the concept of logical entailment", in *Intensional Logic and the Logical Structure of Theories* (in Russian), Tbilisi, 1985.

[2] Smirnova, E. D., *Logic and Philosophy* (in Russian), Moscow, 1996.

[3] Smirnova, E. D., "An approach to the justification of semantics to paraconsistent logics", pp. 255–262 in *Frontiers of Paraconsistent Logic*, D. Batens, C. Mortensen, G. Priest, J.-P. Van Bendegem (eds.), Studies in Logic and Computation, no. 8, Research Studies Press Ltd., Baldock, England, 2000.

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