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## LOGIC AND GROUPS

### 1. Introduction

Abelian group logic (AGL) — in other words, the logic which is sound and complete w.r.t. Abelian groups — is a non-trivial inconsistent logic, i.e. what some paraconsistent logicians call a “dialethic” logic. AGL entered the arena of paraconsistent research in the late 1980s, when Casari (1989) and Meyer and Slaney (1989), quite independently of each other, first axiomatized it and studied its properties (it must be said, however, that Meyer and Slaney circulated unpublished material about Abelian group logic as early as in 1981, so it seems correct to assign them chronological priority).

Casari, in particular, considered Abelian groups as a borderline case of a more general class of algebraic structures (pregroups), also encompassing MV-algebras and Boolean algebras. Correspondingly, he treated AGL as an inconsistent extension of a logic aimed at formalizing the idea of a “comparative implication” (see also Casari, 1990, 1997 for details). Another paper containing results on Abelian group logic is Restall (1993). In both Casari’s and Meyer-Slaney’s approaches AGL is introduced as the intensional fragment of a wider logic, call it L-AGL, the logic of *lattice-ordered* Abelian groups. Such a system contains, of course, connectives of conjunction and disjunction enjoying lattice properties. However, both papers also devote some attention to the purely intensional system AGL (AI in Meyer-Slaney’s terminology).

In what follows, we shall try to push further the study of AGL, trying to highlight its extreme simplicity and symmetry, properties relatively to which it is matched to a comparable extent only, perhaps, by classical logic. At the close, we shall prove some results concerning L-AGL, too.



Section 2 will (except for Lemma 2) be a partial survey of the work done by Casari on AGL. Section 3 will deal with the gentzenization of such a system. Sections 4 and 5 will be devoted to semantics. Section 6 is about L-AGL.

## 2. Hilbert-style systems

The language  $\mathcal{L}(\text{AGL})$  is a propositional language containing the connectives of negation ( $\neg$ ) and implication ( $\rightarrow$ ). AGL can be axiomatized as follows (cf. Casari, 1989):

- (A1)  $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
- (A2)  $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
- (A3)  $(A \rightarrow A) \rightarrow (B \rightarrow B)$
- (A4)  $\neg(A \rightarrow A)$
- (A5)  $\neg\neg A \rightarrow A$
- (A6)  $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
- (A7)  $\neg(A \rightarrow A) \rightarrow (A \rightarrow A)$
- (A8)  $(\neg(A \rightarrow A) \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$
- (R1)  $A, A \rightarrow B \Rightarrow B$

Equivalently, we can enlarge  $\mathcal{L}(\text{AGL})$  by the propositional constant  $T$  (“true”) and replace A3, A4, A7, A8 by:

- (A3')  $T \rightarrow (A \rightarrow A)$
- (A3'')  $(A \rightarrow A) \rightarrow T$
- (A4')  $\neg T$
- (A7')  $\neg T \rightarrow T$
- (A8')  $(\neg T \rightarrow T) \rightarrow T$

It will turn out useful to have recorded some theses and admissible rules of AGL (we shall henceforth drop the subscript “AGL” near the turnstile whenever convenient).



- LEMMA 1.           (i)  $\vdash A \rightarrow A$ ;  
                 (ii)  $\vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$ ;  
                 (iii)  $\vdash (A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$ ;  
                 (iv)  $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ ;  
                 (v)  $\vdash A \rightarrow \neg\neg A$ ;  
                 (vi)  $\vdash A \Rightarrow \vdash \neg A$ ;  
                 (vii)  $\vdash A \rightarrow B \Rightarrow \vdash B \rightarrow A$ ;  
                 (viii)  $\vdash \neg(A \rightarrow B) \rightarrow (B \rightarrow A)$ ;  
                 (ix)  $\vdash (A \rightarrow B) \rightarrow \neg(B \rightarrow A)$ ;  
                 (x)  $\vdash A, \vdash B \Rightarrow \vdash A \rightarrow B$ ;  
                 (xi)  $\vdash T$ ;  
                 (xii)  $\vdash A \Rightarrow \vdash T \rightarrow A$ .

PROOF. Proofs of (i)–(v) can be found in Casari (1989). As regards (vi)–(xii), it is a trivial exercise to check the soundness of such principles w.r.t. the algebraic semantics which follows. The existence of a proof for each of them follows then from the completeness theorem below (which of course does not depend on them).  $\square$

If VAR is the set of propositional variables of  $\mathcal{L}(\text{AGL})$  and FOR is the free algebra of formulas of AGL, an *algebraic model*  $\mathcal{A}$  is a pair  $\langle \mathcal{G}, \rho \rangle$ , where  $\mathcal{G} = \langle G, +, -, 0 \rangle$  is an Abelian group and the *realization*  $\rho: \text{FOR} \rightarrow \mathcal{G}$  is a homomorphism extending the arbitrary mapping  $\rho^*: \text{VAR} \rightarrow G$  in such a way that:

$$\begin{aligned}\rho(p) &= \rho^*(p); \\ \rho(\neg A) &= -\rho(A); \\ \rho(A \rightarrow B) &= -\rho(A) + \rho(B).\end{aligned}$$

(If the constant T is in the language also:

$$\rho(T) = 0).$$

We say that  $A$  is  $\rho$ -true in  $\mathcal{A}$  ( $\rho \vDash_{\mathcal{A}} A$ ) iff  $\rho(A) = 0$ ; that  $A$  is true in  $\mathcal{A}$  ( $\vDash_{\mathcal{A}} A$ ) iff  $\rho \vDash_{\mathcal{A}} A$  for every  $\rho$  on  $\mathcal{A}$ ; that  $A$  is *logically valid* ( $\vDash_{\text{A-AGL}} A$ ) iff  $\vDash_{\mathcal{A}} A$  for every algebraic model  $\mathcal{A}$ .

THEOREM 1.  $\vdash_{\text{AGL}} A$  iff  $\vDash_{\text{A-AGL}} A$ .

PROOF. See Casari (1989).  $\square$



By an *AGL-theory* we mean a set  $\mathbf{M}$  of formulas of  $\mathcal{L}(\text{AGL})$  s.t. if  $A \in \mathbf{M}$  and  $\vdash_{\text{AGL}} A \rightarrow B$ , then  $B \in \mathbf{M}$ . It is easy to prove:

LEMMA 2. *If  $\mathbf{M}$  is an AGL-theory and, for some  $A$ , both  $\vdash A$  and  $A \in \mathbf{M}$ , then  $\text{AGL} \subseteq \mathbf{M}$ .*

PROOF. Suppose  $\vdash B$  and  $B \in \mathbf{M}$ . Then, if  $C$  is any theorem of AGL, by Lemma 1(x) we have  $\vdash B \rightarrow C$ , whence  $C \in \mathbf{M}$  as  $\mathbf{M}$  is an AGL-theory.  $\square$

### 3. Sequent systems

We now introduce two Gentzen-style versions of AGL. First, we shall consider the two-sided calculus G-AGL, with negation and implication as primitive connectives.

Let  $\Gamma, \Delta, \dots$  stand for possibly empty finite multisets of formulas of  $\mathcal{L}(\text{AGL})$ . The postulates of G-AGL are:

$$\begin{aligned} (\text{Ax}) \quad & \Gamma \Rightarrow \Gamma \\ (\text{Cut}) \quad & \frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \\ (\neg \text{L}) \quad & \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \\ (\neg \text{R}) \quad & \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \\ (\rightarrow \text{L}) \quad & \frac{A, \Gamma \Rightarrow \Delta, B}{B \rightarrow A, \Gamma \Rightarrow \Delta} \\ (\rightarrow \text{R}) \quad & \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \end{aligned}$$

In what follows, we shall sometimes find it more convenient to resort to a one-sided version of Abelian group logic. The calculus O-AGL has therefore primitive literals instead of variables and just one primitive binary connective, “ $\oplus$ ”. Then, generalized negation is introduced as usual (cf. e.g. Girard, 1987), except for the fact that we have to take care of the self-duality of  $\oplus$  in the De Morgan equivalences.  $A \rightarrow B$  is defined as  $\neg A \oplus B$ . Here are the postulates of O-AGL:

$$(\text{Ax}) \quad \Rightarrow A_1, \dots, A_n, \neg A_1, \dots, \neg A_n \quad (n \geq 0)$$



$$(Cut) \quad \frac{\Rightarrow \Gamma, A \quad \Rightarrow \Delta, \neg A}{\Rightarrow \Gamma, \Delta}$$

$$(\oplus) \quad \frac{\Rightarrow \Gamma, A, B}{\Rightarrow \Gamma, A \oplus B}$$

Returning now to our main system G-AGL, we are in a position to prove:

THEOREM 2.  $\vdash_{AGL} A \text{ iff } \vdash_{G-AGL} \Rightarrow A$ .

PROOF. Left to right. We proceed by induction on the proof of  $A$  in AGL. Here are some examples (where in each case it is obvious which rule has been applied):

$$(A2) \quad \frac{\frac{\frac{C, B, A \Rightarrow C, B, A}{B \rightarrow C, B, A \Rightarrow C, A}}{A \rightarrow (B \rightarrow C), B, A \Rightarrow C}}{A \rightarrow (B \rightarrow C), B \Rightarrow A \rightarrow C}}{A \rightarrow (B \rightarrow C) \Rightarrow B \rightarrow (A \rightarrow C)}}{\Rightarrow (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))}$$

$$(A3) \quad \frac{\frac{\frac{A, B \Rightarrow A, B}{A \rightarrow A, B \Rightarrow B}}{A \rightarrow A \Rightarrow B \rightarrow B}}{\Rightarrow (A \rightarrow A) \rightarrow (B \rightarrow B)}$$

$$(A4) \quad \frac{\frac{A \Rightarrow A}{A \rightarrow A \Rightarrow}}{\Rightarrow \neg(A \rightarrow A)}$$

$$(A7) \quad \frac{\frac{\frac{A, A \Rightarrow A, A}{A \Rightarrow A, A \rightarrow A}}{\neg(A \rightarrow A), A \Rightarrow A}}{\neg(A \rightarrow A) \Rightarrow A \rightarrow A}}{\Rightarrow \neg(A \rightarrow A) \rightarrow (A \rightarrow A)}$$

Right to left. It is enough to extend to sequents the semantical concepts introduced in Section 2 and show that, if  $\vdash_{G-AGL} \Gamma \Rightarrow \Delta$ , then  $\Gamma \Rightarrow \Delta$  is logically valid. Once this is done in the appropriate way, our claim becomes a consequence of Theorem 1 for  $\Gamma = \emptyset$  and  $\Delta = \{A\}$ .



Hence, let  $\rho(A_1, \dots, A_n \Rightarrow B_1, \dots, B_m) = -\rho(A_1) + \dots + -\rho(A_n) + \rho(B_1) + \dots + \rho(B_m)$  and  $\rho(\Rightarrow) = 0$ . The sequent  $\Gamma \Rightarrow \Delta$  is said to be *logically valid* iff  $\rho(\Gamma \Rightarrow \Delta) = 0$  for every  $\rho$  on every algebraic model  $\mathcal{A}$ . Now we can prove the “if” part of our theorem by a standard induction on the length of the proof of  $\Gamma \Rightarrow \Delta$  in G-AGL. The cases  $(\neg L)$  and  $(\rightarrow R)$  are left to the reader.

(Ad Ax).  $\rho(A_1, \dots, A_n \Rightarrow A_1 \dots, A_n) = -\rho(A_1) + \dots + -\rho(A_n) + \rho(A_1) + \dots + \rho(A_n) = 0 + \dots + 0 = 0$ . Moreover, if  $n = 0$  we are done by definition.

(Ad Cut). Let  $-\rho(\Gamma) + \rho(\Delta) = x$ ,  $-\rho(\Pi) + \rho(\Sigma) = y$ ,  $\rho(A) = z$ . By IH,  $x + z = 0$  and  $-z + y = 0$ . So  $0 = 0 + 0 = -z + z + x + y = 0 + x + y = x + y$ .

(Ad  $\neg R$ ). If  $-\rho(\Gamma) + \rho(\Delta) = x$  and  $\rho(A) = y$ , by IH  $-y + x = 0$ , which is actually what we wanted to prove.

(Ad  $\rightarrow L$ ). If  $-\rho(\Gamma) + \rho(\Delta) = x$ ,  $\rho(A) = y$  and  $\rho(B) = z$ , then by IH  $-y + x + z = 0$ . But then  $-(-z + y) + x = -y + z + x = 0$ .  $\square$

**THEOREM 3.** *G-AGL is cut-free.*

**PROOF.** First, let us show that if  $\mathcal{D}$  is a proof of the following form:

$$\frac{\frac{\mathcal{D}'}{\Gamma \Rightarrow \Delta, A} \quad \frac{\mathcal{D}''}{A, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (Cut)}$$

where  $\mathcal{D}'$  and  $\mathcal{D}''$  are cut-free proofs, then the previous application of cut can be replaced by an application of the following rule:

$$\text{(Elim)} \quad \frac{A, A \Rightarrow \Xi, A}{A \Rightarrow \Xi}$$

yielding the same end sequent. Indeed, if  $\mathcal{D}'$  and  $\mathcal{D}''$  are cut-free, they are chains, since no other rule of G-AGL has two premisses:

$$\frac{\Phi \Rightarrow \Phi}{\Gamma \Rightarrow \Delta, A} \mathcal{D}' \qquad \frac{\Theta \Rightarrow \Theta}{A, \Pi \Rightarrow \Sigma} \mathcal{D}''$$

Hence, we can construct a  $\mathcal{D}'''$  as follows:

$$\frac{\frac{\Phi, \Theta \Rightarrow \Phi, \Theta}{A, \Gamma, \Pi \Rightarrow \Delta, \Sigma, A} \mathcal{D}'''}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (Elim)}$$

A cut-elimination procedure for G-AGL can then be carried out in three steps:



- (A) We replace one by one all applications of (Cut) by applications of (Elim), starting from the maximal sequents in each branch of the proof-tree and descending down to its root. Proof-trees become chains.
- (B) We show that (Elim) is superfluous in proofs containing a single final application of such a rule.
- (C) We extend this result in the standard way to proofs containing a finite arbitrary number of applications of (Elim).

Proof of (B) is a double induction on the rank and the complexity of the principal formula in the application of (Elim) at issue. However, we must suitably adapt to the present case the ordinary definition of rank.

Consider the following proof:

$$\frac{\frac{\mathcal{D}}{\Pi \Rightarrow \Sigma} \text{ (Rule)}}{A, \Gamma \Rightarrow \Delta, A} \text{ (Elim)}}{\Gamma \Rightarrow \Delta} \text{ (Elim)}$$

We stipulate that the left rank of  $A$  ( $rl(A)$ ), i.e. the rank relative to the first occurrence of  $A$ , remains unaltered in passing from  $\Pi \Rightarrow \Sigma$  to  $A, \Gamma \Rightarrow \Delta, A$  if (1) that occurrence of  $A$  was obtained by a rule other than (Ax) and (2) the second occurrence of  $A$  is the principal formula of the application of (Rule); otherwise it increases by one unit. Similar consideration apply to the right rank of  $A$  ( $rr(A)$ ). As usual,  $r(A)$  is defined as  $rl(A) + rr(A)$ .

*Basis* ( $r(A) = 2$ ). Due to the absence of structural rules, there are just three cases to consider (up to trivial simmetries such as permutation of the order of inferences).

First case:

$$\mathcal{D}: \frac{A, \Gamma \Rightarrow \Gamma, A}{\Gamma \Rightarrow \Gamma} \text{ (Elim)} \qquad \mathcal{D}': \Gamma \Rightarrow \Gamma$$

Second case:

$$\mathcal{D}: \frac{\frac{\frac{A, \Gamma \Rightarrow \Delta, A}{\neg A, A, \Gamma \Rightarrow \Delta} \text{ (\neg L)}}{\neg A, \Gamma \Rightarrow \Delta, \neg A} \text{ (\neg R)}}{\Gamma \Rightarrow \Delta} \text{ (Elim)} \qquad \mathcal{D}': \frac{A, \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta} \text{ (Elim)}$$

Third case:

$$\mathcal{D}: \frac{\frac{\frac{A, B, \Gamma \Rightarrow \Delta, A, B}{A \rightarrow B, A, \Gamma \Rightarrow \Delta, B} \text{ (\rightarrow L)}}{A \rightarrow B, \Gamma \Rightarrow \Delta, A \rightarrow B} \text{ (\rightarrow R)}}{\Gamma \Rightarrow \Delta} \text{ (Elim)} \qquad \mathcal{D}': \frac{A, B, \Gamma \Rightarrow \Delta, A, B}{\frac{B, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta} \text{ (Elim)}} \text{ (Elim)}$$



*Step* ( $r(A) > 2$ ). For the same reason as above, the only main case to be treated remains the one concerning inversion of rules. Examples:

$$\begin{array}{l}
 \mathcal{D}: \frac{\frac{A, \Gamma \Rightarrow \Delta, A, B}{A, \neg B, \Gamma \Rightarrow \Delta, A} (\neg\text{L})}{\neg B, \Gamma \Rightarrow \Delta} (\text{Elim}) \\
 \mathcal{D}': \frac{\frac{A, \Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, B} (\text{Elim})}{\neg B, \Gamma \Rightarrow \Delta} (\neg\text{L}) \\
 \mathcal{D}: \frac{\frac{A, B, \Gamma \Rightarrow \Delta, C, A}{A, C \rightarrow B, \Gamma \Rightarrow \Delta, A} (\rightarrow \text{L})}{C \rightarrow B, \Gamma \Rightarrow \Delta} (\text{Elim}) \\
 \mathcal{D}': \frac{\frac{A, B, \Gamma \Rightarrow \Delta, C, A}{B, \Gamma \Rightarrow \Delta, C} (\text{Elim})}{C \rightarrow B, \Gamma \Rightarrow \Delta} (\rightarrow \text{L})
 \end{array}$$

This concludes the proof of our theorem.  $\square$

#### 4. Denotational semantics

Abelian group logic has a semantics of proofs. We can easily obtain it by suitably trivializing some distinctions available in Girard's denotational semantics for linear logic (Girard, 1987; Troelstra, 1992). Remark that in Girard's semantics the multiplicative truth and falsity constants are interpreted by the same coherent space, and this may be seen as a shortcoming of this semantics. In our dialethic setting, however, this feature, far from being a drawback, is indeed a *desideratum*.

An *AGL-coherent space* is an ordered triple  $\mathcal{S} = \langle X, R, S \rangle$ , where  $X$  is a set and  $R, S$  are irreflexive symmetric relations on  $X$  s.t.  $R \cup S \cup I$  is a covering of  $X^2$  (" $I$ " denotes here the identity relation).

We define two operations on AGL-coherent spaces: *orthogonality* (unary) and *sum* (binary).

If  $\mathcal{S} = \langle X, R, S \rangle$ , then

$$\mathcal{S}^\perp \stackrel{\text{df}}{=} \langle X, S, R \rangle.$$

It is easily checked that  $\mathcal{S}^\perp$  is well-defined and  $\mathcal{S}^{\perp\perp} = \mathcal{S}$ .

If  $\mathcal{S} = \langle X, R, S \rangle$  and  $\mathcal{S}' = \langle X', R', S' \rangle$ , then

$$\mathcal{S} + \mathcal{S}' \stackrel{\text{df}}{=} \langle X \times X', R'', S'' \rangle,$$

where for any  $x, y, x', y' \in X$  we have:

$$(x, x') R'' (y, y') \quad \text{iff} \quad x R y \text{ or } x' R' y',$$

and

$$(x, x') S'' (y, y') \quad \text{iff} \quad x S y \text{ or } x' S' y'.$$

Are  $R''$  and  $S''$  well-defined? They are both irreflexive, since, for example,  $(x, x') R'' (x, x')$  iff  $x R x$  or  $x' R' x'$ , i.e. never; symmetry of  $R'', S''$  follows



likewise from symmetry of  $R, S, R', S'$ . It remains to prove that either  $(x, x') R'' (y, y')$  or  $(x, x') S'' (y, y')$  or  $(x = y \text{ and } x' = y')$ , but this is tantamount to: either  $x R y$  or  $x' R' y'$  or  $x S y$  or  $x' S' y'$  or  $(x = y \text{ and } x' = y')$ , and this follows from the properties of  $R, S, R', S'$ .

We can also define another binary operation on AGL-coherent spaces, i.e. *implication*: if  $\mathcal{S} = \langle X, R, S \rangle$  and  $\mathcal{S}' = \langle X', R', S' \rangle$ , then

$$\mathcal{S} \rightarrow \mathcal{S}' \stackrel{\text{df}}{=} \langle X \times X', R'', S'' \rangle,$$

where for any  $x, y, x', y' \in X$  we have:

$$(x, x') R'' (y, y') \quad \text{iff} \quad x S y \text{ or } x' R' y',$$

and

$$(x, x') S'' (y, y') \quad \text{iff} \quad x R y \text{ or } x' S' y'.$$

The following isomorphisms between AGL-coherent spaces are provable:

*De Morgan equalities*:

- (1)  $\mathcal{S} \rightarrow \mathcal{S}' \cong \mathcal{S}^\perp + \mathcal{S}'$ ;
- (2)  $(\mathcal{S} + \mathcal{S}')^\perp \cong \mathcal{S}^\perp + \mathcal{S}'^\perp$ .

*Commutativity isomorphisms*:

- (3)  $\mathcal{S} \rightarrow \mathcal{S}' \cong \mathcal{S}'^\perp \rightarrow \mathcal{S}^\perp$ ;
- (4)  $(\mathcal{S} \rightarrow \mathcal{S}')^\perp \cong \mathcal{S}' \rightarrow \mathcal{S}$ ;
- (5)  $\mathcal{S} + \mathcal{S}' \cong \mathcal{S}' + \mathcal{S}$ .

*Associativity isomorphisms*:

- (6)  $\mathcal{S} \rightarrow (\mathcal{S}' + \mathcal{S}'') \cong (\mathcal{S} \rightarrow \mathcal{S}') + \mathcal{S}''$ ;
- (7)  $\mathcal{S} + (\mathcal{S}' + \mathcal{S}'') \cong (\mathcal{S} + \mathcal{S}') + \mathcal{S}''$ .

As an example we prove (2) and (4), which are not valid in linear logic.

*Ad (2)*:  $(x, x') R (y, y')$  in  $(\mathcal{S} + \mathcal{S}')^\perp$  iff  $(x, x') S (y, y')$  in  $\mathcal{S} + \mathcal{S}'$  iff  $(x S y \text{ in } \mathcal{S} \text{ or } x' S' y' \text{ in } \mathcal{S}')$  iff  $(x R y \text{ in } \mathcal{S}^\perp \text{ or } x' R' y' \text{ in } \mathcal{S}'^\perp)$  iff  $(x, x') R (y, y')$  in  $\mathcal{S}^\perp + \mathcal{S}'^\perp$ . Dually, we can repeat the same reasoning for  $S$ .

*Ad (4)*:  $(x, x') R (y, y')$  in  $(\mathcal{S} \rightarrow \mathcal{S}')^\perp$  iff  $(x, x') S (y, y')$  in  $\mathcal{S} \rightarrow \mathcal{S}'$  iff  $(x R y \text{ in } \mathcal{S} \text{ or } x' S' y' \text{ in } \mathcal{S}')$  iff  $(x, x') R (y, y')$  in  $\mathcal{S}' \rightarrow \mathcal{S}$ . Again, the argument relative to  $S$  is symmetrical.

Let us now return to our one-sided calculus O-AGL of Section 3 and see how it can be interpreted within our semantical frame.

Let SP be an (at least denumerable) set of AGL-coherent spaces, containing the empty one (the empty set with two empty relations on it) and closed



under the operations of orthogonality and sum. If LIT is the set of literals of  $\mathcal{L}(\text{AGL})$  and  $L^c$  is the complementary literal of  $L$ , then given any mapping  $v^*: \text{LIT} \rightarrow \text{SP}$  s.t.  $v^*(L^c) = v^*(L)^\perp$ , a valuation  $v$  transforms sequents of O-AGL into members of SP according to the following clauses:

$$\begin{aligned} v(L) &= v^*(L); \\ v(A \oplus B) &= v(A) + v(B); \\ v(\Rightarrow A_1, \dots, A_n) &= v(A_1) + \dots + v(A_n). \end{aligned}$$

The valuation  $v$  is in itself far from sufficient, since what we are after is a semantics of proofs. So, if  $\mathcal{D}$  is a proof of  $\Rightarrow \Gamma$  in O-AGL, where  $v(\Rightarrow \Gamma) = \langle X, R, S \rangle$ , we interpret it by a mapping  $j$  s.t.  $j(\mathcal{D}) \subseteq X$ . We shall then show that, for every proof  $\mathcal{D}$  of  $\Rightarrow \Gamma$  in O-AGL,  $j(\mathcal{D})$  is a *clique* in  $v(\Rightarrow \Gamma)$ , i.e. that if  $x, y$  both belong to  $j(\mathcal{D})$ , then  $x R y$  or  $x = y$  in  $v(\Rightarrow \Gamma)$  (as a matter of convention, we stipulate that the only subset of  $\Rightarrow$  is a clique in the empty AGL-coherent space).

We inductively define  $j$  as follows (boldface letters stand for  $n$ -tuples):

- $j(\Rightarrow A_1, \neg A_1, \dots, A_n, \neg A_n) = \{\mathbf{x}_1, \mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_n : x_i \in X_i\}$ , where  $v(A_i) = \mathcal{S}_i = \langle X_i, R_i, S_i \rangle$  and  $v(\neg A_j) = v(A_j)^\perp$ ; for  $n = 0$ ,  $j(\Rightarrow A_1, \neg A_1, \dots, A_n, \neg A_n) = \emptyset$ .
- If  $\mathcal{D}$  proves  $\Rightarrow \Gamma, A$ ,  $\mathcal{D}'$  proves  $\Rightarrow \Delta, \neg A$ , and  $\mathcal{D}''$  proves  $\Rightarrow \Gamma, \Delta$  by a cut rule from  $\mathcal{D}$  and  $\mathcal{D}'$ , then  $j(\mathcal{D}'') = \{\mathbf{x}, \mathbf{x}' : \exists y(\mathbf{x}, y \in j(\mathcal{D}) \text{ and } \mathbf{x}', y \in j(\mathcal{D}'))\}$ .
- If  $\mathcal{D}$  proves  $\Rightarrow \Gamma, A, B$  and  $\mathcal{D}'$  proves  $\Rightarrow \Gamma, A \oplus B$  by a plus rule from  $\mathcal{D}$ , then  $j(\mathcal{D}') = \{\mathbf{x}, (y, z) : \mathbf{x}, y, z \in j(\mathcal{D})\}$ .

**THEOREM 4.** *If  $\mathcal{D}$  proves  $\Rightarrow \Gamma$  in O-AGL, then  $j(\mathcal{D})$  is a clique in  $v(\Rightarrow \Gamma)$ .*

**PROOF.** Induction on the length of  $\mathcal{D}$ . Since the inductive step is proved as in Girard (1987), we shall focus on the basis of the induction.

We have to show that either  $(x_1, x_1, \dots, x_n, x_n) R (y_1, y_1, \dots, y_n, y_n)$  or  $(x_1 = y_1 \text{ and } \dots \text{ and } x_n = y_n)$ , i.e. either  $x_1 R_1 y_1$  or  $x_1 \mathcal{S}_1 y_1$  or  $\dots$  or  $x_n R_n y_n$  or  $x_n \mathcal{S}_n y_n$  or  $(x_1 = y_1 \text{ and } \dots \text{ and } x_n = y_n)$ . But this follows easily from the fact that for each  $i \leq n$ ,  $R_i \cup S_i \cup I$  is a covering of  $X_i^2$  (if  $n = 0$ , we are OK by definition).

In fact, our axioms of the form  $\Rightarrow A_1, \neg A_1, \dots, A_n, \neg A_n$  are nothing else than generalized excluded thirds actually embodying a composition rule — a restricted form of weakening which is known to be sound in Girard's semantics (cf. e.g. Blute and Scott, 1996).  $\square$



## 5. Kripke-style semantics

Coherent space semantics is a semantics of proofs, not provability. Moreover, interesting logics which have denotational models, *in primis* linear logic, are usually shown to be sound, not complete, w.r.t. such an interpretation. So, as we have just seen, does also AGL. It is then desirable to have a more traditional semantics — different from the immediate algebraic one presented in Section 2 — which affords a proper completeness proof.

With such an aim in mind, we now proceed to introduce a relational semantics for the Hilbert-style system AGL, taken in its axiomatization with a primitive propositional constant  $\top$  (see above).

A *G-frame* is an ordered quadruple  $\mathcal{F} = \langle W, R, 1, * \rangle$ , where:

- $W$  is a nonempty set containing  $1$ .
- $R$  is 3-place relation on  $W$  satisfying:

$$\begin{aligned} R1 & \quad R1xx; \\ R2 & \quad Rxyz \Rightarrow Ryxz, \\ R3 & \quad R^2(xy)zw \Rightarrow R^2(xz)yw, \\ R4 & \quad Rxyz \ \& \ w \leq z \Rightarrow Rwyz. \end{aligned}$$

As usual (cf. Dunn, 1986),  $R^2(xy)zw$  is short for  $\exists u(Rxyu \ \& \ Ruzw)$ , whereas  $x \leq y$  stands for  $R1xy$ .

- $*$  is a 1-place operation on  $W$  satisfying:

$$\begin{aligned} *1 & \quad x^{**} = x, \\ *2 & \quad Rxyz \Rightarrow Rxz^*y^*, \\ *3 & \quad 1 \leq x \iff 1 \not\leq x^*. \end{aligned}$$

A *G-model* for AGL is a pair  $\mathcal{M} = \langle \mathcal{F}, \vDash \rangle$ , where  $\mathcal{F}$  is a G-frame and  $\vDash \subseteq W \times \text{FOR}$  is a relation satisfying:

$$\begin{aligned} \vDash 1 & \quad x \vDash p \ \& \ x \leq y \Rightarrow y \vDash p; \\ \vDash 2 & \quad x \vDash \top \iff 1 \leq x; \\ \vDash 3 & \quad x \vDash \neg A \iff x^* \not\vDash A; \\ \vDash 4 & \quad x \vDash A \rightarrow B \iff \forall yz(Rxyz \ \& \ y \vDash A \Rightarrow z \vDash B). \end{aligned}$$

A G-model  $\mathcal{M}$  is *unit-splitting* (or, for short, *u-splitting*) iff the forcing relation satisfies, for every  $A$ :

$$\vDash 5 \quad 1 \vDash A \iff 1^* \not\vDash A.$$



Of course we have to show that this last constraint is not incompatible with the previous ones, i.e. that there are u-splitting G-models. But this will be proved through our completeness theorem below.

$A$  is said to be *true in  $\mathcal{M}$*  ( $\mathcal{M} \models A$ ) iff  $1 \models A$ ; to be an *Abelian logical law* ( $\models_{K\text{-AGL}} A$ ) iff  $\mathcal{M} \models A$  for every u-splitting G-model  $\mathcal{M}$ .

LEMMA 3. *In every u-splitting G-model  $\mathcal{M}$ :*

- (i)  $x \models A \ \& \ x \leq y \Rightarrow y \models A$ . (Heredity)
- (ii)  $\forall x(x \models A \Rightarrow x \models B) \Rightarrow 1 \models A \rightarrow B$ . (Verification)
- (iii)  $1 \models A \iff 1 \models \neg A$ .
- (iv)  $1 \models A \iff 1^* \models \neg A$ .
- (v) *If, for some  $A$ ,  $1 \models A$  and  $x \models A$ , then  $1 \leq x$ .*

PROOF. (i)–(ii) See Anderson, Belnap and Dunn (1992), Dunn (1986) or Routley-Meyer (1972).

(iii) Left to right:  $1 \models A \Rightarrow (\models 5) \ 1^* \not\models A \Rightarrow (\models 3) \ 1 \models \neg A$ . Right to left:  $1 \models \neg A \Rightarrow (\models 5) \ 1^* \not\models \neg A \Rightarrow (\models 3, *1) \ 1 \models A$ .

(iv) Symmetrical.

(v) Suppose  $1 \models B$ ,  $x \models B$  and  $1 \not\leq x$ . By (\*1) and (\*3), this last implies  $1 \leq x^*$ .  $1 \models B$  implies instead  $1 \models \neg B$  by (iii) above. Hence, by (i),  $x^* \models \neg B$ , i.e., in virtue of ( $\models 3$ ) and (\*1),  $x \not\models B$ , which is a contradiction.  $\square$

THEOREM 5.  $\vdash_{\text{AGL}} A$  implies  $\models_{K\text{-AGL}} A$ .

PROOF. Standard induction on the length of derivations. In particular, (A1), (A2), (A5), (A6), and (R1) are verified as in Dunn (1986) or in Routley-Meyer (1972). We now argue for the rest of the postulates. Lemma 3(ii) will be used without special mention in what follows.

(A3'). Suppose  $x \models T$ . Then, by ( $\models 2$ ),  $1 \leq x$ . Now assume  $Rxyz$  and  $y \models A$ ; by (R4), then,  $y \leq z$ . Hence Lemma 3(i) implies  $z \models A$ .

(A3''). It is easy to check that  $1 \models A \rightarrow A$ . Suppose now  $x \models A \rightarrow A$ ; by Lemma 3(v) we conclude that  $1 \leq x$ , i.e.  $x \models T$ .

(A4'). Since  $1 \leq 1$ , by (\*3) it is not the case that  $1^* \leq 1^*$ , i.e. ( $\models 2$ )  $1^* \not\models T$ . By ( $\models 3$ ), then  $1 \models \neg T$ .

(A7'). Suppose  $x \models \neg T$ . Then  $x^* \not\models T$ , according to ( $\models 3$ ). Since  $1 \models T$  (as  $1 \leq 1$ ), by Lemma 3(i) it is not the case that  $1 \leq x^*$ . But this amounts to  $1 \leq x$  in virtue of (\*3). Hence ( $\models 2$ ),  $x \models T$ .

(A8'). This axiom is verified exactly like (A3''), since  $1 \models \neg T \rightarrow T$ , as we have just seen.  $\square$



THEOREM 6.  $\models_{K\text{-AGL}} A$  implies  $\vdash_{\text{AGL}} A$ .

PROOF. We prove the contrapositive: assuming that it is not the case that  $\vdash A$ , we show that there is an u-splitting G-model (the canonical model of AGL) s.t.  $1 \not\models A$ .

Our canonical model  $\mathcal{C} = \langle \langle W, R, 1, * \rangle, \models \rangle$  is constructed as follows:

- $W$  is the set of all AGL-theories;
- $Rxyz$  holds iff  $A \rightarrow B \in x$  and  $A \in y$  jointly imply  $B \in z$ ;
- $1$  is AGL;
- $x^* = \{A : \neg A \notin x\}$ ;
- $x \models A$  holds iff  $A \in x$ .

Since  $W$  contains AGL, it is of course a nonempty set containing  $1$ .

That  $R$  satisfies R1–R4 can be shown as in Dunn (1986), exploiting (A2) and Lemma 1(i)–(iii).

By (A5), (A6) and Lemma 1(iv)–(v) the operation  $*$  maps theories to theories and satisfies (\*1) and (\*2).

As to (\*3), we first prove that  $1 \leq x$  implies  $1 \not\leq x^*$ . Suppose  $1 \leq x$ , which is easily seen to mean that  $x$  extends AGL. We have to show that for some  $A$  and  $B$ ,  $B \notin x^*$  although  $\vdash A$  and  $\vdash A \rightarrow B$ . But  $x^* = \{C : \neg C \notin x\}$ ; now, take  $A = D \rightarrow D$  and  $B = E \rightarrow E$ ; you have  $\vdash A$ ,  $\vdash A \rightarrow B$  and  $\vdash \neg B$ , hence, by our hypothesis,  $\neg B \in x$ , i.e.  $B \notin x^*$ .

For the converse implication, suppose that there are  $A$  and  $B$  s.t.  $\vdash A$ ,  $\vdash A \rightarrow B$  and  $B \notin x^*$ , i.e.,  $\neg B \in x$ . It follows from  $\vdash A$  and  $\vdash A \rightarrow B$  that  $\vdash B$ , hence (Lemma 1(vi))  $\vdash \neg B$ . Then, in virtue of Lemma 2,  $x$  extends AGL, i.e.,  $1 \leq x$ .

Now we have to show that  $\models$  is a well-behaved forcing relation. The reader is once again referred to Dunn (1986) or Routley-Meyer (1972) for proofs that  $\models$  meets the criteria ( $\models$ 1), ( $\models$ 3) and ( $\models$ 4). As to ( $\models$ 2), suppose  $1 \leq x$ . But if  $x$  extends AGL, then it surely contains T in virtue of Lemma 1(xi). Conversely, suppose that  $x$  contains T. Now, by Lemma 1(xii), given any theorem  $A$  of AGL,  $T \rightarrow A$  is a theorem of AGL too; hence  $\in x$ . It follows that  $x$  extends AGL.

The very last thing left to prove is that  $\mathcal{C}$  is u-splitting. Remember that  $1^* = \{A : \neg A\}$ . But  $1 \models A$  implies  $1^* \not\models A$ , since  $\vdash A$  implies  $\vdash \neg A$  by Lemma 1(vi) and so  $\neg A \in 1$ , i.e.  $1^* \not\models A$ . Conversely, if  $\neg A \in 1$  then by Lemma 1(vi) again  $A \in 1$ . Hence  $1^* \not\models A$  implies  $1 \models A$ .

This concludes the proof of our theorem.  $\square$



$$\begin{array}{c}
\frac{A, A \Rightarrow A, A}{A \Rightarrow A, (A \rightarrow A)} \text{ } (\rightarrow\text{R}) \qquad \frac{A, A \Rightarrow A, A}{A \Rightarrow \neg A, A, A} \text{ } (\neg\text{R}) \\
\frac{A \Rightarrow A, (A \rightarrow A)}{A \Rightarrow A \vee \neg A, A \rightarrow A} \text{ } (\vee\text{R}) \qquad \frac{A \Rightarrow \neg A, A, A}{A \Rightarrow A \vee \neg A, A \rightarrow A, A} \text{ } (\vee\text{R}) \\
\frac{A \Rightarrow A \vee \neg A, A \rightarrow A}{A \Rightarrow A \vee \neg A, A \rightarrow A} \text{ } (\vee\text{R}) \qquad \frac{A \Rightarrow A \vee \neg A, A, A}{A \rightarrow A \Rightarrow A \vee \neg A, A, (A \rightarrow A)} \text{ } (\rightarrow\text{L}) \\
\frac{A, (A \rightarrow A) \Rightarrow A, (A \rightarrow A)}{\Rightarrow A, (A \rightarrow A) \rightarrow A, (A \rightarrow A)} \text{ } (\rightarrow\text{R}) \qquad \frac{A, (A \rightarrow A) \Rightarrow A \vee \neg A, A \rightarrow A}{A, (A \rightarrow A) \rightarrow A, (A \rightarrow A) \Rightarrow A \vee \neg A} \text{ } (\rightarrow\text{L}) \\
\frac{\Rightarrow A, (A \rightarrow A) \rightarrow A, (A \rightarrow A)}{\Rightarrow A \vee \neg A} \text{ } (\text{Cut})
\end{array}$$

Figure 1. Excluded third without contraction

## 6. From Abelian group logic to Abelian $l$ -group logic

### 6.1. Proof theory

If we extend our language with the connective “&” and add to AGL the standard semilattice axioms for conjunction and the adjunction rule:

- (A9)  $A \& B \rightarrow A$   
(A10)  $A \& B \rightarrow B$   
(A11)  $(A \rightarrow B) \& (A \rightarrow C) \rightarrow (A \rightarrow B \& C)$   
(R2)  $A, B \Rightarrow A \& B$

we get L-AGL, i.e. Abelian  $l$ -group logic. Disjunction can now be defined as usual via the De Morgan laws. The corresponding Gentzen-style system (GL-AGL) can be obtained from G-AGL by adding the standard rules for additive (lattice-theoretical) conjunction. Algebraic semantics (w.r.t.  $l$ -groups) for the new systems can easily be recovered from the one presented in Section 1, by stipulating that  $\rho(A \& B) = \rho(A) \wedge \rho(B)$ , and that  $A$  is  $\rho$ -true in  $\mathcal{A}$  iff  $0 \leq \rho(A)$ . It is possible to prove:

**THEOREM 7.**  $\vdash_{\text{L-AGL}} A$  iff  $\vdash_{\text{AL-AGL}} \Rightarrow A$  iff  $\vDash_{\text{AL-AGL}} A$ .  $\square$

The addition of lattice connectives to Abelian group logic has its pros and cons. One of the advantages is that it affords nice contraction-free proofs of classical tautologies essentially depending on contraction, e.g. the excluded third and the law of distribution (cf. Fig. 1 and Fig. 2 on p. 123).

Among the disadvantages, there is loss of cut elimination.

**THEOREM 8.** *GL-AGL is not cut-free.*



$$\begin{array}{c}
 \frac{\frac{B, A, A \Rightarrow A, A, B}{B, A, A \Rightarrow A, A, B \vee C} (\vee R)}{\frac{B, A, A \& (B \vee C) \Rightarrow A, A, B \vee C}{B, A, A \& (B \vee C) \Rightarrow A \& B, A, B \vee C} (\& L)} (\vee R) \quad \frac{\frac{B, A, B \vee C \Rightarrow B, A, B \vee C}{B, A, A \& (B \vee C) \Rightarrow B, A, B \vee C} (\& L)}{\frac{B, A, A \& (B \vee C) \Rightarrow A \& B, A, B \vee C}{B, A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C), A, B \vee C} (\vee R)} (\& R) \quad \frac{\frac{\frac{C, A, A \Rightarrow A, A, C}{C, A, A \Rightarrow A, A, B \vee C} (\vee R)}{\frac{C, A, A \& (B \vee C) \Rightarrow A, A, B \vee C}{C, A, A \& (B \vee C) \Rightarrow A \& C, A, B \vee C} (\& L)} (\& R)} (\& L) \quad \frac{\frac{C, A, B \vee C \Rightarrow C, A, B \vee C}{C, A, A \& (B \vee C) \Rightarrow A, AB \vee C} (\& L)}{\frac{C, A, A \& (B \vee C) \Rightarrow A \& C, A, B \vee C}{C, A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C), A, B \vee C} (\vee R)} (\& R) \\
 \frac{\frac{B \vee C \Rightarrow B \vee C}{\Rightarrow (B \vee C) \rightarrow (B \vee C)} (\rightarrow R)}{\frac{A \Rightarrow A}{\Rightarrow A \rightarrow A} (\rightarrow R)} (\rightarrow L) \quad \frac{\frac{B \vee C, A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C), A, B \vee C}{(B \vee C) \rightarrow (B \vee C), A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C), A} (\rightarrow L)}{\frac{A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C), A}{A \rightarrow A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C)} (\text{Cut})} (\rightarrow L) \\
 \frac{\frac{A \Rightarrow A}{\Rightarrow A \rightarrow A} (\rightarrow R)}{\frac{A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C)}{A \rightarrow A, A \& (B \vee C) \Rightarrow (A \& B) \vee (A \& C)} (\text{Cut})} (\rightarrow L)
 \end{array}$$

Figure 2. Distribution without contraction



PROOF. As we have seen, the excluded third is provable in our system. Were GL-AGL cut-free, by the subformula property its atomic instances should be provable using just the conjunction and negation rules and the rule (Ax). But, as it can be seen by inspection, there is no combination of such rules yielding the desired result. Hence GL-AGL is not cut-free.  $\square$

## 6.2. Phase semantics

The relational semantics of Section 5 is no good for L-AGL, since it rests upon characteristic properties of AGL that are not shared by the full system. However, we can provide a relational semantics for L-AGL by constructing appropriate *phase models* (cf. Girard, 1987).

Remember that a *phase structure* is a pair  $\mathcal{F} = \langle \mathcal{M}, \perp \rangle$ , where  $\mathcal{M} = \langle M, \cdot, 1 \rangle$  is an Abelian monoid and  $\perp$  is a distinguished subset of  $M$ . Instead of  $x \cdot y$  we shall usually write  $xy$ . We define, as usual, for  $X, Y \subseteq M$ :

$$\begin{aligned} XY &= \{xy : x \in X \ \& \ y \in Y\}, \\ X^\perp &= \{x : \forall y (y \in X \Rightarrow xy \in \perp)\}, \\ X \oplus Y &= (X^\perp Y^\perp)^\perp. \end{aligned}$$

The operation  $c(A) = A^{\perp\perp}$  is a closure operation on  $M$ . We define

$$C(M) = \{X \subseteq M : X = c(X)\}.$$

$\mathcal{F}$  is an *Abelian phase structure* iff:

- (a)  $\langle \perp, \cdot, 1 \rangle$  is a submonoid of  $\mathcal{M}$ ;
- (b) for every  $y \in M$  and every  $X \in C(M)$ ,  $X\{y\} \subseteq X$  implies  $\perp\{y\} \subseteq \perp$ .

LEMMA 4. *In every Abelian phase structure  $\mathcal{F} = \langle \mathcal{M}, \perp \rangle$ , for every  $X \in C(M)$ : (i)  $\perp\perp = \perp$ , (ii)  $\perp = \perp^\perp$ , (iii)  $X \oplus X^\perp = \perp$ , (iv)  $X \oplus \perp = X$ .*

PROOF. (i)  $\perp\perp \subseteq \perp$  since  $\langle \perp, \cdot, 1 \rangle$  is a submonoid of  $\mathcal{M}$ ; for the same reason  $1 \in \perp$ , hence if  $x \in \perp$ ,  $x = x \in \perp\perp$ .

(ii)  $\perp^\perp = \{x : \forall y (y \in \perp \Rightarrow xy \in \perp)\}$ . If  $x \in \perp$ , then, by (a),  $x \in \perp^\perp$ . Conversely, if  $\forall y (y \in \perp \Rightarrow xy \in \perp)$ , choose  $y = 1$  to obtain  $x \in \perp$ .

(iii) We have to prove that  $(X^\perp X)^\perp = \perp$ . The inclusion from right to left follows from standard phase semantics. Suppose now  $w \in (X^\perp X)^\perp$ , which by ordinary theory of phase semantics means  $X\{w\} \subseteq X$ . Then, by (b),  $\perp\{w\} \subseteq \perp$ , which amounts to  $w \in \perp$  by (ii) above.

(iv) As regards  $X \oplus \perp \subseteq X$ , suppose  $x \in X \oplus \perp$ , i.e.,  $\forall y (\forall z (z \in \perp \Rightarrow zy \in \perp) \Rightarrow xy \in X)$ . Let  $y = 1$ . Since the antecedent is trivially satisfied,



$x \in X$ . Conversely, if  $x \in X$  we have to prove that  $\forall y(\forall z(z \in \perp \Rightarrow zy \in \perp) \Rightarrow xy \in X)$ . By (ii) and (iii) above,  $X \oplus X^\perp = \perp^\perp$ , so we can replace  $\forall z(z \in \perp \Rightarrow zy \in \perp)$  by  $\forall z(z \in X \Rightarrow zy \in X)$ , whence our conclusion follows.  $\square$

Remark that an *affine phase structure* (Lafont, 1997) is a phase structure where  $X \in \mathbf{C}(M)$  implies  $\perp X \subseteq \perp$ . In any affine phase structure,  $X \in \mathbf{C}(M)$  implies  $\perp \subseteq X \subseteq \perp^\perp = M$ .

LEMMA 5. *If  $\mathcal{F}$  is an affine Abelian phase structure, then  $M = \perp$  and  $\mathbf{C}(M) = \{M\}$ .*

PROOF. Since  $M$  is closed,  $\perp M \subseteq \perp$ . But  $1 \in \perp$ , so for every  $x$  in  $M$ ,  $x = 1x \in \perp$ . Since  $X \in \mathbf{C}(M)$  implies  $M = \perp \subseteq X \subseteq \perp^\perp = M$ , then  $\mathbf{C}(M) = \{M\}$ .  $\square$

THEOREM 9. *Let  $\Pi \subseteq \mathbf{C}(M)$  be closed w.r.t.  $^\perp, \oplus$ , and contain  $\perp$ .<sup>1</sup> Then  $\mathcal{S} = \langle \Pi, \oplus, ^\perp, \perp, \subseteq \rangle$  is an Abelian *po*-group. If  $\Pi$  is closed w.r.t. intersection, then  $\mathcal{S}$  is an Abelian *l*-group.*

PROOF.  $\langle \Pi, \oplus \rangle$  is an Abelian *po*-semigroup by standard phase semantics. By Lemma 4(iv)  $\perp$  is a zero and by Lemma 4(iii)  $^\perp$  is an inverse operation. If  $\Pi$  is closed w.r.t. intersection, then by ordinary phase semantics  $\langle \Pi, \subseteq \rangle$  is a lattice where joins are represented by  $(X \cup Y)^{\perp\perp}$ .  $\square$

THEOREM 10. *Every Abelian *po*-group  $\mathcal{G} = \langle G, +, -, 0, \leq \rangle$  is isomorphic to an Abelian *po*-group  $\mathcal{S}$  of sets. Moreover, if  $\mathcal{G}$  is lattice-ordered,  $\mathcal{S}$  is lattice-ordered.*

PROOF. Let  $\mathcal{G} = \langle G, +, -, 0, \leq \rangle$  be an Abelian *po*-group. Then  $\mathcal{G}^* = \langle G, +, 0 \rangle$  is an Abelian monoid and  $\mathbf{I}(0) = \{x : x \leq 0\}$  is a distinguished subset of  $G$ . Hence  $\mathcal{F} = \langle \mathcal{G}^*, \mathbf{I}(0) \rangle$  is a phase structure. We can thus define on it operations of generalized product, orthogonality, and sum exactly as above. Notice that  $X^\perp = \{y : \forall x(x \in X \Rightarrow x + y \leq 0)\}$ ; since  $x + y \leq 0$  iff  $0 \leq -(x + y) = -x + -y$  iff  $x \leq -y$ , we have that  $X^\perp = \{y : \forall x(x \in X \Rightarrow x \leq -y)\}$ .

Let also  $\mathbf{I}(x) = \{y : y \leq x\}$  and  $\Pi = \{\mathbf{I}(x) : x \in G\}$ . Now we prove:

- (a)  $\Pi \subseteq \mathbf{C}(G)$ ;
- (b)  $\Pi$  contains  $\perp$ ;

---

<sup>1</sup> Hence  $\perp^\perp$ ; not necessarily, however,  $M$  and  $\emptyset$ .



- (c)  $\Pi$  is closed w.r.t.  $\perp$  and  $\oplus$  (also set-theoretical intersection if  $\mathcal{G}$  is lattice-ordered);
- (d)  $\mathcal{F}$  is an Abelian phase structure.

*Ad* (a):  $\mathbf{I}(y)^{\perp\perp} = \{x : \forall z(\forall w(w \leq y \Rightarrow w \leq -z) \Rightarrow x \leq -z)\}$ . We have to show that  $x \in \mathbf{I}(y)^{\perp\perp} \Rightarrow x \leq y$ . Choose  $z = -y$  to get the desired conclusion.

*Ad* (b):  $\perp = \mathbf{I}(0)$  belongs to  $\Pi$ .

*Ad* (c): We show that:  $\mathbf{I}(x)^{\perp} = \mathbf{I}(-x)$ ;  $\mathbf{I}(x) \oplus \mathbf{I}(y) = \mathbf{I}(x+y)$ ;  $\mathbf{I}(x) \cap \mathbf{I}(y) = \mathbf{I}(x \wedge y)$  (if binary meets exist everywhere). In the first place, remember that  $\mathbf{I}(x)^{\perp} = \{y : \forall z(z \leq x \Rightarrow z \leq -y)\}$ . If  $y \leq -x$ , i.e.  $x \leq -y$ , and  $z \leq x$ , then  $z \leq -y$  by transitivity. Conversely, if  $\forall z(z \leq x \Rightarrow z \leq -y)$ , choose  $z = x$  to get  $x \leq -y$ , i.e.  $y \leq -x$ .

As regards sum, by what we have just proved it is enough to show that  $\mathbf{I}(x+y) = (\mathbf{I}(-x)\mathbf{I}(-y))^{\perp}$ . Suppose then  $z \leq x+y$ ,  $z' \leq -x$ ,  $z'' \leq -y$ . We have to show that  $z' + z'' \leq -z$ . But  $z' + z'' \leq -x - y = -(x+y)$ . Hence  $z \leq x+y \leq -(z' + z'')$ . Contraposing,  $z' + z'' \leq -z$ . Conversely, suppose  $\forall w(w = w' + w'' \ \& \ w' \leq -x \ \& \ w'' \leq -y \Rightarrow w \leq -z)$ . Choose  $w = -x - y$ . You get  $-x - y \leq -z$ , that is  $z \leq -(-x - y) = x + y$ .

As for meets, if  $\langle \mathcal{G}, \leq \rangle$  is a lattice, then  $\mathbf{I}(x)$  is the principal  $l$ -ideal generated in  $\mathcal{G}$  by  $x$ , and we know from lattice theory that  $\mathbf{I}(x) \cap \mathbf{I}(y) = \mathbf{I}(x \wedge y)$ .

*Ad* (d): First of all, remark that  $0$  belongs to  $\mathbf{I}(0)$  and that  $x \leq 0$ ,  $y \leq 0$  imply  $x + y \leq 0 + 0 = 0$ . Moreover,  $X\{y\} \subseteq X$  implies  $\perp\{y\} \subseteq \perp$ , i.e. if  $z \leq x$  implies  $z \leq y = x$ , then  $z \leq 0$  implies  $z + y \leq 0$ . In fact, if  $z \leq 0$ , then  $z \leq x - x$ ; adding  $x$  on both sides,  $z + x \leq x$ . Hence  $z + x + y \leq x$ . Adding  $-x$  on both sides,  $z + y \leq 0$ .

So, by Theorem 9,  $\mathcal{S} = \langle \Pi, \oplus, \perp, \mathbf{I}(0), \subseteq \rangle$  is an Abelian  $po$ -group of sets. Moreover, the map turning  $x$  into  $\mathbf{I}(x)$  is clearly an order-preserving bijection and, as we have seen, preserves inverses, sums and meets. Hence  $\mathcal{G}$  is isomorphic to  $\mathcal{S}$ .  $\square$

Now, we can define a *relational model* for L-AGL as a triple  $\mathcal{R} = \langle \mathcal{F}, \Pi, v \rangle$ , where  $\mathcal{F} = \langle M, \cdot, 1, \perp \rangle$  is an Abelian phase structure (called *frame*),  $\Pi$  is a subset of  $\mathbf{C}(M)$  closed w.r.t. the phase-semantical operations defined as above, and  $v$  is a map assigning to every variable of the language of L-AGL an element of  $\Pi$ , extended to a homomorphism by the clauses:

$$\begin{aligned} v(\neg A) &= v(A)^{\perp}, \\ v(A \ \& \ B) &= v(A) \cap v(B), \\ v(A \rightarrow B) &= v(A)^{\perp} \oplus v(B). \end{aligned}$$



This allows to define a binary accessibility relation on  $M$  setting  $Rxy \stackrel{\text{df}}{\iff} xy \notin \perp$ . Notice that, like in the semantics for intuitionistic logic,  $x \in v(\neg A)$  iff  $\forall z(Rxz \Rightarrow z \notin v(A))$ .

We stipulate that  $A$  is *v-true* in  $\mathcal{R}$  ( $v \vDash_{\mathcal{R}} A$ ) iff  $1 \in v(A)$ ; that  $A$  is *true* in  $\mathcal{F}$  ( $\vDash_{\mathcal{F}} A$ ) iff  $v \vDash_{\mathcal{R}} A$  for every  $v$  on  $\mathcal{F}$ ; that  $A$  is *logically valid* ( $\vDash_{L\text{-AGL}} A$ ) iff  $v \vDash_{\mathcal{R}} A$  for every relational model  $\mathcal{R}$ .

**THEOREM 11.** *For every algebraic model  $\mathcal{A} = \langle \mathcal{G}, \rho \rangle$  there is a relational model  $\mathcal{R}^{\mathcal{A}} = \langle \mathcal{F}, \Pi, v \rangle$  such that for every formula  $A$ ,  $\rho \vDash_{\mathcal{A}} A$  iff  $v \vDash_{\mathcal{R}^{\mathcal{A}}} A$ . Likewise, for every relational model  $\mathcal{R} = \langle \mathcal{F}, \Pi, v \rangle$  there is an algebraic model  $\mathcal{A}^{\mathcal{R}} = \langle \mathcal{G}, \rho \rangle$  such that for every formula  $A$ ,  $v \vDash_{\mathcal{R}} A$  iff  $\rho \vDash_{\mathcal{A}^{\mathcal{R}}} A$ .*

**PROOF.** As regards the first statement, given  $\mathcal{A} = \langle \mathcal{G}, \rho \rangle$ , let  $\mathcal{F}$  be  $\langle G, +, 0, I(0) \rangle$ ,  $\Pi$  be the set of all principal  $l$ -ideals of  $\mathcal{G}$ , and  $v(A) = \{x \in G : x \leq \rho(A)\}$ . By Theorem 10,  $\mathcal{F}$  is a frame and it is easy to check (since the map  $x \mapsto I(x)$  preserves the operations of the  $l$ -group) that  $v(\neg A) = v(A)^{\perp}$ ,  $v(A \rightarrow B) = v(A)^{\perp} \oplus v(B)$  and  $v(A \& B) = v(A) \cap v(B)$ . Thus,  $v$  is well defined. Moreover,  $\rho \vDash_{\mathcal{A}} A$  iff  $0 \leq \rho(A)$  iff  $0 \in v(A)$  iff  $v \vDash_{\mathcal{R}^{\mathcal{A}}} A$ .

For the second part of the theorem, given  $\mathcal{R} = \langle \mathcal{F}, \Pi, v \rangle$  with  $\mathcal{F} = \langle M, \cdot, 1, \perp \rangle$ , let  $\mathcal{G} = \langle \Pi, \oplus, \perp, \perp, \subseteq \rangle$ , constructed as in Theorem 9, and  $\rho(A) = v(A)$ . By Theorem 9,  $\mathcal{G}$  is an  $l$ -group and  $\rho$  is well defined, according to our definitions. Moreover,  $v \vDash_{\mathcal{R}} A$  iff  $1 \in v(A)$  iff  $\perp = \perp^{\perp} \subseteq v(A)$  (by standard phase semantics, since  $v(A)$  is a  $c$ -closed subset of  $M$ ) iff  $\perp \subseteq \rho(A)$  iff  $\rho \vDash_{\mathcal{A}^{\mathcal{R}}} A$ .  $\square$

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